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# Finite-Horizon Ruin Probabilities in a Risk-Switching Sparre Andersen Model

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**Abstract** After implementation of Solvency II, insurance companies can use internal risk models. In this paper, we show how to calculate finite-horizon ruin probabilities and prove for them new upper and lower bounds in a risk-switching Sparre Andersen model. Due to its flexibility, the model can be helpful for calculating some regulatory capital requirements. The model generalizes several discrete time- as well as continuous time risk models. A Markov chain is used as a 'switch' changing the amount and/or respective wait time distributions of claims while the insurer can adapt the premiums in response. The envelopes of generalized moment generating functions are applied to bound insurer's ruin probabilities.

**Keywords** Risk operators · Risk-switching models · Ruin probabilities · Mgf's envelopes · Risk management based on internal models · Solvency II

**Mathematics Subject Classification (2010)** 91B30 · 60J20 · 60J22

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## 1 Introduction

Solvency II changes the insurance industry completely, not only in the EU but also in the USA and Asia. In accordance with Motive (68) of this EU directive (see Solvency II 2009, p. L 335/7), each insurance company can calculate the Solvency Capital Requirement (SCR) with the help of its own tailor-made partial or full internal model. According to Motive (64) of the directive, SCR should be large enough in order to ensure that ruin occurs no more often than once in every 200 cases over the following 12 months. Thus, the probability of default should be analyzed by insurers and supervisors when SCR is to be established, and the model and methods investigated in the present paper can be found useful from this perspective.

To be more concrete, we will show how to calculate finite-horizon ruin probabilities and introduce lower and upper bounds for them in the following risk-switching Sparre Andersen model. Let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{R}$ - the real line. Set $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}^1 = \mathbb{N}\backslash\{1\}$, $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_+^0 = [0, \infty)$. All stochastic objects considered in the paper are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let a random variable $X_k$ denote the amount of the $k$th claim, $T_1$ - the moment when the first claim appears and $T_k$ - the time between the $(k-1)$th claim and the $k$th one, $k \in \mathbb{N}^1$. Let $A_n$ denote the moment when the $n$th claim appears. With this notation, $A_n = T_1 + \ldots + T_n,\ n \in \mathbb{N}^0$, under the convention that $A_0 = 0$. A random variable $C_k$ will denote the insurance premium rate at the time interval $[A_{k-1},\ A_k)$. We assume that the random variables $C_k$, $T_k$ and $X_k$ are positive (a.s.) for each $k \in \mathbb{N}$ and that their distributions have no singular parts.

Let $\{I_k\}_{k\in\mathbb{N}^0}$ be a Markov chain with

1. a finite state space $S = \{1, 2, \ldots, s\},\ s \in \mathbb{N}$;
2. an initial distribution $(p_i)_{i\in S}$ such that the probabilities $p_i = \mathbb{P}(I_0 = i)$ are positive for each $i \in S$;
3. a transition matrix $P = \big(p_{ij}\big)_{i,j\in S}$ such that the probabilities $p_{ij} = \mathbb{P}\big(I_{k+1} = j | I_k = i\big)$ are positive for all $i,\ j \in S$.

We assume that the probabilities $p_{ij}$ are independent of $k \in \mathbb{N}^0$. The jump from $I_{k-1}$ to $I_k$ can change the distribution of $T_k$ and/or $X_k$ at the moment $A_k$ only, so we can treat $\{I_k\}_{k\in\mathbb{N}^0}$ as 'switches'. The insurance premium rate is assumed to satisfy $C_k = c(I_{k-1})$, where $c$ is a known positive function defined on $S$.

Special cases of particular interest of the above risk-switching model are: a discrete time risk-switching model (see Section 4), a risk-switching model with exponentially distributed wait times, the Sparre Andersen model, the classical Cramér-Lundberg model and a discrete time risk model without a switch. Other Markov switching models can be found e.g. in Frühwirth-Schnatter (2006). Markov additive processes are studied in Asmussen (2003) or Feng and Shimizu (2014) among others. The Markov-modulated Poisson processes are investigated e.g. in Reinhard (1984), Asmussen (1989), Asmussen and Albrecher (2010) or Guillou et al. (2013). To the best of our knowledge, Taylor (1976) was the first researcher who used the operator approach to evaluate ruin probabilities. This method was further developed and generalized by Gajek (2005) and Gajek and Rudź (2013, 2017, 2018). In the latter article, more references on the operator approach in ruin theory can be found.

The continuous-time risk theory in a Markovian environment is presented e.g. in Asmussen and Albrecher (2010). This approach dates back to Reinhard (1984) and Asmussen (1989), where the detailed references to queuing theory can be found. Since then, several regime-switching models have been studied (see e.g. Xu et al. 2017; Wang et al. 2016; Landriault et al. 2015; Chen et al. 2014; Guillou et al. 2013).

Denote $Z_k = X_k - c(I_{k-1})T_k, k \in \mathbb{N}$. Let $u \geqslant 0$ denote the insurer's surplus at 0 and $U_n = U(n, u)$ - at the moment $A_n$, respectively. The *surplus process* (*risk process*) $\{U_n\}_{n\in\mathbb{N}}$ is defined in the following way:

$$U_n = u - \sum_{k=1}^{n} Z_k . \tag{1}$$

The *time of ruin* $\tau$ is the first moment when the insurer's surplus falls below zero, i.e.

$$\tau = \tau(u) = \inf\{n \in \mathbb{N} : U(n, u) < 0\}, \tag{2}$$

where $\inf \emptyset$ means $\infty$. The conditional probability that $\tau(u)$ is not greater than $n$, given the initial state $i$, is called the probability of ruin at or before the $n$th claim. Let us denote it by $\Psi_n^i(u)$. By definition,

$$\Psi_0^i(u) = 0, \quad i \in S,\ u \geqslant 0 . \tag{3}$$

In a risk-switching model, we have to consider $s$ ruin probabilities $\Psi_n^1$, ..., $\Psi_n^s$ and a vector formed by them. So let us denote

$$\mathbf{\Psi}_n(u) = (\Psi_n^1(u), ..., \Psi_n^s(u)) \tag{4}$$

where $n \in \mathbb{N}^0$ and $u \geqslant 0$.

