Meccanica (2022) 57:2553–2565
https://doi.org/10.1007/s11012-022-01589-3

The evolution of the law of random processes in the analysis of dynamic systems

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Received: 17 January 2022 / Accepted: 4 September 2022 / Published online: 23 September 2022

Abstract The paper presents a method for determining the evolution of the cumulative distribution function of random processes which are encountered in the study of dynamic systems with some uncertainties in the characterizing parameters. It is proved that these distribution functions are the solution of a partial differential equation, whose coefficients can be determined once the dynamic system has been solved, and whose numerical solution can be obtained with the finite difference method. Two simple problems are solved here both explicitly and numerically, then the obtained results are compared with each other.

Keywords Dynamic system · Probability · Random variable · Stochastic process

1 Introduction

The analysis of structures subjected to dynamic loads is now generally conducted using refined mechanical models and adequate numerical techniques. However, even when the problem is well-posed so that both the existence and the uniqueness of the solution are guaranteed, some parameters used to describe the geometric and mechanical characteristics of the structure and the external actions are generally affected by uncertainties that must be taken into account in the analysis. These input parameters are thus represented by random variables, which are defined on an appropriate probability space and whose law is supposed to be known. A stochastic process is thus obtained which is parameterized over time and has the Euclidean space as state space.

In many applications, it is necessary to determine the probability distribution of some output quantities. It is a question of determining the evolution over time of the law of some random process which is defined in terms of the solution of the dynamic system. This objective is generally achieved using the Monte Carlo method which, at least in principle, gives the possibility to consider complicated models and geometries without having to resort to unrealistic simplifying hypotheses. On the other hand, this method can require long computation times which may not be compatible with the required precision [1].

A different way to deal with the problem is to use the generalized density evolution equation, which is a consequence of the "principle of preservation of probability" [2]. This method leads to writing a linear PDE whose solution gives the probability density function for the quantities of interest. The coefficients of this equation at each instant depend on the value of the state variables and then can be obtained from the (deterministic) solution of the dynamic system. The method of the generalized density evolution equation has been implemented into the MADY code, which has already

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the routines for dynamic analysis of plane, three-dimensional, or beam and shell-based structures [3], and has been applied to the study of some masonry constructions [4, 5]. For these structures, the uncertainties have particular relevance, due to the constitutive characteristics of the material and the geometry.

In this paper, we deduce an equation similar to the one proposed in [2] but which allows calculating the cumulative distribution function directly, instead of the density function, of the chosen random variable. Indeed, while the former is always a locally integrable function, it may happen that the latter is defined only in a distributional sense. The deduction is made using classical theorems of the measure theory; although most notions discussed below can be presented in their intuitive sense, we prefer to give explicit definitions and proofs to avoid misunderstandings. Thus, both the approach proposed in this paper and the one proposed in [2] are rigorously justified. Moreover, with the help of the differential forms and the coarea formula, relations are deduced for the explicit computation of both the probability density function and the cumulative distribution function; these are especially useful if the number of random input parameters is greater than one.

Finally, two examples are presented in which the evolution of the probability distribution of a chosen output parameter is explicitly calculated. The solution is then compared with the numerical one obtained with the MADY code, which has meanwhile been updated with the new numerical procedures presented in this paper.

2 Background and notations

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Here Ω is the set of the *outcomes*, $\mathcal{A}$ is the σ-algebra on Ω made of all the *events* and $\mathbb{P} : \mathcal{A} \to [0,1]$ is a probability measure, i.e. a positive measure on $(\Omega, \mathcal{A})$ such that $\mathbb{P}(\Omega) = 1$. Moreover, let $\mathbb{R}^n$ and $\mathcal{B}(\mathbb{R}^n)$ be the n-dimensional Euclidean space and its corresponding Borel σ-algebra, respectively.

A map $\mathbf{X} : \Omega \to \mathbb{R}^n$ is said to be a (*vector*) *random variable* if it is $\mathcal{B}(\mathbb{R}^n)$-measurable, i.e. if

$$\{\mathbf{X} \in B\} \in \mathcal{A}, \text{ for each } B \in \mathcal{B}(\mathbb{R}^n),$$

where, as usual, $\{\mathbf{X} \in B\}$ denotes $\mathbf{X}^{-1}(B)$. The measure μ_X denotes the *law* of $\mathbf{X}$ or the *image* of measure $\mathbb{P}$ under $\mathbf{X}$ [6], i.e. the measure defined on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ by

$$\mu_X(B) = \mathbb{P}(\{\mathbf{X} \in B\}), \text{ for each } B \in \mathcal{B}(\mathbb{R}^n)$$

and the function $F_X : \mathbb{R}^n \to [0,1]$, defined by

$$F_X(a_1, a_2, \dots, a_n) = \mu_X(\{\mathbf{x} \in \mathbb{R}^n : -\infty < x_i \le a_i,\ i = 1, 2, \dots, n\}),$$

is called *cumulative distribution function* of $\mathbf{X}$.

Let $\mathcal{L}^n$ be the Lebesgue measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Measure μ_X is said to be *absolutely continuous* (with respect to $\mathcal{L}^n$) if $\mu_X(B) = 0$ for every set $B \in \mathcal{B}(\mathbb{R}^n)$) with $\mathcal{L}^n(B) = 0$. In this case, the Radon-Nikodym theorem [6] guarantees the existence of a positive integrable function p_X such that

$$\mu_X(B) = \int_B p_X(\mathbf{x}) d\mathbf{x} \tag{1}$$

for every $B \in \mathcal{B}(\mathbb{R}^n)$). Function p_X is called the *(joint) probability density function* of the vector random variable $\mathbf{X}$ and it holds

$$F_X(x_1, x_2, \dots, x_n) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} p_X(\xi_1, \xi_2, \dots, \xi_n) d\xi_1, d\xi_2, \dots, d\xi_n$$

or

$$p_X(x_1, x_2, \dots, x_n) = \frac{\partial F_X(x_1, x_2, \dots, x_n)}{\partial x_1 \partial x_2 \dots \partial x_n}. \tag{2}$$

If μ_X is not absolutely continuous, then its probability density function does not exist as an integrable function. However, it can be defined as a generalized function by interpreting (2) as a distributional derivative [7].

Let $\mathbf{X}$ be a vector random variable and $f : \mathbb{R}^n \to \mathbb{R}^m$ a Borel function; then $\mathbf{Y} = f \circ \mathbf{X}$ is again a vector random variable and, for each $B \in \mathcal{B}(\mathbb{R}^n)$,

$$\mu_Y(B) = \mu_X\big(f^{-1}(B)\big). \tag{3}$$

If $\mathbf{X}$ has a probability density function p_X, then

$$\mu_Y(B) = \mu_X\big(f^{-1}(B)\big) = \int_{f^{-1}(B)} p_X(\mathbf{x}) d\mathbf{x} \tag{4}$$

by (1) and (3). Moreover, if f is such that [8]

$$\mathcal{L}^m(B) = 0 \Rightarrow \mathcal{L}^n\big(f^{-1}(B)\big) = 0$$

for every Borel subset B of $\mathbb{R}^n$, then, in view of (4), μ_Y is (finite and) absolutely continuous with respect to $\mathcal{L}^m$ and thus it admits a probability density function p_Y. In the particular case when $n = m$ and f is a diffeomorphism, i.e. a bijective application such that both f and f^{-1} are continuously differentiable, then

$$p_Y(\mathbf{y}) = p_X\big(f^{-1}(\mathbf{y})\big)\left|Jf^{-1}(\mathbf{y})\right|$$

where Jf^{-1} denotes the Jacobian of f^{-1}.

