Asymptotic Expansions for Stationary Distributions of Nonlinearly Perturbed Semi-Markov Processes. 1

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Abstract New algorithms for construction of asymptotic expansions for stationary distributions of nonlinearly perturbed semi-Markov processes with finite phase spaces are presented. These algorithms are based on a special technique of sequential phase space reduction, which can be applied to processes with an arbitrary asymptotic communicative structure of phase spaces. Asymptotic expansions are given in two forms, without and with explicit upper bounds for remainders.

Keywords Markov chain · Semi-Markov process · Nonlinear perturbation · Stationary distribution · Expected hitting time · Laurent asymptotic expansion

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1 Introduction

In this paper, we present new algorithms for construction of asymptotic expansions for stationary distributions of nonlinearly perturbed semi-Markov processes with a finite phase space.

We consider models, where the phase space is one class of communicative states, for embedded Markov chains of pre-limiting perturbed semi-Markov processes, while it can

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possess an arbitrary communicative structure, i.e., can consist of one or several closed classes of communicative states and, possibly, a class of transient states, for the limiting embedded Markov chain.

The initial perturbation conditions are formulated in the forms of Taylor and Laurent asymptotic expansions, respectively, for transition probabilities (of embedded Markov chains) and expectations of sojourn times, for perturbed semi-Markov processes. Two variants of these expansions are considered, with remainders given without and with explicit upper bounds.

The algorithms are based on special time-space screening procedures for sequential phase space reduction and algorithms for re-calculation of asymptotic expansions and upper bounds for remainders, which constitute perturbation conditions for semi-Markov processes with reduced phase spaces.

The final asymptotic expansions for stationary distributions of nonlinearly perturbed semi-Markov processes are given in the form of Taylor asymptotic expansions with remainders given in two variants, without (in Part 1 of the paper) and with explicit upper bounds (in Part 2).

Models of perturbed Markov chains and semi-Markov processes, in particular, for the most difficult cases of perturbed processes with absorption and so-called singularly perturbed processes, attracted attention of researchers in the middle of the 20th century. An interest to these models has been stimulated by applications to control and queuing systems, information networks, epidemic models and models of mathematical genetics and population dynamics. As a rule, Markov-type processes with singular perturbations appear as natural tools for mathematical analysis of multi-component systems with weakly interacting components. One of the natural and important problems in this area is connected with analysis of asymptotics for stationary distributions of the corresponding perturbed processes.

We would like to mention the first works related to asymptotical problems for the above models that are, Meshalkin (1958), Simon and Ando (1961), Hanen (1963), Seneta (1967), Schweitzer (1968), and Korolyuk (1969) and, then, to refer to books, which contains parts devoted to perturbed Markov chains and semi-Markov processes and problems related to asymptotic expansions for the above models. These are, Korolyuk and Turbin (1976, 1978), Courtois (1977), Seneta (2006), Stewart and Sun (1990), Korolyuk and Swishchuk (1995), Kartashov (1996), Stewart (1998, 2001), Yin and Zhang (1998, 2005, 2013), Korolyuk and Korolyuk (1999), Konstantinov et al. (2003), Bini et al. (2005), Koroliuk and Limnios (2005), Gyllenberg and Silvestrov (2008) and Avrachenkov et al. (2013). We also refer to the recent paper by Silvestrov and Silvestrov (2016a) where one can find an extended bibliography of works in the area.

We formulate perturbation conditions in terms of asymptotic expansions for transition characteristics of perturbed semi-Markov processes. The remainders in these expansions and, thus, the transition characteristics of perturbed semi-Markov processes can be non-analytic functions of perturbation parameter. This makes a difference with the results for models with linear, polynomial and analytic perturbations.

An important novelty of our studies also is that we consider asymptotic expansions with remainders given not only in a standard form of $o(\cdot)$, but, also, in a more advanced form, with explicit power-type upper bounds for remainders, asymptotically uniform with respect to a perturbation parameter.

Semi-Markov processes are a natural generalization of Markov chains, important theoretically and essentially extending applications of Markov-type models. The asymptotic results

obtained in the paper are a good illustration for this statement. In particular, they automatically yield analogous asymptotic results for nonlinearly perturbed discrete and continuous time Markov chains.

Part 1 of the paper includes six sections.

In Section 2, we present explicit formulas for computing parameters, coefficients and remainders for expansions obtained as results of multiplication by constant, summation, multiplication and division operations with asymptotic Laurent expansions (Lemmas 1 – 4).

In Section 3, we introduce a model of perturbed semi-Markov processes, formulate basic perturbation conditions given in form of asymptotic expansions for transition probabilities of embedded Markov chains and expectations of transition times. We also describe a special time-space procedure of one-state reduction of phase space for semi-Markov processes, get explicit formulas for transition characteristics of reduced semi-Markov processes in the form of rational functions of the corresponding transition characteristics for initial semi-Markov processes, and prove invariance of hitting times with respect to the above time-space procedure (Theorem 1).

In Section 4, we prove that the above reduced semi-Markov processes satisfy the same type perturbation conditions as the initial ones and describe algorithms for re-computing parameters, coefficients and remainders in these conditions for reduced semi-Markov processes in terms of the corresponding parameters, coefficients and remainders appearing in perturbation conditions for the initial semi-Markov processes (Theorems 2 and 3). These algorithms are based on application of operational rules for Laurent asymptotic expansions presented in Section 1 to the above rational functions representing transition characteristics of reduced semi-Markov processes.

In Section 5, we describe a recurrent multi-step time-space screening procedure of phase space reduction, which is based on recurrent repetition of the above one-step time-space screening procedure up to reduction of all states $r \neq i$, except one preliminary chosen state i . We describe the corresponding recurrent algorithms for computing asymptotic expansions for transition characteristics for reduced semi-Markov processes obtained as the result of sequential application of the above one-step time-space screening procedure. The resulting semi-Markov process has the one-state phase space $\{i\}$. The above invariance property of hitting times holds for every one-state reduction step. This implies that the return time for state i is the same for the initial semi-Markov process and for the final reduced one-state semi-Markov process. In the latter case, the return time to state i coincides with the time of first jump, which play the role of transition time. Thus, the Laurent asymptotic expansion for the expectation of return time for state i obtained for the reduced one-state semi-Markov process also yields the Laurent asymptotic expansion for the expectation of the return time to state i for the initial semi-Markov process (Theorem 4). We also prove in this theorem that the resulting asymptotic expansion is invariant with respect to the order, in which the states $r \neq i$ are excluded from the initial phase space.

In Section 6, we get the asymptotic expansion for the stationary probabilities of perturbed semi-Markov processes using the well known representation of stationary probabilities $\pi_i(\varepsilon)$ as quotients of expectations for sojourn times and expectation of return times. Laurent asymptotic expansions for expectations of sojourn times can be easily obtained by application of summation rule to the Laurent asymptotic expansions for expectations of transition times appearing in initial perturbation conditions while Laurent asymptotic expansions expectations of return times are given by Theorem 4. The application of the division

rule to the quotient of these expansions yield, finally, the asymptotic expansions for the stationary probabilities for nonlinearly perturbed semi-Markov processes (Theorem 5), which are the main object of interest in the present paper.

As was mentioned above, we present analogs of the above results in the form of asymptotic expansions with explicit upper bounds for remainders in Part 2 of the paper. Also, examples, which illustrate theoretical results obtained in the paper, are presented in Part 2.

We would like to conclude the introduction with the remark that the present paper is a shorten version of the report Silvestrov and Silvestrov (2016b), where one can find some additional details of proofs, comments and references.

2 Laurent Asymptotic Expansions

In this section, we present so-called operational rules for Laurent asymptotic expansions. The corresponding proofs and comments are given in Appendix A, in Part II of the paper.

Let $A(\varepsilon)$ be a real-valued function defined on an interval $(0, \varepsilon_0]$, for some $0 < \varepsilon_0 \leq 1$, and given on this interval by a Laurent asymptotic expansion,

$$A(\varepsilon) = a_{h_A} \varepsilon^{h_A} + \dots + a_{k_A} \varepsilon^{k_A} + o_A(\varepsilon^{k_A}), \quad (1)$$

where **(a)** $-\infty < h_A \leq k_A < \infty$ are integers, **(b)** coefficients $a_{h_A}, \dots, a_{k_A}$ are real numbers, **(c)** function $o_A(\varepsilon^{k_A})/\varepsilon^{k_A} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

We refer to such Laurent asymptotic expansion as a (h_A, k_A) -expansion.

We say that (h_A, k_A) -expansion $A(\varepsilon)$ is pivotal if it is known that $a_{h_A} \neq 0$.

Let us explain, why we can restrict consideration by the case, where parameter ε takes only positive values. As a matter of fact, if function $A(\varepsilon)$ is also defined on some interval $[\varepsilon'_0, 0)$ and is given on this interval by a Laurent asymptotic expansion $A(\varepsilon) = A'(\varepsilon) = a'_{h'_A} \varepsilon^{h'_A} + \dots + a'_{k'_A} \varepsilon^{k'_A} + o'_A(\varepsilon^{k'_A})$ analogous to the above one given in relation (1), then $A'(\varepsilon)$, $\varepsilon \in [\varepsilon'_0, 0)$ can always be rewritten as a function of positive parameter $-\varepsilon \in (0, -\varepsilon'_0]$ using the formula, $A'(\varepsilon) = A'(-(-\varepsilon)) = (-1)^{h'_A} a'_{h'_A} \cdot (-\varepsilon)^{h'_A} + \dots + (-1)^{k'_A} a'_{k'_A} \cdot (-\varepsilon)^{k'_A} + o'_A((-1)^{k'_A} (-\varepsilon)^{k'_A})$. Thus, the operational analysis of function $A(\varepsilon)$, in particular computing of coefficients and estimation of remainder for the corresponding asymptotic expansion defined at the two-sided neighborhood of 0 can be reduced to analysis of two functions defined at positive one-sided neighborhoods of 0.

Lemma 1 *If function $A(\varepsilon) = a'_{h'_A} \varepsilon^{h'_A} + \dots + a'_{k'_A} \varepsilon^{k'_A} + o'_A(\varepsilon^{k'_A}) = a''_{h''_A} \varepsilon^{h''_A} + \dots + a''_{k''_A} \varepsilon^{k''_A} + o''_A(\varepsilon^{k''_A})$, $\varepsilon \in (0, \varepsilon_0]$ can be represented as, respectively, (h'_A, k'_A) - and (h''_A, k''_A) -expansion, then the asymptotic expansion for function $A(\varepsilon)$ can be represented in the following the most informative form $A(\varepsilon) = a_{h_A} \varepsilon^{h_A} + \dots + a_{k_A} \varepsilon^{k_A} + o_A(\varepsilon^{k_A})$, $\varepsilon \in (0, \varepsilon_0]$ of (h_A, k_A) -expansion, with parameters $h_A = h'_A \vee h''_A$, $k_A = k'_A \vee k''_A$, and coefficients $a_{h_A}, \dots, a_{k_A}$, and remainder $o_A(\varepsilon^{k_A})$ given by the following relations:*

- (i) $a'_l = 0$, for $h'_A \leq l < h_A$ and $a''_l = 0$, for $h''_A \leq l < h_A$;
- (ii) $a_l = a'_l = a''_l$, for $h_A \leq l \leq \tilde{k}_A = k'_A \wedge k''_A$;
- (iii) $a_l = a''_l$, for $\tilde{k}_A = k'_A < l \leq k_A$ if $k'_A < k''_A$;
- (iv) $a_l = a'_l$, for $\tilde{k}_A = k''_A < l \leq k_A$ if $k''_A < k'_A$;

- (v) $o'_A(\varepsilon^{k'_A}) + \sum_{\tilde{k}_A < l \leq k'_A} a'_l \varepsilon^l = o''_A(\varepsilon^{k''_A}) + \sum_{\tilde{k}_A < l \leq k''_A} a''_l \varepsilon^l$, $\varepsilon \in (0, \varepsilon_0]$ and $o_A(\varepsilon^{k_A})$ coincides, for $\varepsilon \in (0, \varepsilon_0]$, with $o''_A(\varepsilon^{k''_A})$ if $k'_A < k''_A$; $o'_A(\varepsilon^{k'_A}) = o''_A(\varepsilon^{k''_A})$ if $k'_A = k''_A$; or $o'_A(\varepsilon^{k'_A})$ if $k'_A > k''_A$.