Let $F^{ij}$ (respectively $G^{ij}$) denote the conditional distribution of $X_1$ (respectively $T_1$), given the initial state $i$ and the state $j$ at the moment $A_1$. Let

$$\overline{m}^i(r) = \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{c(i)t}^\infty e^{-r(c(i)t-x)} dF^{ij}(x) dG^{ij}(t) , \tag{5}$$

$$M^i(r) = \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_0^\infty e^{-r(c(i)t-x)} dF^{ij}(x) dG^{ij}(t) \tag{6}$$

and

$$\underline{m}^i(r) = M^i(r) - \overline{m}^i(r) \tag{7}$$

for all $i \in S$ and $r \in \mathbb{R}$. Assume that positive constants $r_0^1$, ..., $r_0^s$ satisfy the following equations:

$$M^i(r_0^i) = 1, \quad i \in S , \tag{8}$$

forming a vector-type counterpart of the so-called *adjustment coefficient*. It will be called an *adjustment vector*. In Section 3, we will prove the following new upper bound on $\Psi_n^i(u)$:

$$\Psi_n^i(u) \leqslant \inf_{r\in(0,\, r_0^*)} \left\{ e^{-ru} m^*(r) \frac{1 - [M^*(r)]^n}{1 - M^*(r)} \right\}, \quad i \in S,\ n \in \mathbb{N},\ u \geqslant 0 , \tag{9}$$

using the envelopes $m^*(r) = \max\{\overline{m}^i(r) : i \in S\}$ and $M^*(r) = \max\{M^i(r) : i \in S\}$, where $r \geqslant 0$ and $r_0^* = \min\{r_0^i : i \in S\}$ (see Theorem 3 for details). We compare (9) with a generalized Gerber's inequality in Example 2. We present also a numerical example in which (9) turns out to be attainable (see Example 1 for details).

Let $\mathcal{R}$ denote the set of all measurable functions defined on $\mathbb{R}_+^0$ and taking values in $[0, 1]$. The set $\{(\rho_1, ..., \rho_s) : \rho_i \in \mathcal{R}$ for every $i \in S\}$ will be denoted by $\mathcal{R}^s$. The elements of $\mathcal{R}^s$ will be written in bold. We call $\mathbf{L} = (\mathrm{L}_1, ..., \mathrm{L}_s) : \mathcal{R}^s \to \mathcal{R}^s$ *the operator generated by the risk process* (in short *risk operator*) if

$$\mathbf{L}\boldsymbol{\rho}(u) = (\mathrm{L}_1\boldsymbol{\rho}(u), ..., \mathrm{L}_s\boldsymbol{\rho}(u)), \quad u \geqslant 0, \tag{10}$$

where, under the convention that $\int\limits_a^\infty$ means $\int\limits_{(a,\infty)}$ for every $a \geqslant 0$,

$$\begin{aligned} \mathrm{L}_i\boldsymbol{\rho}(u) &= \sum_{j=1}^{s} p_{ij} \int\limits_0^\infty \int\limits_{(0,\,u+c(i)t]} \rho_j(u + c(i)t - x) dF^{ij}(x) dG^{ij}(t) \\ &\quad + \sum_{j=1}^{s} p_{ij} \int\limits_0^\infty \int\limits_{u+c(i)t}^\infty dF^{ij}(x) dG^{ij}(t), \quad i \in S. \end{aligned} \tag{11}$$

Let $R_0^i(u, r) = e^{-ru}$ and $R_n^i(u, r)$ be the $i$th coordinate of the $n$th iteration of the associated risk operator $\mathbf{L}$ on $\mathbf{R}_0(u, r) = (R_0^1(u, r), ..., R_0^s(u, r))$. The following new two-sided bound holds for all $i \in S$, $n \in \mathbb{N}$ and $u \geqslant 0$:

$$\begin{aligned} \sup_{r\in\mathbb{R}_+} \{R_n^i(u, r) - e^{-ru} M^i(r)[M^*(r)]^{n-1}\} &\leqslant \Psi_n^i(u) \\ &\leqslant \inf_{r\in\mathbb{R}_+} \{R_n^i(u, r) - e^{-ru} \underline{m}^i(r)[m_*(r)]^{n-1}\} \end{aligned} \tag{12}$$

where $m_*(r) = \min\{\underline{m}^i(r) : i \in S\}, r \geqslant 0$ (see Section 2 and Theorem 4 for details). As Example 3 shows, (12) can be attainable.

## 2 Auxiliary Results

In this section, we provide some basic facts concerning the risk operator $\mathbf{L}$. Let $B \in \mathcal{F}$. We will use the following notation:

$$\begin{aligned} \mathbb{P}^i(B) &= \mathbb{P}(B|I_0 = i), \\ \mathbb{P}^{ij}(B) &= \mathbb{P}(B|I_0 = i, I_1 = j), \\ H^{ij}(t, x) &= \mathbb{P}^{ij}(T_1 \leqslant t, X_1 \leqslant x), \end{aligned}$$

where $i, j \in S$ and $t, x \in \mathbb{R}_+$. With this notation,

$$\begin{aligned} F^{ij}(x) &= \mathbb{P}^{ij}\big(X_1 \leqslant x\big), \\ G^{ij}(t) &= \mathbb{P}^{ij}(T_1 \leqslant t), \\ \Psi_n^i(u) &= \mathbb{P}^i(\tau(u) \leqslant n), \qquad (13) \\ \Psi^i(u) &= \mathbb{P}^i(\tau(u) < \infty). \qquad (14) \end{aligned}$$

Let us define $\ell_i : \mathcal{R}^s \to \mathcal{R}$ by

$$\ell_i\boldsymbol{\rho}(u) = \sum_{j=1}^{s} p_{ij} \int\limits_0^\infty \int\limits_{(0,\,u+c(i)t]} \rho_j(u + c(i)t - x) dF^{ij}(x) dG^{ij}(t), \quad i \in S, \ u \geqslant 0. \tag{15}$$

