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Comparisons of the Expectations of System and Component Lifetimes in the Failure Dependent Proportional Hazard Model

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Abstract

In the failure dependent proportional hazard model, it is assumed that identical components work jointly in a system. At the moments of consecutive component failures the hazard rates of still operating components can change abruptly due to a change of the load acting on each component. The modification of the hazard rate consists in multiplying the original rate by a positive constant factor. Under the knowledge of the system structure and parameters of the failure dependent proportional hazard model, we determine tight lower and upper bounds on the expected differences between the system and component lifetimes, measured in various scale units based on the central absolute moments of the component lifetime. The results are specified for the systems with unimodal Samaniego signatures.

Keywords Coherent system · Failure dependent proportional hazard model · Generalized order statistics · Samaniego signature · Sharp bound

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1 Introduction

We consider an arbitrarily fixed coherent system composed of n elements with a structure function $\varphi : \{0, 1\}^n \mapsto \{0, 1\}$. It has the Samaniego structural signature $\mathbf{s} = (s_1, \ldots, s_n) \in [0, 1]^n$ whose coordinates are determined as follows

$$s_i = \frac{1}{\binom{n}{i-1}} \sum_{\sum_{j=1}^n x_j = n-i+1} \varphi(x_1, \ldots x_n) - \frac{1}{\binom{n}{i}} \sum_{\sum_{j=1}^n x_j = n-i} \varphi(x_1, \ldots x_n), \quad i = 1, \ldots, n. \tag{1}$$

The notion of the signature was introduced in Samaniego (1985), and formula (1) is due to Boland (2001) (see also Marichal et al. 2011).

We assume that nonnegative random variables $T_1, \ldots, T_n$ are the lifetimes of the system components, and their joint distribution satisfies conditions of the failure dependent proportional hazard model (see, e.g., Hollander and Peña 1995; Aki and Hirano 1997; Burkschat 2009; Navarro and Burkschat 2011). It follows that $T_1, \ldots, T_n$ are exchangeable and the respective order statistics $T_{1:n}, \ldots, T_{n:n}$ satisfy assumptions of the generalized order statistics model proposed by Kamps (1995a) (see also Kamps 1995b, 2016) with some baseline distribution function F and a vector of positive parameters $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$. Note that we obtain the same joint distribution of $T_{1:n}, \ldots, T_{n:n}$ if we replace F and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$ with $F_\alpha = 1 - (1 - F)^\alpha$ and $\boldsymbol{\gamma}_\alpha = \left(\frac{\gamma_1}{\alpha}, \ldots, \frac{\gamma_n}{\alpha}\right)$ for some $\alpha > 0$. For convenience of interpretation we choose the model parameters so that $\gamma_1 = n$. Then the baseline distribution function F represents the lifetime distribution of a component which operates under optimal conditions without outer stress acting on it. The stress that working components undergo after consecutive failures of other components is described by means of the parameters $\gamma_1, \ldots, \gamma_n$ (see, e.g., Kamps 1995a; Cramer and Kamps 2001; Cramer 2016).

The marginal distribution function of consecutive component failure times $T_{r:n}$ can be written as $G_{\boldsymbol{\gamma},r} \circ F$, $r = 1, \ldots, n$, where $G_{\boldsymbol{\gamma},r}$ is the distribution function of the rth generalized order statistic with the standard uniform baseline distribution. It depends on the first r coordinates of the parameter vector $\boldsymbol{\gamma}$ by means of a complicated formula which strongly depends on the multiplicities of parameter values. Respective expressions can be found in Cramer and Kamps (2003). Exchangeability of the component lifetimes in the failure dependent proportional hazard model implies that the common distribution function of every component is the uniform mixture of the distribution functions of order statistics

$$H_1(t) = \frac{1}{n} \sum_{r=1}^n \mathbb{P}(T_{r:n} \le t) = \frac{1}{n} \sum_{r=1}^n G_{\boldsymbol{\gamma},r}(F(t)) = \tilde{G}_{\boldsymbol{\gamma}}(F(t)) \tag{2}$$

(see Rychlik 1993). By the same reason, the distribution function of the system lifetime is the convex combination of order statistics distribution functions

$$H(t) = \sum_{r=1}^n s_r \mathbb{P}(T_{r:n} \le t) = \sum_{r=1}^n s_r G_{\boldsymbol{\gamma},r}(F(t)) = G_{\boldsymbol{\gamma},\mathbf{s}}(F(t)), \tag{3}$$

where the occurring coefficients coincide with the elements of the Samaniego signature vector (see Navarro et al. 2008, and Marichal et al. 2011). Formula (3) is called Samaniego representation, because it was first presented in Samaniego (1985) under the restriction to i.i.d. component lifetimes with a continuous parent distribution.

The purpose of our paper is to evaluate the expected lifetime of the system $\mathbb{E}T$ with fixed structure φ working in given circumstances represented by the parameter vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$ when the baseline distribution function F of the component lifetime is unknown. In the following, we always assume that F and therefore also H_1 is non-degenerate. Moreover, for any distribution function F we denote by F^{-1} the right-continuos version of the corresponding quantile function. We compare $\mathbb{E}T$ with the mean value of the single component lifetime $\mu_1 = \mathbb{E}T_1$. The difference is gauged in the scale units based on central absolute moments of component lifetime $\sigma_1(p) = (\mathbb{E}|T_1 - \mathbb{E}T_1|^p)^{1/p}$ of various orders $p \geq 1$ under the conditions that they are finite. Moreover, distributions attaining derived bounds, with a support in the non-negative real numbers, are given. For the parent distributions with finite support, we also consider the scale measure $\sigma_1(\infty) = \max\{H_1^{-1}(1) - \mu_1, \mu_1 - H_1^{-1}(0)\}$ being the distance of the mean from the furthest point of the support. For the comparison of ET with the mean of the underlying distribution function F in the case of k-out-of-n systems, which reduces to the derivation of bounds for single generalized order statistics, the reader is referred to Cramer et al. (2002, 2004) and Goroncy (2014).

The paper is organized as follows. In Section 2 we describe general results. In Section 3, we deliver more specific evaluations for the systems with unimodal signatures, and present exemplary numerical evaluation for k-out-of-n systems working under a practically justified load-sharing regime.

2 Main Results

We observe that every distribution function $G_{\boldsymbol{\gamma},r}$ is a differentiable, strictly increasing function on [0, 1] that maps the interval onto itself. The properties are shared by the convex combinations $\tilde{G}_{\boldsymbol{\gamma}}$ and $G_{\boldsymbol{\gamma},\mathbf{s}}$ whose derivatives we denote by $\tilde{g}_{\boldsymbol{\gamma}}$ and $g_{\boldsymbol{\gamma},\mathbf{s}}$, respectively. Obviously, if $g_{\boldsymbol{\gamma},r}$ denotes the density corresponding to $G_{\boldsymbol{\gamma},r}$, then

$$g_{\boldsymbol{\gamma},\mathbf{s}}(u) = \sum_{r=1}^{n} s_r g_{\boldsymbol{\gamma},r}(u) \quad \text{and} \quad \tilde{g}_{\boldsymbol{\gamma}}(u) = \frac{1}{n}\sum_{r=1}^{n} g_{\boldsymbol{\gamma},r}(u). \tag{4}$$

Moreover, the inverses $\tilde{G}_{\boldsymbol{\gamma}}^{-1}$ and $G_{\boldsymbol{\gamma},\mathbf{s}}^{-1}$ are well-defined functions from [0, 1] onto [0, 1], and have positive derivatives $\frac{1}{\tilde{g}_{\boldsymbol{\gamma}} \circ \tilde{G}_{\boldsymbol{\gamma}}^{-1}}$ and $\frac{1}{g_{\boldsymbol{\gamma},\mathbf{s}} \circ G_{\boldsymbol{\gamma},\mathbf{s}}^{-1}}$, respectively.