More generally, suppose that there exists a countable family $(B_i)_{i\in I}$ of pairwise disjoint open sets of $\mathbb{R}^n$, such that (i) the event

$$B = \bigcup_{i\in I}\{\mathbf{Y} \in B_i\}$$

has probability $\mathbb{P}(B)$ equal to 1; (ii) for each $i \in I$, the restriction f_i of f to B_i is a diffeomorphism of B_i onto an open set $C_i \subset \mathbb{R}^n$. Under this assumption, define

$$(p_Y)_i(\mathbf{y}) = p_X\big(f_i^{-1}(\mathbf{y})\big)\left|Jf_i^{-1}(\mathbf{y})\right|\chi_{C_i}(\mathbf{y})$$

where

$$\chi_{C_i}(\mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{y} \in C_i \\ 0 & \text{otherwise} \end{cases}$$

is the indicator function of C_i. Then

$$p_{\mathbf{y}} = \sum_{i\in I}(p_{\mathbf{y}})_i.$$

A *stochastic process* $\{\mathbf{X}_t, t \in D\}$, with $D = [0, \bar{t}]$ a real interval, is a family of vector random variables indexed by a parameter t, defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and all with their values in $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, which is called the *state space*. By definition, for each $t \in D$, $\mathbf{X}_t$ is an $\mathcal{A}$-misurable function and, for each $\omega \in \Omega$, $\{\mathbf{X}_t(\omega), t \in D\}$ is a function defined in D that is called *sample function, realizzation* or *trajectory* of the process.

Let be $z \in \mathbb{R}$ and $f : \mathbb{R}^m \to \mathbb{R}$ a smooth function. Then, the set

$$\{z = f\} = \big\{\big\{\mathbf{x} \in \mathbb{R}^m : z - f(\mathbf{x}) = 0 \text{ and } |\nabla f(\mathbf{x})| > 0\},$$

if it is not empty, is a regular hypersurface of $\mathbb{R}^m$ whose orientation is determined by the unit normal vector $\mathbf{n} = \nabla f/|\nabla f|$. Let us put

$$f_{x_j} = \frac{\partial f}{\partial x_j}$$

and denote by μ_S the *volume form* on $\{z = f\}$ [9], i.e. the differential form

$$\mu_S = \sum_{j=1}^{n}\frac{(-1)^{j-1}}{|\nabla f|}f_{x_j}dx_1 \wedge \ldots \wedge d\hat{x}_j \wedge \ldots \wedge dx_n,$$

(where $d\hat{x}_j$ means "omit the factor dx_j"). Recalling that on the hypersurface $\{z = f\}$ we have

$$df = \sum_{j=1}^{n} f_{x_j}dx_j = 0,$$

it is easy to verify that, for each $j = 1, \ldots, n$ it turns out that [7]

$$\mu_S = \frac{(-1)^{j-1}|\nabla f|}{f_{x_j}}dx_1 \wedge \ldots \wedge d\hat{x}_j \wedge \ldots \wedge dx_n, \tag{5}$$

everywhere $f_{x_j} \neq 0$ holds.

For each $\alpha \leq n$, let $\mathcal{H}^\alpha$ be the Hausdorff measure in $\mathbb{R}^n$ [10], defined in such a way that $\mathcal{H}^n = \mathcal{L}^n$. Then for every function g which is integrable on $\{z = f\}$ we have

$$\int_{\{z=f\}} g\,d\mathcal{H}^{n-1} = \int_{\{z=f\}} g\,\mu_S. \tag{6}$$

The following useful result is a consequence of the *coarea formula*. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function such that $|\nabla f(\mathbf{x})| > 0$, and $g : \mathbb{R}^m \to \mathbb{R}$ be an integrable function. Moreover, let us put

$$\{z > f\} = \{\mathbf{x} \in \mathbb{R}^m : z - f(\mathbf{x}) > 0\}.$$

Then

$$\int_{\{z>f\}} g\,d\mathbf{x} = \int_{-\infty}^{z} d\zeta \int_{\{\zeta=f\}} \frac{g}{|\nabla f|}d\mathcal{H}^{m-1} \tag{7}$$

and, in particular,

$$\frac{d}{dz}\int_{\{z>f\}} g\,d\mathbf{x} = \int_{\{z=f\}} \frac{g}{|\nabla f|}d\mathcal{H}^{m-1}. \tag{8}$$

3 Stochastic dynamic system

The equation of motion of a body, discretized by the finite element method with respect to the space variable is reduced to a system of ODEs

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{f}(\dot{\mathbf{y}}, \mathbf{y}) = \mathbf{B}(\mathbf{y}, t)\xi(t), \qquad \mathbf{y}(t_0) = \mathbf{y}_0, \qquad \dot{\mathbf{y}}(t_0) = \dot{\mathbf{y}}_0, \tag{9}$$

where $t \in D$ is the time and $\mathbf{y}, \dot{\mathbf{y}}$ and $\ddot{\mathbf{y}}$ are the displacement, velocity and acceleration vector, respectively, which are defined on D and take their values in $\mathbb{R}^n$. $\mathbf{M}$ is the mass matrix, $\mathbf{f}$ is the internal force vector (including damping and restoring force), $\mathbf{B}$ is the input force influence matrix, $\boldsymbol{\xi}$ is the external excitation vector and $\mathbf{y}_0$ and $\dot{\mathbf{y}}_0$ are the initial displacement and velocity vector, respectively. By introducing the *state vector*

$$\mathbf{x} = \left\{ \begin{array}{c} \dot{\mathbf{y}} \\ \mathbf{y} \end{array} \right\},$$

Equation (9) can be rewritten as

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}, t) + \mathbf{B}(\mathbf{x}, t)\xi(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{10}$$

where

$$\mathbf{A}(\mathbf{x}, t) = \begin{pmatrix} -\mathbf{M}^{-1}\mathbf{f}(\mathbf{x}) \\ \dot{\mathbf{y}} \end{pmatrix}, \qquad \mathbf{B}(\mathbf{x}, t) = \begin{pmatrix} \mathbf{M}^{-1}\mathbf{B}(\mathbf{x}, t) \\ \mathbf{0} \end{pmatrix}.$$

If randomness is present, coming from the initial conditions, the excitations, or the properties of the system, a *random state equation* can be written as [2]

$$\dot{\mathbf{x}} = \mathbf{G}(\boldsymbol{\theta}, \mathbf{x}_0, t) \tag{11}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots \theta_m) \in \Lambda \subset \mathbb{R}^m$ is the vector of all random parameters, i.e. the values assumed by the vector random variable $\boldsymbol{\Theta} : \Omega \to \mathbb{R}^m$, that is supposed to be time-independent and to have a (joint) probability density function $p_{\boldsymbol{\Theta}}$.