Clearly, $\ell_i$ is a linear component of the affine operator $\mathrm{L}_i$ and $\boldsymbol{\ell\rho} = (\ell_1\boldsymbol{\rho}, ..., \ell_s\boldsymbol{\rho})$ is a linear operator transforming $\mathcal{R}^s$ into $\mathcal{R}^s$. We will use the convention

$$\boldsymbol{\ell}^0\boldsymbol{\rho}(u) = \boldsymbol{\rho}(u), \quad \boldsymbol{\ell}^1\boldsymbol{\rho}(u) = \boldsymbol{\ell\rho}(u) = (\ell_1\boldsymbol{\rho}(u), ..., \ell_s\boldsymbol{\rho}(u)),$$

$$\mathbf{L}^0\boldsymbol{\rho}(u) = \boldsymbol{\rho}(u), \quad \mathbf{L}^1\boldsymbol{\rho}(u) = \mathbf{L}\boldsymbol{\rho}(u) = (\mathrm{L}_1\boldsymbol{\rho}(u), ..., \mathrm{L}_s\boldsymbol{\rho}(u)),$$

where $\boldsymbol{\rho} \in \mathcal{R}^s$ and $u \geqslant 0$. For all $n \in \mathbb{N}$, $\boldsymbol{\rho} \in \mathcal{R}^s$ and $u \geqslant 0$, the following properties hold:

$$\boldsymbol{\ell}^n\boldsymbol{\rho}(u) = (\ell_1\boldsymbol{\ell}^{n-1}\boldsymbol{\rho}(u), ..., \ell_s\boldsymbol{\ell}^{n-1}\boldsymbol{\rho}(u)) \tag{16}$$

and

$$\mathbf{L}^n\boldsymbol{\rho}(u) = (\mathrm{L}_1\mathbf{L}^{n-1}\boldsymbol{\rho}(u), ..., \mathrm{L}_s\mathbf{L}^{n-1}\boldsymbol{\rho}(u)).$$

By the ideas of Gajek (2005), one can verify the following relationship concerning $\boldsymbol{\Psi}_{n+1}$, $\boldsymbol{\Psi}_1$ and the risk operator $\mathbf{L}$ (for the proof, see Gajek and Rudź 2018):

**Theorem 1** *Let the following assumptions hold for all $i, j \in S$, $k \in \mathbb{N}^1$ and $t, x \in \mathbb{R}_+$:*

(i) *The conditional distribution of the random variables $Z_2, ..., Z_k$, given $\big(I_0 = i,\ I_1 = j,\ T_1 = t,\ X_1 = x\big)$, is the same as the conditional distribution of the random variables $Z_1, ..., Z_{k-1}$, given $I_0 = j$;*

(ii) *$H^{ij}(t, x) = F^{ij}(x)G^{ij}(t)$.*

*Then*

$$\boldsymbol{\Psi}_{n+1}(u) = \mathbf{L}\boldsymbol{\Psi}_n(u) = \mathbf{L}^n\boldsymbol{\Psi}_1(u) \tag{17}$$

*for all $n \in \mathbb{N}^0$ and $u \geqslant 0$.*

Let $M^*(r) = \max\{M^i(r) : i \in S\}$, where $M^i$ are defined by (6). The following generalization of Gerber's inequality for ruin probabilities $\Psi_n^i$ comes from Gajek and Rudź (2017), where the detailed proof can be found.

**Theorem 2** *Let the assumptions of Theorem 1 hold and assume that the adjustment vector $(r_0^1, ..., r_0^s)$ with positive coordinates exists. Then, for all $i \in S$, $n \in \mathbb{N}$ and $u \geqslant 0$,*

$$\Psi_n^i(u) \leqslant \inf_{r \geqslant r_0^*} \{e^{-ru}M^i(r)[M^*(r)]^{n-1}\} \tag{18}$$

*where $r_0^* = \min\{r_0^i : i \in S\}$.*

In the next section, we will prove a new upper bound on $\Psi_n^i$ and compare it with (18).

## 3 Main Results

Below, we provide an upper bound for $\Psi_n^i$ which uses the envelopes $m^*(r) = \max\{\overline{m}^i(r) : i \in S\}$ and $M^*(r) = \max\{M^i(r) : i \in S\}$, where $\overline{m}^i$ and $M^i$ are defined by (5) and (6), respectively. Recall also that $r_0^* = \min\{r_0^i : i \in S\}$.

**Theorem 3** *Let the assumptions of Theorem 2 hold. Then*

$$\Psi_n^i(u) \leqslant \inf_{r \in (0, r_0^*)} \left\{e^{-ru}m^*(r)\frac{1 - [M^*(r)]^n}{1 - M^*(r)}\right\}, \quad i \in S,\ n \in \mathbb{N},\ u \geqslant 0. \tag{19}$$

*Proof* Let $r \in (0,\ r_0^*)$ be fixed. Consider the following inequality:

$$\Psi_n^i(u) \leqslant e^{-ru} m^*(r) \frac{1-[M^*(r)]^n}{1-M^*(r)}\,. \tag{20}$$

Note that (20) holds for $n = 1$. Indeed,

$$\Psi_1^i(u) \leqslant \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{u+c(i)t}^\infty e^{-r(u+c(i)t-x)} dF^{ij}(x) dG^{ij}(t) \leqslant e^{-ru} m^*(r) \tag{21}$$

by Theorem 1. Assume that (20) holds for some $n \in \mathbb{N}$. We will show that it holds for $n+1$ as well. Indeed, note that Theorem 1 and (21) yield

$$\begin{aligned}
\Psi_{n+1}^i(u) &= \mathrm{L}_i \boldsymbol{\Psi}_n(u) \\
&= \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{(0,\,u+c(i)t]} \Psi_n^j(u+c(i)t-x) dF^{ij}(x) dG^{ij}(t) + \Psi_1^i(u) \\
&\leqslant m^*(r) \frac{1-[M^*(r)]^n}{1-M^*(r)} \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{(0,\,u+c(i)t]} e^{-r(u+c(i)t-x)} dF^{ij}(x) dG^{ij}(t) \\
&\quad + e^{-ru} m^*(r) \\
&\leqslant e^{-ru} m^*(r) \Big(1 + M^*(r) \frac{1-[M^*(r)]^n}{1-M^*(r)}\Big) = e^{-ru} m^*(r) \frac{1-[M^*(r)]^{n+1}}{1-M^*(r)}\,.
\end{aligned}$$

By the induction principle, (20) holds for every $n \in \mathbb{N}$. Since $r \in (0,\ r_0^*)$ is arbitrary, (19) holds. □

Let $\mathrm{I}_A(x) = 1$ if $x \in A$ and 0 otherwise.