The asymptotical expansion $A(\varepsilon)$ is pivotal if and only if $a_{h_A} = a'_{h_A} = a''_{h_A} \neq 0$.

It is also useful to mention that a constant a can be interpreted as function $A(\varepsilon) \equiv a$. Thus, 0 can be represented, for any integer $-\infty < h \leq k < \infty$, as the (h, k) -expansion, $0 = 0\varepsilon^h + \dots + 0\varepsilon^k + o(\varepsilon^k)$, with remainder $o(\varepsilon^k) \equiv 0$. Also, 1 can be represented, for any integer $0 \leq k < \infty$, as the $(0, k)$ -expansion, $1 = 1 + 0\varepsilon + \dots + 0\varepsilon^k + o(\varepsilon^k)$, with remainder $o(\varepsilon^k) \equiv 0$.

Let us consider four Laurent asymptotic expansions, $A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \dots + a_{k_A}\varepsilon^{k_A} + o_A(\varepsilon^{k_A})$, $B(\varepsilon) = b_{h_B}\varepsilon^{h_B} + \dots + b_{k_B}\varepsilon^{k_B} + o_B(\varepsilon^{k_B})$, $C(\varepsilon) = c_{h_C}\varepsilon^{h_C} + \dots + c_{k_C}\varepsilon^{k_C} + o_C(\varepsilon^{k_C})$, and $D(\varepsilon) = d_{h_D}\varepsilon^{h_D} + \dots + d_{k_D}\varepsilon^{k_D} + o_D(\varepsilon^{k_D})$ defined for $0 < \varepsilon \leq \varepsilon_0$, for some $0 < \varepsilon_0 \leq 1$.

The following lemma presents operational rules for Laurent asymptotic expansions.

Lemma 2 *The following operational rules take place for Laurent asymptotic expansions:*

- (i) *If $A(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a (h_A, k_A) -expansion and c is a constant, then $C(\varepsilon) = cA(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a (h_C, k_C) -expansion such that:*
- (a) $h_C = h_A$, $k_C = k_A$;
 - (b) $c_{h_C+r} = ca_{h_C+r}$, $r = 0, \dots, k_C - h_C$;
 - (c) $o_C(\varepsilon^{k_C}) = co_A(\varepsilon^{k_A})$.

This expansion is pivotal if and only if $c_{h_C} = ca_{h_A} \neq 0$.

- (ii) *If $A(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a (h_A, k_A) -expansion and $B(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a (h_B, k_B) -expansion, then $C(\varepsilon) = A(\varepsilon) + B(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a (h_C, k_C) -expansion such that:*

- (a) $h_C = h_A \wedge h_B$, $k_C = k_A \wedge k_B$;
- (b) $c_{h_C+r} = a_{h_C+r} + b_{h_C+r}$, $r = 0, \dots, k_C - h_C$, where $a_{h_C+r} = 0$, for $0 \leq r < h_A - h_C$, and $b_{h_C+r} = 0$, for $0 \leq r < h_B - h_C$;
- (c) $o_C(\varepsilon^{k_C}) = \sum_{k_C < i \leq k_A} a_i \varepsilon^i + \sum_{k_C < j \leq k_B} b_j \varepsilon^j + o_A(\varepsilon^{k_A}) + o_B(\varepsilon^{k_B})$.

This expansion is pivotal if and only if $c_{h_C} = a_{h_C} + b_{h_C} \neq 0$.

- (iii) *If $A(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a (h_A, k_A) -expansion and $B(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a (h_B, k_B) -expansion, then $C(\varepsilon) = A(\varepsilon) \cdot B(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a (h_C, k_C) -expansion such that:*

- (a) $h_C = h_A + h_B$, $k_C = (k_A + h_B) \wedge (k_B + h_A)$;
- (b) $c_{h_C+r} = \sum_{0 \leq i \leq r} a_{h_A+i} b_{h_B+r-i}$, $r = 0, \dots, k_C - h_C$;
- (c) $o_C(\varepsilon^{k_C}) = \sum_{k_C < i+j, h_A \leq i \leq k_A, h_B \leq j \leq k_B} a_i b_j \varepsilon^{i+j} + \sum_{h_A \leq i \leq k_A} a_i \varepsilon^i o_B(\varepsilon^{k_B}) + \sum_{h_B \leq j \leq k_B} b_j \varepsilon^j o_A(\varepsilon^{k_A}) + o_A(\varepsilon^{k_A}) o_B(\varepsilon^{k_B})$.

This expansion is pivotal if and only if $c_{h_C} = a_{h_C} b_{h_B} \neq 0$;

- (iv) *If $B(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a pivotal (h_B, k_B) -expansion, then there exists $0 < \varepsilon'_0 \leq \varepsilon_0$ such that $B(\varepsilon) \neq 0$, $\varepsilon \in (0, \varepsilon'_0]$, and $C(\varepsilon) = \frac{1}{B(\varepsilon)}$, $\varepsilon \in (0, \varepsilon'_0]$ is a pivotal (h_C, k_C) -expansion such that:*

- (a) $h_C = -h_B$, $k_C = k_B - 2h_B$;
- (b) $c_{h_C} = b_{h_B}^{-1}$, $c_{h_C+r} = -b_{h_B}^{-1} \sum_{1 \leq i \leq r} b_{h_B+i} c_{h_C+r-i}$, $r = 1, \dots, k_C - h_C$;
- (c) $o_C(\varepsilon^{k_C}) = -\frac{\sum_{k_B-h_B < i+j, h_B \leq i \leq k_B, h_C \leq j \leq k_C} b_i c_j \varepsilon^{i+j} + \sum_{h_C \leq j \leq k_C} c_j \varepsilon^j o_B(\varepsilon^{k_B})}{b_{h_B} \varepsilon^{h_B} + \dots + b_{k_B} \varepsilon^{k_B} + o_B(\varepsilon^{k_B})}$.

(v) If $A(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a (h_A, k_A) -expansion, and $B(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is a pivotal (h_B, k_B) -expansion, then, there exists $0 < \varepsilon'_0 \leq \varepsilon_0$ such that $B(\varepsilon) \neq 0$, $\varepsilon \in (0, \varepsilon'_0]$, and $D(\varepsilon) = \frac{A(\varepsilon)}{B(\varepsilon)}$, $\varepsilon \in (0, \varepsilon'_0]$ is a (h_D, k_D) -expansion such that:

- (a) $h_D = h_A - h_B$, $k_D = (k_A - h_B) \wedge (k_B - 2h_B + h_A)$;
- (b) $d_{h_D+r} = b_{h_B}^{-1}(a_{h_A+r} - \sum_{1 \leq i \leq r} b_{h_B+i} d_{h_D+r-i})$, $r = 0, \dots, k_D - h_D$;
- (c)
$$o_D(\varepsilon^{k_D}) = \frac{\sum_{k_A \wedge (k_B + h_A - h_B) < l \leq k_A} a_l \varepsilon^l + o_A(\varepsilon^{k_A})}{b_{h_B} \varepsilon^{h_B} + \dots + b_{k_B} \varepsilon^{k_B} + o_B(\varepsilon^{k_B})} - \frac{\sum_{k_A \wedge (k_B + h_A - h_B) < i+j, h_B \leq i \leq k_B, h_D \leq j \leq k_D} b_i d_j \varepsilon^{i+j} + \sum_{h_D \leq j \leq k_D} d_j \varepsilon^j o_B(\varepsilon^{k_B})}{b_{h_B} \varepsilon^{h_B} + \dots + b_{k_B} \varepsilon^{k_B} + o_B(\varepsilon^{k_B})}.$$

This expansion is pivotal if and only if $d_{h_D} = a_{h_A} c_{h_C} = a_{h_A} / b_{h_B} \neq 0$.

Remark 1 The Laurent asymptotic expansions for function $D(\varepsilon)$, given by formulas (a) – (c) in proposition (v) of Lemma 2, coincide with the expansions given by formulas (a) – (c) in propositions (iv) of Lemma 2, if $A(\varepsilon) \equiv 1$. In this case, 1 should be interpreted as the $(0, k_B - h_B)$ -expansion, $1 = 1 + 0\varepsilon + \dots + 0\varepsilon^{k_B - h_B} + o(\varepsilon^{k_B - h_B})$, with remainder $o(\varepsilon^{k_B - h_B}) \equiv 0$.

The following multiple summation and multiplication operational rules for Laurent asymptotic expansions are direct corollaries of the corresponding rules given in Lemma 2.

Lemma 3 Let $A_m(\varepsilon) = a_{h_{A_m}, m} \varepsilon^{h_{A_m}} + \dots + a_{k_{A_m}, m} \varepsilon^{k_{A_m}} + o(\varepsilon^{k_{A_m}})$, $\varepsilon \in (0, \varepsilon_0]$ be a (h_{A_m}, k_{A_m}) -expansion, for $m = 1, \dots, N$. In this case:

(i) $B_n(\varepsilon) = A_1(\varepsilon) + \dots + A_n(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is, for every $n = 1, \dots, N$, a (h_{B_n}, k_{B_n}) -expansion, where:

- (a) $h_{B_n} = \min(h_{A_1}, \dots, h_{A_n})$, $k_{B_n} = \min(k_{A_1}, \dots, k_{A_n})$.
- (b) $b_{h_{B_n}+l, n} = a_{h_{A_1}+l, 1} + \dots + a_{h_{A_n}+l, n}$, $l = 0, \dots, k_{B_n} - h_{B_n}$, where $a_{h_{A_m}+l, m} = 0$, for $0 \leq l < h_{A_m} - h_{B_n}$, $m = 1, \dots, n$.
- (c) $o_{B_n}(\varepsilon^{k_{B_n}}) = \sum_{1 \leq m \leq n} (\sum_{k_{B_n} < l \leq k_{A_m}} a_{l, m} \varepsilon^l + o_{A_m}(\varepsilon^{k_{A_m}}))$.

Expansion $B_n(\varepsilon)$ is pivotal if and only if $b_{h_{B_n}, n} = a_{h_{A_1}, 1} + \dots + a_{h_{A_n}, n} \neq 0$.

(ii) $C_n(\varepsilon) = A_1(\varepsilon) \times \dots \times A_n(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ is, for every $n = 1, \dots, N$, a (h_{C_n}, k_{C_n}) -expansion, where:

- (a) $h_{C_n} = h_{A_1} + \dots + h_{A_n}$, $k_{C_n} = \min(k_{A_l} + \sum_{1 \leq r \leq n, r \neq l} h_{A_r}, l = 1, \dots, n)$.
- (b) $c_{h_{C_n}+l, n} = \sum_{l_1 + \dots + l_n = l, 0 \leq l_m \leq k_{A_m} - h_{A_m}, 1 \leq m \leq n} \prod_{1 \leq m \leq n} a_{h_{A_m}+l_m, m}$, $l = 0, \dots, k_{C_n} - h_{C_n}$.
- (c) $o_{C_n}(\varepsilon^{k_{C_n}}) = \sum_{k_{C_n} < l_1 + \dots + l_n, h_{A_m} \leq l_m \leq k_{A_m}, 1 \leq m \leq n} \prod_{m=1}^n a_{l_m, m} \varepsilon^{l_1 + \dots + l_n} + \sum_{r=1}^n \sum_{1 \leq m_1 < \dots < m_r \leq n} (\prod_{m \neq m_1, \dots, m_r, 1 \leq m \leq n} (\sum_{h_{A_m} \leq l \leq k_{A_m}} a_{l, m} \varepsilon^l)) \times \prod_{i=1}^r o_{A_{m_i}}(\varepsilon^{k_{A_{m_i}}})$.

Expansion $C_n(\varepsilon)$ is pivotal if and only if $c_{h_{C_n}, n} = a_{h_{A_1}, 1} \times \dots \times a_{h_{A_n}, n} \neq 0$.

(iii) Asymptotic expansions for functions $B_n(\varepsilon) = A_1(\varepsilon) + \dots + A_n(\varepsilon)$, $n = 1, \dots, N$ and $C_n(\varepsilon) = A_1(\varepsilon) \times \dots \times A_n(\varepsilon)$, $n = 1, \dots, N$ are invariant with respect to any permutation, respectively, of summation and multiplication order in the above formulas.

The following lemma summarizes some basic algebraic properties of Laurent asymptotic expansions. It is a corollary of Lemmas 1 and 2.