Theorem 1 *Suppose that the system signature* $\mathbf{s} = (s_1, \ldots, s_n)$*, and the parameters of the failure dependent proportional hazard model* $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$ *are fixed. Let* $\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}, \underline{h}_{\boldsymbol{\gamma},\mathbf{s}} : [0,1] \mapsto [0,1]$ *denote the derivatives of the smallest concave majorant and the greatest convex minorant of* $G_{\boldsymbol{\gamma},\mathbf{s}} \circ \tilde{G}_{\boldsymbol{\gamma}}^{-1}$*, respectively. The functions are monotone, continuous, and bounded density functions supported on* [0, 1]. *Moreover, for every fixed* $1 < p < \infty$*, and for every distribution function* H_1 *of the component lifetime* T_1 *such that* $\mathbb{E}|T_1|^p < \infty$ *we have*

$$-\infty < b_{\boldsymbol{\gamma},\mathbf{s}}(p) = -\left[\int_0^1 \left|\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - \overline{h}_{\boldsymbol{\gamma},\mathbf{s}}(\overline{c}_p)\right|^{\frac{p}{p-1}} dx\right]^{\frac{p-1}{p}}$$

$$\leq \frac{\mathbb{E}T - \mu_1}{\sigma_1(p)} \leq B_{\boldsymbol{\gamma},\mathbf{s}}(p) = \left[\int_0^1 \left|\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(\underline{c}_p)\right|^{\frac{p}{p-1}} dx\right]^{\frac{p-1}{p}} < +\infty, \tag{5}$$

where $\mu_1 = \mathbb{E}T_1$ *and* $\sigma_1(p) = (\mathbb{E}|T_1 - \mathbb{E}T_1|^p)^{1/p}$, *and* $\overline{c}_p, \underline{c}_p \in (0,1)$ *are the solutions to equations*

$$\int_0^c [\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}(c) - \overline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x)]^{1/(p-1)} dx = \int_c^1 [\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - \overline{h}_{\boldsymbol{\gamma},\mathbf{s}}(c)]^{1/(p-1)} dx,$$
$$\int_0^c [\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(c) - \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x)]^{1/(p-1)} dx = \int_c^1 [\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(c)]^{1/(p-1)} dx, \tag{6}$$

respectively. Furthermore, if $G_{\boldsymbol{\gamma},\mathbf{s}}(x) > \tilde{G}_{\boldsymbol{\gamma}}(x)$ ($G_{\boldsymbol{\gamma},\mathbf{s}}(x) < \tilde{G}_{\boldsymbol{\gamma}}(x)$, *respectively), for some* $0 \le x \le 1$, *then the lower (upper, respectively) bound is attained by the distribution functions* $\overline{H}_1$ ($\underline{H}_1$, *respectively) satisfying*

$$\frac{\overline{H}_1^{-1}(x) - \mu_1}{\sigma_1(p)} = \left| \frac{\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - \overline{h}_{\boldsymbol{\gamma},\mathbf{s}}(\overline{c}_p)}{-b_{\boldsymbol{\gamma},\mathbf{s}}(p)} \right|^{1/(p-1)} \operatorname{sgn}\{\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}(\overline{c}_p) - \overline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x)\},$$

provided that

$$\mu_1 + \sigma_1(p) \left(\frac{\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0) - \overline{h}_{\boldsymbol{\gamma},\mathbf{s}}(\overline{c}_p)}{-b_{\boldsymbol{\gamma},\mathbf{s}}(p)} \right)^{1/(p-1)} \ge 0,$$

and

$$\frac{\underline{H}_1^{-1}(x) - \mu_1}{\sigma_1(p)} = \left| \frac{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(\underline{c}_p)}{B_{\boldsymbol{\gamma},\mathbf{s}}(p)} \right|^{1/(p-1)} \operatorname{sgn}\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(\underline{c}_p)\}, \tag{7}$$

provided that

$$\mu_1 - \sigma_1(p) \left(\frac{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(\underline{c}_p) - \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)}{B_{\boldsymbol{\gamma},\mathbf{s}}(p)} \right)^{1/(p-1)} \ge 0, \tag{8}$$

respectively.

Proof The actual component lifetime distribution function under the failure dependent proportional hazard regime has representation (2), where F is the baseline distribution function of the generalized order statistics model. Accordingly, the expectation and the pth absolute central moment take on the forms

$$\mu_1 = \mathbb{E}T_1 = \int_0^\infty x\, \tilde{G}_{\boldsymbol{\gamma}}(F(dx)) = \int_0^1 F^{-1}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))\, dx, \tag{9}$$

$$\begin{aligned}\sigma_1^p(p) &= \mathbb{E}|T_1 - \mu_1|^p = \int_0^\infty |x - \mu_1|^p \tilde{G}_{\boldsymbol{\gamma}}(F(dx)) \\ &= \int_0^1 |F^{-1}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x)) - \mu_1|^p dx,\end{aligned} \tag{10}$$

respectively. By Eq. 3, the expectation of the system lifetime for the system with signature $\mathbf{s}$ can be written as

$$\begin{aligned}\mathbb{E}T &= \int_0^\infty x\, G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(\tilde{G}_{\boldsymbol{\gamma}}(F(dx)))) = \int_0^1 F^{-1}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))\, G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(dx)) \\ &= \int_0^1 F^{-1}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))\, \frac{g_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}{\tilde{g}_{\boldsymbol{\gamma}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}\, dx.\end{aligned} \tag{11}$$

Since $\frac{g_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}{\tilde{g}_{\boldsymbol{\gamma}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}$ is the density function of distribution function $G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))$ supported on [0, 1], we have clearly

$$\mathbb{E}T_1 = \int_0^1 \mu_1 \frac{g_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}{\tilde{g}_{\boldsymbol{\gamma}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))} dx. \tag{12}$$

Also, by Eq. 9, for any $c \in \mathbb{R}$

$$\int_0^1 c[F^{-1}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x)) - \mu_1]\, dx = 0. \tag{13}$$

Combining (11), (12) and (13), we obtain

$$\mathbb{E}T - \mathbb{E}T_1 = \int_0^1 [F^{-1}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x)) - \mu_1] \left[\frac{g_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}{\tilde{g}_{\boldsymbol{\gamma}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))} - c \right] dx \tag{14}$$

for arbitrary real c.

We now use formula (14) for proving the upper bound in Eq. 5 and conditions of its attainability. Since the first factor in the integral of Eq. 14 is non-decreasing by assumption, we can write

$$\mathbb{E}T - \mathbb{E}T_1 \le \int_0^1 [F^{-1}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x)) - \mu_1] \left[\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - c \right] dx, \tag{15}$$

because $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x)$ is the derivative of the greatest convex minorant $\underline{H}_{\boldsymbol{\gamma},\mathbf{s}}(x)$, say, of the antiderivative $G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))$ of $\frac{g_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}{\tilde{g}_{\boldsymbol{\gamma}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}$ (see, e.g., Moriguti 1953; Rychlik 2001). By definition and monotonicity of $G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))$, the derivative $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x)$ of the greatest convex minorant is non-decreasing and non-negative. It satisfies $\int_0^1 \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x)dx = \underline{H}_{\boldsymbol{\gamma},\mathbf{s}}(1) - \underline{H}_{\boldsymbol{\gamma},\mathbf{s}}(0) = G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(1)) - G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(0)) = 1$. The greatest convex minorant coincides with $G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))$ on some subintervals of [0, 1], and is linear on the others, but each linear part is tangent to $G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))$ at the respective interval end-points. This implies that the derivative $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x)$ is continuous. We finally check that it is bounded as well. Note that the original density function $\frac{g_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}{\tilde{g}_{\boldsymbol{\gamma}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}$ is bounded, because by Eq. 4 we have

$$\frac{g_{\boldsymbol{\gamma},\mathbf{s}}(x)}{\tilde{g}_{\boldsymbol{\gamma}}(x)} \le n \max_{1\le r\le n} s_r, \qquad 0 < x < 1.$$

The property is shared by $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x)$, because it is equal to $\frac{g_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}{\tilde{g}_{\boldsymbol{\gamma}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}$ on the intervals where $\underline{H}_{\boldsymbol{\gamma},\mathbf{s}}(x) = G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))$, and to

$$\frac{G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(b)) - G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(a))}{b-a} \le \sup_{a<x<b} \frac{g_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}{\tilde{g}_{\boldsymbol{\gamma}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))}$$

on each interval (a, b), say, such that $\underline{H}_{\boldsymbol{\gamma},\mathbf{s}}(x) < G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))$ there.

