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# The four-dimensional Yang–Mills partition function in the vicinity of the vacuum

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**Abstract**

The partition function of four-dimensional Euclidean, non-supersymmetric SU(2) Yang–Mills theory is calculated in the perturbative and weak coupling regime, i.e., in a small open ball about the flat connection and when the gauge coupling constant acquires a small but finite value. The computation is based on various known inequalities, valid only in four dimensions, providing two-sided estimates for the exponentiated Yang–Mills action in terms of the $L^2$-norm of the derivative of the gauge potential only; these estimates then give rise to Gaußian-like infinite- dimensional formal integrals involving the Laplacian and hence can be computed via zeta-function and heat kernel techniques. It then turns out that these formal integrals give a sharp value for the partition function in the aforementioned perturbative and weak coupling regime of the theory. In the resulting expression for the partition function, the original classical value of the coupling constant is shifted to a smaller one which can be interpreted as the manifestation, in this approach, of a non-trivial $\beta$-function and asymptotic freedom in pure non-Abelian gauge theories.

**Keywords** Non-supersymmetric Yang–Mills partition function · Zeta-function regularization · Heat kernel · Asymptotic freedom

**Mathematics Subject Classification** Primary: 81T13; Secondary: 81Q30 · 57M50 · 35K08

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## 1 Introduction and summary

Computing the partition function is a central problem of Yang–Mills theory. For in Feynman's path integral quantization framework it is intrinsically equivalent with the highly non-trivial task of taking summation over all vacuum Feynman graphs, the computation of the partition function is the first and most difficult step toward the construction of the underlying relativistic quantum field theory. In the exposition of the problem mainly found in physicist's textbooks (cf., e.g., [5, 11, 13, 27]), the difficulties are usually attributed to the presence of a huge (namely gauge) symmetry of the theory alone; however, the troubles have certainly much deeper roots related, e.g., with our problematic 18-19th century concept of the continuum [1, 28] and the non-existence of a good measure theory in infinite dimensions [12], too. Nevertheless, because of its central importance, permanent efforts have been made to calculate the partition function during the past decades. These are based upon taming the partition function in order to increase its computational accessibility by using either discretization, i.e., lattice methods (e.g., [2, 3]) or yet working with the continuum but introducing additional structures. Very roughly speaking, these latter approaches hit the field in three powerful waves: in the 1970-1980s various *supersymmetric* and *higher-dimensional extensions* of pure Yang–Mills theory have been introduced making it possible to calculate their corresponding partition functions via Atiyah–Bott-like localization techniques, cf. [17] (especially [17, Chapter 10]). Then, *topological twisting*, an additional modification, was introduced by Witten [29] which together with many other ideas such as the Chern–Simons and conformal field theory correspondence and various duality conjectures, etc. led in the 1990s to revolutionary discoveries connecting quantum field theories and low-dimensional differential topology [20, 29, 30] thereby clearly demonstrating the indeed deep, not only physical but even mathematical, relevance of Yang–Mills partition functions. However, eventually together with Nekrasov's $\Omega$-*deformation* approach [15] from the early 2000s, these supersymmetric twisting and deformation techniques, as a price for computability, gradually converted the Yang–Mills partition function, an originally certainly highly analytical object, into a rather purely combinatorial structure, in this way at least in part having covered or mixed the original physical content of Yang–Mills theory with auxiliary mathematical structures.

In this paper, as a continuation of our earlier work on the Abelian case [8], we make an attempt to return to the original setup and compute the partition function of the non-supersymmetric, non-twisted, etc. but surely non-Abelian four-dimensional Euclidean pure (i.e., without fermions and scalars) gauge theory. The sacrifice we make for not using any supersymmetric, etc. support is that unfortunately we shall neglect all non-perturbative (like instanton, etc.) effects which are, however, certainly key features of non-Abelian gauge theories; that is we shall consider the perturbative regime only. It is worth briefly mentioning here that part of our approach which in our opinion is the most interesting (and well-known) because works only in four dimensions. The curvature of a connection $\nabla = \mathrm{d} + A$ looks like $F_\nabla = \mathrm{d}A + A \wedge A$, i.e., consists of a derivative (dynamical) $\mathrm{d}A$ and a quadratic (interacting) term $A \wedge A$ of the gauge potential. In four dimensions, there is a delicate balance between these terms as a consequence of the Sobolev embedding $L^2_1 \subset L^4$ which is on the borderline in four