Lemma 4 *The summation and multiplication operations for Laurent asymptotic expansions defined in Lemma 2 possess the following algebraic properties, which should be understood as identities for the corresponding Laurent asymptotic expansions (i.e., identities for the corresponding parameters h, k , coefficients and remainders) of functions represented in two alternative forms in the functional identities given below:*

- (i) *The summation and multiplication operations for Laurent asymptotic expansions satisfy the “elimination” identities that are implied by the corresponding functional identities, $A(\varepsilon) + 0 \equiv A(\varepsilon)$, $A(\varepsilon) \cdot 1 \equiv A(\varepsilon)$, $A(\varepsilon) - A(\varepsilon) \equiv 0$ and $A(\varepsilon) \cdot A(\varepsilon)^{-1} \equiv 1$.*
- (ii) *The summation operation for Laurent asymptotic expansions is commutative and associative that is implied by the corresponding functional identities, $A(\varepsilon) + B(\varepsilon) \equiv B(\varepsilon) + A(\varepsilon)$ and $(A(\varepsilon) + B(\varepsilon)) + C(\varepsilon) \equiv A(\varepsilon) + (B(\varepsilon) + C(\varepsilon))$.*
- (iii) *The multiplication operation for Laurent asymptotic expansions is commutative and associative that is implied by the corresponding functional identities, $A(\varepsilon) \cdot B(\varepsilon) \equiv B(\varepsilon) \cdot A(\varepsilon)$ and $(A(\varepsilon) \cdot B(\varepsilon)) \cdot C(\varepsilon) \equiv A(\varepsilon) \cdot (B(\varepsilon) \cdot C(\varepsilon))$.*
- (iv) *The summation and multiplication operations for Laurent asymptotic expansions possess distributive property that is implied by the corresponding functional identity, $(A(\varepsilon) + B(\varepsilon)) \cdot C(\varepsilon) \equiv A(\varepsilon) \cdot C(\varepsilon) + B(\varepsilon) \cdot C(\varepsilon)$.*

Remark 2 In proposition (i) of Lemma 4, 0 should be interpreted as the (h_A, k_A) -expansion, $0 = 0 + 0\varepsilon^{h_A} + \dots + 0\varepsilon^{k_A} + o(\varepsilon^{k_A})$, with remainder $o(\varepsilon^{k_A}) \equiv 0$, and 1 as $(0, k_A - h_A)$ -expansion, $1 = 1 + 0\varepsilon + \dots + 0\varepsilon^{k_A - h_A} + o(\varepsilon^{k_A - h_A})$, with remainder $o(\varepsilon^{k_A - h_A}) \equiv 0$.

Remark 3 The Laurent asymptotic expansion $A(\varepsilon)$ is assumed to be pivotal, in the elimination identity implied by functional identity $A(\varepsilon) \cdot A(\varepsilon)^{-1} \equiv 1$, and to hold, for $0 < \varepsilon \leq \varepsilon'_0$ such that $A(\varepsilon) \neq 0$, $\varepsilon \in (0, \varepsilon'_0]$.

The proofs of Lemmas 1–4 are given in Appendix A of Part 2 of the paper.

3 Nonlinearly Perturbed Semi-Markov Processes

Let $\mathbb{X} = \{1, \dots, N\}$ and $(\eta_n^{(\varepsilon)}, \kappa_n^{(\varepsilon)})$, $n = 0, 1, \dots$ be, for every $\varepsilon \in (0, 1]$, a Markov renewal process, i.e., a homogeneous Markov chain with the phase space $\mathbb{X} \times [0, \infty)$, an initial distribution $\bar{p}^{(\varepsilon)} = \langle p_i^{(\varepsilon)} = \mathbf{P}\{\eta_0^{(\varepsilon)} = i, \kappa_0^{(\varepsilon)} = 0\} = \mathbf{P}\{\eta_0^{(\varepsilon)} = i\}, i \in \mathbb{X}$ and transition probabilities,

$$Q_{ij}^{(\varepsilon)}(t) = \mathbf{P}\{\eta_1^{(\varepsilon)} = j, \kappa_1^{(\varepsilon)} \leq t / \eta_0^{(\varepsilon)} = i, \kappa_0^{(\varepsilon)} = s\}, (i, s), (j, t) \in \mathbb{X} \times [0, \infty). \quad (2)$$

In this case, the random sequence $\eta_n^{(\varepsilon)}$ is also a homogeneous (embedded) Markov chain with the phase space $\mathbb{X}$ and the transition probabilities,

$$p_{ij}(\varepsilon) = \mathbf{P}\{\eta_1^{(\varepsilon)} = j / \eta_0^{(\varepsilon)} = i\} = Q_{ij}^{(\varepsilon)}(\infty), i, j \in \mathbb{X}. \quad (3)$$

The following condition plays an important role in what follows:

- A:** There exist sets $\mathbb{Y}_i \subseteq \mathbb{X}$, $i \in \mathbb{X}$ and $\varepsilon_0 \in (0, 1]$ such that: **(a)** probabilities $p_{ij}(\varepsilon) > 0$, $j \in \mathbb{Y}_i$, $i \in \mathbb{X}$, for $\varepsilon \in (0, \varepsilon_0]$; **(b)** probabilities $p_{ij}(\varepsilon) = 0$, $j \in \bar{\mathbb{Y}}_i$, $i \in \mathbb{X}$, for $\varepsilon \in$

$(0, \varepsilon_0]$; (c) there exists, for every pair of states $i, j \in \mathbb{X}$, an integer $n_{ij} \geq 1$ and a chain of states $i = l_{ij,0}, l_{ij,1}, \dots, l_{ij,n_{ij}} = j$ such that $l_{ij,1} \in \mathbb{Y}_{l_{ij,0}}, \dots, l_{ij,n_{ij}} \in \mathbb{Y}_{l_{ij,n_{ij}-1}}$.

We refer to sets $\mathbb{Y}_i, i \in \mathbb{X}$ as transition sets. Conditions **A** implies that all sets $\mathbb{Y}_i \neq \emptyset, i \in \mathbb{X}$.

Condition **A** also implies that the phase space $\mathbb{X}$ of Markov chain $\eta_n^{(\varepsilon)}$ is one class of communicative states, for every $\varepsilon \in (0, \varepsilon_0]$.

We also assume that the following condition excluding instant transitions holds:

B: $Q_{ij}^{(\varepsilon)}(0) = 0, i, j \in \mathbb{X}$, for every $\varepsilon \in (0, \varepsilon_0]$.

Let us now introduce a semi-Markov process,

$$\eta^{(\varepsilon)}(t) = \eta_{\nu^{(\varepsilon)}(t)}^{(\varepsilon)}, \quad t \geq 0, \quad (4)$$

where $\nu^{(\varepsilon)}(t) = \max(n \geq 0 : \zeta_n^{(\varepsilon)} \leq t)$ is a number of jumps in the time interval $[0, t]$, for $t \geq 0$, and $\zeta_n^{(\varepsilon)} = \kappa_1^{(\varepsilon)} + \dots + \kappa_n^{(\varepsilon)}, n = 0, 1, \dots$, are sequential moments of jumps, for the semi-Markov process $\eta^{(\varepsilon)}(t)$.

If $Q_{ij}^{(\varepsilon)}(t) = I(t \geq 1)p_{ij}(\varepsilon), t \geq 0, i, j \in \mathbb{X}$, then $\eta^{(\varepsilon)}(t) = \eta_{[t]}^{(\varepsilon)}, t \geq 0$ is a discrete time homogeneous Markov chain embedded in continuous time.

If $Q_{ij}^{(\varepsilon)}(t) = (1 - e^{-\lambda_i(\varepsilon)t})p_{ij}(\varepsilon), t \geq 0, i, j \in \mathbb{X}$ (here, $0 < \lambda_i(\varepsilon) < \infty, i \in \mathbb{X}$), then $\eta^{(\varepsilon)}(t), t \geq 0$ is a continuous time homogeneous Markov chain.

Let us also introduce expectations of sojourn times,

$$e_{ij}(\varepsilon) = E_i \kappa_1^{(\varepsilon)} I(\eta_1^{(\varepsilon)} = j) = \int_0^\infty t Q_{ij}^{(\varepsilon)}(dt), \quad i, j \in \mathbb{X}. \quad (5)$$

Here and henceforth, notations P_i and E_i are used for conditional probabilities and expectations under condition $\eta^{(\varepsilon)}(0) = i$.

We also assume that the following condition holds:

C: $e_{ij}(\varepsilon) < \infty, i, j \in \mathbb{X}$, for $\varepsilon \in (0, \varepsilon_0]$.

In the case of discrete time Markov chain, $e_{ij}(\varepsilon) = p_{ij}(\varepsilon), i, j \in \mathbb{X}$.

In the case of continuous time Markov chain, $e_{ij}(\varepsilon) = \frac{p_{ij}(\varepsilon)}{\lambda_i(\varepsilon)}, i, j \in \mathbb{X}$.

Conditions **A** (a) – (b) and **B** imply that, for every $\varepsilon \in (0, \varepsilon_0]$, expectations $e_{ij}(\varepsilon) > 0$, for $j \in \mathbb{Y}_i, i \in \mathbb{X}$, and $e_{ij}(\varepsilon) = 0$, for $j \in \bar{\mathbb{Y}}_i, i \in \mathbb{X}$.

Let us assume that the following perturbation condition, based on Taylor asymptotic expansions, holds:

D: $p_{ij}(\varepsilon) = \sum_{l=l_{ij}^-}^{l_{ij}^+} a_{ij}[l] \varepsilon^l + o_{ij}(\varepsilon^{l_{ij}^+}), \varepsilon \in (0, \varepsilon_0]$, for $j \in \mathbb{Y}_i, i \in \mathbb{X}$, where (a) $0 \leq l_{ij}^- \leq l_{ij}^+ < \infty$ are integers, coefficients $a_{ij}[k], l_{ij}^- \leq k \leq l_{ij}^+$ are real numbers, and $a_{ij}[l_{ij}^-] > 0$, for $j \in \mathbb{Y}_i, i \in \mathbb{X}$; (b) function $o_{ij}(\varepsilon^{l_{ij}^+})/\varepsilon^{l_{ij}^+} \rightarrow 0$ as $\varepsilon \rightarrow 0$, for $j \in \mathbb{Y}_i, i \in \mathbb{X}$.

We also assume that the following perturbation condition, based on Laurent asymptotic expansions, holds:

E: $e_{ij}(\varepsilon) = \sum_{l=m_{ij}^-}^{m_{ij}^+} b_{ij}[l] \varepsilon^l + o_{ij}(\varepsilon^{m_{ij}^+}), \varepsilon \in (0, \varepsilon_0]$, for $j \in \mathbb{Y}_i, i \in \mathbb{X}$, where (a) $-\infty < m_{ij}^- \leq m_{ij}^+ < \infty$ are integers, coefficients $b_{ij}[l], m_{ij}^- \leq l \leq m_{ij}^+$ are real

numbers, and $b_{ij}[m_{ij}^-] > 0$, for $j \in \mathbb{Y}_i, i \in \mathbb{X}$; **(b)** function $\dot{o}_{ij}(\varepsilon^{m_{ij}^+})/\varepsilon^{m_{ij}^+} \rightarrow 0$ as $\varepsilon \rightarrow 0$, for $j \in \mathbb{Y}_i, i \in \mathbb{X}$.

The above perturbation conditions can be interpreted as linear, if the asymptotic expansions appearing in them are of the first order, i.e., parameters $l_{ij}^+ - l_{ij}^- = m_{ij}^+ - m_{ij}^- = 1, i \in \mathbb{Y}_j, j \in \mathbb{X}$. Otherwise, these perturbation conditions can be interpreted as nonlinear.