Observe that the left-hand side of Eq. 6 is continuous in c, and monotonously non-decreasing from 0 at 0 to a finite value at 1. The right-hand side is continuously non-increasing from $\int_0^1 \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{1/(p-1)}(x)dx < \infty$ at $c = 0$ to 0 at $c = 1$. Accordingly, there exists a non-empty, possibly degenerate interval consisting of solutions to Eq. 6. Plugging

$c = \underline{h}_{\gamma,\mathbf{s}}(\underline{c}_p)$ with any $\underline{c}_p$ that solves (6) into (15), and then applying the Hölder inequality we get

$$\mathbb{E}T - \mathbb{E}T_1 \leq \left(\int_0^1 |F^{-1}(\tilde{G}_{\gamma}^{-1}(x)) - \mu_1|^p dx\right)^{\frac{1}{p}} \left(\int_0^1 \left|\underline{h}_{\gamma,\mathbf{s}}(x) - \underline{h}_{\gamma,\mathbf{s}}(\underline{c}_p)\right|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}, \tag{16}$$

which is equivalent to the upper evaluation in Eq. 5 by Eq. 10.

For some parameters γ and signatures $\mathbf{s}$, it may happen that $G_{\gamma,\mathbf{s}}(x) \geq \tilde{G}_{\gamma}(x)$ and so $G_{\gamma,\mathbf{s}}(\tilde{G}_{\gamma}^{-1}(x)) \geq x$ for all $0 \leq x \leq 1$. Then $\underline{H}_{\gamma,\mathbf{s}}(x) = x$ and $\underline{h}_{\gamma,\mathbf{s}}(x) = 1$ if $0 \leq x \leq 1$. Equality (6) is satisfied for any $0 \leq c \leq 1$, and, in consequence, $B_{\gamma,\mathbf{s}}(p) = 0$ for every $1 < p < \infty$. We do not analyze here if these zero bounds are optimal.

Now we check that Eq. 7 with (8) determine the conditions for attainability of the bound when $G_{\gamma,\mathbf{s}}(x) < \tilde{G}_{\gamma}(x)$ for some $0 < x < 1$ so that $\underline{h}_{\gamma,\mathbf{s}}(x)$ is not a constant function. The equality in (15) is achieved when $H_1^{-1}(x) = F^{-1}(\tilde{G}_{\gamma}^{-1}(x))$ is constant on every interval where the greatest convex minorant $\underline{H}_{\gamma,\mathbf{s}}(x)$ differs from (i.e., is smaller than) the antiderivative $G_{\gamma,\mathbf{s}}(\tilde{G}_{\gamma}^{-1}(x))$ (see Moriguti 1953; or Rychlik 2001, Lemma 3, p. 34). The minorant is linear on these intervals, and its derivative $\underline{h}_{\gamma,\mathbf{s}}(x)$ is constant there. Relation (7) guarantees constancy of $\underline{H}_1^{-1}(x)$ on these intervals. The equality in the Hölder inequality (16) holds when $\underline{H}_1^{-1}(x) - \mu_1$ and $\left|\underline{h}_{\gamma,\mathbf{s}}(x) - \underline{h}_{\gamma,\mathbf{s}}(\underline{c}_p)\right|^{1/(p-1)} \operatorname{sgn}\{\underline{h}_{\gamma,\mathbf{s}}(x) - \underline{h}_{\gamma,\mathbf{s}}(\underline{c}_p)\}$ are proportional with a nonnegative proportionality coefficient α, and here $\alpha = \frac{\sigma_1(p)}{B_{\gamma,\mathbf{s}}^{1/(p-1)}(p)} > 0$, as desired. It remains to show that Eqs. 7 and 8 define the quantile function of a lifetime distribution with assumed moments. Firstly, $\underline{H}_1^{-1}$ is a nondecreasing function by Eq. 7 and non-negative by Eq. 8. The respective distribution function has the expectation equal to μ_1, when (7) integrates over $(0, 1)$ to 0, which is guaranteed by condition (6). It has the pth absolute central moment $\sigma_1(p)$ which follows from the equations

$$\int_0^1 \left|\frac{\underline{H}_1^{-1}(x) - \mu_1}{\sigma_1(p)}\right|^p dx = \int_0^1 \left|\frac{\underline{h}_{\gamma,\mathbf{s}}(x) - \underline{h}_{\gamma,\mathbf{s}}(\underline{c}_p)}{B_{\gamma,\mathbf{s}}(p)}\right|^{p/(p-1)} dx = 1.$$

The proof for the lower bounds is analogous. □

For $p = 2$, relations $\overline{h}_{\gamma,\mathbf{s}}(\overline{c}_2) = \underline{h}_{\gamma,\mathbf{s}}(\underline{c}_2) = 1$ hold, and all the formulas of Theorem 1 simplify significantly. In particular, we have

$$\begin{aligned} b_{\gamma,\mathbf{s}}(2) &= -\left[\int_0^1 \overline{h}^2_{\gamma,\mathbf{s}}(x)dx - 1\right]^{1/2}, \\ B_{\gamma,\mathbf{s}}(2) &= \left[\int_0^1 \underline{h}^2_{\gamma,\mathbf{s}}(x)dx - 1\right]^{1/2}. \end{aligned} \tag{17}$$

The extreme cases $p = 1$ and $p = \infty$ are treated separately.

Theorem 2 *Under the assumptions and notation of* Theorem 1, *for* $p = 1$ *and every distribution function of component lifetime* T_1 *with finite expectation* μ_1, *we get*

$$b_{\gamma,\mathbf{s}}(1) = -\frac{\overline{h}_{\gamma,\mathbf{s}}(0) - \overline{h}_{\gamma,\mathbf{s}}(1)}{2} \leq \frac{\mathbb{E}T - \mu_1}{\sigma_1(1)} \leq B_{\gamma,\mathbf{s}}(1) = \frac{\underline{h}_{\gamma,\mathbf{s}}(1) - \underline{h}_{\gamma,\mathbf{s}}(0)}{2}, \tag{18}$$

where $\sigma_1(1)$ *denotes the mean absolute deviation from the mean of* T_1.

If $G_{\boldsymbol{\gamma},\mathbf{s}}(x) > \tilde{G}_{\boldsymbol{\gamma}}(x)$ ($G_{\boldsymbol{\gamma},\mathbf{s}}(x) < \tilde{G}_{\boldsymbol{\gamma}}(x)$, respectively) for some $0 \le x \le 1$, then the lower (upper, respectively) bound is non-zero and optimal. In particular, when $\lambda(\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{0\}))$ and $\lambda(\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{1\}))$ ($\lambda(\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{0\}))$ and $\lambda(\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{1\}))$, respectively), with λ denoting the Lebesgue measure on the real line, are positive, then the lower (upper) bound is attained by three-point distribution

$$\mathbb{P}\left(T_1 = \mu_1 - \frac{\sigma_1(1)}{2\lambda(\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{0\}))}\right) = \lambda(\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{0\})),$$
$$\mathbb{P}(T_1 = \mu_1) = \lambda(\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}((0,1))),$$
$$\mathbb{P}\left(T_1 = \mu_1 + \frac{\sigma_1(1)}{2\lambda(\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{1\}))}\right) = \lambda(\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{1\})),$$

when $\mu_1 - \frac{\sigma_1(1)}{2\lambda(\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{0\}))} \ge 0$, *and*

$$\mathbb{P}\left(T_1 = \mu_1 - \frac{\sigma_1(1)}{2\lambda(\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{0\}))}\right) = \lambda(\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{0\})),$$
$$\mathbb{P}(T_1 = \mu_1) = \lambda(\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}((0,1))),$$
$$\mathbb{P}\left(T_1 = \mu_1 + \frac{\sigma_1(1)}{2\lambda(\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{1\}))}\right) = \lambda(\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{1\})),$$

when $\mu_1 - \frac{\sigma_1(1)}{2\lambda(\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{0\}))} \ge 0$, *respectively. If any of these sets has the Lebesgue measure zero, then the bound is attained in the limit when we shift probability mass $\varepsilon > 0$ to all corresponding probabilities, and let ε tend to 0.*

Observe that in cases when $\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{0\})$ and $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{0\})$ have Lebesgue measures zero, the approximations proposed in the last statement of Theorem 2 impose increasing restrictions $\sigma_1(1) \le 2\mu_1\varepsilon$ on the relation between the dispersion and the mean of T_1.

