



Topological Manin pairs and (n, s) -type series

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Abstract

Lie subalgebras of $L = \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n \mathfrak{g}[x]$, complementary to the diagonal embedding Δ of $\mathfrak{g}[[x]]$ and Lagrangian with respect to some particular form, are in bijection with formal classical r -matrices and topological Lie bialgebra structures on the Lie algebra of formal power series $\mathfrak{g}[[x]]$. In this work we consider arbitrary subspaces of L complementary to Δ and associate them with so-called series of type (n, s) . We prove that Lagrangian subspaces are in bijection with skew-symmetric (n, s) -type series and topological quasi-Lie bialgebra structures on $\mathfrak{g}[[x]]$. Using the classification of Manin pairs we classify up to twisting and coordinate transformations all quasi-Lie bialgebra structures. Series of type (n, s) , solving the generalized classical Yang-Baxter equation, correspond to subalgebras of L . We discuss their possible utility in the theory of integrable systems.

Keywords Lie bialgebras · quasi-Lie bialgebras · Manin pairs · Yang-Baxter equations · r -matrices · Lie algebra splittings

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Dedicated to the memory of Yuri Manin.

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1 Introduction

Let F be an algebraically closed field of characteristic 0 equipped with the discrete topology and \mathfrak{g} be a simple Lie algebra over F . We define the Lie algebra $\mathfrak{g}[[x]]$ to be the space $\mathfrak{g} \otimes F[[x]]$ with the bracket

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg$$

and we equip it with the (x) -adic topology. The continuous dual of $\mathfrak{g}[[x]]$ is denoted by $\mathfrak{g}[[x]]'$ and it is endowed with the discrete topology.

A topological Manin pair is a pair $(L, \mathfrak{g}[[x]])$ where

1. L is a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form B ;
2. $\mathfrak{g}[[x]] \subset L$ is a Lagrangian subalgebra with respect to B ;
3. for any continuous functional $T: \mathfrak{g}[[x]] \rightarrow F$ there is $f \in L$ such that $T = B(f, -)$.

Topological Manin pairs were classified in [1] using the tools from [8]. More precisely, if $(L, \mathfrak{g}[[x]])$ is a topological Manin pair, then L is isomorphic, as a Lie algebra with form, to either $L(\infty)$ or $L(n, \alpha)$. In this paper we consider only the "non-degenerate" case, namely $L \cong L(n, \alpha)$.

As a Lie algebra

$$L(n, \alpha) = \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n \mathfrak{g}[x].$$

The bilinear form B on $L(n, \alpha)$ is completely determined by the sequence $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$. For example, when $n = 0$ the form is given by

$$B(a \otimes f, b \otimes g) = \kappa(a, b) \text{res}_0 \{ \alpha(x) fg \},$$

where κ is the Killing form on \mathfrak{g} and $\alpha(x) := 1 + \alpha_{-2}x + \alpha_{-3}x^2 + \dots \in F((x))$. In case $n > 0$ the form is given by a similar formula; see Sect. 2.

It was established in [1], that the following objects are in one-to-one correspondence

- Lagrangian subalgebras $W \subseteq L(n, 0), 0 \leq n \leq 2$, complementary to the diagonal

$$\Delta := \{ (f, [f]) \mid f \in \mathfrak{g}[[x]] \},$$

i.e. $\Delta \dot{+} W = L(n, 0)$;

- non-degenerate topological Lie bialgebra structures on $\mathfrak{g}[[x]]$ and
- formal solutions to the classical Yang-Baxter equation (CYBE) in the form

$$\frac{y^n \Omega}{x - y} + g(x, y) = \Omega \sum_{k \geq 0} x^{-k-1} y^{k+n} + g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]], \tag{1}$$

where $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the quadratic Casimir element and $g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$.

Furthermore, the proof of the above-mentioned correspondence reveals that series Eq. (1) can be viewed as a generating series for the corresponding subalgebra W . The present paper can be thus considered as a continuation of [1], where we extend the preceding correspondence using series of type (n, s) .

To define a series of type (n, s) fix a basis $\{b_i\}_{i=1}^d$ of \mathfrak{g} , orthonormal with respect to its Killing form κ , and interpret $y^n\Omega/(x - y)$ as a series

$$\frac{y^n\Omega}{x - y} = \sum_{k=0}^{\infty} \sum_{i=1}^d w_{k,i} \otimes b_i y^k \in ((\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]) \otimes \mathfrak{g})[[y]]. \tag{2}$$

This expression might be understood as a Taylor series expansion. Elements $w_{k,i} \in \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$ are presented explicitly in Eq. (20). A series of type (n, s) is a series of the form

$$\frac{s(x)y^n\Omega}{x - y} + g(x, y) \in ((\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]) \otimes \mathfrak{g})[[y]], \tag{3}$$

where $s \in F[[x]]^\times$ and $g \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$; See Definition 3.2. For each series r of type (n, s) we define another series \bar{r} of the same type as follows

$$\bar{r} := \frac{s(y)x^n\Omega}{x - y} - \tau(g(y, x)), \tag{4}$$

where τ is the $F[[x, y]]$ -linear extension of the map $a \otimes b \mapsto b \otimes a$.

Each series of type (n, s) produces a subspace of $\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$ complementary to the diagonal embedding Δ of $\mathfrak{g}[[x]]$. The following results generalize the above-mentioned correspondence from [1].

Theorem A *Let $n \in \mathbb{Z}_{\geq 0}$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$ be an arbitrary sequence with the corresponding series $\alpha(x) := x^{-n} + \alpha_{n-2}x^{-n+1} + \dots + \alpha_0x^{-1} + \dots \in F((x))$. For any (n, s) -type series*

$$r = \sum_{k=0}^{\infty} \sum_{i=1}^d f_{k,i} \otimes b_i y^k \in ((\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]) \otimes \mathfrak{g})[[y]] \tag{5}$$

define the space

$$W(r) := \text{span}_F\{f_{k,i} \mid k \geq 0, 1 \leq i \leq d\} \subseteq \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]. \tag{6}$$

The following results are true:

1. W defines a bijection between series of type $(n, \frac{1}{x^n\alpha(x)})$ and subspaces $V \subset L(n, \alpha)$ complementary to the diagonal Δ , i.e. $L(n, \alpha) = \Delta \dot{+} V$;
2. For any series r of type $(n, \frac{1}{x^n\alpha(x)})$ we have $W(r)^\perp = W(\bar{r})$ inside $L(n, \alpha)$;

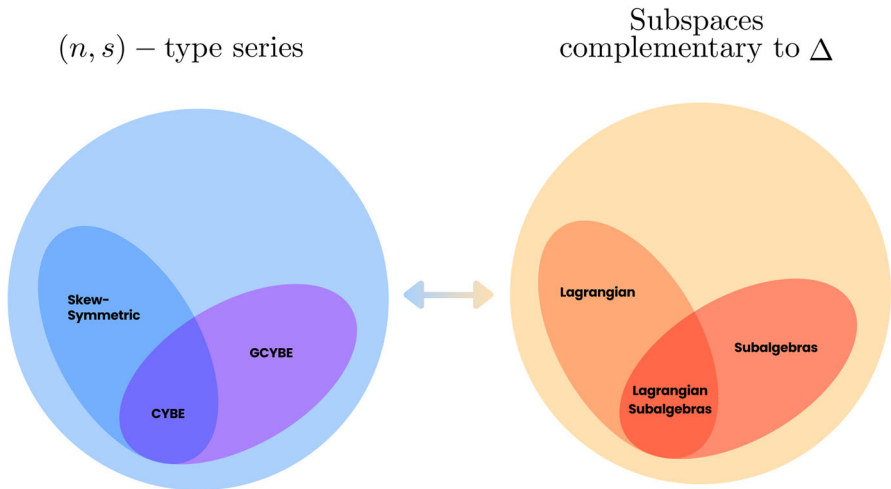


Fig. 1 Series-subspaces correspondence

3. Any series r of type $(n, \frac{1}{x^n \alpha(x)})$ satisfies $GCYB(r) = \psi$ (see Definition 3.5 for the meaning of $GCYB(r)$), where $\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x_1, x_2, x_3]]$ is defined by

$$B(v_1 \otimes v_2 \otimes v_3, \psi) = B(v_1, [v_2, v_3])$$

for all $v_1 \in W(\bar{r})$, $v_2, v_3 \in W(r)$.

In particular, considering the case when r is skew-symmetric, meaning $r = \bar{r}$, or when $\psi = 0$ we get the following correspondences.

Corollary B Let $n \in \mathbb{Z}_{\geq 0}$, $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$ and W be the map from Theorem A. Then

1. W defines a bijection between skew-symmetric $(n, \frac{1}{x^n \alpha(x)})$ -type series and Lagrangian subspaces $V \subseteq L(n, \alpha)$, complementary to the diagonal Δ ;
2. W defines a bijection between $(n, \frac{1}{x^n \alpha(x)})$ -type series solving the GCYBE and subalgebras $V \subseteq L(n, \alpha)$ complementary to the diagonal Δ .

Observe that an (n, s) -type series produces a subspace of $L(n, \alpha)$ for any sequence α . However, to obtain the compatibility with the form, given by α , we need the equality $s(x) = 1/(x^n \alpha(x))$. In this case, the components $f_{k,i}$ and $b_i y^k$ of the series become dual bases for $W(r)$ and Δ respectively.

The requirement on a series r of type (n, s) to solve the CYBE is equivalent to being skew-symmetric and to solve the GCYBE. Together with Corollary B this implies that Lagrangian subalgebras $W \subset L(n, \alpha)$, satisfying $W \dot{+} \Delta = L(n, \alpha)$, are in bijection with $(n, 1/(x^n \alpha(x)))$ -type series solving the classical Yang-Baxter equation. These correspondences are schematically depicted in Fig. 1.

Remark 1.1 Let r be a series of type (n, s) . Applying the projection $(a, b) \otimes c \mapsto a \otimes c$ onto the left component to r we obtain the series

$$r_{\text{proj}} = \frac{s(x)y^n \Omega}{x - y} + g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})(x) \llbracket y \rrbracket. \tag{7}$$

Conversely, starting with a series r_{proj} of the form Eq. (7), we can obtain an (n, s) -type series r by taking two Taylor series expansions of r_{proj} at $x = 0$ and $y = 0$ respectively and then constructing r by combining the coefficients of $b_i y^k, k \geq 0$, in these expansions. These two constructions are inverse to each other and hence both r and its projection r_{proj} contain exactly the same information. Consequently, all the statements made for (n, s) -type series can be stated in terms of their projections onto the left component and vice versa. In contrast with [1], in this paper we give preference to series of type (n, s) rather than to their projections, because the statement that series of type (n, s) generate subspaces of $L(n, \alpha)$ becomes transparent.