*Example 1* We will show that (19) is attainable even for $n = s = 1$. Consider the discrete time risk model without a switch. Denote the aggregated amount of premiums received each period by $\gamma$. Let $F$ be the common distribution function of the sums of claims in successive time periods. By Theorem 3,

$$\Psi_n(u) \leqslant \inf_{r \in (0,\, r_0)} \left\{ e^{-ru} m(r) \frac{1-[M(r)]^n}{1-M(r)} \right\}.$$

Assume that $\gamma = 12, u = 0.01, n = 1$ and $F(x) = (1-e^{-\beta x})\, \mathrm{I}_{(0,\,\infty)}(x)$, where $\beta = 0.1$. Then, the adjustment coefficient $r_0 \approx 0.031$. In this case, $\Psi_n(u) = e^{-\beta(u+\gamma)} = 0.301$. The graph of $e^{-ru} m(r)(1-[M(r)]^n)/(1-M(r))$, as a function of $r$, is presented in Fig. 1. We see that

$$\inf_{r \in (0,\, r_0)} \left\{ e^{-ru} m(r) \frac{1-[M(r)]^n}{1-M(r)} \right\} = 0.301 = \Psi_n(u)\,,$$

so the upper bound resulting from Theorem 3 is attainable to at least three decimal places.

The next example shows that (18) can be more accurate than (19) for relatively small $\gamma$, while for large $\gamma$ the opposite is true.

*Example 2* Consider the model discussed in Example 1. Assume that $n = 10$, $u = 0.1$ and $F(x) = (1-e^{-\beta x})\, \mathrm{I}_{(0,\,\infty)}(x)$, where $\beta = 6$.

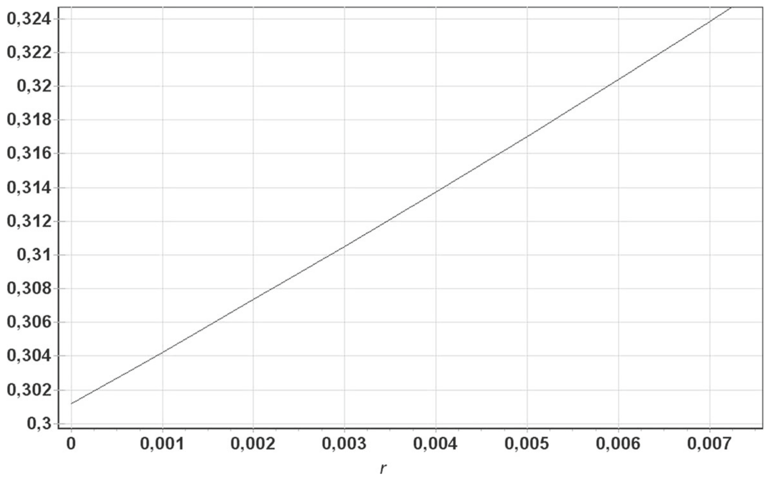


**Fig. 1** $(1-[M(r)]^n)e^{-ru}m(r)/(1-M(r))$, as a function of $r$ (zoom of an exponential-like curve), for an exponential distribution with scale parameter 0.1, amount of aggregated premiums $\gamma = 12$, initial surplus $u = 0.01$ and $n = 1$

*Case 1* $\gamma = 0.3$. Then, the adjustment coefficient $r_0 \approx 4.395$ and

$$\inf_{r \geqslant r_0} \{e^{-ru}[M(r)]^n\} = 0.644 < 0.985 = \inf_{r\in(0,\, r_0)} \left\{ e^{-ru} m(r) \frac{1-[M(r)]^n}{1-M(r)} \right\}.$$

Hence (18) is more accurate than (19).

*Case 2* $\gamma = 0.5$. Then, the adjustment coefficient $r_0 \approx 5.643$ and

$$\inf_{r \geqslant r_0} \{e^{-ru}[M(r)]^n\} = 0.57 > 0.13 = \inf_{r\in(0,\, r_0)} \left\{ e^{-ru} m(r) \frac{1-[M(r)]^n}{1-M(r)} \right\}.$$

Hence (19) is better than (18).

The graphs of $e^{-ru}m(r)\frac{1-[M(r)]^n}{1-M(r)}$ and $e^{-ru}[M(r)]^n$, as functions of $r$, are presented in Figs. 2–3.

The above example shows that none of the inequalities (18) and (19) is better than the other. However, note that these inequalities can be unified in the following way. Let

$$\mu^i(r) = m^*(r)\, \mathrm{I}_{(0,\, r_0^*)}(r) + M^i(r)\, \mathrm{I}_{[r_0^*,\, \infty)}(r), \quad i \in S,\ r > 0\,.$$

From Theorems 2 and 3, we get the following

**Corollary 1** *Let the assumptions of Theorem 2 hold. Then*

$$\Psi_n^i(u) \leqslant \inf_{r\in\mathbb{R}_+} \left\{ e^{-ru} \mu^i(r) \frac{[M^*(r)]^n - \mathrm{I}_{(0,\, r_0^*)}(r)}{M^*(r) - \mathrm{I}_{(0,\, r_0^*)}(r)} \right\}, \quad i \in S,\ n \in \mathbb{N},\ u \geqslant 0\,. \tag{22}$$