Let us, for the moment, exclude sub-condition **(a)** from condition **A**. Conditions **D** and **E** imply that there exists $\tilde{\varepsilon}_0 \in (0, \varepsilon_0]$ such that $p_{ij}(\varepsilon) = \sum_{l=l_{ij}^-}^{l_{ij}^+} a_{ij}[l]\varepsilon^l + o_{ij}(\varepsilon^{l_{ij}^+}) > 0$ and $e_{ij}(\varepsilon) = \sum_{l=m_{ij}^-}^{m_{ij}^+} b_{ij}[l]\varepsilon^l + \dot{o}_{ij}(\varepsilon^{m_{ij}^+}) > 0$, for $j \in \mathbb{Y}_i, i \in \mathbb{X}, \varepsilon \in (0, \tilde{\varepsilon}_0]$. We can, just, decrease parameter ε_0 and to take the new $\varepsilon_0 = \tilde{\varepsilon}_0$. Condition **A (a)** holds for this new value of ε_0 . An actual value of parameter $\varepsilon_0 \in (0, 1]$ is not important in propositions concerned asymptotic expansions with remainders given in form of $o(\cdot)$. We, however, do prefer to include sub-condition **(a)** in condition **A**, in order to have a clear description for the communicative structure of the phase space $\mathbb{X}$, in one condition. In this case, the above inequalities hold for $\tilde{\varepsilon}_0 = \varepsilon_0$. Conditions **D** and **E** are consistent with condition **A (a)**, according to the above remarks.

Matrix $\|p_{ij}(\varepsilon)\|$ is stochastic, for every $\varepsilon \in (0, \varepsilon_0]$. This model stochasticity assumption holds by the default.

Condition **D** should, also, be consistent with this model stochasticity assumption.

Condition **D** and proposition **(i)** (the multiple summation rule) of Lemma 3 imply that sum $\sum_{j \in \mathbb{Y}} p_{ij}(\varepsilon)$ can, for every subset $\mathbb{Y} \subseteq \mathbb{Y}_i$ and $i \in \mathbb{X}$, be represented in the form of the following Laurent asymptotic expansion,

$$\sum_{j \in \mathbb{Y}} p_{ij}(\varepsilon) = \sum_{l=l_{i,\mathbb{Y}}^-}^{l_{i,\mathbb{Y}}^+} a_{i,\mathbb{Y}}[l]\varepsilon^l + o_{i,\mathbb{Y}}(\varepsilon^{l_{i,\mathbb{Y}}^+}), \quad (6)$$

where: (a) $l_{i,\mathbb{Y}}^\pm = \min_{j \in \mathbb{Y}} l_{ij}^\pm$, (b) $a_{i,\mathbb{Y}}[l] = \sum_{j \in \mathbb{Y}} a_{ij}[l]$, $l = l_{i,\mathbb{Y}}^-, \dots, l_{i,\mathbb{Y}}^+$, where $a_{ij}[l] = 0$, for $0 \leq l < l_{ij}^-, j \in \mathbb{Y}$, (c) $o_{i,\mathbb{Y}}(\varepsilon^{l_{i,\mathbb{Y}}^+}) = \sum_{j \in \mathbb{Y}} (\sum_{l_{i,\mathbb{Y}}^+ < l \leq l_{ij}^+} a_{ij}[l]\varepsilon^l + o_{ij}(\varepsilon^{l_{ij}^+}))$.

Let us introduce the following condition, which presents additional links between the asymptotic expansions appearing in condition **D**, which are caused by the above model stochasticity assumption:

F: **(a)** $a_{i,\mathbb{Y}_i}[l] = \sum_{j \in \mathbb{Y}_i} a_{ij}[l] = \mathbf{I}(l = 0), 0 = l_{i,\mathbb{Y}_i}^- \leq l \leq l_{i,\mathbb{Y}_i}^+, i \in \mathbb{X}$, where $a_{ij}[l] = 0$, for $0 \leq l < l_{ij}^-, j \in \mathbb{Y}_i, i \in \mathbb{X}$; **(b)** $o_{i,\mathbb{Y}_i}(\varepsilon^{l_{i,\mathbb{Y}_i}^+}) = 0, \varepsilon \in (0, \varepsilon_0], i \in \mathbb{X}$.

Lemma 5 *Let conditions **A (a) – (b)** and **D** hold. In this case, condition **F** is equivalent to the model stochasticity assumption that matrix $\|p_{ij}(\varepsilon)\|$ is stochastic, for every $\varepsilon \in (0, \varepsilon_0]$.*

Proof The model stochasticity assumption for matrices $\|p_{ij}(\varepsilon)\|, \varepsilon \in (0, \varepsilon_0]$, takes, under conditions **A (a) – (b)**, the form of the following identity, which should hold for every $i \in \mathbb{X}$,

$$\sum_{j \in \mathbb{Y}_i} p_{ij}(\varepsilon) = 1, \varepsilon \in (0, \varepsilon_0]. \quad (7)$$

Condition **D** and Lemma 3 let us write down the asymptotic expansion (6) for the case $\mathbb{Y} = \mathbb{Y}_i$. Constant 1 also can be interpreted as the asymptotic expansion $1 = 1 + 0\varepsilon + \dots +$

$0\varepsilon^k + o(\varepsilon^k)$ for $k = l_i, \gamma_i$ and $o(\varepsilon^k) \equiv 0$. Then, identity (7) let one apply Lemma 1 to the described above two asymptotic expansions and get relations appearing in condition **F**.

Conditions **A (a) – (b)** imply that $p_{ij}(\varepsilon) \geq 0$ for $i, j \in \mathbb{X}$. In this case, conditions **D** and **F** obviously imply that $\sum_{j \in \mathbb{X}} p_{ij}(\varepsilon) = 1$, for $i \in \mathbb{X}$. Thus, matrix $\|p_{ij}(\varepsilon)\|$ is stochastic. $\square$

It is also worse to note that, under the assumption of holding condition **A (a)**, the perturbation conditions **D** and **E** are independent.

To see this, let us take arbitrary positive functions $p_{ij}(\varepsilon)$, $j \in \mathbb{X}_i, i \in \mathbb{X}$ and $e_{ij}(\varepsilon)$, $j \in \mathbb{Y}_i, i \in \mathbb{X}$ satisfying, respectively, conditions **D** and **E**, and, also, the corresponding stochasticity identities (7). Then, there exist semi-Markov transition probabilities $Q_{ij}^{(\varepsilon)}(t)$, $t \geq 0$, $j \in \mathbb{Y}_i, i \in \mathbb{X}$ such that $Q_{ij}^{(\varepsilon)}(\infty) = p_{ij}(\varepsilon)$, $j \in \mathbb{Y}_i, i \in \mathbb{X}$ and $\int_0^\infty t Q_{ij}^{(\varepsilon)}(dt) = e_{ij}(\varepsilon)$, $j \in \mathbb{Y}_i, i \in \mathbb{X}$, for every $\varepsilon \in (0, \varepsilon_0]$. It is readily seen that, for example, semi-Markov transition probabilities $Q_{ij}^{(\varepsilon)}(t) = I(t \geq e_{ij}(\varepsilon)/p_{ij}(\varepsilon))p_{ij}(\varepsilon)$, $t \geq 0$, $j \in \mathbb{Y}_i, i \in \mathbb{X}$ satisfy the above relations.

Conditions **A–C** imply that, for every $\varepsilon \in (0, \varepsilon_0]$, the semi-Markov process $\eta^{(\varepsilon)}(t)$ is also ergodic, and its stationary distribution $\bar{\pi}^{(\varepsilon)} = \langle \pi_1(\varepsilon), \dots, \pi_N(\varepsilon) \rangle$ is given by the following ergodic relation,

$$\mu_i^{(\varepsilon)}(t) = \frac{1}{t} \int_0^t I(\eta^{(\varepsilon)}(s) = i) ds \xrightarrow{P} \pi_i(\varepsilon) \text{ as } t \rightarrow \infty, \text{ for } i \in \mathbb{X}. \quad (8)$$

This ergodic relation holds for any initial distribution $\bar{p}^{(\varepsilon)}$, and the stationary distribution $\bar{\pi}^{(\varepsilon)}$ does not depend on the initial distribution. Also, $\pi_i(\varepsilon) > 0$, $i \in \mathbb{X}$ and $\sum_{i \in \mathbb{X}} \pi_i(\varepsilon) = 1$, for every $\varepsilon \in (0, \varepsilon_0]$.

Let us define hitting times, which are random variables given by the following relation, for $j \in \mathbb{X}$,

$$\tau_j^{(\varepsilon)} = \sum_{n=1}^{v_j^{(\varepsilon)}} \kappa_n^{(\varepsilon)}, \text{ where } v_j^{(\varepsilon)} = \min(n \geq 1 : \eta_n^{(\varepsilon)} = j). \quad (9)$$

Let us denote,

$$E_{ij}(\varepsilon) = E_i \tau_j^{(\varepsilon)}, \quad i, j \in \mathbb{X}. \quad (10)$$

As is known, conditions **A – C** imply that, expectations of hitting times $0 < E_{ij}(\varepsilon) < \infty$, $i, j \in \mathbb{X}$, for every $\varepsilon \in (0, \varepsilon_0]$,

The following well known relation for stationary probabilities (which holds for every $\varepsilon \in (0, \varepsilon_0]$) plays an important role in what follows,

$$\pi_i(\varepsilon) = \frac{e_i(\varepsilon)}{E_{ii}(\varepsilon)}, \quad i \in \mathbb{X}, \quad (11)$$

where

$$e_i(\varepsilon) = E_i \kappa_1^{(\varepsilon)} = \sum_{j \in \mathbb{Y}_i} e_{ij}(\varepsilon), \quad i \in \mathbb{X}. \quad (12)$$

Condition **D** implies that there exists $\lim_{\varepsilon \rightarrow 0} p_{ij}(\varepsilon) = p_{ij}(0)$, which equals to $a_{ij}[0] \in (0, 1]$ if $l_{ij}^- = 0$, $j \in \mathbb{Y}_i, i \in \mathbb{X}$; or to 0 if $l_{ij}^- > 0$, $j \in \mathbb{Y}_i, i \in \mathbb{X}$ or $j \in \bar{\mathbb{Y}}_i, i \in \mathbb{X}$. Condition **E** implies that there exists $\lim_{\varepsilon \rightarrow 0} e_{ij}(\varepsilon) = e_{ij}(0)$, which equals to ∞ if $m_{ij}^- < 0$, $j \in \mathbb{Y}_i, i \in \mathbb{X}$; or to $b_{ij}[0] \in (0, \infty)$ if $m_{ij}^- = 0$, $j \in \mathbb{Y}_i, i \in \mathbb{X}$; or to 0 if $m_{ij}^- > 0$, $j \in \mathbb{Y}_i, i \in \mathbb{X}$ or $j \in \bar{\mathbb{Y}}_i, i \in \mathbb{X}$.

Matrix $\|p_{ij}(\varepsilon)\|$ is stochastic, for every $\varepsilon \in (0, \varepsilon_0]$, and, thus, matrix $\|p_{ij}(0)\|$ is also stochastic. Let $\eta_n^{(0)}$ be a Markov chain with the phase space $\mathbb{X}$ and the matrix of transition

probabilities $\|p_{ij}(0)\|$. It is possible that matrix $\|p_{ij}(0)\|$ has more zero elements than matrices $\|p_{ij}(\varepsilon)\|$ and, thus, $\mathbb{X}$ can consist of one or several closed classes of communicative states plus, possibly, a class of transient states, for the Markov chain $\eta_n^{(0)}$.

Our goal is to design an effective algorithm for construction of asymptotic expansions for stationary probabilities $\pi_i(\varepsilon)$, $i \in \mathbb{X}$, under the assumption that conditions **A** – **E** hold. As we shall see, the proposed algorithm can be applied to models with an arbitrary asymptotic communicative structure of phase spaces.

The models of nonlinearly perturbed discrete and continuous Markov chains are particular cases of the above model of nonlinearly perturbed semi-Markov processes.

If $\eta^{(\varepsilon)}(t)$ is a discrete time Markov chain, condition **D** implies condition **E**, since, in this case, expectations $e_{ij}(\varepsilon) = p_{ij}(\varepsilon)$, $j \in \mathbb{Y}_i$, $i \in \mathbb{X}$.

If $\eta^{(\varepsilon)}(t)$ is a continuous time Markov chain, condition **E** can be replaced by an analogous condition, which assumes that expectations $e_i(\varepsilon) = \lambda_i(\varepsilon)^{-1}$, $i \in \mathbb{X}$ can be represented in the form of pivotal Laurent asymptotic expansions. This condition and condition **D** would imply condition **E**, with the corresponding Laurent asymptotic expansions obtained by application proposition (ii) (the multiplication rule) of Lemma 2 to the products $e_{ij}(\varepsilon) = \lambda_i(\varepsilon)^{-1} p_{ij}(\varepsilon)$, $j \in \mathbb{Y}_i$, $i \in \mathbb{X}$.