Reinterpreting the results of [1] in terms of (n, s) -type series we see that skew-symmetric series of type $(n, 1/(x^n \alpha(x)))$, that also solve the GCYBE, exist only for $n = 0, 1$ and $n = 2$ with $\alpha_0 = 0$.

Lagrangian subalgebras of $L(n, \alpha)$, complementary to Δ , correspond to topological Lie bialgebra structures on $\mathfrak{g} \llbracket x \rrbracket$. If we instead consider Lagrangian subspaces (not necessarily subalgebras) of $L(n, \alpha)$, we get so called (non-degenerate) topological quasi-Lie bialgebra structures on $\mathfrak{g} \llbracket x \rrbracket$. A topological quasi-Lie bialgebra structure on $\mathfrak{g} \llbracket x \rrbracket$ consists of

- a skew-symmetric continuous linear map $\delta: \mathfrak{g} \llbracket x \rrbracket \rightarrow (\mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y \rrbracket$ and
- a skew-symmetric element $\varphi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y, z \rrbracket$,

which are subject to the following three conditions

- $\delta([a, b]) = [a \otimes 1 + 1 \otimes a, \delta(b)] - [b \otimes 1 + 1 \otimes b, \delta(a)]$, i.e. δ is a 1-cocycle;
- $\frac{1}{2} \text{Alt}((\delta \otimes 1)\delta(a)) = [a \otimes 1 \otimes 1 + 1 \otimes a \otimes 1 + 1 \otimes 1 \otimes a, \varphi]$;
- $\text{Alt}((\delta \otimes 1 \otimes 1)\varphi) = 0$,

where $\text{Alt}(x_1 \otimes \dots \otimes x_n) := \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$.

Following [5] we prove the following direct relation between δ, φ and skew-symmetric (n, s) -type series r .

Proposition C *There is a bijection between topological quasi-Lie bialgebras and skew-symmetric (n, s) -type series. Let r be the (n, s) -type series corresponding to $(\mathfrak{g} \llbracket x \rrbracket, \delta, \varphi)$, then, under the identification $\mathfrak{g} \llbracket x \rrbracket \cong \Delta$, we have the following identities:*

- $[a \otimes 1 + 1 \otimes a, r] = -\delta(a)$ for any $a \in \mathfrak{g} \llbracket x \rrbracket$ and
- $\text{CYB}(r) = -\varphi$.

The same is true if r is interpreted as an element in $(\mathfrak{g} \otimes \mathfrak{g})(x) \llbracket y \rrbracket$.

In view of this result we call skew-symmetric (n, s) -type series quasi- r -matrices.

Repeating the ideas from [7] and [5] we show that topological quasi-Lie bialgebras can be twisted similar to topological Lie bialgebras. More precisely, if δ is a quasi-Lie bialgebra structure on $\mathfrak{g} \llbracket x \rrbracket$, given by the Lagrangian subspace W , and $s :=$

$\sum_i a_i \otimes b^i \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ is an arbitrary skew-symmetric tensor, then

$$W_s := \left\{ \sum_i B(b^i, w)a_i - w \mid w \in W \right\} \tag{8}$$

is another (twisted) Lagrangian subspace complementary to the diagonal. This observation implies, that in order to classify all topological quasi-Lie bialgebra structures on $\mathfrak{g}[[x]]$ up to twisting it is enough to find one single Lagrangian subspace within each $L(n, \alpha)$. Moreover, it was shown in [1] that substitutions of the form $x \mapsto x + a_2x^2 + a_3x^3 + \dots, a_i \in F$, allow us to assume that α has the form

$$\alpha = (\dots, 0, \alpha_0, 0, \dots, 0).$$

Lagrangian subspaces for such $L(n, \alpha)$ are constructed in Sect. 4.1.

Using Theorem A and Proposition C we explain how twisting of a Lagrangian subspace $W \subset L(n, \alpha)$ is seen at the level of δ and the corresponding quasi- r -matrix r .

Corollary D *Let $(\mathfrak{g}[[x]], \delta, \varphi)$ be a topological quasi-Lie bialgebra structure corresponding to the quasi- r -matrix r . If we twist $W(r)$ with a skew-symmetric tensor s we obtain another topological quasi-Lie bialgebra $(\mathfrak{g}[[x]], \delta_s, \varphi_s)$, such that*

1. $W(r)_s = W(r - s)$;
2. $\delta_s = \delta + ds$;
3. $\varphi_s = \varphi + \text{CYB}(s) - \frac{1}{2}\text{Alt}((\delta \otimes 1)s)$.

Therefore, to describe all quasi- r -matrices up to twisting it is enough to find one single quasi- r -matrix for each $L(n, \alpha)$. We achieve that goal in Sect. 4.2 by writing out explicitly series of type (n, s) for subspaces from Sect. 4.1.

The results above, in particular, show that if r is a quasi- r -matrix and $\delta(a) := [a \otimes 1 + 1 \otimes a, r]$, then the condition

$$\text{Alt}((\delta \otimes 1 \otimes 1)\text{CYB}(r)) = 0 \tag{9}$$

is trivially satisfied.

We conclude the paper by using Theorem A for construction of Lie algebra splittings $\Delta \dot{+} W = L(n, \alpha)$ and the corresponding (n, s) -type series, which we call generalized r -matrices. These constructions are important in the theory of integrable systems because of their use in the Adler-Konstant-Symes (AKS) scheme and the so-called r -matrix approach; see [4, 6]. The subalgebra splittings of $L(0, 0)$ as well as their physical applications were considered in e.g. [9, 10].

Our first result shows that in order to obtain new generalized r -matrices from subalgebra splittings $L(n, \alpha) = \Delta \dot{+} W$ with $n > 2$, the subalgebra W must be unbounded. Otherwise the situation can be reduced to the splitting of $L(2, \alpha)$.

Proposition E *Let $L(n, \alpha) = \Delta \dot{+} W$ for some subalgebra $W \subset L(n, \alpha)$ and $n > 2$. Assume W is bounded, i.e. there is an integer $N > 0$ such that*

$$x^{-N} \mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N \mathfrak{g}[x^{-1}],$$

where W_+ is the projection of $W \subset L(n, \alpha) = \mathfrak{g}((x)) \oplus \mathfrak{g}[x]/x^n \mathfrak{g}[x]$ on the first component $\mathfrak{g}((x))$. Then we have the inclusion

$$\{0\} \times [x^2] \mathfrak{g}[x]/x^n \mathfrak{g}[x] \subseteq W$$

and the image \tilde{W} under the canonical projection $L(n, \alpha) \rightarrow L(2, \alpha)$ is a subalgebra satisfying $L(2, \alpha) = \Delta + \tilde{W}$.

Despite this result we think that bounded subalgebras $W \subset L(n, \alpha)$ complementary to Δ are still interesting, because in the case $\alpha \neq 0$ they lead to unbounded orthogonal complements W^\perp which are also important in view of the AKS scheme. We give examples of subalgebras of $L(n, \alpha)$ with unbounded orthogonal complements.

2 Topological Manin pairs

Let F be an algebraically closed field of characteristic 0, \mathfrak{g} be a finite-dimensional simple F -Lie algebra and $\mathfrak{g}[[x]] := \mathfrak{g} \otimes F[[x]]$ be the Lie algebra with the bracket defined by

$$[a \otimes f, b \otimes g] := [a, b] \otimes fg,$$

for all $a, b \in \mathfrak{g}$ and $f, g \in F[[x]]$. From now on, we always endow F with the discrete topology and view $\mathfrak{g}[[x]]$ as a topological Lie algebra with the (x) -adic topology.

A *topological Manin pair* is a pair $(L, \mathfrak{g}[[x]])$, where L is a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form B , such that

1. $\mathfrak{g}[[x]] \subseteq L$ is a Lagrangian Lie subalgebra with respect to B ;
2. for any continuous functional $T: \mathfrak{g}[[x]] \rightarrow F$ there exists an element $f \in L$ such that $T = B(f, -)$.

The statements of [8, Proposition 2.9] and [1, Proposition 3.12] give a description of all topological Manin pairs. For precise formulation we need to repeat the definitions of some specific Lie algebras with forms from [1, Section 3.2] and [8, Section 2].

Definition 2.1 We define the Lie algebra $L(\infty) := \mathfrak{g} \otimes A(\infty)$, where $A(\infty)$ is the unital commutative algebra with underlying space $\sum_{i \geq 0} Fa_i + F[[x]]$ and multiplication given by

$$a_i a_j := 0, \quad a_i x^j := a_{i-j} \text{ for } i \geq j \text{ and } a_i x^j := 0 \text{ otherwise.}$$

Let $t: A \rightarrow F$ be the functional, given by $t(a_0) := 1, t(a_i) := 0, i \geq 1$ and $t(F[[x]]) := 0$. We equip $L(\infty)$ with the symmetric non-degenerate invariant bilinear form

$$B \left(a \otimes \left(\sum_{i \geq 0} c_i a_i, f(x) \right), b \otimes \left(\sum_{i \geq 0} t_i a_i, g(x) \right) \right)$$

$$:= \kappa(a, b) \mathfrak{t} \left(g(x) \sum_{i \geq 0} c_i a_i + f(x) \sum_{i \geq 0} t_i a_i \right). \tag{10}$$

Definition 2.2 Let $n \geq 1$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$ be an arbitrary sequence. Consider the algebra

$$A(n, \alpha) := F((x)) \oplus F[x]/(x^n).$$

Abusing the notation we denote the element $x^{-n} + \alpha_{n-2}x^{-n+1} + \dots + \alpha_0x^{-1} + \dots \in F((x))$ with the same letter α . Define the functional $\mathfrak{t}: A(n, \alpha) \rightarrow F$ by

$$\mathfrak{t}(f, [p]) := \text{res}_0 \{ \alpha(f - p) \}.$$

Taking the tensor product of $A(n, \alpha)$ with \mathfrak{g} we get the Lie algebra

$$L(n, \alpha) := \mathfrak{g} \otimes A(n, \alpha) \tag{11}$$

which we equip with the form

$$B(a \otimes (f, [p]), b \otimes (g, [q])) := \kappa(a, b) \mathfrak{t}(fg, [pq]). \tag{12}$$

It is known that the bilinear form B is symmetric non-degenerate and invariant.

Definition 2.3 Take an arbitrary sequence $\alpha = (\alpha_i \in F \mid -\infty < i \leq -2)$ and let $A(0, \alpha) := F((x))$. We define the functional $\mathfrak{t}: A(0, \alpha) \rightarrow F$ by

$$\mathfrak{t}(f) := \text{res}_0 \{ \alpha f \},$$

where $\alpha = 1 + \alpha_{-2}x + \dots \in F((x))$. We equip the Lie algebra $L(0, \alpha) := \mathfrak{g} \otimes A(0, \alpha)$ with the bilinear form

$$B(a \otimes f, b \otimes g) := \kappa(a, b) \mathfrak{t}(fg), \tag{13}$$

which is again symmetric non-degenerate and invariant. From now on we identify $F((x))$ with $F((x)) \times \{0\}$ and write $(f, 0)$ for elements in $A(0, \alpha)$.