Proof We focus on upper bounds, because arguments for deriving lower ones are similar. Recalling (15), we get

$$\begin{aligned}\mathbb{E}T - \mathbb{E}T_1 &\le \int_0^1 [F^{-1}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x)) - \mu_1]\left[\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - c\right] dx \\ &\le \sup_{0\le x\le 1} |\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - c| \int_0^1 |F^{-1}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x)) - \mu_1| dx \\ &= \sigma_1(1) \sup_{0\le x\le 1} |\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - c| \end{aligned} \tag{19}$$

for every $c \in \mathbb{R}$. Since

$$\begin{aligned}\sup_{0\le x\le 1} |\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - c| &= \sup_{0\le x\le 1} \max\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - c, c - \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x)\} \\ &= \max\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1) - c, c - \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)\},\end{aligned}$$

the supremum is minimal when $c = \frac{1}{2}[\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1) + \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)]$, and it amounts to $\frac{1}{2}[\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1) - \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)]$. This proves our upper estimate in Eq. 18.

Determining the attainability conditions, we omit the case when $G_{\boldsymbol{\gamma},\mathbf{s}}(x) \geq \tilde{G}_{\boldsymbol{\gamma}}(x)$ for all $0 \leq x \leq 1$ which leads to $B_{\boldsymbol{\gamma},\mathbf{s}}(1) = 0$. In the opposite case, we have $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1)$. Suppose first that $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)$ for some $x > 0$ and $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1)$ for some $x < 1$. Then we get the equality in the latter inequality of Eq. 19 with $c = \frac{1}{2}[\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1) + \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)]$ iff

$$H_1^{-1}(x) - \mu_1 = F^{-1}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x)) - \mu_1 \begin{cases} \leq 0, & \text{if } \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0), \\ = 0, & \text{if } \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1), \\ \geq 0, & \text{if } \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1). \end{cases}$$

For $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)$ and $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1)$ on two non-degenerate intervals, it is possible that $\underline{H}_{\boldsymbol{\gamma},\mathbf{s}}(x) < G_{\boldsymbol{\gamma},\mathbf{s}}(\tilde{G}_{\boldsymbol{\gamma}}^{-1}(x))$ on their interiors. In order to get the equality in the first inequality of Eq. 19, we require that H_1^{-1} is constant there. The respective distribution is non-degenerate with mean μ_1 if the two constants are non-zero. Relations

$$H_1^{-1}(x) - \mu_1 = \begin{cases} -a, & \text{if } \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0), \\ 0, & \text{if } \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1), \\ b, & \text{if } \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1), \end{cases}$$

for some $a, b > 0$ combined with the moment constraints imply that $a = \frac{\sigma_1(1)}{2\lambda(\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{0\}))}$ and $b = \frac{\sigma_1(1)}{2\lambda(\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}^{-1}(\{1\}))}$. The component lifetime is non-negative if $\mu_1 \geq a$.

Suppose now that $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x)$ is strictly increasing in some neighborhoods of 0 and 1. Take $\varepsilon > 0$ sufficiently small so that the intervals $(0, \varepsilon)$ and $(1-\varepsilon, 1)$ are contained in these neighborhoods. Define distribution functions $H_{1,\varepsilon}(x) = \tilde{G}_{\boldsymbol{\gamma}}(F_\varepsilon(x))$ such that the respective quantile functions satisfy

$$H_{1,\varepsilon}^{-1}(x) - \mu_1 = \begin{cases} -\frac{\sigma_1(1)}{2\varepsilon}, & 0 < x \leq \varepsilon, \\ 0, & \varepsilon < x \leq 1-\varepsilon, \\ \frac{\sigma_1(1)}{2\varepsilon}, & 1-\varepsilon \leq x < 1. \end{cases}$$

They have common expectation μ_1 and mean absolute deviation from the mean $\sigma_1(1)$. Moreover,

$$\begin{aligned} &\frac{1}{\sigma_1(1)} \int_0^1 [H_{1,\varepsilon}^{-1}(x) - \mu_1] \left[\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - \frac{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1) + \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)}{2} \right] dx \\ &= \frac{1}{2\varepsilon} \left\{ \int_0^\varepsilon \left[\frac{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1) + \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)}{2} - \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) \right] dx \right. \\ &\quad \left. + \int_{1-\varepsilon}^1 \left[\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) - \frac{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1) + \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)}{2} \right] dx \right\} \\ &\to \frac{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1) - \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)}{2} \end{aligned}$$

as $\varepsilon \to 0$, by boundedness and continuity of $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}$. The interval $(\varepsilon, 1-\varepsilon)$ where each $H_{1,\varepsilon}^{-1}$ is constant and equal to μ_1 contains all the possible subintervals of $(0, 1)$ where $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}$ is constant and $\underline{H}_{\boldsymbol{\gamma},\mathbf{s}} < G_{\boldsymbol{\gamma},\mathbf{s}} \circ \tilde{G}_{\boldsymbol{\gamma}}^{-1}$. This implies that for every sufficiently small ε the first inequality in Eq. 19 becomes equality.

The attainability proof in the cases when $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}$ is constant on either of the neighborhoods of 0 and 1, and is strictly increasing on the other is much the same, and therefore we omit it here. □

Theorem 3 *Under the assumptions and notation of* Theorem 1, *for $p = \infty$ and all bounded component lifetime distributions we have*

$$b_{\boldsymbol{\gamma},\mathbf{s}}(\infty) = -\left[1 - 2\overline{H}_{\boldsymbol{\gamma},\mathbf{s}}\left(\frac{1}{2}\right)\right] \le \frac{\mathbb{E}T - \mu_1}{\sigma_1(\infty)} \le B_{\boldsymbol{\gamma},\mathbf{s}}(\infty) = 1 - 2\underline{H}_{\boldsymbol{\gamma},\mathbf{s}}\left(\frac{1}{2}\right), \tag{20}$$

where $\mu_1 = \mathbb{E}T_1$ and $\sigma_1(\infty) = \operatorname{ess\,sup}|T_1 - \mu_1|$. If $G_{\boldsymbol{\gamma},\mathbf{s}}(x) > \tilde{G}_{\boldsymbol{\gamma}}(x)$ ($G_{\boldsymbol{\gamma},\mathbf{s}}(x) < \tilde{G}_{\boldsymbol{\gamma}}(x)$, respectively) for some $0 \le x \le 1$, then the lower (upper, respectively) bound is non-zero and optimal.