If the (deterministic) problem (10) is well posed, then for each choice of $\boldsymbol{\theta}$ and $\mathbf{x}_0$, Eq. (11) has one and only one solution

$$\mathbf{x} = \mathbf{H}(\boldsymbol{\theta}, \mathbf{x}_0, t), \qquad \mathbf{H}(\boldsymbol{\theta}, \mathbf{x}_0, 0) = \mathbf{x}_0$$

with $\mathbf{H} : \Lambda \times \mathbb{R}^{2n} \times D \to \mathbb{R}^{2n}$ a suitable smooth function.

Below, for sake of simplicity, we omit to explicitly indicate the dependence on $\mathbf{x}_0$ and write

$$\mathbf{x}_t = \mathbf{H}(\boldsymbol{\theta}, t)$$

to denote the solution of (11).

In applications, we are interested in considering stochastic processes of the type

$$Z_t(\boldsymbol{\theta}) = Z(\boldsymbol{\theta}, t) = \psi \circ \mathbf{H}(\boldsymbol{\theta}, t),$$

with

$$Z(\boldsymbol{\theta}, 0) = \psi(\mathbf{x}_0) = z_0, \tag{12}$$

and in determining the evolution of the law of Z_t. Here $\psi : \mathbb{R}^{2n} \to \mathbb{R}$ is a deterministic smooth function. To this aim, for each $t \in D$, let us consider the family of measures $\{\mu_t^\theta : \boldsymbol{\theta} \in \Lambda\}$ which are defined on $\mathbb{R}$ by

$$\mu_t^\theta(B) = \chi_B \circ Z_t = \begin{cases} 1 & \text{if } Z_t(\boldsymbol{\theta}) \in B, \\ 0 & \text{otherwise}, \end{cases} \tag{13}$$

for every Borel subset B of $\mathbb{R}$, with χ_B the indicator function of B. Measure μ_t^θ is the law of the conditional probability of Z_t, given $\boldsymbol{\Theta}$, and sometimes it is denoted by $p_{Z\boldsymbol{\Theta}}(z|\boldsymbol{\Theta} = \boldsymbol{\theta}, t)$. Indeed, once $\boldsymbol{\theta}$ and t are fixed, μ_t^θ is the Dirac measure on $\mathbb{R}$, concentrated in the point $Z_t(\boldsymbol{\theta})$. The cumulative distribution function of μ_t^θ is the real function

$$F_t^\theta(z) = \mu_t^\theta((-\infty, z]) = \begin{cases} 1 & \text{if } z \geq Z_t(\boldsymbol{\theta}), \\ 0 & \text{otherwise}. \end{cases} \tag{14}$$

Let $\mathcal{C}_0$ be the space of the continuous real functions that are compactly supported in $\mathbb{R}$, with the maximum norm $|\cdot|_{\mathcal{C}_0}$. For every function $f \in \mathcal{C}_0$ we have

$$\int_{\mathbb{R}} f(z)\mu_t^\theta(dz) = f(Z_t(\boldsymbol{\theta})) \tag{15}$$

(this is true for the indicator functions of measurable sets by (13) and then for simple functions so that the general case follows from an approximation argument). Therefore, the map $\boldsymbol{\theta} \to \int_{\mathbb{R}} f(z)\mu_t^\theta(dz)$ is measurable on Λ.

Proposition *(i) For fixed $t \in D$, there is one and one measure μ_t on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} f(z)\mu_t(dz) = \int_\Lambda p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} \int_{\mathbb{R}} f(z)\mu_t^\theta(dz) \tag{16}$$

for each $f \in \mathcal{C}_0$.

(ii) μ_t *is the law of* Z_t, *i.e., for every Borel subset B of* $\mathbb{R}$,

$$\mu_t(B) = \mathbb{P}\big(Z_t^{-1}(B)\big).$$

Proof (i) Let be

$$\Phi(f) = \int_\Lambda p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} \int_{\mathbb{R}} f(z)\mu_t^\theta(dz).$$

Then, noting that

$$\int_{\mathbb{R}} \mu_t^\theta(dz) = \mu_t^\theta(\mathbb{R}) = 1 \qquad \text{and} \qquad \int_\Lambda p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} = \mathbb{P}(\Omega) = 1$$

we obtain

$$\begin{aligned}|\Phi(f)| &\le \int_\Lambda p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} \int_{\mathbb{R}} |f(z)|\mu_t^\theta(dz) \\ &\le |f|_{\mathcal{C}_0} \int_\Lambda p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} \int_{\mathbb{R}} \mu_t^\theta(dz) = |f|_{\mathcal{C}_0}.\end{aligned}$$

Thus, Φ is a linear and continuous functional on $\mathcal{C}_0$ and (i) follows from the Riesz representation theorem [11].

(ii) For every Borel subset B of $\mathbb{R}$ we have

$$\begin{aligned}\mu_t(B) &= \int_{\mathbb{R}} \chi_B d\mu_t(dz) = \int_\Lambda p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} \int_{\mathbb{R}} \chi_B(z)\mu_t^\theta(dz) \\ &= \int_\Lambda \chi_B(Z_t)(\boldsymbol{\theta}))p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} \\ &= \int_\Lambda p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\chi_{Z_t^{-1}(B)}(\boldsymbol{\theta})d\boldsymbol{\theta}) \\ &= \int_{Z_t^{-1}(B)} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} = \mathbb{P}(Z_t^{-1}(B)),\end{aligned} \tag{17}$$

where the second identity follows from (16), the third follows from (15) and the fourth follows from the relation $\chi_B \circ Z_t = \chi_{Z_t^{-1}(B)}$. □

If we denote by $F_Z(z,t)$ the cumulative distribution function of μ_t, with the help of (16) and (14) we can write

$$\begin{aligned}F_Z(z,t) = \mu_t((-\infty,z]) &= \int_{\mathbb{R}} \chi_{(-\infty,z]}\mu_t(z)(dz) \\ &= \int_\Lambda p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} \int_{\mathbb{R}} \chi_{(-\infty,z]}\mu_t^\theta(dz) =\end{aligned}$$

$$\begin{aligned}\int_\Lambda p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\mu_t^\theta((-\infty,z])d\boldsymbol{\theta} &= \int_\Lambda p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\chi_{\{Z_t\le z\}}d\boldsymbol{\theta} \\ &= \int_{\{Z_t\le z\}} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta},\end{aligned} \tag{18}$$

where

$$\{Z_t \le z\} = \{\boldsymbol{\theta} \in \Lambda : z - Z_t(\boldsymbol{\theta}) \ge 0\}. \tag{19}$$

Moreover, under the hypothesis that, for each $t \in D$, Z_t is a smooth function with

$$\inf_{\theta\in\Lambda} |\nabla Z_t(\boldsymbol{\theta})| > 0, \tag{20}$$

from (18) to (7) it follows

$$F_Z(z,t) = \int_{-\infty}^z d\zeta \int_{\{Z_t=\zeta\}} \frac{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{|\nabla Z_t(\boldsymbol{\theta})|} d\mathcal{H}^{m-1}(\boldsymbol{\theta}), \tag{21}$$

where

$$\{Z_t = \zeta\} = \{\boldsymbol{\theta} \in \Lambda : \zeta - Z_t(\boldsymbol{\theta}) = 0\}.$$