Definition 2.4 A series of the form $\varphi = x + a_2x^2 + a_3x^3 + \dots \in F[[x]]$ is called a *coordinate transformation*. Coordinate transformations form a group $\text{Aut}_0 F[[x]]$ under substitution which we view as a subgroup of automorphisms of $F[[x]]$.

An element $\varphi \in \text{Aut}_0 F[[x]]$ induces an automorphism of $A(n, \alpha)$ by $f/g \mapsto \varphi(f)/\varphi(g)$ and $[p] \mapsto [\varphi(p)]$ that changes the functional \mathfrak{t} to $\mathfrak{t} \circ \varphi$. We write $A(n, \alpha)^{(\varphi)}$ for the algebra $A(n, \alpha)$ with the functional $\mathfrak{t} \circ \varphi$. It is not hard to see that for any $\varphi \in \text{Aut}_0 F[[x]]$ there is a sequence β such that $A(n, \alpha)^{(\varphi)} = A(n, \beta)$.

Let $(L, \mathfrak{g}[[x]])$ be a topological Manin pair. According to [8, Proposition 2.9] as a Lie algebra with form $L \cong L(\infty)$ or $L \cong L(n, \alpha)$, for some $n \geq 0$ and some sequence α . Here we identify $\mathfrak{g}[[x]]$ with the diagonal

$$\Delta := \{(f, [f]) \mid f \in \mathfrak{g}[[x]]\} \subset L(n, \alpha).$$

Moreover, we can assume that all the elements α_i in the sequence α , except maybe α_0 , are 0 by virtue of the following result.

Proposition 2.5 [1, Proposition 3.12] *Let $n \geq 0$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ be a sequence. There exists a $\varphi \in \text{Aut}_0 F[[x]]$ such that $A(n, \alpha) \cong A(n, \beta)^{(\varphi)}$, where β is the sequence satisfying $\beta_i = 0$ for all $i \neq 0$ and $\beta_0 = \alpha_0$.*

Remark 2.6 Observe that the result of Proposition 2.5 can be interpreted in terms of a formal differential equation. Consider an arbitrary $\alpha(x) = x^{-n} + \alpha_{n-2}x^{-n+1} + \dots + \alpha_0x^{-1} + \dots \in F((x))$ and $\beta(x) = x^{-n} + \alpha_0x^{-1}$. Then the functionals t_α and t_β defined on $A(n, \alpha)$ and $A(n, \beta)$ respectively are given by

$$t_\alpha(f, [p]) = \text{res}_0\{\alpha(f - p)\} \quad \text{and} \quad t_\beta(f, [p]) = \text{res}_0\{\beta(f - p)\}$$

The equality $A(n, \alpha)^{(\varphi)} = A(n, \beta)$ can be expressed as

$$\text{res}_0\{\beta(x)f(x)\} = \text{res}_0\{\alpha(x)f(\varphi(x))\} = \text{res}_0\{\alpha(\psi(x))f(x)\psi'(x)\}, \tag{14}$$

where $\psi \in \text{Aut}_0(F[[x]])$ is the compositional inverse of φ , i.e. $\varphi(\psi(x)) = x$. Since the residue pairing is non-degenerate on $F((x))$, we obtain

$$\alpha(\psi(x))\psi'(x) = \beta(x). \tag{15}$$

In particular, the transformation φ is the compositional inverse of the solution to Eq. (15).

3 Series of type (n, s) and subspaces of $L(n, \alpha)$

Let $\{b_i\}_{i=1}^d$ be an orthonormal basis of \mathfrak{g} with respect to the Killing form κ . We write Ω for the quadratic Casimir element $\sum_{i=1}^d b_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$. It satisfies the identity $[a \otimes 1 + 1 \otimes a, \Omega] = 0$ for all $a \in \mathfrak{g}$.

In this section we describe a bijection between subspaces $W \subset L(n, \alpha)$ complementary to Δ and certain series. The following definition introduces convenient spaces containing these series.

Definition 3.1 We put $A_1(n, \alpha) := A(n, \alpha) = F((x_1)) \oplus F[x_1]/(x_1^n)$ and then define inductively the algebras

$$A_m(n, \alpha) := A_{m-1}(n, \alpha)((x_m)) \oplus A_{m-1}(n, \alpha)[x_m]/x_m^n A_{m-1}(n, \alpha), \quad m > 1. \tag{16}$$

The functional t defined on $A(n, \alpha)$ extends inductively to a functional on $A_m(n, \alpha)$. More precisely,

$$t \left(\sum_{k \geq -N} f_k x_m^k, \sum_{\ell=0}^{n-1} [g_\ell x_m^\ell] \right) := \sum_{k \geq -N} t(f_k) t(x_m^k, 0) + \sum_{\ell=0}^{n-1} t(g_\ell) t(0, [x_m]^\ell), \tag{17}$$

where $f_k, g_\ell \in A_{m-1}(n, \alpha)$. Since $t(x^n F[[x]]) = 0$, the sum on the right-hand side of Eq. (17) is finite and well-defined. This allows us to extend the form B on $L(n, \alpha)$ to a symmetric non-degenerate bilinear form on the \mathfrak{g} -module

$$L_m(n, \alpha) := \mathfrak{g}^{\otimes m} \otimes A_m(n, \alpha) \tag{18}$$

by letting

$$B((a_1 \otimes \dots \otimes a_m) \otimes f, (b_1 \otimes \dots \otimes b_m) \otimes g) := t(fg) \prod_{k=1}^m \kappa(a_k, b_k), \tag{19}$$

for all $a_1, \dots, a_m, b_1, \dots, b_m \in \mathfrak{g}$ and $f, g \in A_m(n, \alpha)$.

Fix some integer $n \geq 0$. We interpret the quotient $y^n \Omega / (x - y)$ in the following way

$$\begin{aligned} \frac{y^n \Omega}{x - y} &= \sum_{k=0}^{n-1} \sum_{i=1}^d b_i(0, -[x]^{(n-1)-k}) \otimes b_i(y^k, [y]^k) \\ &+ \sum_{k=n}^{\infty} \sum_{i=1}^d b_i(x^{(n-1)-k}, 0) \otimes b_i(y^k, 0) \\ &= \sum_{k=0}^{\infty} \sum_{i=1}^d w_{k,i} \otimes b_i(y^k, [y]^k) \in (L(n, \alpha) \otimes \mathfrak{g}) [[y, [y]]] \subset L_2(n, \alpha), \end{aligned} \tag{20}$$

where α is an arbitrary sequence and we write $b_i(x^\ell, [x]^m)$ meaning $b_i \otimes (x^\ell, [x]^m)$.

Definition 3.2 Since $(L(n, \alpha) \otimes \mathfrak{g}) [[y, [y]]]$ is an $F[[x]] \cong F[[x, [x]]]$ -module and

$$(\mathfrak{g} \otimes \mathfrak{g})[[x, y]] \cong (\Delta \otimes \mathfrak{g})[[y, [y]]] \subset (L(n, \alpha) \otimes \mathfrak{g}) [[y, [y]]]$$

the series

$$r(x, y) = \frac{s(x)y^n \Omega}{x - y} + g(x, y), \tag{21}$$

where $g \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ and $s \in F[[x]]^\times$, is also inside $(L(n, \alpha) \otimes \mathfrak{g}) [[y, [y]]]$. Series of the form Eq. (21) are called *series of type (n, s)* .

Remark 3.3 Every series

$$r(x, y) = \frac{h(x, y)\Omega}{x - y} + g(x, y) \in L_2(n, \alpha),$$

where $h \in F[[x, y]]$, $h(x, x) \neq 0$ and $g \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$, has a unique representation as a series of type (n, s) . Indeed, write $h(x, x) = x^n s(x)$ for some $s \in F[[x]]^\times$. Then $h(x, y) - y^n s(x) = (x - y)f(x, y)$ for some $f \in F[[x, y]]$. This implies that we can rewrite r in the (n, s) form

$$r(x, y) = \frac{s(x)y^n\Omega}{x - y} + f(x, y)\Omega + g(x, y). \tag{22}$$

In the construction of f we are using the fact that for any F -vector space V and any element $h \in V[[x, y]]$

$$h(z, z) = 0 \implies h(x, y) = (x - y)f(x, y) \tag{23}$$

for some $f \in V[[x, y]]$.

Definition 3.4 For each series r of type (n, s) we define another series \bar{r} of the same type (n, s) by

$$\bar{r}(x, y) := \frac{s(y)x^n\Omega}{x - y} - \tau(g(y, x)) \in (L(n, \alpha) \otimes \mathfrak{g})[[y, [y]]], \tag{24}$$

where τ is the $F[[x, y]]$ -linear extension of the map $a \otimes b \mapsto b \otimes a$. To see that this is an (n, s) -type series its enough to apply the argument from Remark 3.3. Series of type (n, s) , satisfying $r = \bar{r}$, are called *skew-symmetric*.

Definition 3.5 The *generalized classical Yang-Baxter equation (GCYBE)* is the equation for an (n, s) -type series of the form

$$\begin{aligned} \text{GCYB}(r) := & [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] \\ & + [r^{13}(x_1, x_3), \bar{r}^{23}(x_2, x_3)] = 0. \end{aligned} \tag{25}$$

Here $(-)^{13}: L_2(n, \alpha) \rightarrow (U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})) \otimes A_3(n, \alpha)$ is the inclusion map given by

$$\begin{aligned} a \otimes b \otimes & \left(\sum_{k \geq -N} F(x_1, [x_1])x_2^k, \sum_{m=0}^{n-1} G(x_1, [x_1])[x_2]^m \right) \\ \mapsto a \otimes 1 \otimes b \otimes & \left(\sum_{k \geq -N} F(x_1, [x_1])x_3^k, \sum_{m=0}^{n-1} G(x_1, [x_1])[x_3]^m \right). \end{aligned}$$

Other inclusions are defined in a similar manner. The commutators are then taken in the associative $A_3(n, \alpha)$ - algebra $(U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})) \otimes A_3(n, \alpha)$.

Before formulating the main theorem of the section we note that if $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$ is an arbitrary sequence and $\alpha(x) = x^{-n} + \alpha_{n-2}x^{-n+1} + \dots + \alpha_0x^{-1} + \dots \in F((x))$ is the corresponding series, then $x^n\alpha(x) \in F[[x]]^\times$.