The upper bound is attained by the following discrete distributions. If $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}\left(\frac{1}{2}\right) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1)$, then the equality in the latter inequality in Eq. 20 *holds when*

$$\begin{aligned}
&\mathbb{P}\left(T_1 = \mu_1 - \sigma_1(\infty)\right) = \lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}\left(\frac{1}{2}\right)\right\}\right),\\
&\mathbb{P}\left(T_1 = \mu_1 + \sigma_1(\infty)\frac{\lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}\left(\frac{1}{2}\right)\right\}\right) - \lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) > \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}\left(\frac{1}{2}\right)\right\}\right)}{\lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}\left(\frac{1}{2}\right)\right\}\right)}\right)\\
&= \lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}\left(\frac{1}{2}\right)\right\}\right),\\
&\mathbb{P}\left(T_1 = \mu_1 + \sigma_1(\infty)\right) = \lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) > \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}\left(\frac{1}{2}\right)\right\}\right)
\end{aligned} \tag{21}$$

(note that if $\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}\left(\frac{1}{2}\right)\right\}$ consists only of the single point $\frac{1}{2}$, then the denominator in the formula describing the middle support point vanishes, but so do the numerator and the probability value, and this line can be simply dropped). If $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}\left(\frac{1}{2}\right) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1)$, then the equality holds if

$$\begin{aligned}
\mathbb{P}\left(T_1 = \mu_1 - \sigma_1(\infty)\frac{\lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) > \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)\right\}\right)}{\lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)\right\}\right)}\right) &= \lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)\right\}\right),\\
\mathbb{P}\left(T_1 = \mu_1 + \sigma_1(\infty)\right) &= \lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) > \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)\right\}\right).
\end{aligned} \tag{22}$$

If $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}\left(\frac{1}{2}\right) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1)$, then the equality is attained when

$$\begin{aligned}
\mathbb{P}\left(T_1 = \mu_1 - \sigma_1(\infty)\right) &= \lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1)\right\}\right),\\
\mathbb{P}\left(T_1 = \mu_1 + \sigma_1(\infty)\frac{\lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) < \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1)\right\}\right)}{\lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1)\right\}\right)}\right) &= \lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(1)\right\}\right).
\end{aligned} \tag{23}$$

The conditions for nonnegativity of component lifetimes T_1 defined above are $\mu_1 \ge \sigma_1(\infty)\frac{\lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) > \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)\right\}\right)}{\lambda\left(\left\{\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(x) = \underline{h}_{\boldsymbol{\gamma},\mathbf{s}}(0)\right\}\right)}$ in case (22), *and $\mu_1 \ge \sigma_1(\infty)$ for* Eqs. 21 *and* 23.

In order to describe the lower bound attainability conditions, it suffices to replace $\underline{h}_{\boldsymbol{\gamma},\mathbf{s}}$ by $\overline{h}_{\boldsymbol{\gamma},\mathbf{s}}$, and reverse the inequalities between them in the previous paragraph.

Proof Again, we confine ourselves to considering positive upper bounds only under the condition $G_{\gamma,\mathbf{s}} \ngeq \tilde{G}_{\gamma}$. For every $c \in \mathbb{R}$

$$\int_0^1 [F^{-1}(\tilde{G}_{\gamma}^{-1}(x)) - \mu_1] \left[\underline{h}_{\gamma,\mathbf{s}}(x) - c\right] dx$$
$$\leq \sup_{0\leq x\leq 1} |F^{-1}(\tilde{G}_{\gamma}^{-1}(x)) - \mu_1| \int_0^1 |\underline{h}_{\gamma,\mathbf{s}}(x) - c| dx,$$

and the integral in the second line is minimized at $c = \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)$, because

$$\begin{aligned}
&\int_0^1 \left|\underline{h}_{\gamma,\mathbf{s}}(x) - \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)\right| dx \\
&= \int_0^{1/2} \left[\underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right) - \underline{h}_{\gamma,\mathbf{s}}(x)\right] dx + \int_{1/2}^1 \left[\underline{h}_{\gamma,\mathbf{s}}(x) - \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)\right] dx \\
&= \int_0^{1/2} [c - \underline{h}_{\gamma,\mathbf{s}}(x)]\, dx + \int_{1/2}^1 [\underline{h}_{\gamma,\mathbf{s}}(x) - c]\, dx \\
&\leq \int_0^1 \left|\underline{h}_{\gamma,\mathbf{s}}(x) - c\right| dx.
\end{aligned}$$

For the choice $c = \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)$, we have

$$\int_0^1 [H_1^{-1}(x) - \mu_1] \left[\underline{h}_{\gamma,\mathbf{s}}(x) - \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)\right] dx$$
$$\leq \sigma_1(\infty) \int_0^1 \left|\underline{h}_{\gamma,\mathbf{s}}(x) - \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)\right| dx = \sigma_1(\infty) \left[1 - 2\underline{H}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)\right]. \tag{24}$$

We get the equality in Eq. 24 if $H_1^{-1}(x) - \mu_1 = -\sigma_1(\infty)$ when $\underline{h}_{\gamma,\mathbf{s}}(x) < \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)$, and $H_1^{-1}(x) - \mu_1 = +\sigma_1(\infty)$ when $\underline{h}_{\gamma,\mathbf{s}}(x) > \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)$. On the set $\left\{\underline{h}_{\gamma,\mathbf{s}}(x) = \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)\right\}$, the only restriction is that the function in nondecreasing and ranges over the interval $[-\sigma_1(\infty), \sigma_1(\infty)]$. However, in order to get equality in Eq. 15, $H_1^{-1}(x)$ should be constant there. Summing up, the upper bound in Eq. 20 is attained if $H_1^{-1}(x) - \mu_1$ is equal to $-\sigma_1(\infty)$, $a \in [-\sigma_1(\infty), \sigma_1(\infty)]$, and $\sigma_1(\infty)$ on the sets where $\underline{h}_{\gamma,\mathbf{s}}(x)$ is less than, equal to, and greater than $\underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)$, respectively. There are three possibilities in general $\underline{h}_{\gamma,\mathbf{s}}(0) < \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right) < \underline{h}_{\gamma,\mathbf{s}}(1)$, $\underline{h}_{\gamma,\mathbf{s}}(0) = \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right) < \underline{h}_{\gamma,\mathbf{s}}(1)$ and $\underline{h}_{\gamma,\mathbf{s}}(0) < \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right) = \underline{h}_{\gamma,\mathbf{s}}(1)$. So it may happen that the sets $\left\{\underline{h}_{\gamma,\mathbf{s}}(x) < \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)\right\}$ and $\left\{\underline{h}_{\gamma,\mathbf{s}}(x) > \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)\right\}$ are empty, and $\left\{\underline{h}_{\gamma,\mathbf{s}}(x) = \underline{h}_{\gamma,\mathbf{s}}\left(\frac{1}{2}\right)\right\}$ is degenerate. Combining these requirements with the first moment condition $\int_0^1 [H_1^{-1}(x) - \mu_1] dx = 0$, by simple calculations we arrive at formulas (21)–(23). □

The attainability conditions in Theorems 1–3 can be also expressed in terms of distribution functions F of the nominal component lifetime. To this end it suffices to apply transformation $F = \tilde{G}_{\gamma}^{-1} \circ H_1$. We also notice that we obtain zero upper bounds for all $1 \leq p \leq \infty$ when $G_{\gamma,\mathbf{s}} \geq \tilde{G}_{\gamma}$. This means that for any nominal distribution function F of the component lifetime, the system lifetime T is stochastically less than the actual component lifetime, and inequality $\mathbb{E}T \leq \mathbb{E}T_1$ is evident. Similarly $G_{\gamma,\mathbf{s}} \leq \tilde{G}_\gamma$ always implies $\mathbb{E}T \geq \mathbb{E}T_1$. Even a minor violation of these stochastic orderings between $G_{\gamma,\mathbf{s}}$ and $\tilde{G}_\gamma$ implies that we obtain nontrivial positive upper bounds, and negative lower ones, respectively.