In many applications of interest, for each $t \in D$ and $B \in \mathcal{B}(\mathbb{R})$, Z_t is such that

$$\mathcal{L}(B) = 0 \Rightarrow \mathcal{L}^m(Z_t^{-1}(B)) = 0,$$

so that in view of (17), it also results $\mu_t(B) = 0$ and therefore μ_t is absolutely continuous with respect to $\mathcal{L}$ [and $F_Z(\cdot,t)$ is an absolutely continuous real function] [6]. Then, by the Radon-Nikodym theorem there exists a probability density function $p_Z(z,t)$ of μ_t, i.e. for every integrable function f it holds

$$\int_{\mathbb{R}} f(z)\mu_t(dz) = \int_{\mathbb{R}} f(z)p_Z(z)dz.$$

Thus, in this case from (21) to (8), we deduce

$$p_Z(z,t) = \frac{dF_Z(z,t)}{dz} = \int_{\{Z_t=z\}} \frac{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{|\nabla Z_t(\boldsymbol{\theta})|} d\mathcal{H}^{m-1}(\boldsymbol{\theta}). \tag{22}$$

For all z and t, $\{Z_t = z\}$ is a regular hypersurface of Λ by (20). Moreover, in view of (5) and (6), we have

$$\begin{aligned}&\int_{\{Z_t=z\}} \frac{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{|\nabla Z_t(\boldsymbol{\theta})|} d\mathcal{H}^{m-1}(\boldsymbol{\theta}) \\ &\quad = \int_{\{Z_t=z\}} (-1)^{j-1} \frac{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{\frac{\partial Z_t}{\partial \theta_j}} d\theta_1 \wedge \ldots \wedge d\hat{\theta}_j \wedge \ldots \wedge d\theta_m\end{aligned} \tag{23}$$

if on the hypersurface $\{Z_t = z\}$ the inequality $\frac{\partial Z_t}{\partial \theta_j} \neq 0$ holds.

As will be shown later in the examples, Eqs. (18), (21) and (22), (23) can be used for the explicit calculation of the cumulative distribution function and the probability density function, respectively, even if the dimension of Λ is greater than one.

4 The evolution of the law of Z_t

In [2] the authors obtain a linear PDE in the unknown

$$p_{Z\Theta}(z, \boldsymbol{\theta}, t) = p_{Z\Theta}(z|\Theta = \boldsymbol{\theta}, t)p_\Theta(\boldsymbol{\theta})$$

which once solved allows to determine the probability density function $p_Z(z, t)$ of Z_t, by integrating $p_{Z\Theta}$ over Λ, with respect to $\boldsymbol{\theta}$. Below we propose a similar method to directly determine the cumulative distribution function $F_Z(z, t)$ of Z_t. Let's start with some notation.

Let be $N = \mathbb{R} \times \Lambda \times D \subset \mathbb{R}^{m+2}$, $G(z, \boldsymbol{\theta}, t) = z - Z(\boldsymbol{\theta}, t)$ and χ_K be the indicator function of the region

$$K = \{(z, \boldsymbol{\theta}, t) \in N : G \geq 0\}$$

that is

$$\chi_K(z, \boldsymbol{\theta}, t) = \begin{cases} 1 & \text{if } z - Z(\boldsymbol{\theta}, t) \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

(note that, for each pair z and t, region $\{Z_t \leq z\}$ that has been defined in (19) is a subset of K). Let us denote by $\mathcal{D}$ the space of infinitely differentiable real functions with compact support in N.

Consider the two distributions [7] which, for every $\phi \in \mathcal{D}$, are defined in N by

$$< \chi_K, \phi >= \int_N \chi_K(\mathbf{x})\phi((\mathbf{x})d\mathbf{x} = \int_K \phi(\mathbf{x})d\mathbf{x},$$

with $\mathbf{x}$=$(z, \boldsymbol{\theta}, t)$, and

$$< \delta(G), \phi >= \int_N \delta(G(\mathbf{x}))\phi(\mathbf{x})d\mathbf{x} = \int_{\mathcal{I}} \frac{\phi}{|\nabla G|} d\mathcal{H}^{m+1},$$

where

$$\mathcal{I} = \{(z, \boldsymbol{\theta}, t) \in N : G(z, \boldsymbol{\theta}, t) = 0\},$$

is a regular hypersurface, because in N we have $|\nabla G| = \sqrt{1 + |\nabla Z|^2} \geq 1$. Note that while the first distribution is regular, the second is concentrated on a set that is negligible with respect to the Lebesgue measure.

It can be proved [7] that $\delta(G)$ is the distributional derivative of χ_K, i.e. the distribution such that

$$\frac{\partial \chi_K}{\partial z} = \delta(G) \qquad \frac{\partial \chi_K}{\partial \theta_j} = -\frac{\partial Z}{\partial \theta_j}\delta(G) \qquad \frac{\partial \chi_K}{\partial t} = -\frac{\partial Z}{\partial t}\delta(G) \tag{25}$$

or also

$$\nabla \chi_K = \delta(G)\nabla G.$$

From $(25)_1$ to $(25)_3$ we deduce the PDE

$$\frac{\partial \chi_K(z, \boldsymbol{\theta}, t)}{\partial t} + \frac{\partial Z(\boldsymbol{\theta}, t)}{\partial t}\frac{\partial \chi_K(z, \boldsymbol{\theta}, t)}{\partial z} = 0 \tag{26}$$

with the initial conditions

$$\chi_K(z, \boldsymbol{\theta}, 0) = \begin{cases} 1 & \text{if } z \geq z_0 \\ 0 & \text{otherwise} \end{cases}$$

which follows from (24) to (12).

Once χ_K has been calculated, F_Z is obtained from the relation

$$F_Z(z, t) = \int_\Lambda \chi_K(z, \boldsymbol{\theta}, t)p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta}. \tag{27}$$

In a similar way, [7] it is defined the distributional derivative δ' of δ and it is shown that it results

$$\frac{\partial \delta(G)}{\partial z} = \delta'(G), \qquad \frac{\partial \delta(G)}{\partial \theta_j} = -\frac{\partial Z}{\partial \theta_j}\delta'(G), \qquad \frac{\partial \delta(G)}{\partial t} = -\frac{\partial Z}{\partial t}\delta'(G)$$

from which the PDE that has been proposed in [2] can be obtained, i.e.

$$\frac{\partial p_{Z\boldsymbol{\Theta}}(z, \boldsymbol{\theta}, t)}{\partial t} + \frac{\partial Z(\boldsymbol{\theta}, t)}{\partial t}\frac{\partial p_{Z\boldsymbol{\Theta}}(z, \theta, t)}{\partial z} = 0. \tag{28}$$

The corresponding initial condition is

$$p_{Z\boldsymbol{\Theta}}(z, \boldsymbol{\theta}, 0) = \delta(z - z_0)p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$$
$$\text{or} \qquad p_{Z\boldsymbol{\Theta}}(z, \theta, 0) = p_{Z_0}(z)p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}),$$

depending on whether z_0 is deterministic or not. Once $p_{Z\boldsymbol{\Theta}}$ has been calculated, p_Z is obtained from the relation

$$p_Z(z,t) = \int_\Lambda p_{Z\boldsymbol{\Theta}}(z,\boldsymbol{\theta},t)d\boldsymbol{\theta}. \tag{29}$$

Thus χ_K and $p_{Z\boldsymbol{\Theta}}$ satisfy the same distributional PDE but with different initial conditions.