4 Semi-Markov Processes with Reduced Phase Spaces

Let us choose some state $r \in \mathbb{X}$ and consider the reduced phase space ${}_r\mathbb{X} = \mathbb{X} \setminus \{r\}$, with the state r excluded from the phase space $\mathbb{X}$.

Let us assume that the initial distributions satisfy the following assumption,

$$p_r^{(\varepsilon)} = P\{\eta_0^{(\varepsilon)} = r\} = 0, \quad \varepsilon \in (0, \varepsilon_0]. \quad (13)$$

Let us define the sequential moments of hitting the reduced space ${}_r\mathbb{X}$ by the embedded Markov chain $\eta_n^{(\varepsilon)}$,

$${}_r\xi_n^{(\varepsilon)} = \min(k > {}_r\xi_{n-1}^{(\varepsilon)}, \eta_k^{(\varepsilon)} \in {}_r\mathbb{X}), \quad n = 1, 2, \dots, \quad {}_r\xi_0^{(\varepsilon)} = 0. \quad (14)$$

Now, let us define the random sequence,

$$({}_r\eta_n^{(\varepsilon)}, {}_r\kappa_n^{(\varepsilon)}) = \begin{cases} (\eta_0^{(\varepsilon)}, 0) & \text{for } n = 0, \\ (\eta_{{}_r\xi_n^{(\varepsilon)}}^{(\varepsilon)}, \sum_{k={}_r\xi_{n-1}^{(\varepsilon)}+1}^{{}_r\xi_n^{(\varepsilon)}} \kappa_k^{(\varepsilon)}) & \text{for } n = 1, 2, \dots \end{cases} \quad (15)$$

This sequence is also a Markov renewal process with a phase space ${}_r\mathbb{X} \times [0, \infty)$, the initial distribution ${}_r\bar{p}^{(\varepsilon)} = \langle {}_r p_i^{(\varepsilon)} = p_i^{(\varepsilon)}, i \in {}_r\mathbb{X} \rangle$ (recall that $p_r^{(\varepsilon)} = 0$), and transition probabilities defined for $(i, s), (j, t) \in {}_r\mathbb{X} \times [0, \infty)$,

$${}_rQ_{ij}^{(\varepsilon)}(t) = P\{{}_r\eta_1^{(\varepsilon)} = j, {}_r\kappa_1^{(\varepsilon)} \leq t / {}_r\eta_0^{(\varepsilon)} = i, {}_r\kappa_0^{(\varepsilon)} = s\}. \quad (16)$$

Respectively, one can define the transformed semi-Markov process with the reduced phase space ${}_r\mathbb{X}$,

$${}_r\eta^{(\varepsilon)}(t) = {}_r\eta_{{}_r\nu^{(\varepsilon)}(t)}^{(\varepsilon)}, \quad t \geq 0, \quad (17)$$

where ${}_r\nu^{(\varepsilon)}(t) = \max(n \geq 0 : {}_r\xi_n^{(\varepsilon)} \leq t)$ is a number of jumps at time interval $[0, t]$, for $t \geq 0$, and ${}_r\xi_n^{(\varepsilon)} = {}_r\kappa_1^{(\varepsilon)} + \dots + {}_r\kappa_n^{(\varepsilon)}$, $n = 0, 1, \dots$ are sequential moments of jumps, for the semi-Markov process ${}_r\eta^{(\varepsilon)}(t)$.

The transition probabilities ${}_r Q_{ij}^{(\varepsilon)}(t)$ are expressed via the transition probabilities $Q_{ij}^{(\varepsilon)}(t)$ by the following formula, for $t \geq 0$, $i, j \in {}_r\mathbb{X}$,

$${}_r Q_{ij}^{(\varepsilon)}(t) = Q_{ij}^{(\varepsilon)}(t) + \sum_{n=0}^{\infty} Q_{ir}^{(\varepsilon)}(t) * Q_{rr}^{(\varepsilon)*n}(t) * Q_{rj}^{(\varepsilon)}(t). \quad (18)$$

Here, symbol $*$ is used to denote the convolution of distribution functions (possibly improper), and $Q_{rr}^{(\varepsilon)*n}(t)$ is the n times convolution of the distribution function $Q_{rr}^{(\varepsilon)}(t)$.

Relation (18) directly implies the following formula for transition probabilities of the reduced embedded Markov chain ${}_r\eta_n^{(\varepsilon)}$, for $i, j \in {}_r\mathbb{X}$,

$$\begin{aligned} {}_r p_{ij}(\varepsilon) &= {}_r Q_{ij}^{(\varepsilon)}(\infty) = p_{ij}(\varepsilon) + \sum_{n=0}^{\infty} p_{ir}(\varepsilon) p_{rr}(\varepsilon)^n p_{rj}(\varepsilon) \\ &= p_{ij}(\varepsilon) + p_{ir}(\varepsilon) \frac{p_{rj}(\varepsilon)}{1 - p_{rr}(\varepsilon)}. \end{aligned} \quad (19)$$

Note that condition **A** implies that probabilities $p_{rr}(\varepsilon) \in [0, 1)$, $r \in \mathbb{X}$, $\varepsilon \in (0, \varepsilon_0]$.

Let us introduce sets, $\mathbb{Y}_{ir}^- = \{j \in {}_r\mathbb{X} : j \in \mathbb{Y}_r\}$ if $r \in \mathbb{Y}_i$, or $\emptyset$ if $r \notin \mathbb{Y}_i$, and $\mathbb{Y}_{ir}^+ = \{j \in {}_r\mathbb{X} : j \in \mathbb{Y}_i\}$, for $i, r \in \mathbb{X}$.

We omit the proof of the following simple lemma.

Lemma 6 *Condition A, assumed to hold for the Markov chains $\eta_n^{(\varepsilon)}$, also holds for the Markov chains ${}_r\eta_n^{(\varepsilon)}$, with the same parameter ε_0 and transition sets ${}_r\mathbb{Y}_i$ defined by the following relation, for $i \in {}_r\mathbb{X}$,*

$${}_r\mathbb{Y}_i = \{j \in {}_r\mathbb{X} : {}_r p_{ij}(\varepsilon) > 0, \varepsilon \in (0, \varepsilon_0]\} = \mathbb{Y}_{ir}^- \cup \mathbb{Y}_{ir}^+. \quad (20)$$

Let us introduce expectations,

$${}_r e_{ij}(\varepsilon) = \int_0^{\infty} t {}_r Q_{ij}^{(\varepsilon)}(dt), \quad i, j \in {}_r\mathbb{X}. \quad (21)$$

Relation (18) directly implies the following formula for expectations of sojourn times for the reduced semi-Markov process ${}_r\eta^{(\varepsilon)}(t)$, for $i, j \in {}_r\mathbb{X}$,

$$\begin{aligned} {}_r e_{ij}(\varepsilon) &= e_{ij}(\varepsilon) + \sum_{n=0}^{\infty} (e_{ir}(\varepsilon) p_{rj}(\varepsilon) + (n+1) e_{rr}(\varepsilon) p_{ir}(\varepsilon) p_{rj}(\varepsilon) \\ &\quad + e_{rj}(\varepsilon) p_{ir}(\varepsilon)) p_{rr}(\varepsilon)^n = e_{ij}(\varepsilon) + e_{ir}(\varepsilon) \frac{p_{rj}(\varepsilon)}{1 - p_{rr}(\varepsilon)} \\ &\quad + e_{rr}(\varepsilon) \frac{p_{ir}(\varepsilon)}{1 - p_{rr}(\varepsilon)} \frac{p_{rj}(\varepsilon)}{1 - p_{rr}(\varepsilon)} + e_{rj}(\varepsilon) \frac{p_{ir}(\varepsilon)}{1 - p_{rr}(\varepsilon)}. \end{aligned} \quad (22)$$

The following simple lemma is the direct corollary of relation (22).

Lemma 7 *Conditions B and C, assumed to hold for the semi-Markov processes $\eta^{(\varepsilon)}(t)$, also hold for the semi-Markov processes ${}_r\eta^{(\varepsilon)}(t)$.*

The following theorem presents the result, similar to those given in recent papers by Silvestrov and Manca (2015) and Silvestrov and Silvestrov (2016a, b). It plays an important role in what follows.

Theorem 1 *Let conditions **A** – **C** hold for semi-Markov processes $\eta^{(\varepsilon)}(t)$. Then, for any state $j \in {}_r\mathbb{X}$, the first hitting times $\tau_j^{(\varepsilon)}$ and ${}_r\tau_j^{(\varepsilon)}$ to the state j , respectively, for semi-Markov processes $\eta^{(\varepsilon)}(t)$ and ${}_r\eta^{(\varepsilon)}(t)$, coincide, and, thus, the expectations of hitting times $E_{ij}(\varepsilon) = E_i\tau_j^{(\varepsilon)} = E_i{}_r\tau_j^{(\varepsilon)}$, for any $i, j \in {}_r\mathbb{X}$ and $\varepsilon \in (0, \varepsilon_0]$.*

Proof The first hitting times to a state $j \in {}_r\mathbb{X}$ are connected for Markov chains $\eta_n^{(\varepsilon)}$ and ${}_r\eta_n^{(\varepsilon)}$ by the following relation,

$$v_j^{(\varepsilon)} = \min(n \geq 1 : \eta_n^{(\varepsilon)} = j) = \min({}_r\xi_n^{(\varepsilon)} \geq 1 : {}_r\eta_n^{(\varepsilon)} = j) = {}_r\xi_{v_j^{(\varepsilon)}}^{(\varepsilon)}, \quad (23)$$

where ${}_rv_j^{(\varepsilon)} = \min(n \geq 1 : {}_r\eta_n^{(\varepsilon)} = j)$.

Relation (23) implies that the following relation holds for the first hitting times to a state $j \in {}_r\mathbb{X}$, for the semi-Markov processes $\eta^{(\varepsilon)}(t)$ and ${}_r\eta^{(\varepsilon)}(t)$,

$$\tau_j^{(\varepsilon)} = \sum_{n=1}^{v_j^{(\varepsilon)}} \kappa_n^{(\varepsilon)} = \sum_{n=1}^{{}_r\xi_{v_j^{(\varepsilon)}}^{(\varepsilon)}} \kappa_n^{(\varepsilon)} = \sum_{n=1}{{}_rv_j^{(\varepsilon)}} {}_r\kappa_n^{(\varepsilon)} = {}_r\tau_j^{(\varepsilon)}. \quad (24)$$

The equality of expectations is an obvious corollary of relation (24). $\square$

We would like to preface lemmas and theorems presenting algorithms of for constriction of asymptotic expansions for nonlinearly perturbed semi-Markov processes by comments clarifying slightly unusual references to descriptions of algorithms in the proofs.

All lemmas and theorems formulated below, contain proofs of propositions that the corresponding functionals for perturbed reduced semi-Markov processes can be represented in the form of asymptotic expansions. These proofs are based on recurrent application of operational formulas for Laurent asymptotic expansions presented in Section 1 to the reduced semi-Markov processes constructed with the use of the corresponding recurrent time-space screening procedures of phase space reduction. In fact, one should correctly describe to which functions, in which order, and which operational rules should be applied for getting the corresponding expansions (their parameters, coefficients and remainders) as well as to indicate some particular cases, where the corresponding computational steps should be modified. This is exactly what is done in all proofs of the corresponding lemmas and theorems.

An explicit writing down corresponding operational formulas representing the above recurrent algorithms (which could be given, say, as corollaries of these lemmas and theorems) would, in fact, replicate the above proofs in the formal form, require implementation of huge number of intermediate notations, take too much space, etc., but would not add any new essential information about the corresponding algorithms. That is why the decision was made, just, to say in each theorem that the description of the corresponding algorithms are given in their proofs. This makes formulations slightly unusual. But, as we think, this is the most compact way for presentation of the corresponding asymptotic results and algorithms.