3 Unimodal Signatures

In order to evaluate the bounds in Theorems 1, 2 and 3, derivatives of the smallest concave majorant and the greatest convex minorant of the distribution function $G_{\gamma,\mathbf{s}} \circ \tilde{G}_\gamma^{-1}$ should be determined. For corresponding unimodal density functions, there is a well-known procedure to obtain these derivatives (see, e.g., Moriguti 1953; Rychlik 2001). Namely, if $g : [0, 1] \to [0, \infty)$ is any unimodal density function with the corresponding distribution function G, then the derivative of the greatest convex minorant is obtained as follows. If g is decreasing or $g(0) \geq 1$, then $\underline{g}(x) = 1$ for $x \in [0, 1]$. Otherwise, the derivative is given by

$$\underline{g}(x) = \begin{cases} g(x), & \text{for } 0 \leq x < \underline{u}_*, \\ g(\underline{u}_*), & \text{for } \underline{u}_* \leq x \leq 1. \end{cases} \tag{25}$$

If g is increasing, then $\underline{u}_* = 1$. Otherwise, $0 < \underline{u}_* < 1$ is the unique solution to the so–called Moriguti equation

$$g(u) = \frac{1 - G(u)}{1 - u}.$$

The derivative of the smallest concave majorant can be derived similarly. If g is increasing or $g(1) \geq 1$, then $\overline{g}(x) = 1$ for $x \in [0, 1]$. Otherwise, the derivative has the form

$$\overline{g}(x) = \begin{cases} g(\overline{u}_*), & \text{for } 0 \leq x < \overline{u}_*, \\ g(x), & \text{for } \overline{u}_* \leq x \leq 1. \end{cases} \tag{26}$$

If g is decreasing, then $\overline{u}_* = 0$. Otherwise, $\overline{u}_*$ coincides with the unique solution in $(0, 1)$ to

$$g(u) = \frac{G(u)}{u}.$$

The next lemma shows that unimodality of the signature is sufficient to get unimodality of the density. The assumption is not very restrictive, because coherent systems with non-unimodal signatures are very rare. An example of a system of size 5 with bimodal signature was presented in Jasiński et al. (2009), and the construction was extended to higher dimensions by Bieniek and Burkschat (2018). In the proof, we apply a characterization of unimodality based on sign change behavior (cf., e.g., Marshall and Olkin 2007, proof of Proposition B.2., p. 99).

Theorem 4 *If the signature* $\mathbf{s} = (s_1, \ldots, s_n)$ *is unimodal, i.e., there exists* $1 \leq k \leq n$ *such that*

$$s_1 \leq \cdots \leq s_k, s_k \geq \cdots \geq s_n,$$

then the density function $\frac{g_{\gamma,\mathbf{s}} \circ \tilde{G}_\gamma^{-1}}{\tilde{g}_\gamma \circ \tilde{G}_\gamma^{-1}}$ *of the distribution function* $G_{\gamma,\mathbf{s}} \circ \tilde{G}_\gamma^{-1}$ *is also unimodal, i.e., there exists a mode* $m \in [0, 1]$ *such that* $\frac{g_{\gamma,\mathbf{s}} \circ \tilde{G}_\gamma^{-1}}{\tilde{g}_\gamma \circ \tilde{G}_\gamma^{-1}}$ *is increasing on* $(0, m)$ *and decreasing on* $(m, 1)$.

Proof Recall that the density function $\frac{g_{\gamma,\mathbf{s}}\circ\tilde{G}_{\gamma}^{-1}}{\tilde{g}_{\gamma}\circ\tilde{G}_{\gamma}^{-1}}$ is positive and continuous on $(0, 1)$ (cf. Cramer et al. 2004). Moreover, the density is a bounded function on $(0, 1)$ (see the proof of Theorem 1). Therefore, $\frac{g_{\gamma,\mathbf{s}}\circ\tilde{G}_{\gamma}^{-1}}{\tilde{g}_{\gamma}\circ\tilde{G}_{\gamma}^{-1}}$ defines a non-negative, continuous (and bounded) function on the interval $[0, 1]$. Clearly, it is unimodal iff the function $g = \frac{g_{\gamma,\mathbf{s}}}{\tilde{g}_{\gamma}}$ is unimodal. Let

$$A_c = \{x \in [0, 1] : g(x) \geq c\}, \quad c \geq 0.$$

Observe that g is unimodal on $[0, 1]$ iff A_c is an interval (possibly empty or degenerate) in $[0, 1]$ for every $c \in [0, \infty)$. Every element of the set

$$M = \bigcap_{c\geq 0, A_c\neq\emptyset} A_c = A_{\max\{f(x):x\in[0,1]\}}$$

can be chosen as mode m. Moreover, the condition on the sets A_c holds iff

$$g(x) - c \text{ changes sign at most twice for every } c \in [0, \infty)$$
$$\text{and, if there are two sign changes, then the sequence must be } - + -. \quad (27)$$

Equivalently, we can consider $g_{\gamma,\mathbf{s}}(x) - c\tilde{g}_{\gamma}(x)$ instead of $g(x) - c$ in Eq. 27. Due to Eqs. 2 and 3, it follows that

$$g_{\gamma,\mathbf{s}}(x) - c\tilde{g}_{\gamma}(x) = \sum_{r=1}^{n} \left(s_r - \frac{c}{n}\right) g_{\gamma,r}(x), \quad x \in (0, 1),$$

where $g_{\gamma,r}$ denotes the density function of the rth uniform generalized order statistic. Since the signature $\mathbf{s}$ is unimodal, for every $c \geq 0$ there are at most two sign changes in the sequence $(s_1 - \frac{c}{n}, \ldots, s_n - \frac{c}{n})$ and, if there are exactly two changes, then the pattern is $-+-$. Thus, the variation diminishing property of the densities of uniform generalized order statistics (see Bieniek 2007) yields that condition (27) is satisfied. Consequently, $g = \frac{g_{\gamma,\mathbf{s}}}{\tilde{g}_{\gamma}}$ is unimodal. □

Corollary 1 *If the sequence $s_1, \ldots, s_n$ is increasing (decreasing), then the density function $\frac{g_{\gamma,\mathbf{s}}\circ\tilde{G}_{\gamma}^{-1}}{\tilde{g}_{\gamma}\circ\tilde{G}_{\gamma}^{-1}}$ is also increasing (decreasing) on* $(0, 1)$.

Proof The statement can be proven analogously to Theorem 4 by noting that the function $g = \frac{g_{\gamma,\mathbf{s}}}{\tilde{g}_{\gamma}}$ is increasing (decreasing) iff the following condition holds: $g(x) - c$ changes sign at most once for every $c \in [0, \infty)$ and, if there is a sign change, then the sequence must be $- +$ $(+-)$. □

For coherent systems with two or more components, the density function $\frac{g_{\gamma,\mathbf{s}}\circ\tilde{G}_{\gamma}^{-1}}{\tilde{g}_{\gamma}\circ\tilde{G}_{\gamma}^{-1}}$ cannot be a constant function on an interval. This follows from the next lemma, because every coherent system with $n \geq 2$ components has a signature vector with $s_1 = 0$ or $s_n = 0$ (see Miziuła and Rychlik 2015, Remark 3).

Lemma 1 *Let $\mathbf{s} = (s_1, \ldots, s_n)$ be a vector with non-negative entries and $s_1 + \cdots + s_n = 1$. If $\mathbf{s} \neq (\frac{1}{n}, \ldots, \frac{1}{n})$, then the density function $\frac{g_{\gamma,\mathbf{s}}\circ\tilde{G}_{\gamma}^{-1}}{\tilde{g}_{\gamma}\circ\tilde{G}_{\gamma}^{-1}}$ is not constant on any subinterval of* $(0, 1)$.

Proof The proof proceeds by contradiction. Assume that there exists an open interval $I \subset (0, 1)$ and a constant $c > 0$ such that

$$\frac{g_{\boldsymbol{\gamma},\mathbf{s}}(x)}{\tilde{g}_{\boldsymbol{\gamma}}(x)} = c, \quad x \in I.$$

We conclude that

$$0 = g_{\boldsymbol{\gamma},\mathbf{s}}(x) - c\tilde{g}_{\boldsymbol{\gamma}}(x) = \sum_{r=1}^{n} \left(s_r - \frac{c}{n}\right) g_{\boldsymbol{\gamma},r}(x), \quad x \in I.$$

This violates Lemma 2 of Bieniek (2007) which yields that the number of zeroes of the function $g_{\boldsymbol{\gamma},\mathbf{s}} - c\tilde{g}_{\boldsymbol{\gamma}}$ in (0, 1) does not exceed $n - 1$, so it must be finite. □

The following theorem yields values of the considered density at the boundaries of its domain. Note that the limit in $x = 1$ can be also explicitly derived for arbitrary $\gamma_1, \ldots, \gamma_n > 0$ (see Bieniek 2007, Lemma 3, Burkschat and Navarro 2013, Lemma 2.2).