dimensions. Indeed, this embedding allows one to compare the $L^2$-norm of the $\mathrm{d}A$ and $A \wedge A$ terms. Physically speaking, this means that precisely in four dimensions the energy content in the Yang–Mills field strength is equally distributed between its dynamical and interacting terms.[1] From the mathematical aspect, the existence of $L^2_1 \subset L^4$ allows one to estimate the $L^2$-norm of the curvature of a connection from both below and above by various, at most quartic, expressions involving the $L^2$-norm of the derivative part of the gauge potential alone. These estimates can be re-written as Gaußian-like expressions for the Laplacian and hence can be formally Feynman integrated using $\zeta$-function and heat kernel techniques providing a two-sided estimate for the partition function. After adjusting the physical and technical parameters involved in this procedure, this "scissor" about the partition function closes up giving rise to an expression for it.

For clarity, we emphasize that our forthcoming calculations and assertions are supposed to be mathematically rigorous *except* precisely the mathematical definition of Feynman integration itself (which of course is a crucial point); this latter thing will be rather treated only formally throughout the text but in the standard way by using $\zeta$-function regularization. We also emphasize that what we are going to write throughout the text as

$$Z_\varepsilon(\mathbb{R}^4, \tau)$$

and want to calculate is *not* an approximation of the full partition function $Z(\mathbb{R}^4, \tau)$ of four-dimensional non-supersymmetric Yang–Mills theory (containing all instanton and other non-perturbative contributions) but a contribution of the *vicinity of the vacuum*, i.e., the complete perturbative regime in the weak coupling limit to the full partition function. Of course, an important question is whether or not $Z_\varepsilon(\mathbb{R}^4, \tau)$ already gives rise to the leading contribution to $Z(\mathbb{R}^4, \tau)$, i.e., whether or not by some (hidden) localization mechanism already $Z(\mathbb{R}^4, \tau) \approx Z_\varepsilon(\mathbb{R}^4, \tau)$. The answer for this question is certainly negative because on the one hand localization phenomena are expected to occur only in supersymmetrized Yang–Mills theories [17] (and we are not dealing with them here) and on the other hand instantons with nonzero topological numbers surely give further relevant contributions to the full partition function $Z(\mathbb{R}^4, \tau)$ hopefully rendering it a nice modular form in its (probably quantum corrected) $\tau \in \mathbb{C}^+$ variable as indicated by various $S$-duality conjectures (far from being complete cf., e.g., [16, 21, 26, 31]). Nevertheless, $Z_\varepsilon(\mathbb{R}^4, \tau)$ already alone is expected to reveal something from the quantum behavior of gauge theory.

After these careful circumscriptions, limitations and clarifications our main *formal* result can be summarized as follows. For the very technical details, we refer to Sects. 3 and 1.

**Theorem 1.1** *Consider a non-supersymmetric pure* SU(2) *gauge theory with complex coupling constant* $\tau \in \mathbb{C}^+$ *over the Euclidean* 4*-space* $(\mathbb{R}^4, \eta)$. *Take a constant* $0 < \varepsilon < \sqrt{8}\pi$ *and consider those* SU(2) *connections* $\nabla$ *which are close to the flat*

[1] One is tempted to say that although in dimensions different from four classical Yang–Mills theory can be formulated, its underlying quantum theory will be governed by $\mathrm{d}A$ or $A \wedge A$ alone; hence, it exhibits a different, perhaps less complex, behavior.

*connection* $\nabla^0$ *in the sense that* $\|F_\nabla\|_{L^2(\mathbb{R}^4)} < \varepsilon$. *Let* $Z_\varepsilon(\mathbb{R}^4, \tau)$ *denote the corresponding* truncated partition function *of the theory obtained by formally Feynman integrating the exponentiated Yang–Mills action over gauge equivalence classes of* SU(2) *connections close to the flat connection against a formal measure provided by the round sphere* $(S^4, g_R)$ *of radius* $R$ *which is a one-point conformal compactification of* $(\mathbb{R}^4, \eta)$ *(hence this formal measure and thus* $Z_\varepsilon(\mathbb{R}^4, \tau)$ *itself may in principle depend on* $R$*).*