As was mentioned above, condition **A** implies that sets $\mathbb{Y}_{rr}^+ \neq \emptyset$, $r \in \mathbb{X}$ and the non-absorption probability $\bar{p}_{rr}(\varepsilon) = 1 - p_{rr}(\varepsilon) > 0$, for $r \in \mathbb{X}$, $\varepsilon \in (0, \varepsilon_0]$. This probability satisfies the following relation, for every $r \in \mathbb{X}$ and $\varepsilon \in (0, \varepsilon_0]$,

$$\bar{p}_{rr}(\varepsilon) = 1 - p_{rr}(\varepsilon) = \sum_{j \in \mathbb{Y}_{rr}^+} p_{rj}(\varepsilon). \quad (25)$$

Lemma 8 *Let conditions **A** and **D** hold. Then, the pivotal $(\bar{l}_{rr}^-, \bar{l}_{rr}^+)$ -expansions for the non-absorption probabilities $\bar{p}_{rr}(\varepsilon)$, $r \in \mathbb{X}$ are given by the algorithm described below, in the proof of the lemma.*

Proof Let $r \in \mathbb{Y}_r$. First, proposition (i) (the multiple summation rule) of Lemma 3 should be applied to the sum $\sum_{j \in \mathbb{Y}_{rr}^+} p_{rj}(\varepsilon)$. Second, propositions (i) (the multiplication by constant -1) and (ii) (the summation with constant 1) of Lemma 2 should be applied to the asymptotic expansion for probability $p_{rr}(\varepsilon)$ given in condition **B**, in order to get the asymptotic expansion for function $1 - p_{rr}(\varepsilon)$. Third, Lemma 1 should be applied to the asymptotic expansion for function $\bar{p}_{rr}(\varepsilon)$ given in two alternative forms by relation (25). Note that condition **F** holds also for the above case, where the asymptotic expansion for probability $\bar{p}_{rr}(\varepsilon)$, obtained at the second step, is replaced by the improved version of this expansion, obtained with the use of Lemma 1 at the third step. The case $r \notin \mathbb{Y}_r$ is trivial, since, in this case, probability $\bar{p}_{rr}(\varepsilon) \equiv 1$. According to Lemmas 1–3, $(\bar{l}_{rr}^-, \bar{l}_{rr}^+)$ -expansions $\bar{p}_{rr}(\varepsilon) = \sum_{l=\bar{l}_{rr}^-}^{\bar{l}_{rr}^+} \bar{a}_{rr}[l]\varepsilon^l + \bar{o}_{rr}(\varepsilon^{\bar{l}_{rr}^+})$, $\varepsilon \in (0, \varepsilon_0]$, $r \in \mathbb{X}$, yielded by the above algorithm, are pivotal. $\square$

Let us now describe an algorithm for construction of asymptotic expansions for transition probabilities ${}_r p_{ij}(\varepsilon)$ given by relation (19).

Theorem 2 *Conditions **A** and **D**, assumed to hold for the Markov chains $\eta_n^{(\varepsilon)}$, also hold for the reduced Markov chains ${}_r \eta_n^{(\varepsilon)}$, with the same parameter ε_0 and the transition sets ${}_r \mathbb{Y}_i$, $i \in {}_r \mathbb{X}$, given by relation (20). The pivotal $({}_r l_{ij}^-, {}_r l_{ij}^+)$ -expansions appearing in condition **D** are given for transition probabilities ${}_r p_{ij}(\varepsilon)$, $j \in {}_r \mathbb{Y}_i$, $i \in {}_r \mathbb{X}$, $r \in \mathbb{X}$ by the algorithm described below, in the proof of the theorem.*

Proof Lemma 6 implies that condition **A** holds for the Markov chains ${}_r \eta_n^{(\varepsilon)}$, with the same parameter ε_0 as for the Markov chains $\eta_n^{(\varepsilon)}$, and the transition sets ${}_r \mathbb{Y}_i$, $i \in {}_r \mathbb{X}$ given by relation (20).

Let us prove that condition **D** also holds for the Markov chains ${}_r \eta_n^{(\varepsilon)}$, with the same parameter ε_0 and the transition sets ${}_r \mathbb{Y}_i$, $i \in {}_r \mathbb{X}$ given by relation (20). In order to do this, let us construct the corresponding asymptotic expansions appearing in this condition. Let $j, r \in \mathbb{Y}_i \cap \mathbb{Y}_r$. First, proposition (v) (the division rule) of Lemma 2 should be applied to the quotient $\frac{p_{rj}(\varepsilon)}{1 - p_{rr}(\varepsilon)}$. Second, proposition (iii) (the multiplication rule) of Lemma 2 should be applied to the product $p_{ir}(\varepsilon) \cdot \frac{p_{rj}(\varepsilon)}{1 - p_{rr}(\varepsilon)}$. Third, proposition (ii) (the summation rule) of Lemma 2 should be applied to sum ${}_r p_{ij}(\varepsilon) = p_{ij}(\varepsilon) + p_{ir}(\varepsilon) \cdot \frac{p_{rj}(\varepsilon)}{1 - p_{rr}(\varepsilon)}$. The asymptotic expansions for probabilities $p_{ij}(\varepsilon)$, $p_{ir}(\varepsilon)$, and $p_{rj}(\varepsilon)$, given in condition **B**, and probability $1 - p_{rr}(\varepsilon)$, given in Lemma 8, should be used. If $j \notin \mathbb{Y}_i$ then $p_{ij}(\varepsilon) \equiv 0$; if $j \notin \mathbb{Y}_r$ then $p_{rj}(\varepsilon) \equiv 0$; if $r \notin \mathbb{Y}_i$ then $p_{ir}(\varepsilon) \equiv 0$; if $r \notin \mathbb{Y}_r$ then $1 - p_{rr}(\varepsilon) \equiv 1$. In these cases, the above algorithm is readily simplified with the use of Lemma 4. Note that parameter ε_0 does not change in the multiplication and summation steps as well as in the division step, since $1 - p_{rr}(\varepsilon) > 0$, $\varepsilon \in (0, \varepsilon_0]$. According to Lemma 2, the $({}_r l_{ij}^-, {}_r l_{ij}^+)$ -expansions ${}_r p_{ij}(\varepsilon) = \sum_{l={}_r l_{ij}^-}^{{}_r l_{ij}^+} {}_r a_{ij}[l]\varepsilon^l + {}_r o_{ij}(\varepsilon^{{}_r l_{ij}^+})$, $\varepsilon \in (0, \varepsilon_0]$, $j \in {}_r \mathbb{Y}_i$, $i \in {}_r \mathbb{X}$, $r \in \mathbb{X}$, yielded by the above algorithm, are pivotal. $\square$

Remark 4 The matrix of transition probabilities $\|_r p_{ij}(\varepsilon)\|$ is stochastic, for every $\varepsilon \in (0, \varepsilon_0]$. Thus, under conditions of Theorem 2, condition **F** holds for the asymptotic expansions of transition probabilities ${}_r p_{ij}(\varepsilon)$, $j \in {}_r Y_i$, $i \in {}_r X$, given in this theorem.

Let us now describe an algorithm for construction of asymptotic expansions for expectations ${}_r e_{ij}(\varepsilon)$ given by relation (22).

Theorem 3 *Conditions A – E, assumed to hold for the semi-Markov processes $\eta^{(\varepsilon)}(t)$, also hold for the reduced semi-Markov processes ${}_r \eta^{(\varepsilon)}(t)$. Parameter ε_0 , in conditions A, D and E, is the same for processes $\eta^{(\varepsilon)}(t)$ and ${}_r \eta^{(\varepsilon)}(t)$. The transition sets ${}_r Y_i$, $i \in {}_r X$ are given for processes ${}_r \eta^{(\varepsilon)}(t)$ by relation (20). The pivotal $({}_r m_{ij}^-, {}_r m_{ij}^+)$ -expansions appearing in condition E are given for expectations ${}_r e_{ij}(\varepsilon)$, $j \in {}_r Y_i$, $i \in {}_r X$ by the algorithm described below, in the proof of the theorem.*

Proof Lemma 6 and Theorem 2 imply that conditions **A** and **D** hold for the semi-Markov processes ${}_r \eta^{(\varepsilon)}(t)$, with the same parameter ε_0 as for the semi-Markov processes $\eta^{(\varepsilon)}(t)$, and the transition sets ${}_r Y_i$, $i \in {}_r X$ given by relation (20). Also, conditions **B** and **C** hold for the semi-Markov processes ${}_r \eta^{(\varepsilon)}(t)$, by Lemma 7.

In order to prove that condition **E** also holds for the semi-Markov processes ${}_r \eta^{(\varepsilon)}(t)$, with the same parameter ε_0 and the transition sets ${}_r Y_i$, $i \in {}_r X$ given by relation (20), let us construct the corresponding asymptotic expansions appearing in this condition. Let $j, r \in Y_i \cap Y_r$. First, proposition (v) (the division rule) of Lemma 2 should be applied to the quotients $\frac{p_{rj}(\varepsilon)}{1-p_{rr}(\varepsilon)}$ and $\frac{p_{ir}(\varepsilon)}{1-p_{rr}(\varepsilon)}$. Second, proposition (iii) (the multiplication rule) of Lemma 2 should be applied to the products $e_{ir}(\varepsilon) \cdot \frac{p_{rj}(\varepsilon)}{1-p_{rr}(\varepsilon)}$ and $e_{rj}(\varepsilon) \cdot \frac{p_{ir}(\varepsilon)}{1-p_{rr}(\varepsilon)}$, and proposition (ii) (the multiple multiplication rule) of Lemma 3 to the product $e_{rr}(\varepsilon) \cdot \frac{p_{ir}(\varepsilon)}{1-p_{rr}(\varepsilon)} \cdot \frac{p_{rj}(\varepsilon)}{1-p_{rr}(\varepsilon)}$. Third, proposition (i) (the multiple summation rule) of Lemma 3 should be applied to sum ${}_r e_{ij}(\varepsilon) = e_{ij}(\varepsilon) + e_{ir}(\varepsilon) \cdot \frac{p_{rj}(\varepsilon)}{1-p_{rr}(\varepsilon)} + e_{rr}(\varepsilon) \cdot \frac{p_{ir}(\varepsilon)}{1-p_{rr}(\varepsilon)} \cdot \frac{p_{rj}(\varepsilon)}{1-p_{rr}(\varepsilon)} + e_{rj}(\varepsilon) \cdot \frac{p_{ir}(\varepsilon)}{1-p_{rr}(\varepsilon)}$. The asymptotic expansions for probabilities $p_{ij}(\varepsilon)$, $p_{ir}(\varepsilon)$ and $p_{rj}(\varepsilon)$, given in condition **D**, probability $1 - p_{rr}(\varepsilon)$, given in Lemma 8, and expectations $e_{ij}(\varepsilon)$, $e_{ir}(\varepsilon)$, $e_{rr}(\varepsilon)$ and $e_{rj}(\varepsilon)$, given in condition **E**, should be used. If $j \notin Y_i$ then $p_{ij}(\varepsilon) \equiv 0$ and $e_{ij}(\varepsilon) \equiv 0$; if $j \notin Y_r$ then $p_{rj}(\varepsilon) \equiv 0$ and $e_{rj}(\varepsilon) \equiv 0$; if $r \notin Y_i$ then $p_{ir}(\varepsilon) \equiv 0$ and $e_{ir}(\varepsilon) \equiv 0$; if $r \notin Y_r$ then $1 - p_{rr}(\varepsilon) \equiv 1$ and $e_{rr}(\varepsilon) \equiv 0$. In these cases, the above algorithm is readily simplified with the use of Lemma 4. As in Theorem 2, parameter ε_0 does not change in the multiplication and summation steps as well as in the division step, since $1 - p_{rr}(\varepsilon) > 0$, $\varepsilon \in (0, \varepsilon_0]$.

According to Lemmas 2 and 3, the $({}_r m_{ij}^-, {}_r m_{ij}^+)$ -expansions ${}_r e_{ij}(\varepsilon) = \sum_{{}_r m_{ij}^-}^{{}_r m_{ij}^+} {}_r b_{ij}[l] \varepsilon^l + {}_r \dot{o}_{ij}(\varepsilon^{{}_r m_{ij}^+})$, $\varepsilon \in (0, \varepsilon_0]$, $j \in {}_r Y_i$, $i \in {}_r X$, $r \in X$, yielded by the above algorithm, are pivotal. $\square$

5 Sequential Reduction of the Phase Space

In what follows, let $\bar{r}_{i,N} = \langle r_{i,1}, \dots, r_{i,N} \rangle = \langle r_{i,1}, \dots, r_{i,N-1}, i \rangle$ be a permutation of the sequence $\langle 1, \dots, N \rangle$ such that $r_{i,N} = i$, and let $\bar{r}_{i,n} = \langle r_{i,1}, \dots, r_{i,n} \rangle$, $n = 1, \dots, N$ be the corresponding chain of growing sequences of states from space X .