In applications, system (11) and Eqs. (26) [or (28)] are both solved numerically, after which $F_Z(z,t)$ is determined by the relation (27) [or $p_Z(z,t)$ is determined by (29)]. Two examples are presented below where the whole procedure can be followed explicitly. Let's briefly mention the solution technique for Eq. (26). The analogue for Eq. (28) is dealt with in detail in [2].

Using the method of characteristics [12] for this purpose, we can write the following system of ODEs, with the corresponding initial conditions

$$\begin{array}{ll} \frac{dz}{ds} = \frac{\partial Z(\boldsymbol{\theta},t)}{\partial t} & z(0)=\tau, \\ \frac{dt}{ds} = 1 & t(0) = 0, \\ \frac{d\chi_K}{ds} = 0 & \chi_K(\tau,\boldsymbol{\theta},0) = \begin{cases} 1 & \text{if } \tau \geq z_0, \\ 0 & \text{otherwise}. \end{cases} \end{array} \tag{30}$$

From $(30)_2$ we obtain $t = s$, which in view of $(30)_1$ provides the following two implications

$$\begin{array}{c} \frac{dz}{dt} = \frac{\partial Z(\boldsymbol{\theta},t)}{\partial t} \Rightarrow \\ z(s,\tau) = Z(\boldsymbol{\theta},s) + \tau - Z(\boldsymbol{\theta},0) \Rightarrow \\ \tau = z - Z(\boldsymbol{\theta},t) + Z(\boldsymbol{\theta},0). \end{array} \tag{31}$$

From $(30)_3$ to (31), taking into account that $Z(\boldsymbol{\theta},0) = z_0$ by (12), we deduce

$$\chi_K(z,\boldsymbol{\theta},t) = \chi_K(\tau,\boldsymbol{\theta},0) = \begin{cases} 1 & \text{if } \tau \geq Z(\boldsymbol{\theta},0), \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1 & \text{if } z - Z(\boldsymbol{\theta},t) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, from (27) we obtain

$$F_Z(z,t) = \int_\Lambda \chi_K(z,\boldsymbol{\theta},t)p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} = \int_{\{Z_t \leq z\}} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} \tag{32}$$

which coincides with (18).

5 Examples

5.1 The motion of a material point

Let us consider the rectilinear motion of a material point whose velocity is proportional to the traveled distance. The proportionality coefficient and the initial position are expressed by the two independent positive-valued random variables Θ_1 and Θ_2, respectively. Distance x is measured in meters and time t in seconds, and a superimposed dot indicates derivation with respect to time. We can write

$$\dot{x} = \theta_1 x, \qquad x(0) = \theta_2$$

and

$$\boldsymbol{\theta} = (\theta_1,\theta_2) \in \Lambda = \mathbb{R}^+ \times \mathbb{R}^+, \qquad t \in D = [0,\infty).$$

By separating the variables we obtain

$$\ln\left(\frac{x}{\theta_2}\right) = \theta_1 t,$$

from which, assuming that ψ is the identity function,

$$Z(\boldsymbol{\theta},t) = x(\boldsymbol{\theta},t) = \theta_2 e^{\theta_1 t} \qquad \text{and} \qquad \frac{\partial Z}{\partial t} = \theta_1\theta_2 e^{\theta_1 t}$$

follow. As

$$\nabla Z(\boldsymbol{\theta},t) = \left(\frac{\partial Z}{\partial \theta_1}, \frac{\partial Z}{\partial \theta_2}\right) = e^{\theta_1 t}(\theta_2 t, 1),$$

from (21) we obtain the cumulative distribution function of Z_t

$$F_Z(z,t) = \int_{-\infty}^{z} d\zeta \int_{\{Z_t=\zeta\}} \frac{p_{\boldsymbol{\Theta}}}{e^{\theta_1 t}\sqrt{1+\theta_2^2 t^2}} d\mathcal{H}^1(\boldsymbol{\theta}),$$

where $p_{\boldsymbol{\Theta}}$ is the joint probability density function of the random variables Θ_1 and Θ_2.

Alternatively, relation (18) can be used,

$$F_Z(z,t) = \int_{\{Z_t \leq z\}} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta}.$$

Particularly, we assume that Θ_1 and Θ_2 are two independent variables, each uniformly distributed on the interval [0, 1], so that $p_{\boldsymbol{\Theta}}$ is the indicator function of the square $Q = [0,1] \times [0,1]$. Then, we obtain (Fig. 1)

$$F_Z(z,t) = \int_{\{Z_t \le z\}\cap Q} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}$$
$$= \begin{cases} \int_0^1 z e^{-\theta_1 t} d\theta_1 = \frac{z}{t}(1-e^{-t}) & \text{if } 0 < z \le 1, \\ \ln z/t + \int_{\frac{\ln z}{t}}^1 z e^{-\theta_1 t} d\theta_1 = \frac{1}{t}(\ln z + 1 - z e^{-t}) & \text{if } 1 < z \le e^t, \\ 1 & \text{if } z > e^t, \end{cases}$$

from which, deriving with respect to z, we deduce the probability density function

$$p_Z(z,t) = \begin{cases} \frac{1}{t}(1-e^{-t}) & \text{if } 0 < z \le 1, \\ \frac{1}{t}(\frac{1}{z} - e^{-t}) & \text{if } 1 < z \le e^t, \\ 0 & \text{if } z > e^t. \end{cases} \quad (33)$$

Obviously, the same result is obtained by directly using the relations (22) and (23). Thus for example, for $0 < z \le 1$, we have (Fig. 1)

$$p_Z(z,t) = \int_{\{Z_t = z\}} \frac{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{\nabla Z_t(\boldsymbol{\theta})} d\mathcal{H}^1(\theta) = \int_{\{Z_t=z\}} \frac{d\theta_2}{\frac{\partial Z_t}{\partial \theta_1}}$$
$$= \int_{ze^{-t}}^{z} \frac{d\theta_2}{tz} = \frac{1}{t}(1-e^{-t}),$$

according to $(33)_1$.

Equation (26) and the corresponding initial conditions take the form

$$\frac{\partial \chi_K(z,\boldsymbol{\theta},t)}{\partial t} + \theta_1\theta_2 e^{\theta_1 t}\frac{\partial \chi_K(z,\boldsymbol{\theta},t)}{\partial z} = 0$$

and

$$\chi_K(z,\boldsymbol{\theta},0) = \begin{cases} 1 & \text{if } z \ge \theta_2, \\ 0 & \text{otherwise}, \end{cases}$$

respectively.

The numerical solution can be accomplished by means of the following steps.