Theorem 3.6 *Let $n \in \mathbb{Z}_{\geq 0}$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$ be an arbitrary sequence with the corresponding series $\alpha(x) \in F((x))$. Consider the map*

$$W : L_2(n, \alpha) \longrightarrow \{V \subset L(n, \alpha) \mid V \text{ is a subspace}\}$$

given by

$$\begin{aligned} & \sum_{i,j} b_i \otimes b_j \otimes \left(\sum_{k \geq -N_i} (f_k^{ij}, [p_k^{ij}])x^k, \sum_{m=0}^{n-1} (g_m^{ij}, [q_m^{ij}])[x]^m \right) \\ & \mapsto \text{span}_F \{ b_i (f_k^{ij}, [p_k^{ij}]) \mid k \geq -N, 1 \leq i, j \leq d \}. \end{aligned}$$

The following results are true:

1. W defines a bijection between series of type $(n, \frac{1}{x^n\alpha(x)})$ and subspaces $V \subseteq L(n, \alpha)$ complementary to the diagonal Δ , i.e. $L(n, \alpha) = \Delta \dot{+} V$;
2. For any series r of type $(n, \frac{1}{x^n\alpha(x)})$ we have $W(r)^\perp = W(\bar{r})$ inside $L(n, \alpha)$;
3. Any series r of type $(n, \frac{1}{x^n\alpha(x)})$ satisfies $\text{GCYB}(r) = \psi$, where

$$\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x_1, [x_1]], (x_2, [x_2]), (x_3, [x_3])]]$$

is defined by

$$B(v_1 \otimes v_2 \otimes v_3, \psi) = B(v_1, [v_2, v_3])$$

for all $v_1 \in W(\bar{r})$, $v_2, v_3 \in W(r)$.

Proof Fix an $(n, \frac{1}{x^n\alpha(x)})$ -type series inside $(L(n, \alpha) \otimes \mathfrak{g})[[y, [y]]]$

$$\begin{aligned} r(x, y) &= \frac{1}{x^n\alpha(x)} \frac{y^n \Omega}{x - y} + g(x, y) \\ &= \sum_{k=0}^{\infty} \sum_{i=1}^d s_{k,i} \otimes b_i(y^k, [y]^k) + \sum_{k=0}^{\infty} \sum_{i=1}^d g_{k,i} \otimes b_i(y^k, [y]^k). \end{aligned}$$

It is easy to see that

$$U := \text{span}_F \{w_{k,i} \mid k \geq 0, 1 \leq k \leq d\} \subset L(n, \alpha),$$

where $w_{k,i}$ are defined in Eq. (20), satisfies the condition $\Delta \dot{+} U = L(n, \alpha)$. Since $s := \frac{1}{x^n \alpha(x)}$ is invertible, we have $sU \dot{+} s\Delta = sU \dot{+} \Delta = L(n, \alpha)$. In other words, the space

$$sU = \text{span}_F \{s_{k,i} = sw_{k,i} \mid k \geq 0, 1 \leq k \leq d\} \subset L(n, \alpha) \tag{26}$$

is also complementary to the diagonal. Finally, since $g_{k,i} \in \Delta$ the space

$$W(r) = \text{span}_F \{sw_{k,i} + g_{k,i} \mid k \geq 0, 1 \leq k \leq d\} \subset L(n, \alpha)$$

is complementary to the diagonal. Conversely, if $V \subset L(n, \alpha)$ satisfies $V \dot{+} \Delta = L(n, \alpha)$, then for each $k \geq 0$ and $1 \leq i \leq d$ we can find a unique $g_{k,i} \in \Delta$ such that $sw_{k,i} + g_{k,i} \in V$. Define the (n, s) series r_V by

$$r_V(x, y) = \sum_{k \geq 0} \sum_{i=1}^d (sw_{k,i} + g_{k,i}) \otimes b_i(y^k, [y]^k).$$

It is now clear, that $W(r_V) = V$. These constructions establish the bijection in part 1.

To prove the second statement, observe that

$$B(sw_{k,i}, b_j(y^\ell, [y]^\ell)) = \delta_{i,j} \delta_{k,\ell}. \tag{27}$$

Furthermore, the straightforward calculation shows that

$$B(sw_{k,i}, sw_{\ell,j}) = \begin{cases} -\text{res}_0 \{sx^{(n-1)-k-\ell-1}\} & \text{if } i = j \text{ and } 0 \leq k, \ell \leq n - 1, \\ \text{res}_0 \{sx^{(n-1)-k-\ell-1}\} & \text{if } i = j \text{ and } k, \ell \geq n, \\ 0 & \text{otherwise,} \end{cases}$$

$$= \begin{cases} -s_{k+\ell-n+1} & \text{if } i = j, 0 \leq k, \ell \leq n - 1 \text{ and } k + \ell \geq n - 1, \\ s_{k+\ell-n+1} & \text{if } i = j \text{ and } k, \ell \geq n, \\ 0 & \text{otherwise,} \end{cases}$$

where $s(x) = \sum_{k=0}^\infty s_k x^k$. We write

$$\begin{aligned} \bar{r}(x, y) &= \frac{s(y)x^n \Omega}{x - y} - \tau(g(y, x)) = \frac{s(x)y^n \Omega}{x - y} - \frac{(s(x)y^n - s(y)x^n)\Omega}{x - y} - \tau(g(y, x)) \\ &= \sum_{k \geq 0} \sum_{i=1}^d (sw_{k,i} + \bar{g}_{k,i}) \otimes b_i(y^k, [y]^k). \end{aligned}$$

Consider the quotient

$$\begin{aligned} \frac{(s(x)y^n - s(y)x^n)\Omega}{x - y} &= \frac{y^n(s(x) - s(y))\Omega}{x - y} - \frac{s(y)(x^n - y^n)\Omega}{x - y} \\ &= \sum_{k \geq 0} \sum_{i=1}^d s_k \left(\sum_{\ell=1}^k b_i(x^{k-\ell}, [x]^{k-\ell}) \otimes b_i(y^{(n-1)+\ell}, [y]^{(n-1)+\ell}) \right. \\ &\quad \left. - \sum_{\ell=1}^n b_i(x^{n-\ell}, [x]^{n-\ell}) \otimes b_i(y^{k+\ell-1}, [y]^{k+\ell-1}) \right). \end{aligned}$$

The coefficient of $b_i(x^k, [x]^k) \otimes b_i(y^\ell, [y]^\ell)$ in the expression above is

$$\begin{aligned} &-s_{k+\ell-(n-1)} \text{ if } 0 \leq k, \ell \leq n - 1 \text{ and } k + \ell \geq n - 1, \\ &s_{k+\ell-(n-1)} \text{ if } k, \ell \geq n, \end{aligned}$$

which coincides with $B(sw_{k,i}, sw_{\ell,i})$. If we now expand the coefficients $g_{k,i}$ in the following way

$$g_{k,i} = \sum_{\ell \geq 0} \sum_{j=1}^d g_{k,i}^{\ell,j} b_j(x^\ell, [x]^\ell),$$

the coefficients $\bar{g}_{k,i}$ can be rewritten as

$$\bar{g}_{k,i} = - \sum_{\ell \geq 0} \sum_{j=1}^d (g_{\ell,j}^{k,i} + B(sw_{k,i}, sw_{\ell,j})) b_j(x^k, [x]^k) \otimes b_j(y^\ell, [y]^\ell).$$

Combining all the results above we obtain the desired equality

$$\begin{aligned} B(sw_{k,i} + g_{k,i}, sw_{\ell,j} + \bar{g}_{\ell,j}) &= B(sw_{k,i}, sw_{\ell,j}) + B(sw_{k,i}, \bar{g}_{\ell,j}) \\ &\quad + B(g_{k,i}, sw_{\ell,j}) + B(g_{k,i}, \bar{g}_{\ell,j}) \\ &= B(sw_{k,i}, sw_{\ell,j}) + (-g_{k,i}^{\ell,j} - B(sw_{k,i}, sw_{\ell,j})) + g_{k,i}^{\ell,j} + 0 \\ &= 0 \end{aligned}$$

which completes the proof of the second statement.

Using the same technique as in [2, Section 1], one can prove that

$$\psi := \text{GCYB}(r) \in (\Delta \otimes \mathfrak{g} \otimes \mathfrak{g})[[x_2, [x_2]], [x_3, [x_3]]]$$

for any series r of type (n, s) . Define $r_{k,i} := sw_{k,i} + g_{k,i}$ and $\bar{r}_{k,i} := sw_{k,i} + \bar{g}_{k,i}$ and rewrite GCYB(r) as

$$\begin{aligned} \psi &= \sum_{k, \ell \geq 0} \sum_{i, j=1}^d [r_{k,i}, r_{\ell,j}] \otimes b_i(x_2^k, [x_2]^k) \otimes b_j(x_3^\ell, [x_3]^\ell) \\ &+ \sum_{k \geq 0} \sum_{i=1}^d r_{k,i} \otimes \left([b_i(x_2^k, [x_2]^k) \otimes (1, 1), r(x_2, x_3)] \right. \\ &\left. + [(1, 1) \otimes b_i(x_3^k, [x_3]^k), \bar{r}(x_2, x_3)] \right). \end{aligned} \tag{28}$$

Applying $B(\bar{r}_{k_1, i_1} \otimes r_{k_2, i_2} \otimes r_{k_3, i_3}, -)$ to the equation above, we get

$$B(\bar{r}_{k_1, i_1} \otimes r_{k_2, i_2} \otimes r_{k_3, i_3}, \psi) = B(\bar{r}_{k_1, i_1}, [r_{k_2, i_2}, r_{k_3, i_3}]). \tag{29}$$

This gives the last statement because $W(r)$ and $W(\bar{r})$ are generated by $r_{k,i}$ and $\bar{r}_{k,i}$ respectively. □

Corollary 3.7 *Let $n \in \mathbb{Z}_{\geq 0}$, $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$ and W be as in Theorem 3.6. Then*

1. W defines a bijection between skew-symmetric $\left(n, \frac{1}{x^n \alpha(x)}\right)$ -type series and Lagrangian subspaces $V \subseteq L(n, \alpha)$ complementary to the diagonal Δ ;
2. W defines a bijection between $\left(n, \frac{1}{x^n \alpha(x)}\right)$ -type series solving GCYBE and subalgebras $V \subseteq L(n, \alpha)$ complementary to the diagonal Δ .

As we can see from the proof of Theorem 3.6 the element ψ in $\text{GCYB}(r) = \psi$ represents the obstruction for $W(r)$ from being a Lie subalgebra. This observation raises an interesting question that we do not consider in this paper: what elements ψ can appear on the right-hand side of the above-mentioned equation.