*Provided the complex coupling constant* $\tau \in \mathbb{C}^+$ *has large enough imaginary part (the weak coupling regime) and accordingly both the vicinity parameter* $\varepsilon$ *is small enough (the perturbative regime) and the compactification radius* $R$ *is small enough (a technical condition on the formal measure) then, using* $\zeta$*-function regularization and heat kernel techniques, the truncated partition function can be computed and*

$$Z_\varepsilon(\mathbb{R}^4, \tau) = \left(\frac{\mathrm{Im}\tau}{2\pi^2 N^2}\right)^{-\frac{11}{20}} \frac{2^{-\frac{11}{20}}}{\sqrt{\pi}} \cos\left(\tfrac{11\pi}{40}\right) \Gamma\left(\tfrac{9}{40}\right) \mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0) - 3\zeta'_{\Delta_0}(0)}$$

*where* $N$ *is a constant satisfying* $1 \leqq N < \sqrt{2}$ *and* $\Gamma$ *is Euler's Gamma function; moreover,* $\zeta_{\Delta_k}$ *are the* $\zeta$*-functions of Laplacians acting on* $k$*-forms over* $(S^4, g_R)$.

*The truncated partition function* $Z_\varepsilon(\mathbb{R}^4, \tau)$ *depends on* $R$ *only through the formal determinant term* $\mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0) - 3\zeta'_{\Delta_0}(0)}$. *More precisely, provided the radii* $0 < R_1 < R_2$ *are both small enough, hence the corresponding* $(S^4, g_{R_i})$ *are two allowed conformal one-point compactifications of* $(\mathbb{R}^4, \eta)$*; then,*

$$Z^1_\varepsilon(\mathbb{R}^4, \tau) = \left(\frac{R_1}{R_2}\right)^{\frac{11}{10}} Z^2_\varepsilon(\mathbb{R}^4, \tau)$$

*demonstrating that the conformal invariance of classical gauge theory breaks down.*

***Remark*** 1. $\mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0) - 3\zeta'_{\Delta_0}(0)} = 10.710...$ over the unit sphere and this expression of the formal determinant can be further expanded in terms of the derivatives of the standard Riemann and Hurwitz $\zeta$-functions (cf., e.g., [7, 14, 18]); however, the result is not promising hence omitted. One might hope to obtain nicer determinant expressions by introducing Dirac fermions into the theory, too. Also cf. [4].

2. The particular numerical values of the determinant above, the exponent $-\frac{11}{20}$ or the coefficient $N$ in $Z_\varepsilon(\mathbb{R}^4, \tau)$ bears no direct physical meaning for they depend on the particular regularization scheme used to make sense of infinite-dimensional integrals here. Concerning $N$ it is essentially nothing else than a good choice for a constant in Uhlenbeck's gauge fixing theorem [25] (see Lemma 3.1) and the only relevant point is that $N < \sqrt{2}$ must hold in order our method to work (see Lemma 3.3). This is provided by the at least one universal property of $N$, namely that whatever its value is, it is conformally invariant and surely $1 \leqq N$ such that $N \to 1$ as $\varepsilon \to 0$ (see Lemma 3.1).
3. Nevertheless, Theorem 1.1, when compared with the analogous Abelian result, admits an interesting physical interpretation in the context of *asymptotic freedom* which is a key property of non-Abelian gauge theories. The complex coupling

constant is defined as $\tau := \frac{\theta}{2\pi} + \frac{4\pi}{e^2}\sqrt{-1}$ where $\theta$ is the so-called $\theta$-parameter and $e$ is the coupling constant of the gauge theory. It enters the theory at its classical level, i.e., $\tau$ appears already in its defining action. However, it is well known that in a non-supersymmetric four-dimensional gauge theory, meanwhile, $\theta$ is unaffected, hence is a true quantum parameter, $e$ is subject to quantum corrections, i.e., the theory has a *non-trivial $\beta$-function*. Therefore, in our case it is intriguing to physically interpret the appearance of the purely technical–mathematical constant $N$ in Theorem 1.1 as a *quantum correction* of the classical gauge coupling. That is, by recalling from [8] the full partition function over $(S^4, g_R)$ in the U(1) case:

$$Z(\mathbb{R}^4, \tau) = \left(\frac{\mathrm{Im}\tau}{8\pi^2}\right)^{-\frac{11}{60}} \mathrm{e}^{\frac{1}{2}\zeta'_{\Delta_1}(0) - \zeta'_{\Delta_0}(0)}$$

we cannot resist the temptation to re-write the truncated SU(2) partition function computed here as