Now, we show how to approximate ruin probabilities by model-adjusted exponential functions $\mathbf{R}_n(u, r)$. Given $r \in \mathbb{R}_+$, let us denote

$$R_0^i(u, r) = e^{-ru},$$

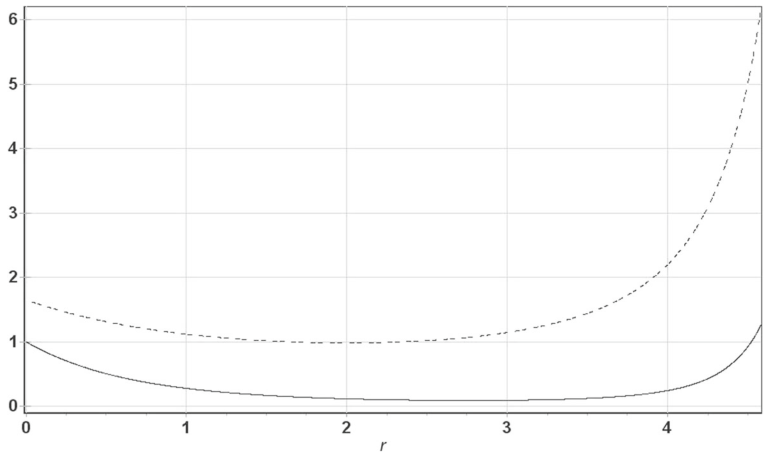


**Fig. 2** $(1 - [M(r)]^n)e^{-ru}m(r)/(1 - M(r))$ (the dashed line) and $e^{-ru}[M(r)]^n$ (the solid line), as functions of $r$, for an exponential distribution with scale parameter 6, amount of aggregated premiums $\gamma = 0.3$, initial surplus $u = 0.1$ and $n = 10$ (case 1)

and

$$\mathbf{R}_0(u, r) = (R_0^1(u, r), ..., R_0^s(u, r)) ,$$

where $i \in S$ and $u \geqslant 0$. For every $i \in S$, define iteratively a sequence $\{R_n^i\}_{n \in \mathbb{N}}$ by

$$R_n^i(u, r) = \mathrm{L}_i \mathbf{R}_{n-1}(u, r), \quad n \in \mathbb{N}, \ r > 0, \ u \geqslant 0 ,$$

where $\mathbf{R}_n(u, r) = (R_n^1(u, r), ..., R_n^s(u, r))$.

Our next result provides a two-sided bound for $\Psi_n^i(u)$ in terms of $M^*(r) = \max\{M^i(r) : i \in S\}$ and $m_*(r) = \min\{\underline{m}^i(r) : i \in S\}$, where $M^i$ and $\underline{m}^i$ are defined by (6) and (7), respectively.

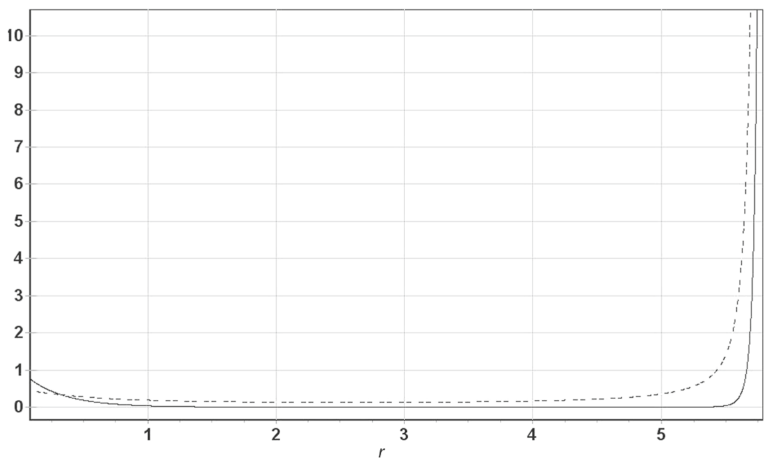


**Fig. 3** $(1 - [M(r)]^n)e^{-ru}m(r)/(1 - M(r))$ (the dashed line) and $e^{-ru}[M(r)]^n$ (the solid line), as functions of $r$, for an exponential distribution with scale parameter 6, amount of aggregated premiums $\gamma = 0.5$, initial surplus $u = 0.1$ and $n = 10$ (case 2)

**Theorem 4** *Let the assumptions of Theorem 2 hold. Then*

$$\inf_{r\in\mathbb{R}_+}\{R_n^i(u,r)-e^{-ru}\underline{m}^i(r)[m_*(r)]^{n-1}\}\geqslant \Psi_n^i(u)\geqslant \sup_{r\in\mathbb{R}_+}\{R_n^i(u,r)-e^{-ru}M^i(r)[M^*(r)]^{n-1}\}$$

*for all* $i\in S$, $n\in\mathbb{N}$ *and* $u\geqslant 0$.

*Proof* It is easy to see that

$$R_n^i(u,r)-\Psi_n^i(u)=\ell_i\boldsymbol{\ell}^{n-1}\mathbf{R}_0(u,r)\,. \tag{23}$$

Consider the following inequality:

$$e^{-ru}\underline{m}^i(r)[m_*(r)]^{n-1}\leqslant \ell_i\boldsymbol{\ell}^{n-1}\mathbf{R}_0(u,r)\leqslant e^{-ru}M^i(r)[M^*(r)]^{n-1} \tag{24}$$

where $i\in S$, $r\in\mathbb{R}_+$ and $u\geqslant 0$. We will show first that (24) holds for $n=1$. Indeed,

$$\begin{aligned}
e^{-ru}\underline{m}^i(r) &= e^{-ru}\sum_{j=1}^{s}p_{ij}\int_0^\infty\int_{(0,\,c(i)t]}e^{-r(c(i)t-x)}dF^{ij}(x)dG^{ij}(t)\\
&\leqslant \ell_i\mathbf{R}_0(u,r)\\
&= \sum_{j=1}^{s}p_{ij}\int_0^\infty\int_{(0,\,u+c(i)t]}e^{-r(u+c(i)t-x)}dF^{ij}(x)dG^{ij}(t)\\
&\leqslant e^{-ru}\sum_{j=1}^{s}p_{ij}\int_0^\infty\int_0^\infty e^{-r(c(i)t-x)}dF^{ij}(x)dG^{ij}(t)\\
&= e^{-ru}M^i(r)\,.
\end{aligned}$$