Observe that if r is a series of type (n, s) and it satisfies

$$\begin{aligned} \text{CYB}(r) &:= [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] \\ &+ [r^{13}(x_1, x_3), r^{23}(x_2, x_3)] = \psi \end{aligned} \tag{30}$$

for some $\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x, y, z]]$, then r is automatically skew-symmetric and hence solves the first equation as well. To prove that one can e.g. repeat the argument from [1, Lemma 5.2]. In other words, for a fixed ψ solutions to $\text{CYB}(r) = \psi$ form a subclass of solutions to $\text{GCYB}(r) = \psi$. In particular, solutions to $\text{CYB}(r) = 0$ are exactly the skew-symmetric solutions to $\text{GCYB}(r) = 0$. We call the equation $\text{CYB}(r) = \psi$ *Manin-Yang-Baxter equation*.

Remark 3.8 As our notation suggest, we could have interpreted $y^n \Omega / (x - y)$ as

$$\frac{y^n \Omega}{x - y} = \sum_{k \geq 0} \sum_{i=1}^d b_i x^{-k-1} \otimes b_i y^{n+k} \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]]$$

and performed all the arithmetic calculations in this form. To restore an (n, s) -type series from

$$\frac{s(x)y^n\Omega}{x - y} + g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]] \tag{31}$$

we can simply view $s(x) \in F[[x]]^\times$ and $g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ as elements in $F[[x, [x]]]^\times$ and $(\mathfrak{g} \otimes \mathfrak{g})[[x, [x]], (y, [y])]$ respectively and reinterpret the singular part $y^n\Omega/(x - y)$ as it was done in Eq. (20).

Conversely, to get a series of the form Eq. (31) from a series of type (n, s) we can just project the latter onto the first component.

In other words, we have a bijection between (n, s) -type series in $L_2(n, \alpha)$ and their projections Eq. (31) onto the first component given by different interpretations of the singular part $y^n\Omega/(x - y)$.

Although, all arithmetic operations can be performed in the form Eq. (31), the construction of $W(r)$ and statements like $\Delta \cap W(r) = 0$ require us to pass to the interpretation Eq. (20). This is our main motivation to work directly with (n, s) -type series in $L_2(n, \alpha)$ instead of their projections.

In view of Remark 3.8, we have a new proof of [1, Corollary 5.5].

Corollary 3.9 *Classical (formal) r -matrices, i.e. skew-symmetric elements*

$$\frac{s(x)y^n\Omega}{x - y} + g(x, y) = \frac{1}{x^n\alpha(x)} \frac{y^n\Omega}{x - y} + g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]], \tag{32}$$

solving GCYBE, are in bijection with skew-symmetric series of type (n, s) solving GCYBE and hence in bijection with Lagrangian Lie subalgebras of $L(n, \alpha)$ complementary to the diagonal Δ .

The result of [1, Theorem 5.6] can be now formulated in the following way.

Corollary 3.10 *Skew-symmetric series of type $(n, \frac{1}{x^n\alpha(x)})$ that also solve GCYBE exist only for $n = 0, 1$ and $n = 2$ with $\alpha_0 = 0$.*

4 Quasi-Lie bialgebra structures on $\mathfrak{g}[[x]]$

We remind that F is a discrete algebraically closed field of characteristic 0 and $\mathfrak{g}[[x]]$ is an F -Lie algebra equipped with the (x) -adic topology.

As we now know, series of type $(n, 1/(x^n\alpha(x)))$ solving CYBE Eq. (30) are in bijection with Lagrangian subalgebras $W \subset L(n, \alpha)$ complementary to the diagonal. On the other hand, such Lagrangian subalgebras are in bijection with non-degenerate topological Lie bialgebra structures. See [1] for their definition and classification.

It turns out, that if we drop the condition on W being a subalgebra, we get so called (non-degenerate) topological quasi-Lie bialgebras. This section is devoted to their classification up to topological twists and coordinate transformations.

Definition 4.1 A topological quasi-Lie bialgebra structure on $\mathfrak{g}[[x]]$ consists of

- a skew-symmetric continuous linear map $\delta: \mathfrak{g}[[x]] \rightarrow (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ and
- a skew-symmetric element $\varphi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x, y, z]]$,

which are subject to the following conditions

1. $\delta([a, b]) = [a \otimes 1 + 1 \otimes a, \delta(b)] - [b \otimes 1 + 1 \otimes b, \delta(a)]$, i.e. δ is a 1-cocycle;
2. $\frac{1}{2}\text{Alt}((\delta \otimes 1)\delta(a)) = [a \otimes 1 \otimes 1 + 1 \otimes a \otimes 1 + 1 \otimes 1 \otimes a, \varphi]$;
3. $\text{Alt}((\delta \otimes 1 \otimes 1)\varphi) = 0$,

where $\text{Alt}(x_1 \otimes \dots \otimes x_n) := \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$.

Lemma 4.2 There is a one-to-one correspondence between triples $(L, \mathfrak{g}[[x]], W)$, where $(L, \mathfrak{g}[[x]])$ is a topological Manin pair and $W \subset L$ is a Lagrangian subspace satisfying $W \dot{+} \mathfrak{g}[[x]] = L$, and quasi-Lie bialgebra structures on $\mathfrak{g}[[x]]$.

Proof We start with a topological Manin pair $(L, \mathfrak{g}[[x]])$. If $W \subset L$ is a Lagrangian subspace complementary to $\mathfrak{g}[[x]]$, then it is easy to see that $W \cong \mathfrak{g}[[x]]'$. Therefore, we have an isomorphism of vector spaces

$$L \cong \mathfrak{g}[[x]] \dot{+} \mathfrak{g}[[x]]'.$$

The form on L under this isomorphism becomes standard evaluation form $\langle -, - \rangle$ on $\mathfrak{g}[[x]] \dot{+} \mathfrak{g}[[x]]'$. We fix such an isomorphism.

Let us define two linear functions

$$p_1: \mathfrak{g}[[x]]' \otimes \mathfrak{g}[[y]]' \rightarrow \mathfrak{g}[[x]] \text{ and } p_2: \mathfrak{g}[[x]]' \otimes \mathfrak{g}[[y]]' \rightarrow \mathfrak{g}[[x]]'$$

by $[f, g] = p_1(f \otimes g) + p_2(f \otimes g)$. We put

$$\delta := p_2^\vee: (\mathfrak{g}[[x]]')^\vee \cong \mathfrak{g}[[x]] \rightarrow (\mathfrak{g}[[x]]' \otimes \mathfrak{g}[[y]]')^\vee \cong (\mathfrak{g} \otimes \mathfrak{g})[[x, y]],$$

and let $\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x, y, z]]$ be the unique element satisfying the condition

$$\langle h, [f, g] \rangle = \langle h, p_1(f \otimes g) \rangle = \langle f \otimes g \otimes h, \psi \rangle \text{ for all } f, g, h \in \mathfrak{g}[[x]]'. \tag{33}$$

The skew-symmetry of p_2 implies the skew-symmetry of δ , whereas the skew-symmetry of p_1 and the invariance of the evaluation form yield the skew-symmetry of ψ .

Next, we observe that for all $a, b \in \mathfrak{g}[[x]]$ and $f, g \in \mathfrak{g}[[x]]'$ we have

$$\begin{aligned} \langle [a, f], g \rangle &= \langle a, [f, g] \rangle = \langle a, p_2(f \otimes g) \rangle = \langle \delta(a), f \otimes g \rangle = \langle (f \otimes 1)\delta(a), g \rangle, \\ \langle [a, f], b \rangle &= -\langle f, [a, b] \rangle = -\langle f \circ \text{ad}_a, b \rangle. \end{aligned}$$

In other words, the invariance of the form forces the following equality to hold

$$[a, f] = -f \circ \text{ad}_a + (f \otimes 1)\delta(a). \tag{34}$$

Using Eq. (34) and non-degeneracy of the form we show that δ is a 1-cocycle:

$$\begin{aligned}
 \langle \delta([a, b]), f \otimes g \rangle &= \langle [a, b], p_2(f \otimes g) \rangle = \langle [a, b], [f, g] \rangle = \langle [[a, b], f], g \rangle \\
 &= \langle -[[b, f], a] - [[f, a], b], g \rangle \\
 &= \langle [f \circ \text{ad}_b - (f \otimes 1)\delta(b), a] - [f \circ \text{ad}_a - (f \otimes 1)\delta(a), b], g \rangle \\
 &= -\langle a, [f \circ \text{ad}_b, g] \rangle + \langle b, [f \circ \text{ad}_a, g] \rangle + \langle (f \otimes \text{ad}_a)\delta(b), g \rangle \\
 &\quad - \langle (f \otimes \text{ad}_b)\delta(a), g \rangle \\
 &= \langle [a \otimes 1 + 1 \otimes a, \delta(b)] - [b \otimes 1 + 1 \otimes b, \delta(a)], f \otimes g \rangle.
 \end{aligned}
 \tag{35}$$

The 1-cocycle condition implies that δ is continuous as it was noted in [1, Remark 3.16].

For conditions 2 and 3 from the definition of a topological quasi-Lie bialgebra consider the Jacobi identity for $f, g, h \in \mathfrak{g}[[x]]'$:

$$\begin{aligned}
 0 &= [p_1(f \otimes g), h] + [p_1(g \otimes h), f] + [p_1(h \otimes f), g] \\
 &\quad + p_1(p_2(f \otimes g) \otimes h) + p_1(p_2(g \otimes h) \otimes f) + p_1(p_2(h \otimes f) \otimes g) \\
 &\quad + p_2(p_2(f \otimes g) \otimes h) + p_2(p_2(g \otimes h) \otimes f) + p_2(p_2(h \otimes f) \otimes g).
 \end{aligned}
 \tag{36}$$

We denote by \circ the summation over circular permutations of symbols f, g and h , e.g. $\circ \langle p_1(f \otimes g), h \rangle = \langle p_1(f \otimes g), h \rangle + \langle p_1(g \otimes h), f \rangle + \langle p_1(h \otimes f), g \rangle$. Applying $\langle -, a \rangle$ to Eq. (36) for an arbitrary $a \in \mathfrak{g}[[x]]'$ gives

$$\begin{aligned}
 \langle p_2(p_2 \otimes 1)(\circ f \otimes g \otimes h), a \rangle &= -\langle \circ [p_1(f \otimes g), h], a \rangle \\
 \langle p_2 \otimes 1(\circ f \otimes g \otimes h), \delta(a) \rangle &= \circ \langle -h \circ \text{ad}_a, p_1(f \otimes g) \rangle \\
 \langle \circ f \otimes g \otimes h, (\delta \otimes 1)\delta(a) \rangle &= \circ \langle f \otimes g \otimes (-h \circ \text{ad}_a), \psi \rangle \\
 \langle f \otimes g \otimes h, \text{Alt}((\delta \otimes 1)\delta(a))/2 \rangle &= -\langle f \otimes g \otimes h, [1 \otimes 1 \otimes a + 1 \otimes a \otimes 1 \\
 &\quad + a \otimes 1 \otimes 1, \psi] \rangle,
 \end{aligned}$$

where the very last identity holds because of the skew-symmetry of ψ . Multiplying this equality by 2 we get the relation

$$\langle f \otimes g \otimes h, \text{Alt}((\delta \otimes 1)\delta(a)) + 2[1 \otimes 1 \otimes a + 1 \otimes a \otimes 1 + a \otimes 1 \otimes 1, \psi] \rangle = 0.$$

Letting $\varphi := -\psi$ we obtain the second identity from the definition of a topological quasi-Lie bialgebra structure. Applying instead $\langle s, - \rangle, s \in \mathfrak{g}[[x]]'$ to the Jacobi identity Eq. (36) we get the desired

$$\text{Alt}((\delta \otimes 1 \otimes 1)\psi) = 0.$$

Therefore, $(\mathfrak{g}[[x]], \delta, \varphi)$ is a topological quasi-Lie bialgebra.