Step 1 Construct a partition of $\Lambda = [0,1]\times[0,1]$, in a finite number N of squares Λ_i with center $\boldsymbol{\theta}^i = (\theta_1^i, \theta_2^i)$ $(i = 1,\dots,N)$, and assign to each $\boldsymbol{\theta}^i$ the corresponding value $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}^i) = \frac{1}{N}$.

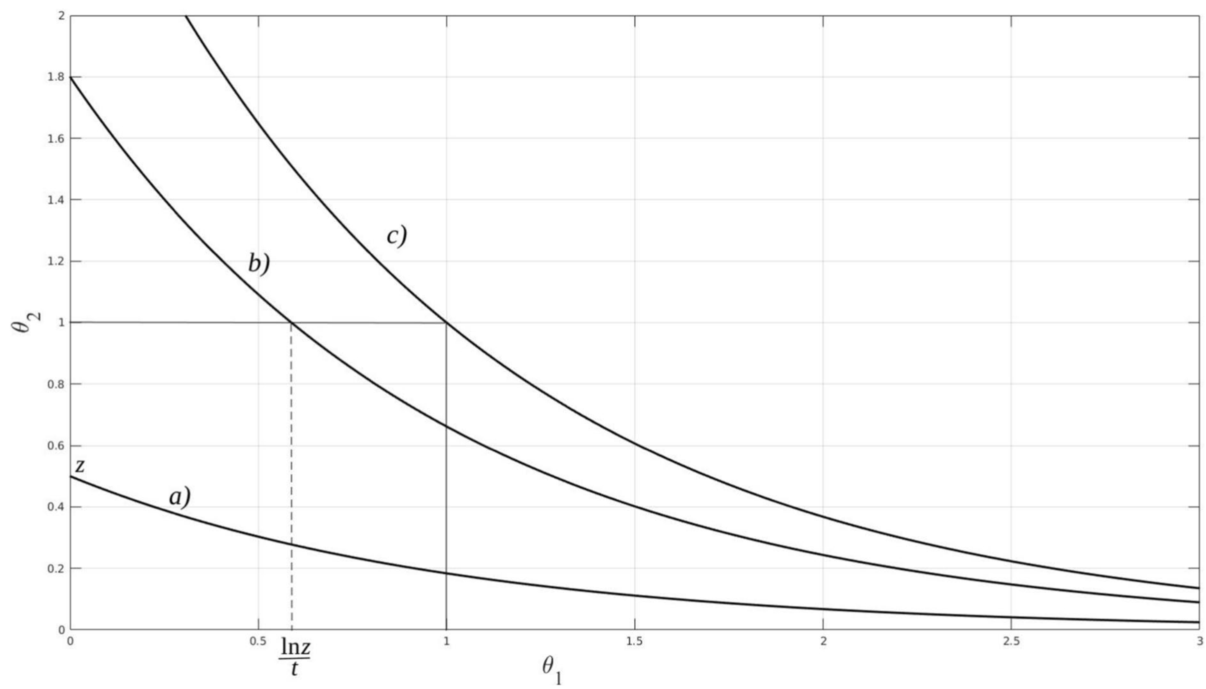

Fig. 1 Graph of $z = Z_t$, for **a** $0 \le z \le 1$, **b** $1 \le z \le e^t$ and **c** $z = e^t$. (Example 1)

Step 2 Discretize the first quadrant of the z, t plane by choosing a mesh width Δz and a time step Δt, and defining the discrete mesh points (z_j, t_n) by

$$z_j = j\Delta z, \qquad t_n = n\Delta t.$$

Then, for each representative point θ^i, solve the equation

$$\frac{\partial \chi_K(z, \boldsymbol{\theta}^i, t)}{\partial t} + \theta_1^i \theta_2^i e^{\theta_1 t} \frac{\partial \chi_K(z, \boldsymbol{\theta}^i, t)}{\partial z} = 0,$$

with the help of the initial condition

$$\chi_K(z, \boldsymbol{\theta}^i, 0) = \begin{cases} 1 & \text{if } z \geq \theta_2^i, \\ 0 & \text{otherwise}, \end{cases}$$

by using the *total variation diminishing (TVD)* method [2, 13].

Step 3 Finally, determine the cumulative distribution function by means of the discretized version of the relation (32), i.e. for each mesh point (z_j, t_n) compute

$$F_Z(z_j, t_n) = \sum_{i=1}^{N} \frac{\chi_K(z_j, \boldsymbol{\theta}^i, t_n)}{N}.$$

Figs. 2 and 3 show the numerical results that have been obtained with $\Delta z = 6.25 \cdot 10^{-3} m$, $\Delta t = 6.25 \cdot 10^{-4} s$. Each side of Λ has been divided into 40 equal subintervals, so that $N = 1600$.

In addition to the cumulative function, the probability density function $p(z, t)$ was also numerically calculated by means of Eqs. (28) and (29), at the same four instants previously considered. The calculations have been made for $N = 40 \times 40$, $N = 80 \times 80$ and $N = 160 \times 160$. We used $\Delta z = 6.25 \cdot 10^{-3} m$ and $\Delta t = 6.25 \cdot 10^{-4} s$ in the first two cases, and $\Delta z = 3.125 \cdot 10^{-3} m$ and $\Delta t = 3.125 \cdot 10^{-4} s$ in the third case. The obtained results are shown in Fig. 4 for $t = t_1$ and $t = t_4$; the other cases are similar.

At each instant t a possible estimate of the relative error is given by the formula

$$\delta(t) = \frac{\sqrt{\sum_{i=1}^{N} (f((z_i, t)) - g(z_i, t))^2}}{\sqrt{\sum_{i=1}^{N} (f(z_i, t))^2}} \tag{34}$$

where the values of $f(z_i, t)$ and $g(z_i, t)$ are obtained explicitly and numerically, respectively.

The error was calculated, for $0 \leq z \leq 4$, and in all the examined cases its order of magnitude turned out to be practically constant over time. In the case of the cumulative function, the maximum error was less than $1.7 \cdot 10^{-3}$. In the case of the density function, it was less than $1.9 \cdot 10^{-1}$ for $N = 40 \times 40$, less than $1.4 \cdot 10^{-1}$ for $N = 80 \times 80$ and less than $.9 \cdot 10^{-1}$ for $N = 160 \times 160$.