Assume that (24) holds for some $n\in\mathbb{N}$. We will show that it holds for $n+1$ as well. Indeed, note that (16) implies that

$$\begin{aligned}
e^{-ru}\underline{m}^i(r)[m_*(r)]^n &= [m_*(r)]^n\sum_{j=1}^{s}p_{ij}\int_0^\infty\int_{(0,\,c(i)t]}e^{-r(u+c(i)t-x)}dF^{ij}(x)dG^{ij}(t)\\
&\leqslant [m_*(r)]^{n-1}\sum_{j=1}^{s}p_{ij}\underline{m}^j(r)\int_0^\infty\int_{(0,\,u+c(i)t]}e^{-r(u+c(i)t-x)}dF^{ij}(x)dG^{ij}(t)\\
&\leqslant \ell_i\boldsymbol{\ell}^n\mathbf{R}_0(u,r)=\ell_i(\ell_1\boldsymbol{\ell}^{n-1}\mathbf{R}_0(u,r),\,...,\,\ell_s\boldsymbol{\ell}^{n-1}\mathbf{R}_0(u,r))\\
&= \sum_{j=1}^{s}p_{ij}\int_0^\infty\int_{(0,\,u+c(i)t]}\ell_j\boldsymbol{\ell}^{n-1}\mathbf{R}_0(u+c(i)t-x,r)dF^{ij}(x)dG^{ij}(t)\\
&\leqslant [M^*(r)]^{n-1}\sum_{j=1}^{s}p_{ij}M^j(r)\int_0^\infty\int_{(0,\,u+c(i)t]}e^{-r(u+c(i)t-x)}dF^{ij}(x)dG^{ij}(t)\\
&\leqslant e^{-ru}[M^*(r)]^n\sum_{j=1}^{s}p_{ij}\int_0^\infty\int_0^\infty e^{-r(c(i)t-x)}dF^{ij}(x)dG^{ij}(t)\\
&= e^{-ru}M^i(r)[M^*(r)]^n\,.
\end{aligned}$$

By the induction principle, (24) holds for every $n \in \mathbb{N}$. Note that

$$\Psi_n^i(u) \geqslant R_n^i(u,r) - e^{-ru} M^i(r)[M^*(r)]^{n-1}$$

because of (23) and (24). Similarly,

$$\Psi_n^i(u) \leqslant R_n^i(u,r) - e^{-ru} \underline{m}^i(r)[m_*(r)]^{n-1} .$$

Since $r \in \mathbb{R}_+$ is arbitrary, the assertion follows. □

*Remark 1* It is easy to see that the upper and lower bounds in Theorem 4 converge geometrically to $\Psi_n^i(u)$.

In the following example, the upper bound introduced in Theorem 4 is shown to be attainable.

*Example 3* Consider the model discussed in Example 1. By Theorem 4,

$$\Psi_n(u) \leqslant \inf_{r\in\mathbb{R}_+} \{R_n(u,r) - e^{-ru}[\underline{m}(r)]^n\} .$$

Assume that $\gamma = 0.50$, $u = 0.11$, $n = 2$ and $F(x) = (1-e^{-\beta x})\,\mathrm{I}_{(0,\,\infty)}(x)$, where $\beta = 5$. In this case, $\Psi_n(u) = e^{-\beta(u+\gamma)}[1 + \beta e^{-\beta\gamma}(u+\gamma)] = 0.059$. The graph of $R_n(u,r) - e^{-ru}[\underline{m}(r)]^n$, as a function of $r$, is presented in Fig. 4. One can easily find the infimum value. We see that

$$\inf_{r\in\mathbb{R}_+} \{R_n(u,r) - e^{-ru}[\underline{m}(r)]^n\} = 0.059 = \Psi_n(u) ,$$

so the upper bound resulting from Theorem 4 is attainable to at least three decimal places.

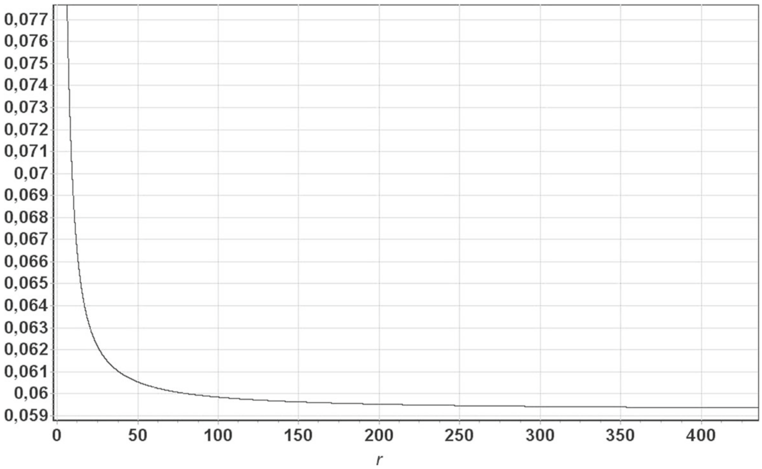


**Fig. 4** $R_n(u,r) - e^{-ru}[\underline{m}(r)]^n$, as a function of $r$, for an exponential distribution with scale parameter 5, amount of aggregated premiums $\gamma = 0.50$, initial surplus $u = 0.11$ and $n = 2$

## 4 Iterating the Risk Operator - Numerical Aspects and Applications

In this section, we discuss a research methodology based on iterating the risk operator $\mathbf{L}$. We are going to present some technical details and difficulties faced in it.

First of all, let us recall that Theorem 1 gives an exact recursive formula for the vector $\boldsymbol{\Psi}_n$ of ruin probabilities in the risk-switching model. In Gajek and Rudź (2013), a fast algorithm to iterate a one-dimensional risk operator was presented showing that this numerical problem is complex even for non-switching risk models.