For the converse direction, we put $L := \mathfrak{g}[[x]] \dot{+} \mathfrak{g}[[x]]'$ with the standard evaluation form; we let p_1 be the unique element in $\text{Hom}_{F\text{-Vect}}(\mathfrak{g}[[x]]' \otimes \mathfrak{g}[[x]]', \mathfrak{g}[[x]])$ satisfying

Eq. (33) with $\psi := -\varphi$; we define $p_2 := \delta'$, i.e. the dual map of δ . The Lie bracket between two elements in $\mathfrak{g}[[x]]'$ is given by the sum $p_1 + p_2$. Defining $[a, f]$ as in Eq. (34) the evaluation form becomes invariant and we get a topological Manin pair $(L, \mathfrak{g}[[x]])$ with the Lagrangian subspace $\mathfrak{g}[[x]]'$. These constructions are clearly inverse to each other. \square

Combining the classification of Manin pairs mentioned in Sect. 2 with Corollary 3.7 and Lemma 4.2 we get the following description of all topological quasi-Lie bialgebra structures on $\mathfrak{g}[[x]]$.

Lemma 4.3 *There is a bijection between topological quasi-Lie bialgebra structures on $\mathfrak{g}[[x]]$ and Lagrangian subspaces $W \subset L(n, \alpha)$ or $L(\infty)$ complementary to the diagonal Δ , where $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$ is an arbitrary sequence and $n \geq 0$. Moreover, such Lagrangian subspaces $W \subset L(n, \alpha)$ are in bijection with skew-symmetric sequences of type $(n, 1/(x^n\alpha(x)))$.*

In view of this result we call skew-symmetric series of type (n, s) as well as their projections onto the first component *quasi-r-matrices*. Quasi-Lie bialgebra structures can also be described using their associated quasi-r-matrices in the following way.

Proposition 4.4 *Assume $(\mathfrak{g}[[x]], \delta, \varphi)$ is a topological quasi-Lie bialgebra and let $r \in L_2(n, \alpha)$ be the corresponding quasi-r-matrix given by the bijection from Lemma 4.3. Under the identification $\mathfrak{g}[[x, [x]]) \cong \mathfrak{g}[[x]]$ we have the following identities:*

- $[a \otimes 1 + 1 \otimes a, r] = -\delta(a)$ for any $a \in \mathfrak{g}[[x]]$ and
- $\text{CYB}(r) = -\varphi$.

The same is true for the projection $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]]$.

Proof We start, as in the proof of Lemma 4.2, by fixing an identification $L(n, \alpha) = \Delta \dot{+} W(r) \cong \mathfrak{g}[[x]] \dot{+} \mathfrak{g}[[x]]'$. Let $\{v_{k,i}\}$ be a basis for $\mathfrak{g}[[x]]'$ dual to $\{\varepsilon_{k,i} := b_i y^k\}$. Then $r = \sum_{k \geq 0} \sum_{i=1}^d v_{k,i} \otimes \varepsilon_{k,i}$ and we have

$$\begin{aligned} [a \otimes 1 + 1 \otimes a, r] &= \sum_{k \geq 0} \sum_{i=1}^d [a, v_{k,i}] \otimes \varepsilon_{k,i} + v_{k,i} \otimes [a, \varepsilon_{k,i}] \\ &= \sum_{k \geq 0} \sum_{i=1}^d (-v_{k,i} \circ \text{ad}_a + (v_{k,i} \otimes 1)\delta(a)) \otimes \varepsilon_{k,i} + v_{k,i} \otimes [a, \varepsilon_{k,i}]. \end{aligned}$$

Applying $\langle v_{\ell,j} \otimes v_{m,t}, - \rangle$ to the equality above we get

$$\begin{aligned} \langle v_{\ell,j} \otimes v_{m,t}, [a \otimes 1 + 1 \otimes a, r] \rangle &= \sum_{k \geq 0} \sum_{i=1}^d \langle v_{\ell,j} \otimes v_{m,t}, (v_{k,i} \otimes 1)\delta(a) \otimes \varepsilon_{k,i} \rangle \\ &= \langle v_{\ell,j}, (v_{m,t} \otimes 1)\delta(a) \rangle \\ &= \langle v_{\ell,j} \otimes v_{m,t}, -\delta(a) \rangle. \end{aligned}$$

Applying instead $\langle \varepsilon_{\ell,j} \otimes v_{m,t}, - \rangle$ to the same equality we obtain

$$\begin{aligned} \langle \varepsilon_{\ell,j} \otimes v_{m,t}, [a \otimes 1 + 1 \otimes a, r] \rangle &= \sum_{k \geq 0} \sum_{i=1}^d \langle \varepsilon_{\ell,j} \otimes v_{m,t}, (-v_{k,i} \circ \text{ad}_a) \otimes \varepsilon_{k,i} \\ &\quad + v_{k,i} \otimes [a, \varepsilon_{k,i}] \rangle \\ &= -\langle \varepsilon_{\ell,j}, v_{m,t} \circ \text{ad}_a \rangle + \langle v_{m,t}, [a, \varepsilon_{\ell,j}] \rangle \\ &= 0. \end{aligned}$$

This implies the desired equality $[a \otimes 1 + 1 \otimes a, r] = -\delta(a)$. The identity $\text{CYB}(r) = -\varphi$ follows from the skew-symmetry of r , Theorem 3.6 and the fact that $\varphi = -\psi$ according to the proof of Lemma 4.2. \square

Remark 4.5 Assume $r \in (\mathfrak{g} \otimes \mathfrak{g})(\llbracket x \rrbracket) \llbracket y \rrbracket$ is a series such that

$$[f(x) \otimes 1 + 1 \otimes f(y), r(x, y)] \in (\mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y \rrbracket \tag{37}$$

for all $f \in \mathfrak{g} \llbracket x \rrbracket$. Write $r = s(x^{-1}, y) + g(x, y)$, where $s \in x^{-1}(\mathfrak{g} \otimes \mathfrak{g}) \llbracket x^{-1} \rrbracket \llbracket y \rrbracket$ and $g \in (\mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y \rrbracket$. Then, because of Eq. (37), we must have

$$[a \otimes 1 + 1 \otimes a, s(x^{-1}, y)] = 0$$

for all $a \in \mathfrak{g}$. Since the \mathfrak{g} -invariant elements of $\mathfrak{g} \otimes \mathfrak{g}$ are precisely the multiples of the quadratic Casimir element Ω , we have the identity $s(x^{-1}, y) = p(x^{-1}, y)\Omega$ for some $p \in x^{-1}F \llbracket x^{-1} \rrbracket \llbracket y \rrbracket$. Furthermore, the condition

$$[ax \otimes 1 + 1 \otimes ay, p(x^{-1}, y)\Omega] = [a(x - y) \otimes 1, p(x^{-1}, y)\Omega] \in (\mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y \rrbracket$$

implies $(x - y)p(x^{-1}, y) \in F \llbracket x, y \rrbracket$, meaning that there exists an $s \in F \llbracket y \rrbracket$ such that $p(x^{-1}, y) = s(y)/(x - y)$. In other words, r has the form Eq. (21). This result can be considered as another motivation to study series of type (n, s) .

Observe that if we know one Lagrangian subspace W_0 inside $L \cong \mathfrak{g} \llbracket x \rrbracket \dot{+} \mathfrak{g} \llbracket x \rrbracket'$ then any other Lagrangian subspace can be constructed from W_0 through twisting. More precisely, if $s = \sum_i a_i \otimes b^i \in (\mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y \rrbracket$ is a skew-symmetric tensor, then we can associate with it a (twisted) Lagrangian subspace

$$W_s := \left\{ \sum_i B(b^i, w)a_i - w \mid w \in W \right\} \subseteq L \tag{38}$$

complementary to $\mathfrak{g} \llbracket x \rrbracket$. The converse is also true; for proof see [3]. In other words, the following statement holds.

Lemma 4.6 *There is a bijection between Lagrangian subspaces $W \subseteq L(n, \alpha)$ or $L(\infty)$ and skew-symmetric tensors in $(\mathfrak{g} \otimes \mathfrak{g}) \llbracket x, y \rrbracket$.*

Combining Lemma 4.4, Eq. (38) and the algorithm for constructing a quasi- r -matrix from a Lagrangian subspace $W \subset L(n, \alpha)$, $W \dot{+} \Delta = L(n, \alpha)$, we obtain the following twisting rules for Lagrangian subspaces, quasi-Lie bialgebra structures and quasi- r -matrices.

Lemma 4.7 *Let $(\mathfrak{g}[[x]], \delta, \varphi)$ be a topological quasi-Lie bialgebra structure corresponding to the quasi- r -matrix r . If we twist $W(r)$ with a skew-symmetric tensor s as described in Eq. (38) we obtain another topological quasi-Lie bialgebra $(\mathfrak{g}[[x]], \delta_s, \varphi_s)$, such that*

1. $W(r)_s = W(r - s)$;
2. $\delta_s = \delta + ds$;
3. $\varphi_s = \varphi + \text{CYB}(s) - \frac{1}{2}\text{Alt}((\delta \otimes 1)s)$,

where $ds(a) := [a \otimes 1 + 1 \otimes a, s]$.

Remark 4.8 Since any quasi- r -matrix r defines a topological quasi-Lie bialgebra structure $\delta(a) = [a \otimes 1 + 1 \otimes a, r]$ on $\mathfrak{g}[[x]]$, the third condition in Definition 4.1 is trivially satisfied. In other words,

$$\text{Alt}((\delta \otimes 1 \otimes 1)\text{CYB}(r)) = 0$$

for any quasi- r -matrix r .