Theorem 4 *Let conditions A – E hold for semi-Markov processes $\eta^{(\varepsilon)}(t)$. Then, for every $i \in X$, the pivotal (M_{ii}^-, M_{ii}^+) -expansion for the expectation of hitting time $E_{ii}(\varepsilon)$ is given*

by the algorithm based on the sequential exclusion of states $r_{i,1}, \dots, r_{i,N-1}$ from the phase space $\mathbb{X}$ of the processes $\eta^{(\varepsilon)}(t)$. This algorithm is described below, in the proof of the theorem. The above (M_{ii}^-, M_{ii}^+) -expansion is invariant with respect to any permutation $\bar{r}_{i,N} = \langle r_{i,1}, \dots, r_{i,N-1}, i \rangle$ of sequence $\langle 1, \dots, N \rangle$.

Proof Let us assume that $p_i^{(\varepsilon)} = 1$. Denote as $\bar{r}_{i,0}\eta^{(\varepsilon)}(t) = \eta^{(\varepsilon)}(t)$, the initial semi-Markov process. Let us exclude state $r_{i,1}$ from the phase space of semi-Markov process $\bar{r}_{i,0}\eta^{(\varepsilon)}(t)$ using the time-space screening procedure described in Section 4. Let $\bar{r}_{i,1}\eta^{(\varepsilon)}(t)$ be the corresponding reduced semi-Markov process. The above procedure can be repeated. The state $r_{i,2}$ can be excluded from the phase space of the semi-Markov process $\bar{r}_{i,1}\eta^{(\varepsilon)}(t)$. Let $\bar{r}_{i,2}\eta^{(\varepsilon)}(t)$ be the corresponding reduced semi-Markov process. By continuing the above procedure for states $r_{i,3}, \dots, r_{i,n}$, we construct the reduced semi-Markov process $\bar{r}_{i,n}\eta^{(\varepsilon)}(t)$.

The process $\bar{r}_{i,n}\eta^{(\varepsilon)}(t)$ has the phase space $\bar{r}_{i,n}\mathbb{X} = \mathbb{X} \setminus \{r_{i,1}, r_{i,2}, \dots, r_{i,n}\}$. The transition probabilities of the embedded Markov chain $\bar{r}_{i,n}p_{i'j'}(\varepsilon)$, $i', j' \in \bar{r}_{i,n}\mathbb{X}$, and the expectations of sojourn times $\bar{r}_{i,n}e_{i'j'}(\varepsilon)$, $i', j' \in \bar{r}_{i,n}\mathbb{X}$ are determined for the semi-Markov process $\bar{r}_{i,n}\eta^{(\varepsilon)}(t)$ by the transition probabilities and the expectations of sojourn times for the process $\bar{r}_{i,n-1}\eta^{(\varepsilon)}(t)$, respectively, via relations (19) and (22).

By Theorem 1, the expectation of hitting time $E_{i'j'}(\varepsilon)$ coincides for the semi-Markov processes $\bar{r}_{i,0}\eta^{(\varepsilon)}(t)$, $\bar{r}_{i,1}\eta^{(\varepsilon)}(t)$, $\dots$, $\bar{r}_{i,n}\eta^{(\varepsilon)}(t)$, for every $i', j' \in \bar{r}_{i,n}\mathbb{X}$.

By Theorems 2 and 3, the semi-Markov processes $\bar{r}_{i,n}\eta^{(\varepsilon)}(t)$ satisfy conditions **B**, **C** and, also, conditions **A**, **D** and **E**, with the same parameter ε_0 as for processes $\bar{r}_{i,n-1}\eta^{(\varepsilon)}(t)$.

The transition sets $\bar{r}_{i,n}\mathbb{Y}_{i'}$, $i' \in \bar{r}_{i,n}\mathbb{X}$ are determined by the transition sets $\bar{r}_{i,n-1}\mathbb{Y}_{i'}$, $i' \in \bar{r}_{i,n-1}\mathbb{X}$, via relation (20) given in Lemma 6. Therefore, the pivotal $(\bar{r}_{i,n}l_{i'j'}^-, \bar{r}_{i,n}l_{i'j'}^+)$ -expansions, $\bar{r}_{i,n}p_{i'j'}(\varepsilon) = \sum_{\bar{r}_{i,n}l_{i'j'}^-}^{\bar{r}_{i,n}l_{i'j'}^+} \bar{r}_{i,n}a_{i'j'}[l]\varepsilon^l + \bar{r}_{i,n}o_{i'j'}(\varepsilon^{\bar{r}_{i,n}l_{i'j'}^+})$, $\varepsilon \in (0, \varepsilon_0]$, $j' \in \bar{r}_{i,n}\mathbb{Y}_{i'}$, $i' \in \bar{r}_{i,n}\mathbb{X}$, and the pivotal $(\bar{r}_{i,n}m_{i'j'}^-, \bar{r}_{i,n}m_{i'j'}^+)$ -expansions, $\bar{r}_{i,n}e_{i'j'}(\varepsilon) = \sum_{\bar{r}_{i,n}m_{i'j'}^-}^{\bar{r}_{i,n}m_{i'j'}^+} \bar{r}_{i,n}b_{i'j'}[l]\varepsilon^l + \bar{r}_{i,n}\dot{o}_{i'j'}(\varepsilon^{\bar{r}_{i,n}m_{i'j'}^+})$, $\varepsilon \in (0, \varepsilon_0]$, $j' \in \bar{r}_{i,n}\mathbb{Y}_{i'}$, $i' \in \bar{r}_{i,n}\mathbb{X}$, can be constructed by applying the algorithms given in Theorems 2 and 3, respectively, to the $(\bar{r}_{i,n-1}l_{i'j'}^-, \bar{r}_{i,n-1}l_{i'j'}^+)$ -expansions for transition probabilities $\bar{r}_{i,n-1}p_{i'j'}(\varepsilon)$, $j' \in \bar{r}_{i,n-1}\mathbb{Y}_{i'}$, $i' \in \bar{r}_{i,n-1}\mathbb{X}$ and to the $(\bar{r}_{i,n-1}m_{i'j'}^-, \bar{r}_{i,n-1}m_{i'j'}^+)$ -expansions for expectations $\bar{r}_{i,n-1}e_{i'j'}(\varepsilon)$, $j' \in \bar{r}_{i,n-1}\mathbb{Y}_{i'}$, $i' \in \bar{r}_{i,n-1}\mathbb{X}$.

The algorithm described above has a recurrent form and should be realized sequentially for the reduced semi-Markov processes $\bar{r}_{i,1}\eta^{(\varepsilon)}(t), \dots, \bar{r}_{i,n}\eta^{(\varepsilon)}(t)$ starting from the initial semi-Markov process $\bar{r}_{i,0}\eta^{(\varepsilon)}(t)$.

For every $j' \in \bar{r}_{i,n}\mathbb{Y}_{i'}$, $i' \in \bar{r}_{i,n}\mathbb{X}$, $n = 1, \dots, N-1$, the asymptotic expansions for the transition probability $\bar{r}_{i,n}p_{i'j'}(\varepsilon)$ and the expectation $\bar{r}_{i,n}e_{i'j'}(\varepsilon)$, resulted by the recurrent algorithm of sequential phase space reduction described above, are invariant with respect to any permutation $\bar{r}'_{i,n} = \langle r'_{i,1}, \dots, r'_{i,n} \rangle$ of sequence $\bar{r}_{i,n} = \langle r_{i,1}, \dots, r_{i,n} \rangle$.

Indeed, for every permutation $\bar{r}'_{i,n}$ of sequence $\bar{r}_{i,n}$, the corresponding reduced semi-Markov process $\bar{r}'_{i,n}\eta^{(\varepsilon)}(t)$ is constructed as the sequence of states for the initial semi-Markov process $\eta^{(\varepsilon)}(t)$ at sequential moments of its hitting into the same reduced phase space $\bar{r}'_{i,n}\mathbb{X} = \mathbb{X} \setminus \{r'_{i,1}, \dots, r'_{i,n}\} = \bar{r}_{i,n}\mathbb{X} = \mathbb{X} \setminus \{r_{i,1}, \dots, r_{i,n}\}$. The times between sequential jumps of the reduced semi-Markov process $\bar{r}'_{i,n}\eta^{(\varepsilon)}(t)$ are the times between sequential hitting of the above reduced phase space by the initial semi-Markov process $\eta^{(\varepsilon)}(t)$.

This implies that the transition probability $\bar{r}_{i,n} p_{i'j'}(\varepsilon)$ and the expectation $\bar{r}_{i,n} e_{i'j'}(\varepsilon)$ are, for every $j' \in \bar{r}_{i,n} \mathbb{Y}_{i'}, i' \in \bar{r}_{i,n} \mathbb{X}, n = 1, \dots, N - 1$, invariant (as functions of ε) with respect to any permutation $\bar{r}'_{i,n}$ of the sequence $\bar{r}_{i,n}$. Moreover, as follows from algorithms presented above, in Lemma 8 and Theorems 2 and 3, the transition probability $\bar{r}_{i,n} p_{i'j'}(\varepsilon)$ is a rational function of the initial transition probabilities $p_{i''j''}(\varepsilon), j'' \in \mathbb{Y}_{i''}, i'' \in \mathbb{X}$, and the expectation $\bar{r}_{i,n} e_{i'j'}(\varepsilon)$ is a rational function of the initial transition probabilities $p_{i''j''}(\varepsilon), j'' \in \mathbb{Y}_{i''}, i'' \in \mathbb{X}$ and the initial expectations of sojourn times $e_{i''j''}(\varepsilon), j'' \in \mathbb{Y}_{i''}, i'' \in \mathbb{X}$ (quotients of sums of products for some of these probabilities and expectations), which, according to the above remarks, are invariant with respect to any permutation $\bar{r}'_{i,n}$ of the sequence $\bar{r}_{i,n}$.

By using identity arithmetical transformations (disclosure of brackets, imposition of a common factor out of the brackets, bringing a fractional expression to a common denominator, permutation of summands or multipliers, elimination of expressions with equal absolute values and opposite signs in the sums and elimination of equal expressions in quotients) the rational functions $\bar{r}'_{i,n} p_{i'j'}(\varepsilon)$ and $\bar{r}'_{i,n} e_{i'j'}(\varepsilon)$ can be transformed, respectively, into the rational functions $\bar{r}_{i,n} p_{i'j'}(\varepsilon)$ and $\bar{r}_{i,n} e_{i'j'}(\varepsilon)$ and vice versa. By Lemma 4, these transformations do not affect the corresponding asymptotic expansions for functions $\bar{r}_{i,n} p_{i'j'}(\varepsilon)$ and $\bar{r}_{i,n} e_{i'j'}(\varepsilon)$ and, thus, these expansions are invariant with respect to any permutation $\bar{r}'_{i,n}$ of the sequence $\bar{r}_{i,n}$.

In fact, one should only check the above invariance propositions for the case, where the permutations $\bar{r}'_{i,n}$ is obtained from the sequence $\bar{r}_{i,n}$ by exchange of a pair of neighbor states $r_{i,k}$ and $r_{i,k+1}$, for some $1 \leq k \leq n - 1$. Then, the proof can be repeated for a pair of neighbor states for the sequence $\bar{r}'_{i,n}$, etc. In this way, the proof can be expanded to the case of an arbitrary permutation $\bar{r}'_{i,n}$ of the sequence $\bar{r}_{i,n}$. The above mentioned proof of pairwise permutation invariance involves processes $\bar{r}_{i,k-1} \eta^{(\varepsilon)}(t)$, $\bar{r}_{i,k} \eta^{(\varepsilon)}(t)$ and $\bar{r}_{i,k+1} \eta^{(\varepsilon)}(t)$. It is absolutely analogous, for $1 \leq k \leq n - 1$. Taking this into account, we just show how this proof can be accomplished for the case $k = 1$.