In a risk-switching case, additional difficulties arise. To compute any coordinate of an arbitrary iteration of $\mathbf{L}$, one has to compute all coordinates of the previous iteration. Moreover, all the iterations should be carefully discretised, see below. We will discuss these problems within the following discrete time risk switching model.

Assume that there exists a number $m \in \mathbb{R}_+$ such that $\mathbb{P}(T_k = m) = 1$ for each $k \in \mathbb{N}$. Under this assumption, each of the random variables $T_1,\ T_2,\ \dots$ denotes a fixed period $m$ (for instance a quarter) and $X_k$ - the sum of claims during the $k$th period. The random variables $X_1,\ X_2,\ \dots$ are assumed to be non-negative (a.s.). Let a positive random variable $\gamma_k = c(I_{k-1})m$, where $c$ is a known function defined on $S$, denote the total amount of premiums during the $k$th period. Thus, denoting the right side of this equation by $\gamma(\cdot)$, the amount of premiums during the first period, $\gamma_1$, given the initial state $i$, equals a positive real $\gamma(i)$. The time of ruin $\tau$ and the probability of ruin up to the end of the $n$th period, $\Psi_n^i$, are defined just as in (2) and (13), respectively (see also Gajek and Rudź 2018).

Let us look at the risk operator generated by the above risk process. We have: $\mathrm{L}_i\,\boldsymbol{\rho}(u) = \ell_i\,\boldsymbol{\rho}(u) + \Psi_1^i(u)$, where

$$\ell_i\,\boldsymbol{\rho}(u) = \sum_{j=1}^{s} p_{ij} \int\limits_{[0,\,u+\gamma(i)]} \rho_j(u+\gamma(i)-x)dF^{ij}(x) \ \text{ and } \ \Psi_1^i(u) = \sum_{j=1}^{s} p_{ij} \int\limits_{u+\gamma(i)}^{\infty} dF^{ij}(x)\,,$$

for all $i \in S$, $\boldsymbol{\rho}= (\rho_1, \dots, \rho_s) \in \mathcal{R}^s$, and $u \geqslant 0$.

To compute the two-sided bound coming from Theorem 4, it is necessary to know the $n$th iteration $\mathbf{R}_n(u,r) = (R_n^1(u,r), \dots, R_n^s(u,r))$ of the risk operator $\mathbf{L}$ on the vector $\mathbf{R}_0(u,r)$. We will present now a numerical procedure for iterating $\mathbf{L}$ on an arbitrary function $\boldsymbol{\rho}^0 = (\rho_1^0, \dots, \rho_s^0) \in \mathcal{R}^s$.

For every $i \in S$, define iteratively a sequence $\{\rho_i^n\}_{n\in\mathbb{N}}$ by

$$\rho_i^n(u) = \mathrm{L}_i\,\boldsymbol{\rho}^{n-1}(u), \quad n \in \mathbb{N},\ u \geqslant 0\,,$$

where $\boldsymbol{\rho}^n(u) = (\rho_1^n(u), \dots, \rho_s^n(u))$. We will show how to calculate iteratively $\boldsymbol{\rho}^k$ using the above equality. Fix $\gamma(i)$ and $F^{ij}$ for all $i, j \in S$. Set $\gamma^* = \max\{\gamma(i) : i \in S\}$.

Let each coordinate $\rho_1^0(u), \dots, \rho_s^0(u)$ of the initial approximation $\boldsymbol{\rho}^0(u)$ be *discretised*[1] with a fixed step $h > 0$ on a large enough interval $[0, \widetilde{u}\,]$, $\widetilde{u} \geqslant k\gamma^*$. To get the first iteration $\boldsymbol{\rho}^1$, one should discretise each of its $s$ coordinates:

$$\rho_i^1(u) = \sum_{j=1}^{s} p_{ij} \int\limits_{[0,\,u+\gamma(i)]} \rho_j^0(u+\gamma(i)-x)dF^{ij}(x) + \Psi_1^i(u), \quad i \in S\,,$$

[1] i.e. $u = lh,\ l = 0, 1, 2, \dots, p$ for some $p = \frac{\widetilde{u}}{h} \in \mathbb{N}$

on the interval $[0\ ,\ \widetilde{u}-\gamma^*]$ with the same step $h$. Similarly, one establishes $\boldsymbol{\rho}^2, \ldots, \boldsymbol{\rho}^{k-1}$. Finally, we arrive at the $k$th iteration $\boldsymbol{\rho}^k$, $k \in \mathbb{N}$, of $\mathbf{L}$ on $\boldsymbol{\rho}^0$. To determine it, one should discretise the following $s$ coordinates:

$$\rho_i^k(u) = \sum_{j=1}^{s} p_{ij} \int\limits_{[0,\, u+\gamma(i)]} \rho_j^{k-1}(u+\gamma(i)-x)dF^{ij}(x) + \Psi_1^i(u), \quad i \in S\,,$$

on the interval $[0\ ,\ \widetilde{u}-k\gamma^*]$ with the step $h$.

However, there are some computational difficulties when iterating vector-valued risk operators. First of all, to compute $\boldsymbol{\rho}^k$ one has to know and save in a file all $s$ coordinates of each previous iterations $\boldsymbol{\rho}^{k-1}, \boldsymbol{\rho}^{k-2}, \ldots, \boldsymbol{\rho}^0$. On the other hand, due to the argument $u+\gamma(i)-x$ of the above integrands, the estimates calculated in each previous iteration should be properly discretised (at the same points) on the interval larger by $\gamma(i)$ (or by $\gamma^*$, in the worst case) than in the actual one. Therefore, the initial interval $[0, \widetilde{u}\,]$ should be large enough.