Lemma 4.6 and Lemma 4.7 state that, in order to obtain a description of topological quasi-Lie bialgebra structures on $\mathfrak{g}[[x]]$ up to twisting it is enough to find a single Lagrangian subspace W_0 , complementary to $\mathfrak{g}[[x]]$, inside $L(\infty)$ and each $L(n, \alpha)$. The same is true for the associated quasi- r -matrices

The case $L(\infty)$ is trivial, because by definition $\mathfrak{g}[[x]]' = \bigoplus_{j \geq 0} \mathfrak{g} \otimes a_j \subseteq L(\infty)$ is a Lagrangian subalgebra (see Lemma 2.1). Similar to the Lie bialgebra case, topological quasi-Lie bialgebras corresponding to the Manin pair $(L(\infty), \mathfrak{g}[[x]])$ are called *degenerate*.

Let us now focus on *non-degenerate* topological quasi-Lie bialgebra structures, i.e. the ones corresponding to the Manin pairs $(L(n, \alpha), \Delta)$. By Proposition 2.5 for each Manin pair $(L(n, \alpha), \Delta)$ there exists an appropriate coordinate transformation that makes it into $(L(n, \beta), \Delta)$, where $\beta_0 = \alpha_0$ and all other $\beta_i = 0$. This means, that to classify all non-degenerate topological quasi-Lie bialgebras on $\mathfrak{g}[[x]]$, up to coordinate transformations and twisting, it is enough to construct a Lagrangian subspace W_0 within each $L(n, \alpha_0) := L(n, (\dots, 0, \alpha_0, 0, \dots, 0))$ complementary to Δ . Equivalently, it is enough to find a quasi- r -matrix of type (n, α_0) for any $n \geq 0$ and $\alpha_0 \in F$.

4.1 Lagrangian subspaces of $L(n, \alpha_0)$

As before we let $\{b_i\}_{i=1}^d$ be an orthonormal basis for \mathfrak{g} with respect to the Killing form κ . The form B on $L(n, \alpha_0)$ has the following explicit form

$$\begin{aligned}
 & B(a \otimes (f, [p]), b \otimes (g, [q])) \\
 &= \begin{cases} \kappa(a, b) \{ \text{coeff}_{n-1}(fg - pq) - \alpha_0 \text{coeff}_0(fg - pq) \} & \text{if } n \geq 2, \\ \kappa(a, b) \text{coeff}_{n-1}(fg - pq) & \text{if } n = 0, 1. \end{cases}
 \end{aligned} \tag{39}$$

We now present an explicit construction for a Lagrangian subspace of $L(n, \alpha_0)$ complementary to Δ for arbitrary $n \geq 0$ and $\alpha_0 \in F$. Using the twisting procedure from Lemma 4.7, this subspace can be twisted in order to obtain all other Lagrangian subspaces of $L(n, \alpha_0)$ complementary to Δ .

$n = 0$: When $n = 0$, the subalgebra $W_0 := x^{-1}\mathfrak{g}[[x^{-1}]] \subseteq \mathfrak{g}((x))$ is known to be Lagrangian.

$n = 1$: For $n = 1$ it is easy to see that the subspace

$$W_0 := \text{span}_F \{ b_i(1, -1), b_i(x^{-k}, 0) \mid k \geq 1, 1 \leq i \leq d \} \subset L(1, \alpha_0) \tag{40}$$

is Lagrangian and complementary to the diagonal Δ .

$n = 2k$: For even $n \geq 2$ and arbitrary $\alpha_0 \in F$ the subspace $W_0 \subset L(n, \alpha_0)$ spanned by the elements

$$\begin{aligned}
 & b_i \left\{ (x^{(n-1)-m}, 0) - \alpha_0 (x^{2(n-1)-m}, 0) + \alpha_0^2 (x^{3(n-1)-m}, 0) - \alpha_0^3 (x^{4(n-1)-m}, 0) + \dots \right\}, \\
 & 0 \leq m \leq \frac{n}{2} - 1, \\
 & b_i \left(0, -[x]^{(n-1)-\ell} \right), \quad \frac{n}{2} \leq \ell < n - 1, \\
 & b_i \left(0, -1 + \frac{\alpha_0}{2} [x]^{n-1} \right), \\
 & b_i(x^{-k}, 0), k \geq 1,
 \end{aligned}$$

is Lagrangian and complementary to the diagonal.

$n = 2k + 1$: Modifying slightly the basis for even case we obtain the following basis for $W_0 \subset L(n, \alpha_0)$ with odd $n \geq 3$:

$$\begin{aligned}
 & b_i \left\{ (x^{(n-1)-m}, 0) - \alpha_0 (x^{2(n-1)-m}, 0) + \alpha_0^2 (x^{3(n-1)-m}, 0) - \alpha_0^3 (x^{4(n-1)-m}, 0) + \dots \right\}, \\
 & 0 \leq m \leq \frac{n-1}{2} - 1, \\
 & b_i \left\{ \left(x^{\frac{n-1}{2}}, -[x]^{\frac{n-1}{2}} \right) - \alpha_0 \left(x^{\frac{3(n-1)}{2}}, 0 \right) + \alpha_0^2 \left(x^{\frac{5(n-1)}{2}}, 0 \right) - \alpha_0^3 \left(x^{\frac{7(n-1)}{2}}, 0 \right) + \dots \right\}, \\
 & b_i(0, -[x]^{(n-1)-\ell}), \quad \frac{n-1}{2} + 1 \leq \ell < n - 1, \\
 & b_i \left(0, -1 + \frac{\alpha_0}{2} [x]^{n-1} \right), \\
 & b_i(x^{-k}, 0), k \geq 1.
 \end{aligned}$$

The subspaces above were constructed by "guessing". However, there is an abstract procedure that produces Lagrangian subspaces for arbitrary n and α . We present it here for completeness.

The easiest skew-symmetric (n, s) -type series is given by

$$\begin{aligned}
 r(x, y) &:= \frac{1}{2} \left(\frac{s(x)y^n \Omega}{x-y} + \frac{s(y)x^n \Omega}{x-y} \right) = \frac{s(x)y^n \Omega}{x-y} + \frac{\Omega}{2} \left(\frac{s(y)x^n - s(x)y^n}{x-y} \right) \\
 &= \frac{s(x)y^n \Omega}{x-y} - \frac{1}{2} \sum_{k,\ell=0}^{\infty} \sum_{i,j=1}^d B(sw_{k,i}, sw_{\ell,j}) b_i(x, [x]^k) \otimes b_j(y, [y]^\ell),
 \end{aligned}$$

where we recall that

$$B(sw_{k,i}, sw_{\ell,j}) = \begin{cases} -s_{k+\ell-n+1} & \text{if } i = j, 0 \leq k, \ell \leq n-1 \text{ and } k + \ell \geq n-1, \\ s_{k+\ell-n+1} & \text{if } i = j \text{ and } k, \ell \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 3.7 the subspace

$$\begin{aligned}
 W(r) &= \text{span}_F \left\{ sw_{k,i} - \frac{1}{2} \sum_{\ell=0}^{\infty} B(sw_{\ell,i}, sw_{k,i}) b_i(x, [x]^\ell) \mid k \geq 0, 1 \leq d \leq n \right\} \\
 &= \text{span}_F \left\{ sw_{k,i} + \frac{1}{2} \left(\sum_{\ell=0}^{n-1} s_{k+\ell-n+1} b_i(x, [x]^\ell) - \sum_{\ell=n}^{\infty} s_{k+\ell-n+1} b_i(x, [x]^\ell) \right) \mid \right. \\
 &\quad \left. k \geq 0, 1 \leq d \leq n \right\}
 \end{aligned}$$

is Lagrangian and complementary to the diagonal. Here we used the convention that $s_k = 0$ for $k < 0$. Calculating the basis explicitly for some particular s requires some effort and it may not look as friendly as the ones given above.

4.2 Quasi- r -matrices

The goal of this section is to describe the quasi- r -matrices corresponding to the Lagrangian subspaces described in the previous section. The twisting procedure from Lemma 4.7 then yields all other quasi- r -matrices.

The proof of Theorem 3.6 gives us an algorithm for constructing a series of type $(n, s(x) := 1/(x^n \alpha(x)))$ from a subspace $W \subset L(n, \alpha)$ complementary to the diagonal. More precisely, the desired series is given by

$$\sum_{k \geq 0} \sum_{i=1}^d v_{k,i} \otimes b_i(y^k, [y]^k), \tag{41}$$

where

$$W = \text{span}_F\{v_{k,i} \mid k \geq 0, 1 \leq i \leq d\} \text{ and } B(v_{k,i}, b_j(y^\ell, [y]^\ell)) = \delta_{i,j} \delta_{k,\ell},$$

i.e. $\{v_{k,i}\}$ is a basis of V dual to $\{b_i(y^k, [y]^k)\}$. Indeed, non-degeneracy of the form B then implies that $v_{k,i}$ has the desired form $v_{k,i} = sw_{k,i} + g_{k,i}$ for some $g_{k,i} \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$.

Applying this idea to W_0 's constructed in the preceding section we get the following series.

$n = 0$: The classical r -matrix (equivalently $(0, 1)$ -type series) corresponding to $W_0 := x^{-1} \mathfrak{g}[[x^{-1}]] \subseteq \mathfrak{g}((x))$ is the Yang's matrix $\Omega/(x - y)$.

$n = 1$: The quasi- r -matrix corresponding to $\text{span}_F\{b_i(1, -1), b_i(x^{-k}, 0) \mid k \geq 1, 1 \leq i \leq d\} \subset L(1, \alpha_0)$ is

$$\begin{aligned} & \frac{y\Omega}{x - y} + \frac{1}{2} \sum_{i=1}^d b_i(1, -1) \otimes b_i(1, 1) \in L_2(1, 1) \text{ with the projection} \\ & \frac{y\Omega}{x - y} + \frac{1}{2} \Omega \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]]. \end{aligned}$$

$n = 2k$: For even $n \geq 2$ and arbitrary $\alpha_0 \in F$ we have the following quasi- r -matrix

$$\begin{aligned} & \frac{1}{1 + \alpha_0 x^{n-1}} \frac{y^n \Omega}{x - y} + \frac{\Omega}{1 + \alpha_0 x^{n-1}} \sum_{0 \leq m < \frac{n}{2}} x^{(n-1)-m} y^m \\ & + \frac{\alpha_0 \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})} \left(y^{2(n-1)} \right. \\ & \left. + \sum_{\frac{n}{2} \leq \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} - \frac{1}{2} x^{n-1} y^{n-1} \right). \end{aligned}$$

$n = 2k + 1$: In the odd case $n \geq 3$ the series corresponding to $W_0 \subset L(n, \alpha_0)$ is

$$\begin{aligned} & \frac{1}{1 + \alpha_0 x^{n-1}} \frac{y^n \Omega}{x - y} + \frac{\Omega}{1 + \alpha_0 x^{n-1}} \left(x^{\frac{n-1}{2}} y^{\frac{n-1}{2}} + \sum_{0 \leq m < \frac{n-1}{2}} x^{(n-1)-m} y^m \right) \\ & + \frac{\alpha_0 \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})} \left(y^{2(n-1)} \right. \\ & \left. + \sum_{\frac{n-1}{2} < \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} - \frac{1}{2} x^{n-1} y^{n-1} \right). \end{aligned}$$

5 Lie algebra splittings of $L(n, \alpha)$ and generalized r -matrices

By Corollary 3.7 we have a bijection between subalgebras of $L(n, \alpha)$ and series of type $(n, 1/(x^n\alpha(x)))$ solving GCYBE. Therefore, we can construct new solutions to GCYBE by finding subalgebras of $L(n, \alpha)$ complementary to the diagonal. However, as the following result shows, the most interesting new solutions should arise from unbounded subalgebras of $L(n, \alpha)$, $n > 2$.