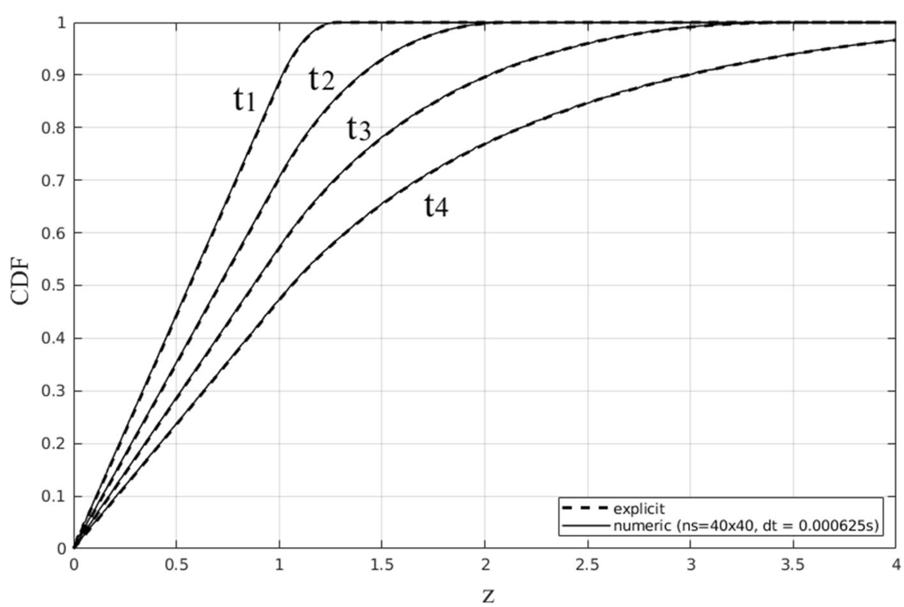

Fig. 2 Cumulative distribution functions at $t_1 = 0.25s, t_2 = 0.75s$, $t_3 = 1.25s$ and $t_4 = 1.75s$ comparison between the explicit and numerical solutions. (Example 1)

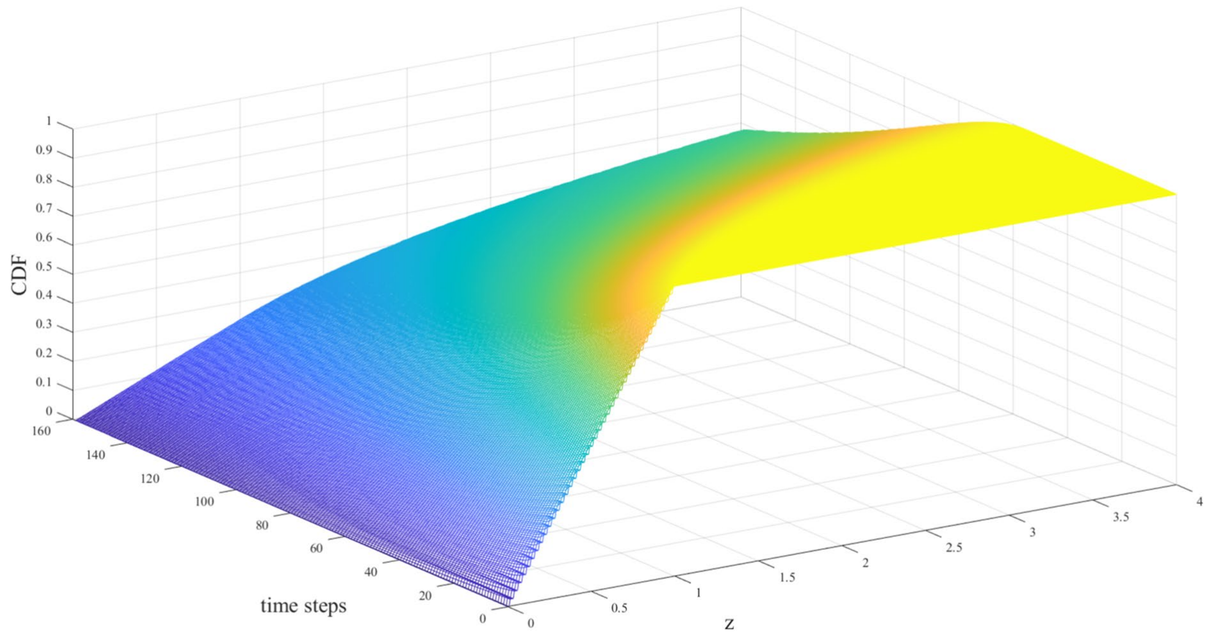

Fig. 3 Graph of the function $F(z,t)$ in the region $[0.4m]\times[0.4s]$; each time interval corresponds to $0.025s$. (Example 1)

5.2 The Klein–Gordon equation

Let $\mathfrak{S}=\{x\in\mathbb{R}:0\le x\le l\}$ be the reference configuration of a string which is made of elastic material having a mass density ρ per unit length. The string is fixed at its ends and is subjected to a constant tension τ and an additional force proportional to the displacement u. The equation of motion for oscillations of small amplitude is

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}+\alpha^2 u=0, \tag{35}$$

where

$$c=\sqrt{\frac{\tau}{\rho}} \tag{36}$$

and $\alpha^2 u$ is the additional force per unit mass. This is also the Klein–Gordon equation of quantum field theory [12, 14].

The function

$$u(x,t)=Asin\left(\frac{\pi x}{l}\right)cos(\omega t) \tag{37}$$

solves Eq. (35) for

$$\omega=\sqrt{\left(\frac{\pi c}{l}\right)^2+\alpha^2}, \tag{38}$$

The boundary conditions

$$u(0,t)=u(l,t)=0$$

and the initial conditions

$$u(x,0)=Asin\left(\frac{\pi x}{l}\right)\quad\text{and}\quad\frac{\partial u(x,0)}{\partial t}=0.$$

We assume that $\pi c/l$ and α are independent random variables, which will be indicated with Θ_1 and Θ_2 respectively, and that they are uniformly distributed over $\Lambda=[\kappa_i,\kappa_f]\times[\alpha_i,\alpha_f]$.

We choose Z as the (dimensionless) displacement $u\left(\frac{l}{2},t\right)/A$ of the middle point of the string, so that it results

$$Z(\boldsymbol{\theta},t)=\cos\left(t\sqrt{\theta_1^2+\theta_2^2}\right).$$

Equation (26) becomes

$$\frac{\partial\chi_t K(z,\boldsymbol{\theta},t)}{\partial t}-\left(\sqrt{\theta_1^2+\theta_2^2}sin\left(t\sqrt{\theta_1^2+\theta_2^2}\right)\right)\frac{\partial\chi_K(z,\boldsymbol{\theta},t)}{\partial z}=0, \tag{39}$$

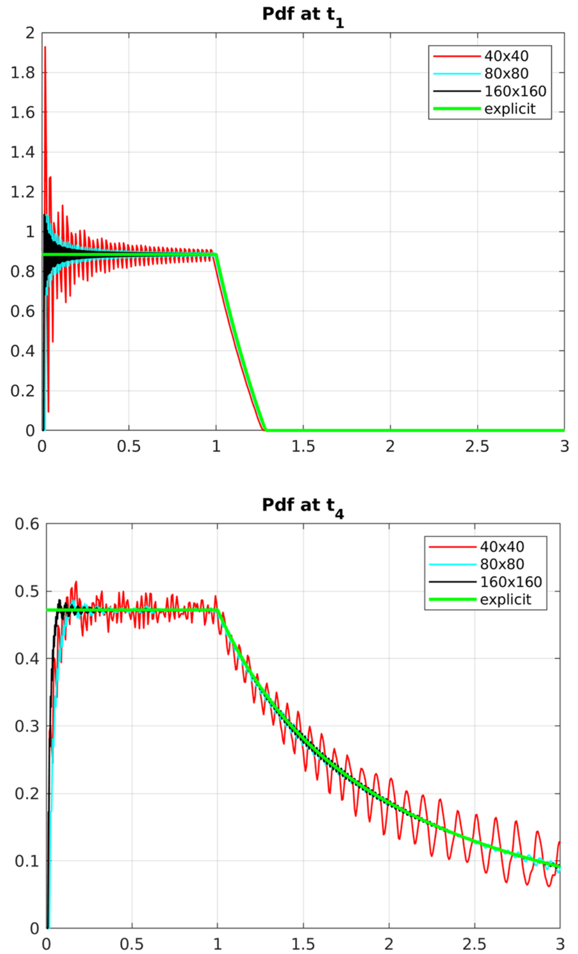

Fig. 4 Probability density function $p(z, t)$ at time $t = t_1$ and $t = t_4$, for $N = 40 \times 40$, $N = 80 \times 80$ and $N = 160 \times 160$; comparison between the explicit and numerical solutions (Example 1)

with the initial condition

$$\chi_K(z, \boldsymbol{\theta}, 0) = \begin{cases} 1 & \text{if } z \geq 1, \\ 0 & \text{otherwise}. \end{cases}$$