*Example 4* Consider the discrete time risk-switching model. Assume that $S = \{1, 2\}$, $p_{11} = 0.95$, $p_{12} = 0.05$, $p_{21} = 0.9$, $p_{22} = 0.1$, $\gamma(1) = 3$, $\gamma(2) = 4$, $F^{11}(x) = F^{21}(x) = (1-e^{-\beta_1 x})\,\mathbf{I}_{(0,\,\infty)}(x)$ and $F^{12}(x) = F^{22}(x) = (1-e^{-\beta_2 x})\,\mathbf{I}_{(0,\,\infty)}(x)$, where $\beta_1 = 1$ and $\beta_2 = 0.6$. Set $h = 0.001$, $k = 3$, $\gamma^* = 4$, $s = 2$, $\widetilde{u} = 20$ and $\boldsymbol{\rho}^0 = (\Psi_1^1, \Psi_1^2)$. The probabilities of ruin up to the end of the first four periods, given the initial states 1 and 2, respectively, are presented in Fig. 5.

The premium $c(i)$ depends on the state $i$ of the Markov chain and the insurer can adapt it to the history in response to any change of the state. As is shown in the next example, the Solvency Capital Requirement (SCR) depends on the state $i$ in the beginning of each year.

*Example 5* Consider the four periods risk-switching model discussed in Example 4. The probabilities of ruin up to the end of the fourth quarter, given the initial states 1 and 2, respectively, are presented in Fig. 6. We see that

$$\Psi_4^1(u^1) \leqslant 0.005 \quad \text{for every} \quad u^1 \geqslant 3.12,$$

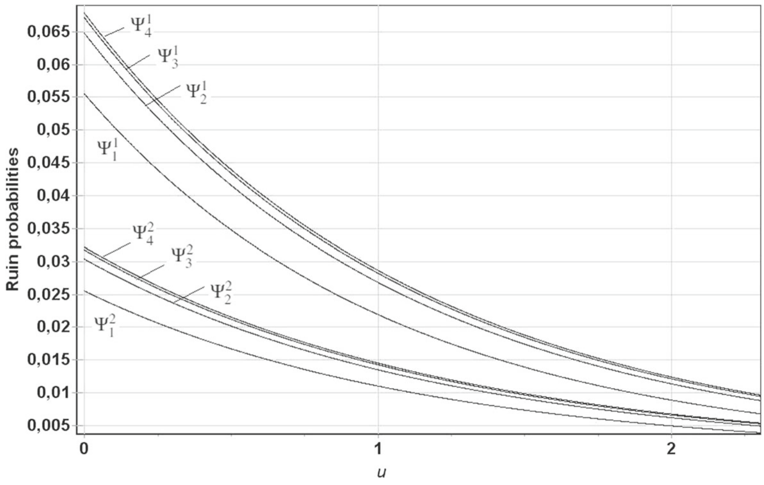


**Fig. 5** The probabilities of ruin up to the end of the first four periods, given the initial states 1 and 2, respectively

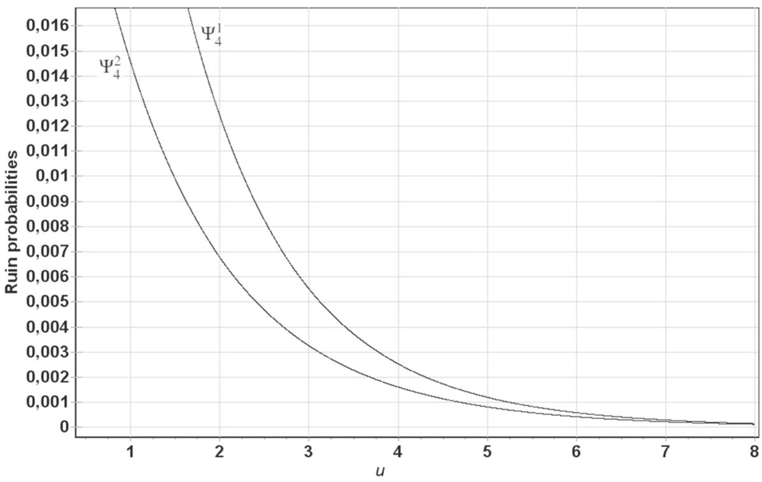


**Fig. 6** The probabilities of ruin within one-year time, given the initial states 1 and 2, respectively

and

$$\Psi_4^2(u^2) \leqslant 0.005 \quad \text{for every} \quad u^2 \geqslant 2.40\,.$$

Therefore the above values $u^1 = 3.12$ and $u^2 = 2.40$ determine SCR in accordance with Solvency II regulations.

## 5 Conclusion

In this paper, we consider a risk-switching Sparre Andersen model described by a vector-type risk process (1) with the risk operator **L** given by (10) and (11). Due to its flexibility, the above model can be useful in the real-world applications. Within this model, we provide the following new upper bound for the vector $\boldsymbol{\Psi}_n$ of ruin probabilities:

$$\Psi_n^i(u) \leqslant \inf_{r\in\mathbb{R}_+} \left\{ e^{-ru} m^*(r) \frac{1-[M^*(r)]^n}{1-M^*(r)} \right\}, \quad i \in S,\ n \in \mathbb{N},\ u \geqslant 0\,. \tag{25}$$

This bound is compared with the generalized Gerber's bound (18). It turns out that which of these inequalities performs better than the other depends on the values of the insurance premiums $c(i)$. Additionally, we prove the following upper and lower bounds:

$$\sup_{r\in\mathbb{R}_+}\{R_n^i(u,r) - e^{-ru} M^i(r)[M^*(r)]^{n-1}\} \leqslant \Psi_n^i(u) \leqslant \inf_{r\in\mathbb{R}_+}\{R_n^i(u,r) - e^{-ru}\underline{m}^i(r)[m_*(r)]^{n-1}\} \tag{26}$$

which converge geometrically to $\Psi_n^i(u)$ (for details, see Theorem 4). Bounds (25) and (26) have been shown to be attainable.

Risk-switching models of an insurer's insolvency can play more and more important role, as the Solvency II directive becomes a standard solvency framework across the EU countries. According to this regulation, SCR should be large enough to reduce the probability $\Psi_n^i$ of default within one-year time period to at most 0.5%. In this context, iterating $n$ times the risk operator (11) is a numerical method which can lead to the exact value of $\Psi_n^i$. To avoid extensive numerical calculations, one can apply directly the bounds (25) and (26) which, at least in some examples, provide acceptable estimations.

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