Proposition 5.1 *Let $L(n, \alpha) = \Delta \dot{+} W$ for some subalgebra $W \subset L(n, \alpha)$ and $n > 2$. Assume W is bounded, i.e. there is an integer $N > 0$ such that*

$$x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N\mathfrak{g}[x^{-1}],$$

where W_+ is the projection of $W \subset L(n, \alpha) = \mathfrak{g}((x)) \oplus \mathfrak{g}[x]/x^n\mathfrak{g}[x]$ on the first component $\mathfrak{g}((x))$. Then there is an element $\sigma \in \text{Aut}_{F[x]-\text{LieAlg}}(\mathfrak{g}[x])$ such that

$$\{0\} \times [x^2]\mathfrak{g}[x]/x^n\mathfrak{g}[x] \subseteq (\sigma \times \sigma)W \subseteq x\mathfrak{g}[x^{-1}] \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$$

and the image \tilde{W} under the canonical projection $L(n, \alpha) \rightarrow L(2, \alpha)$ is a subalgebra satisfying $L(2, \alpha) = \Delta \dot{+} \tilde{W}$.

In the language of (n, s) -type series: Let

$$r = \frac{s(x)y^n\Omega}{x - y} + g(x, y)$$

be the generalized r -matrix corresponding to a bounded $W \subset L(n, \alpha)$, $n \geq 2$. Then there is $p(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[x, y]$ of degree at most one in x and an element $\sigma \in \text{Aut}_{F[x]-\text{LieAlg}}(\mathfrak{g}[x])$ such that

$$(\sigma(x) \otimes \sigma(y))r(x, y) = y^{n-2} \underbrace{\left(\frac{s(x)y^2\Omega}{x - y} + p(x, y) \right)}_{r'(x, y)},$$

where r' is a generalized r -matrix in $L_2(2, \alpha)$.

Proof The condition $x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N\mathfrak{g}[x^{-1}]$ means exactly that W_+ is an order. Moreover, since W is complementary to the diagonal, we have $W_+ + \mathfrak{g}[x] = \mathfrak{g}[x, x^{-1}]$. It was shown in [11] that such orders, up to the action of some $\sigma \in \text{Aut}_{F[x]-\text{LieAlg}}(\mathfrak{g}[x])$, are contained in a maximal order \mathfrak{M} associated to the so called fundamental simplex Δ_{st} . These maximal orders are explicitly described in [11] and satisfy $\mathfrak{M} \subseteq x\mathfrak{g}[x^{-1}]$. Therefore, we have $\sigma W_+ \subseteq \mathfrak{M} \subseteq x\mathfrak{g}[x^{-1}]$. Moreover, we have the identity

$$(\sigma \times \sigma)W \dot{+} \Delta = L(n, \alpha),$$

implying the inclusion $\{0\} \times [x^2]\mathfrak{g}[x]/x^n\mathfrak{g}[x] \subseteq (\sigma \times \sigma)W$. The remaining parts follow straightforward from the construction Theorem 3.6. □

Unfortunately, we have not found a new example of an unbounded subalgebra of $L(n, \alpha)$. However, we present an infinite family of bounded subalgebras. We believe these examples are still interesting because their orthogonal complements, which are important in the view of Adler-Kostant-Symes scheme, are unbounded if $\alpha \neq 0$.

Consider the subspaces of $L(n, \alpha_0)$, $n > 0$:

$$W_0 = \text{span}_F \{b_i(x^{-k}, 0), b_i(1, 0), b_i(0, -[x]^\ell) \mid k \geq 1, 1 \leq \ell \leq n - 1\},$$

$$W_1 = \text{span}_F \{b_i(x^{-k}, 0), b_i(0, -1), b_i(0, -[x]^\ell) \mid k \geq 1, 1 \leq \ell \leq n - 1\}.$$

These are clearly subalgebras. The corresponding generalized r -matrices are

$$r_0 = \frac{1}{1 + \alpha_0 x^{n-1}} \frac{y^n \Omega}{x - y} + \frac{y^{n-1} \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})}$$

$$+ \frac{\alpha_0 \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})} \left(y^{2(n-1)} + \sum_{0 \leq \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} \right)$$

$$= \frac{y^{n-1}}{1 + \alpha_0 y^{n-1}} \left(\frac{y \Omega}{x - y} + \Omega \right),$$

$$r_1 = \frac{1}{1 + \alpha_0 x^{n-1}} \frac{y^n \Omega}{x - y} + \frac{\alpha_0 \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})}$$

$$\times \left(y^{2(n-1)} + \sum_{0 \leq \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} \right)$$

$$= \frac{1}{1 + \alpha_0 y^{n-1}} \frac{y^n \Omega}{x - y}.$$

By considering decompositions $\mathfrak{g} = \mathfrak{s}_1 \dot{+} \mathfrak{s}_2$ of \mathfrak{g} into direct sums of subalgebras we can get an infinite family of generalized r -matrices "in between" r_0 and r_1 . More precisely, let $\{s_{1,i}\}_{i=1}^{d_1}$ and $\{s_{2,j}\}_{j=1}^{d_2}$ be bases for \mathfrak{s}_1 and \mathfrak{s}_2 respectively. Such a decomposition leads to another subalgebra of $L(n, \alpha_0)$:

$$W_{01} := \text{span}_F \left\{ b_i(x^{-k}, 0), s_{1,m}(1, 0), s_{2,j}(0, 1), b_i(0, -[x]^\ell) \mid k \geq 1, 1 \leq \ell \leq n - 1, 1 \leq i \leq d, \right.$$

$$\left. 1 \leq m \leq d_1, 1 \leq j \leq d_2 \right\}.$$

Rewrite the elements b_i in terms of $s_{1,m}$ and $s_{2,j}$:

$$b_i = \sum_{m=1}^{d_1} \lambda_{1,m}^i s_{1,m} + \sum_{j=1}^{d_2} \lambda_{2,j}^i s_{2,j},$$

where $\lambda_{1,m}^i, \lambda_{2,j}^i \in F$. Finding a basis in W_{12} dual to $\{b_i(y^m, [y]^m)\} \subset \Delta$ and then projecting the generating series for W_{01} onto the first component we obtain the following generalized r -matrix

$$\begin{aligned}
 r_{01} &= \frac{1}{1 + \alpha_0 x^{n-1}} \frac{y^n \Omega}{x - y} + \frac{\alpha_0 \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})} \\
 &\times \left(y^{2(n-1)} + \sum_{0 < \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} \right) \\
 &+ \frac{y^{n-1}}{1 + \alpha_0 y^{n-1}} \sum_{i=1}^d \sum_{m=1}^{d_1} \lambda_{1,m}^i s_{1,m} \otimes b_i \\
 &= \frac{y^{n-1}}{1 + \alpha_0 y^{n-1}} \left(\frac{y \Omega}{x - y} + \sum_{i=1}^d \sum_{m=1}^{d_1} \lambda_{1,m}^i s_{1,m} \otimes b_i \right). \tag{42}
 \end{aligned}$$

Clearly r_{01} coincides with r_0 when $\mathfrak{s}_1 = \mathfrak{g}$ and r_1 if $\mathfrak{s}_2 = \mathfrak{g}$. The corresponding orthogonal complements are

$$W_0^\perp = W(\overline{r_0}) = \text{span}_F \left\{ b_i \left(0, [x]^{n-1} \right), b_i \left(\frac{x^{-k(n-1)-m}}{1 + \alpha_0 x^{n-1}}, 0 \right) \mid k \geq -1, 0 < m < n - 1 \right\},$$

$$W_1^\perp = W(\overline{r_1}) = \text{span}_F \left\{ b_i \left(\frac{x^{-k(n-1)-m}}{1 + \alpha_0 x^{n-1}}, 0 \right) \mid k \geq -1, 0 \leq m < n - 1 \right\},$$

$$\begin{aligned}
 W_{01}^\perp = W(\overline{r_{01}}) &= \mathfrak{s}_1^\perp \left(\frac{x^{n-1}}{1 + \alpha_0 x^{n-1}}, 0 \right) \dot{+} \mathfrak{s}_2^\perp \left(0, [x]^{n-1} \right) \\
 &\dot{+} \text{span}_F \left\{ b_i \left(\frac{x^{-k(n-1)-m}}{1 + \alpha_0 x^{n-1}}, 0 \right) \mid k \geq -1, 0 < m < n - 1 \right\}, \tag{43}
 \end{aligned}$$

which are unbounded because of the factor $1/(1 + \alpha_0 x^{n-1})$.

Note that a series of type (n, s) defines a subspace inside $L(n, \alpha)$ for any α , because the subalgebra property is not affected by the form. With the previous examples in mind we can prove the following statement.

Lemma 5.2 *Let B_0 and B_α be the bilinear forms on $L(n, 0)$ and $L(n, \alpha)$ respectively. For a series r of type (n, s) we have*

$$W(r)^\perp_{B_\alpha} = \frac{1}{x^n \alpha(x)} W(r)^\perp_{B_0} \subset L(n, \alpha). \tag{44}$$

Proof Set $u(x) := 1/(x^n \alpha(x))$. Write

$$r = \sum_{k \geq 0} \sum_{i=1}^d (s w_{k,i} + g_{k,i}) \otimes b_i(y^k, [y]^k) \quad \text{and} \quad \overline{r} = \sum_{k \geq 0} \sum_{i=1}^d (s w_{k,i} + \overline{g_{k,i}}) \otimes b_i(y^k, [y]^k).$$

Then by Theorem 3.6 and definition Eq. (12) $B_\alpha(sw_{k,i} + g_{k,i}, u(sw_{\ell,j} + \overline{g_{\ell,j}})) = B_0(sw_{k,i} + g_{k,i}, sw_{\ell,j} + \overline{g_{\ell,j}}) = 0$. \square

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Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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