Theorem 5 *Let* $h_{\boldsymbol{\gamma},\mathbf{s}} = \frac{g_{\boldsymbol{\gamma},\mathbf{s}} \circ \tilde{G}_{\boldsymbol{\gamma}}^{-1}}{\tilde{g}_{\boldsymbol{\gamma}} \circ \tilde{G}_{\boldsymbol{\gamma}}^{-1}}$. *Then*

$$h_{\boldsymbol{\gamma},\mathbf{s}}(0) = \lim_{x \to 0+} h_{\boldsymbol{\gamma},\mathbf{s}}(x) = ns_1$$

and, if $\gamma_1 \geq \cdots \geq \gamma_n$,

$$h_{\boldsymbol{\gamma},\mathbf{s}}(1) = \lim_{x \to 1-} h_{\boldsymbol{\gamma},\mathbf{s}}(x) = ns_n.$$

Proof By applying L'Hôpital's rule, we obtain for $1 \leq r < k \leq n$ (cf. Burkschat and Navarro 2013, Lemma 2.2)

$$\lim_{u \to 0+} \frac{g_{\boldsymbol{\gamma},k}(u)}{g_{\boldsymbol{\gamma},r}(u)} = 0$$

and, if $\gamma_1 \geq \cdots \geq \gamma_n$,

$$\lim_{u \to 1-} \frac{g_{\boldsymbol{\gamma},r}(u)}{g_{\boldsymbol{\gamma},k}(u)} = 0.$$

In particular, we get then

$$h_{\boldsymbol{\gamma},\mathbf{s}}(0) = \lim_{u \to 0+} \frac{g_{\boldsymbol{\gamma},\mathbf{s}}(u)/g_{\boldsymbol{\gamma},1}(u)}{\tilde{g}_{\boldsymbol{\gamma}}(u)/g_{\boldsymbol{\gamma},1}(u)} = \frac{s_1}{\frac{1}{n}}$$

and

$$h_{\boldsymbol{\gamma},\mathbf{s}}(1) = \lim_{u \to 1-} \frac{g_{\boldsymbol{\gamma},\mathbf{s}}(u)/g_{\boldsymbol{\gamma},n}(u)}{\tilde{g}_{\boldsymbol{\gamma}}(u)/g_{\boldsymbol{\gamma},n}(u)} = \frac{s_n}{\frac{1}{n}}.$$

This yields the assertion. □

Remark 1 Theorem 4, Theorem 5 *and* Lemma 1 *yield that the density function* $\frac{g_{\boldsymbol{\gamma},\mathbf{s}} \circ \tilde{G}_{\boldsymbol{\gamma}}^{-1}}{\tilde{g}_{\boldsymbol{\gamma}} \circ \tilde{G}_{\boldsymbol{\gamma}}^{-1}}$ *is strictly unimodal, i.e., there exists a mode* $m \in (0, 1)$ *such that* $\frac{g_{\boldsymbol{\gamma},\mathbf{s}} \circ \tilde{G}_{\boldsymbol{\gamma}}^{-1}}{\tilde{g}_{\boldsymbol{\gamma}} \circ \tilde{G}_{\boldsymbol{\gamma}}^{-1}}$ *is strictly increasing on* $(0, m)$ *and strictly decreasing on* $(m, 1)$*, in the case of a coherent system with* $\gamma_1 \geq \cdots \geq \gamma_n$ *and unimodal signature vector* $\mathbf{s} = (s_1, \ldots, s_n)$*, where* $s_1 = 0$ *and* $s_n = 0$.

We close the paper with exemplary numerical evaluations of the bounds derived in Section 2 for some specific coherent system.

First assume that the density $\frac{g_{\boldsymbol{\gamma},\mathbf{s}}\circ\tilde{G}_{\boldsymbol{\gamma}}^{-1}}{\tilde{g}_{\boldsymbol{\gamma}}\circ\tilde{G}_{\boldsymbol{\gamma}}^{-1}}$ is strictly unimodal. Then the Moriguti equation takes on the form

$$\frac{g_{\boldsymbol{\gamma},\mathbf{s}}\circ\tilde{G}_{\boldsymbol{\gamma}}^{-1}(u)}{\tilde{g}_{\boldsymbol{\gamma}}\circ\tilde{G}_{\boldsymbol{\gamma}}^{-1}(u)}=\frac{1-G_{\boldsymbol{\gamma},\mathbf{s}}\circ\tilde{G}_{\boldsymbol{\gamma}}^{-1}(u)}{1-u}. \tag{28}$$

Equivalently, putting $u=\tilde{G}_{\boldsymbol{\gamma}}(y)$ for $y\in[0,1]$, we obtain

$$\frac{g_{\boldsymbol{\gamma},\mathbf{s}}(y)}{\tilde{g}_{\boldsymbol{\gamma}}(y)}=\frac{1-G_{\boldsymbol{\gamma},\mathbf{s}}(y)}{1-\tilde{G}_{\boldsymbol{\gamma}}(y)}. \tag{29}$$

By Lemma 5 of Bieniek (2008) we have

$$1-G_{\boldsymbol{\gamma},r}(y)=(1-y)\sum_{j=1}^{r}\frac{1}{\gamma_j}g_{\boldsymbol{\gamma},j}(y),$$

so that an easy manipulation yields

$$1-G_{\boldsymbol{\gamma},\mathbf{s}}(y)=(1-y)\sum_{j=1}^{n}\left(\sum_{r=j}^{n}s_r\right)\frac{1}{\gamma_j}g_{\boldsymbol{\gamma},j}(y).$$

In particular, putting $s_r=\frac{1}{n}$, $1\le r\le n$, we get

$$1-\tilde{G}_{\boldsymbol{\gamma}}(y)=\frac{1-y}{n}\sum_{j=1}^{n}\frac{n-j+1}{\gamma_j}g_{\boldsymbol{\gamma},j}(y).$$

Therefore the Moriguti equation (29) turns into

$$\frac{\sum_{j=1}^{n}s_j g_{\boldsymbol{\gamma},j}(y)}{\sum_{j=1}^{n}g_{\boldsymbol{\gamma},j}(y)}=\frac{\sum_{j=1}^{n}\left(\sum_{r=j}^{n}s_r\right)\frac{1}{\gamma_j}g_{\boldsymbol{\gamma},j}(y)}{\sum_{j=1}^{n}\frac{n-j+1}{\gamma_j}g_{\boldsymbol{\gamma},j}(y)}.$$

Under the assumption of strict unimodality of the corresponding density this equation has the unique root $\underline{y}_*\in(0,1)$.

Now we consider a situation where the (constant) overall load on an n-component system is evenly distributed among all still operating units. This load is described by means of constant parameters $\gamma_r=n$, $r=1,\dots,n$, of the generalized order statistics model. Then, at the rth stage between the $(r-1)$st and rth failure, the individual load on the intact components is $\alpha_r=n/(n-r+1)$ (see, e.g., Balakrishnan et al. 2011, Example 1 or Burkschat and Navarro 2013, Remark 2.3). In particular, the failure times can be also interpreted as consecutive rth values of n-records (see Dziubdziela and Kopociński 1976; Kamps 1995a;). Furthermore, in this setting we consider k-out-of-n:F systems with $\mathbf{s}$ given by $s_k=1$ and $s_r=0$ for $r\neq k$.