The transition probabilities $\bar{r}_{i,2} p_{i'j'}(\varepsilon)$ and $\bar{r}'_{i,2} p_{i'j'}(\varepsilon)$ for the sequences $\bar{r}_{i,2} = \langle r_1, r_2 \rangle$ and $\bar{r}'_{i,2} = \langle r_2, r_1 \rangle$ (here, $i, i', j' \neq r_1, r_2$) can be transformed into the same symmetric (with respect to r_1, r_2) rational function of the corresponding transition probabilities, using the identity arithmetical transformations listed above,

$$\begin{aligned} \bar{r}_{i,2} p_{i'j'}(\varepsilon) &= r_1 p_{i'j'}(\varepsilon) + r_1 p_{i'r_2}(\varepsilon) \frac{r_1 p_{r_2j'}(\varepsilon)}{1 - r_1 p_{r_2r_2}(\varepsilon)} \\ &= p_{i'j'}(\varepsilon) + p_{i'r_1}(\varepsilon) \frac{p_{r_1j'}(\varepsilon)}{1 - p_{r_1r_1}(\varepsilon)} \\ &\quad + \frac{(p_{i'r_2}(\varepsilon) + p_{i'r_1}(\varepsilon) \frac{p_{r_1r_2}(\varepsilon)}{1 - p_{r_1r_1}(\varepsilon)})(p_{r_2j'}(\varepsilon) + p_{r_2r_1}(\varepsilon) \frac{p_{r_1j'}(\varepsilon)}{1 - p_{r_1r_1}(\varepsilon)})}{1 - p_{r_2r_2}(\varepsilon) - p_{r_2r_1}(\varepsilon) \frac{p_{r_1r_2}(\varepsilon)}{1 - p_{r_1r_1}(\varepsilon)}} \\ &= p_{i'j'}(\varepsilon) + \frac{p_{i'r_1}(\varepsilon) p_{r_1j'}(\varepsilon) (1 - p_{r_2r_2}(\varepsilon)) + p_{i'r_1}(\varepsilon) p_{r_1r_2}(\varepsilon) p_{r_2j'}(\varepsilon)}{(1 - p_{r_1r_1}(\varepsilon))(1 - p_{r_2r_2}(\varepsilon)) - p_{r_1r_2}(\varepsilon) p_{r_2r_1}(\varepsilon)} \\ &\quad + \frac{p_{i'r_2}(\varepsilon) p_{r_2j'}(\varepsilon) (1 - p_{r_1r_1}(\varepsilon)) + p_{i'r_2}(\varepsilon) p_{r_2r_1}(\varepsilon) p_{r_1j'}(\varepsilon)}{(1 - p_{r_1r_1}(\varepsilon))(1 - p_{r_2r_2}(\varepsilon)) - p_{r_1r_2}(\varepsilon) p_{r_2r_1}(\varepsilon)} \\ &= r_2 p_{i'j'}(\varepsilon) + r_2 p_{i'r_1}(\varepsilon) \frac{r_2 p_{r_1j'}(\varepsilon)}{1 - r_2 p_{r_1r_1}(\varepsilon)} = \bar{r}'_{i,2} p_{i'j'}(\varepsilon). \end{aligned} \quad (26)$$

Therefore, by Lemma 4, the Laurent asymptotic expansions for transition probabilities $\bar{r}_{i,2} p_{i'j'}(\varepsilon)$ and $\bar{r}_{i,2} p_{i'j'}(\varepsilon)$, given by the recurrent algorithm of sequential phase space reduction described above, are identical.

The proof of identity for the Laurent asymptotic expansions of expectations $\bar{r}_{i,2} e_{i'j'}(\varepsilon)$ and $\bar{r}_{i,2} e_{i'j'}(\varepsilon)$, given by the recurrent algorithm of sequential phase space reduction described above, is analogous.

Let us take $n = N - 1$. In this case, the semi-Markov process $\bar{r}_{i,N-1} \eta^{(\varepsilon)}(t)$ has the phase space $\bar{r}_{i,N-1} \mathbb{X} = \mathbb{X} \setminus \{r_{i,1}, r_{i,2}, \dots, r_{i,N-1}\} = \{i\}$, which is a one-state set. The process $\bar{r}_{i,N-1} \eta^{(\varepsilon)}(t)$ returns in state i after every jump. Its transition probability $\bar{r}_{i,N-1} p_{ii}(\varepsilon) = 1$ and the expectation of hitting time $E_{ii}(\varepsilon) = \bar{r}_{i,N-1} e_{ii}(\varepsilon)$.

Thus, the above recurrent algorithm of sequential phase space reduction makes it possible to write down the following pivotal Laurent asymptotic expansion,

$$E_{ii}(\varepsilon) = \sum_{l=M_{ii}^-}^{M_{ii}^+} B_{ii}[l] \varepsilon^l + \ddot{o}_{ii}(\varepsilon^{M_{ii}^+}), \quad \varepsilon \in (0, \varepsilon_0], \quad (27)$$

where (a) $M_{ii}^\pm = \bar{r}_{i,N-1} m_{ii}^\pm$; (b) $B_{ii}[l] = \bar{r}_{i,N-1} b_{ii}[l]$, $l = M_{ii}^-, \dots, M_{ii}^+$; (c) $\ddot{o}_{ii}(\varepsilon^{M_{ii}^+}) = \bar{r}_{i,N-1} \ddot{o}_{ii}(\varepsilon^{M_{ii}^+})$.

By the above remarks, the asymptotic expansion given in relation (27) is invariant with respect to the choice of sequence $\bar{r}_{i,N-1} = \langle r_{i,1}, \dots, r_{i,N-1} \rangle$. This legitimates notations (with omitted index $\bar{r}_{i,N-1}$) used for parameters, coefficients and remainder in the above asymptotic expansion.

The algorithm for construction of the Laurent asymptotic expansion for expectation $E_{ii}(\varepsilon)$, given in relation (27), can be repeated for every $i \in \mathbb{X}$. $\square$

Remark 5 Since matrices $\|\bar{r}_{i,n} p_{i'j'}(\varepsilon)\|$, $\varepsilon \in (0, \varepsilon_0]$, $n = 0, \dots, N - 1$ are stochastic, the asymptotic expansions for transition probabilities $\bar{r}_{i,n} p_{i'j'}(\varepsilon)$, $j' \in \bar{r}_{i,n} \mathbb{Y}_{i'}$, $i' \in \bar{r}_{i,n} \mathbb{X}$ satisfy condition **F**, for every $n = 0, \dots, N - 1$.

6 Asymptotic Expansions for Stationary Distributions

The following theorem is the main new result in Part I of the present paper.

Theorem 5 *Let conditions A – E hold for semi-Markov processes $\eta^{(\varepsilon)}(t)$. Then, for every $i \in \mathbb{X}$, the pivotal (n_i^-, n_i^+) -expansion for the stationary probability $\pi_i(\varepsilon)$ is given by the algorithm based on the sequential exclusion of states $r_{i,1}, \dots, r_{i,N-1}$ from the phase space $\mathbb{X}$ of the processes $\eta^{(\varepsilon)}(t)$. This algorithm is described below, in the proof of the theorem. The above (n_i^-, n_i^+) -expansion is invariant with respect to any permutation $\bar{r}_{i,N} = \langle r_{i,1}, \dots, r_{i,N-1}, i \rangle$ of sequence $\langle 1, \dots, N \rangle$. Relations (1)–(6), given in the proof, hold for these expansions.*

Proof First, condition **E** and proposition (i) (the multiple summation rule) of Lemma 3 make it possible to write down pivotal (m_i^-, m_i^+) -expansions for expectations $e_i(\varepsilon)$, $i \in \mathbb{X}$.

These expansions take the following form, for $i \in \mathbb{X}$,

$$e_i(\varepsilon) = \sum_{j \in \mathbb{Y}_i} e_{ij}(\varepsilon) = \sum_{l=m_i^-}^{m_i^+} b_i[l] \varepsilon^l + \dot{o}_i(\varepsilon^{m_i^+}), \quad \varepsilon \in (0, \varepsilon_0], \quad (28)$$

where (a) $m_i^\pm = \min_{j \in \mathbb{Y}_i} m_{ij}^\pm$; (b) $b_i[m_i^- + l] = \sum_{j \in \mathbb{Y}_i} b_{ij}[m_i^- + l]$, $l = 0, \dots, m_i^+ - m_i^-$, where $b_{ij}[m_i^- + l] = 0$, for $0 \leq l < m_{ij}^- - m_i^-$, $j \in \mathbb{Y}_i$; (c) $\dot{o}_i(\varepsilon^{m_i^+})$ is given by formula (c) from proposition (i) (the multiple summation rule) of Lemma 3, which should be applied to the corresponding Laurent asymptotic expansions given in condition E.

Second, conditions A – E, the asymptotic expansions given in relations (27) and (28), and proposition (v) (the division rule) of Lemma 2 make it possible to write down (n_i^-, n_i^+) -expansions for the stationary probabilities $\pi_i(\varepsilon) = \frac{e_i(\varepsilon)}{E_{ii}(\varepsilon)}$, $i \in \mathbb{X}$. These expansions take the following form, for $i \in \mathbb{X}$,

$$\pi_i(\varepsilon) = \sum_{l=n_i^-}^{n_i^+} c_i[l] \varepsilon^l + o_i(\varepsilon^{n_i^+}), \quad \varepsilon \in (0, \varepsilon_0], \quad (29)$$

where: (a) $n_i^- = m_i^- - M_{ii}^-$, $n_i^+ = (m_i^+ - M_{ii}^-) \wedge (M_{ii}^+ - 2M_{ii}^- + m_i^-)$; (b) $c_i[n_i^- + l] = B_{ii}[M_{ii}^-]^{-1} (b_i[m_i^- + l] - \sum_{1 \leq l' \leq l} B_{ii}[M_{ii}^- + l] c_i[n_i^- + l - l'])$, $l = 0, \dots, n_i^+ - n_i^-$; (c) $o_i(\varepsilon^{n_i^+})$ is given by formula (f) from proposition (v) (the division rule) of Lemma 2, which should be applied to the asymptotic expansions given in relations (27) and (28).

Since the asymptotic expansions given in relations (27) and (28) are pivotal, the expansions given in relation (29) are also pivotal, i.e., $c_i[n_i^-] = b_{ii}[m_i^-] / B_{ii}[M_{ii}^-] \neq 0$, $i \in \mathbb{X}$. Moreover, since $\pi_i(\varepsilon) > 0$, $i \in \mathbb{X}$, $\varepsilon \in (0, \varepsilon_0]$, the following relation takes place, (1) $c_i[n_i^-] > 0$, $i \in \mathbb{X}$. By the definition, $e_i(\varepsilon) \leq E_{ii}(\varepsilon)$, $i \in \mathbb{X}$, $\varepsilon \in (0, \varepsilon_0]$. This implies that parameters $M_{ii}^- \leq m_i^-$, $i \in \mathbb{X}$ and, thus, (2) $n_i^- \geq 0$, $i \in \mathbb{X}$. Since, $\sum_{i \in \mathbb{X}} \pi_i(\varepsilon) = 1$, $\varepsilon \in (0, \varepsilon_0]$, parameters $n_i^\pm$, $i \in \mathbb{X}$ and coefficients $c_i[l]$, $l = n_i^-, \dots, n_i^+$, $i \in \mathbb{X}$ satisfy relations, (3) $n^- = \min_{i \in \mathbb{X}} n_i^- = 0$, and, (4) $\sum_{i \in \mathbb{X}} c_i[l] = I(l = 0)$, $0 \leq l \leq n^+ = \max_{i \in \mathbb{X}} n_i^+$. Moreover, the remainders of asymptotic expansions given in (29) satisfy identity, (5) $\sum_{i \in \mathbb{X}} (\sum_{n^+ < l \leq n_i^+} c_i[l] \varepsilon^l + o_i(\varepsilon^{n_i^+})) = 0$, $\varepsilon \in (0, \varepsilon_0]$. By the above remarks, (6) there exists $\lim_{\varepsilon \rightarrow 0} \pi_i(\varepsilon) = \pi_i(0)$, which equals to $c_i[0] > 0$ if $i \in \mathbb{X}_0$, or 0 if $i \notin \mathbb{X}_0$, where $\mathbb{X}_0 = \{i \in \mathbb{X} : n_i^- = 0\}$. As follows from Theorem 4, the asymptotic expansion (27) for expectation $E_{ii}(\varepsilon)$ and, thus, the asymptotic expansion (29) for stationary probability $\pi_i(\varepsilon)$ is, for every $i \in \mathbb{X}$, invariant with respect to any permutation $\bar{r}_{i,N} = \langle r_{i,1}, \dots, r_{i,N-1}, i \rangle$ of sequence $\langle 1, \dots, N \rangle$. $\square$

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