The calculations were made for $t = .025s$, $t = .05s$, and $t = .075s$; we limit ourselves to describe the first case because the other two are similar. Let us denote by $\varsigma = \alpha_f(\kappa_f - \kappa_i)$ the area of Λ, and by

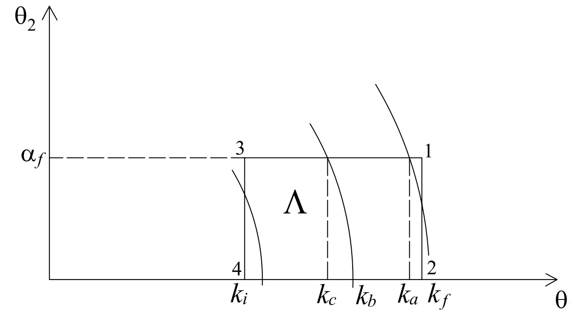

Fig. 5 Circumferences with center at the origin and radius $cos^{-1}(z)/t$, for $z_1 \leq z \leq z_2$, $z_2 < z \leq z_3$ and $z_3 < z \leq z_4$. (Example 2)

$$z_1 = cos\left(t\sqrt{\kappa_f^2 + \alpha_f^2}\right), z_2 = cos\left(t\kappa_f^2\right),$$
$$z_3 = cos\left(t\sqrt{\kappa_i^2 + \alpha_f^2}\right) \text{ and } z_4 = cos(t\kappa_i)$$

The values of z for which the circumference with center at the origin of the plane (θ_1, θ_2) and radius $cos^{-1}(z)/t$ passes through the corresponding vertices of Λ (Fig. 5). Then, with the help of (18), we obtain

(i) For $z \leq z_1$,

$F(z, t) = 0;$

(ii) For $z_1 \leq z \leq z_2$,

$$F(z,t) = \frac{1}{\varsigma}\left((\alpha_f(\kappa_f - \kappa_a) - \int_{\kappa_a}^{\kappa_f} \sqrt{\frac{1}{t^2}\left(cos^{-1}(z)^2 - \theta_1^2\right)}d\theta_1 \right)$$

where $\kappa_a = \sqrt{\frac{1}{t^2}\left(cos^{-1}(z)^2 - \alpha_f^2\right.};$

(iii) For $z_2 \leq z \leq z_3$,

$$F(z,t) = \frac{1}{\varsigma}\left((\alpha_f(\kappa_f - \kappa_a) - \int_{\kappa_c}^{\kappa_b} \sqrt{\frac{1}{t^2}\left(cos^{-1}(z)^2 - \theta_1^2\right)}d\theta_1 \right)$$

where $\kappa_b = \frac{1}{t}cos^{-1}(z)$ and κ_c has the same expression as κ_a;

(iv) For $z_3 \leq z \leq z_4$,

$$F(z,t) = 1 - \frac{1}{\varsigma}\int_{\kappa_i}^{\kappa_b} \sqrt{\frac{1}{t^2}(cos^{-1}(z)^2 - \theta_1^2)}d\theta_1;$$

(v) For $z \geq z_4$

$F(z, t) = 1.$

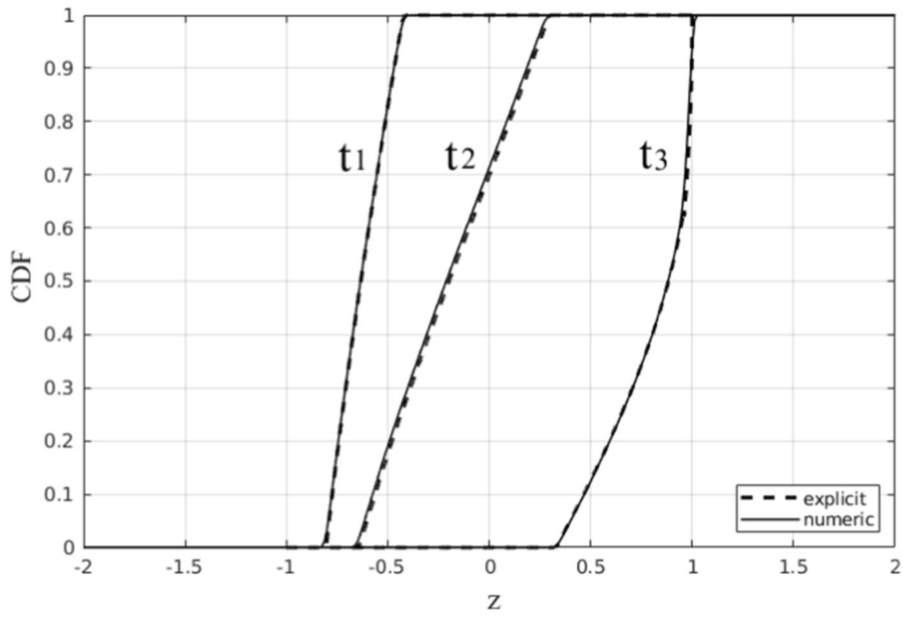

Fig. 6 Cumulative distribution function at $t_1 = .025s$, $t_2 = .05s$ and $t_3 = .075s$; comparison between the explicit and numerical solutions (Example 2)

The following parameter values have been used: $\kappa_i = 80$, $\kappa_f = 100$, $\alpha_i = 0$ and $\alpha_f = 10s^{-1}$, and then $\varsigma = 200s^{-2}$. The intervals $[\kappa_i, \kappa_f]$ and $[\alpha_i, \alpha_f]$ have been divided respectively into 40 and 20 equal subintervals, so that the partition of Λ is made up of $N = 800$ rectangles. Furthermore, $\Delta z = 1.25 \cdot 10^{-2}$ and $\Delta t = 7.81 \cdot 10^{-5}s$ were used. The relative error calculated with the formula (34) is of the order of $4 \cdot 10^{-2}$. Fig. 6 compares the numerical results with those obtained explicitly, at the three instants considered.

6 Conclusions

From the results obtained in the cases examined it appears that the accuracy of the cumulative distribution function, calculated with the proposed method, is good. The relative numerical procedure can be implemented in any code that allows the solution of the dynamic system and can be easily extended to vector-valued random processes.

Funding Open access funding provided by Università degli Studi di Firenze within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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