If $k=1$, then the signature and the corresponding density are decreasing (see Corollary 1), and so the derivative of the greatest convex minorant is just constant equal to 1. Therefore the corresponding upper bounds from Theorems 1, 2 and 3 amount to 0. If $k=n$, then the signature and the corresponding density are increasing, so that the derivative of the greatest convex minorant is equal to the original density.

If $2\le k\le n-1$, then the signature satisfies the condition from Remark 1 and the Moriguti equation takes on the simple form

$$\frac{g_{\boldsymbol{\gamma},k}(y)}{\sum_{j=1}^{n}g_{\boldsymbol{\gamma},j}(y)}=\frac{\sum_{j=1}^{k}g_{\boldsymbol{\gamma},j}(y)}{\sum_{j=1}^{n}(n-j+1)g_{\boldsymbol{\gamma},j}(y)}, \tag{30}$$

where

$$g_{\gamma,r}(y) = \frac{n^r}{(r-1)!}(1-y)^{n-1}\left[-\ln(1-y)\right]^{r-1}, \quad y \in (0,1).$$

Note that Eq. 30 is in fact a polynomial equation with respect to $z = -\log(1-y)$. It has a unique root $\underline{y}_* \in (0,1)$ which is determined numerically. Then $\underline{u}_* = \tilde{G}_{\gamma}(\underline{y}_*)$ is the unique solution to Eq. 28. Denoting $h_{\gamma,\mathbf{s}} = \frac{g_{\gamma,\mathbf{s}}\circ\tilde{G}_{\gamma}^{-1}}{\tilde{g}_{\gamma}\circ\tilde{G}_{\gamma}^{-1}}$ for $2 \le k < n$ we have by Eq. 25

$$\underline{h}_{\gamma,\mathbf{s}}(x) = \begin{cases} h_{\gamma,\mathbf{s}}(x), & \text{for } 0 \le x < \underline{u}_*, \\ h_{\gamma,\mathbf{s}}(\underline{u}_*), & \text{for } \underline{u}_* \le x \le 1. \end{cases}$$

For $k = n$ we simply have $\underline{h}_{\gamma,\mathbf{s}} = h_{\gamma,\mathbf{s}}$, so that we put $u_* = 1$ in this case. By Eq. 17 this easily implies that for $2 \le k \le n$

$$B_{\gamma,\mathbf{s}}(2) = \left[\int_0^{\underline{y}_*} \frac{(g_{\gamma,\mathbf{s}}(y))^2}{\tilde{g}_{\gamma}(y)}dy + \frac{g_{\gamma,\mathbf{s}}(\underline{y}_*)}{\tilde{g}_{\gamma}(\underline{y}_*)}\left(1 - G_{\gamma,\mathbf{s}}(\underline{y}_*)\right) - 1\right]^{1/2}.$$

Moreover, for $2 \le k \le n$ by Eqs. 18 and 20,

$$B_{\gamma,\mathbf{s}}(1) = \frac{1}{2}\frac{g_{\gamma,\mathbf{s}}(\underline{y}_*)}{\tilde{g}_{\gamma}(\underline{y}_*)},$$

$$B_{\gamma,\mathbf{s}}(\infty) = \begin{cases} 1 - 2G_{\gamma,\mathbf{s}}\circ\tilde{G}_{\gamma}^{-1}\left(\frac{1}{2}\right), & \text{if } \tilde{G}_{\gamma}(\underline{y}_*) > \frac{1}{2}, \\ \frac{g_{\gamma,\mathbf{s}}(\underline{y}_*)}{\tilde{g}_{\gamma}(\underline{y}_*)} - 1, & \text{if } \tilde{G}_{\gamma}(\underline{y}_*) \le \frac{1}{2}. \end{cases}$$

Applying the last three formulae we derive numerical values of upper bounds corresponding to k-out-of-n systems in the load sharing setup with $n = 10$ and $k = 2, \ldots, 10$. They are presented in Table 1.

The formulae for calculating corresponding lower bounds are more complicated. For the parallel system with $k = n$, density function $h_{\gamma,\mathbf{s}} = \frac{g_{\gamma,\mathbf{s}}\circ\tilde{G}_{\gamma}^{-1}}{\tilde{g}_{\gamma}\circ\tilde{G}_{\gamma}^{-1}}$ is increasing so that $\overline{h}_{\gamma,\mathbf{s}}(u) = 1, 0 < u < 1$, and respective bounds of Theorems 1–3 are zero. For $2 \le k \le n-1$ the breaking point of Eq. 26 is $\overline{u}_* = \tilde{G}_{\gamma}(\overline{y}_*)$, where $0 < \overline{y}_* < 1$ uniquely solves

$$\left[1 - (1-y)^{10}\sum_{i=0}^{k-1}\frac{[-10\ln(1-x)]^i}{i!}\right]\sum_{i=0}^{9}\frac{[-10\ln(1-x)]^i}{i!}$$
$$= \left[10 - (1-y)^{10}\sum_{i=0}^{9}(10-i)\frac{[-10\ln(1-x)]^i}{i!}\right]\frac{[-10\ln(1-x)]^{k-1}}{(k-1)!},$$

and $\overline{u}_* = \overline{y}_* = 0$ for $k = 1$. Then for $1 \le k \le n-1$ we obtain

$$b_{\gamma,\mathbf{s}}(1) = -\frac{1}{2}\frac{g_{\gamma,\mathbf{s}}(\overline{y}_*)}{\tilde{g}_{\gamma}(\overline{y}_*)},$$

$$b_{\gamma,\mathbf{s}}(2) = -\left[\frac{g_{\gamma,\mathbf{s}}(\overline{y}_*)}{\tilde{g}_{\gamma}(\overline{y}_*)}G_{\gamma,\mathbf{s}}(\underline{y}_*) + \int_{\overline{y}_*}^{1}\frac{(g_{\gamma,\mathbf{s}}(y))^2}{\tilde{g}_{\gamma}(y)}dy - 1\right]^{1/2},$$

$$b_{\gamma,\mathbf{s}}(\infty) = \begin{cases} 2G_{\gamma,\mathbf{s}}\circ\tilde{G}_{\gamma}^{-1}\left(\frac{1}{2}\right) - 1, & \text{if } \tilde{G}_{\gamma}(\overline{y}_*) < \frac{1}{2}, \\ \frac{g_{\gamma,\mathbf{s}}(\overline{y}_*)}{\tilde{g}_{\gamma}(\overline{y}_*)} - 1, & \text{if } \tilde{G}_{\gamma}(\overline{y}_*) \ge \frac{1}{2}. \end{cases}$$

Table 1 Numerical values of $b_{\gamma,\mathbf{s}}(p)$ and $B_{\gamma,\mathbf{s}}(p)$ with $p = 1, 2$ and ∞ for the load–sharing k-out-of-10:F systems

k	$b_{\gamma,\mathbf{s}}(1)$	$B_{\gamma,\mathbf{s}}(1)$	$b_{\gamma,\mathbf{s}}(2)$	$B_{\gamma,\mathbf{s}}(2)$	$b_{\gamma,\mathbf{s}}(\infty)$	$B_{\gamma,\mathbf{s}}(\infty)$
1	-5	0	-2.0000	0	-0.9934	0
2	-1.4921	0.5027	-1.1668	0.0595	-0.9207	0.0054
3	-0.9713	0.5196	-0.8065	0.1725	-0.7545	0.0393
4	-0.7383	0.5502	-0.5574	0.2817	-0.4764	0.1003
5	-0.6095	0.5949	-0.3577	0.3897	-0.2190	0.1899
6	-0.5376	0.6596	-0.1906	0.5027	-0.0752	0.3191
7	-0.5057	0.7588	-0.0620	0.6288	-0.0113	0.5176
8	$-0.5-2.4\cdot10^{-5}$	0.9372	-0.0028	0.7816	$-4.95\cdot10^{-5}$	0.8166
9	$-0.5-4.9\cdot10^{-27}$	1.3807	$-1.53\cdot10^{-14}$	0.9849	$-9.98\cdot10^{-27}$	0.9071
10	0	5	0	1.2688	0	0.9567

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