$$Z_\varepsilon(\mathbb{R}^4, \tau) = \left(\frac{\mathrm{Im}\tau_{\mathrm{eff}}}{8\pi^2}\right)^{-\frac{11}{20}} \frac{2^{-\frac{11}{20}}}{\sqrt{\pi}} \cos\left(\tfrac{11\pi}{40}\right) \Gamma\left(\tfrac{9}{40}\right) \mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0) - 3\zeta'_{\Delta_0}(0)},$$

i.e., absorb $N$ into the classical $\tau$ in this way shifting it to $\tau_{\mathrm{eff}} = \frac{\theta}{2\pi} + \frac{4\pi}{e^2_{\mathrm{eff}}}\sqrt{-1}$ where $e_{\mathrm{eff}} := \frac{N}{2}e$ is considered as an effective, i.e., perturbatively quantum corrected coupling constant (the inessential numerical term $\frac{2^{-\frac{11}{20}}}{\sqrt{\pi}} \cos\left(\frac{11\pi}{40}\right) \Gamma\left(\frac{9}{40}\right) = 1.013...$ rather looks like a non-Abelian correction to the formal determinant). However, the key property of $N$, i.e., that $1 \leqq N < \sqrt{2}$, makes sure that $e_{\mathrm{eff}} < e$ rendering the effective gauge coupling constant *smaller* than its classical value. This is qualitatively consistent with our picture on asymptotic freedom in pure non-Abelian gauge theories, the net effect of a highly counter-intuitive Yang–Mills-charge-anti-screening-mechanism generated by virtual charged gauge bosons floating around the real ones. In addition, it is well known (cf., e.g., [11]) that the presence of a non-trivial $\beta$-function in Yang–Mills theory is in conjunction with the breakdown of its classical conformal symmetry at the quantum level introduced by the formal integration measure lacking conformal invariance; hence, our physical interpretation of Theorem 1.1 is consistent from this angle as well.

4. We can also make a comment regarding $S$-duality [26]. In Theorem 1.1, it is assumed that $\tau$ has large (but finite!) imaginary part, that is, the gauge coupling $e$ is small. This assumption is physically clear because in this weak coupling regime the existence of convergent perturbation series is reasonable. The weak and the strong coupling regimes of a gauge theory are related by $S$-duality transformations. Supposing that $\tau_{\mathrm{eff}}$ is already meaningful at the quantum level, more precisely after taking into account at least small perturbative quantum corrections as in Theorem 1.1, and recalling the identity $\mathrm{Im}\left(-\frac{1}{\tau_{\mathrm{eff}}}\right) = \frac{1}{\tau_{\mathrm{eff}}\,\overline{\tau}_{\mathrm{eff}}}\mathrm{Im}\tau_{\mathrm{eff}}$, we recognize that the truncated partition function is a modular form with (holomorphic and anti-holomorphic) weight $(\frac{11}{20}, \frac{11}{20})$; hence, $Z_\varepsilon(\mathbb{R}^4, \tau_{\mathrm{eff}})$ has a promising behavior under

$S$-duality transformations [26]. Of course to say something more definitive on this topic (for instance, what about the modular properties of the full partition function with some meaningful $\tau_{\text{eff}}$ and how SU(2) is replaced with its Langlands dual group SO(3), etc.), one would need to calculate the complete partition function $Z(\mathbb{R}^4, \tau_{\text{eff}})$ consisting of all instanton, etc. corrections; this is, however, far beyond our technical skills at this stage of the art.

5. Finally, for future work we record here without proof that essentially by verbatim repeating the calculations below the partition function can also be computed in the *vicinity of an (anti-)instanton* $\nabla^k$ with instanton number $k \in \mathbb{Z}$ as well. It takes the shape $\mathrm{e}^{-\sqrt{-1}\pi k\tau} Z_{\varepsilon,k}(\mathbb{R}^4, \tau)$ if $k \geqq 0$ or similarly $\mathrm{e}^{\sqrt{-1}\pi k\overline{\tau}} Z_{\varepsilon,k}(\mathbb{R}^4, \tau)$ if $k \leqq 0$ where $Z_{\varepsilon,k}(\mathbb{R}^4, \tau)$ is an expression analogous to $Z_{\varepsilon,0}(\mathbb{R}^4, \tau) := Z_\varepsilon(\mathbb{R}^4, \tau)$ in Theorem 1.1 such that various ordinary Laplacians $\Delta_i = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ and their corresponding functions $\zeta_{\Delta_i}$ are to be replaced with the twisted ones $\Delta_i^k := \mathrm{d}_{\nabla^k}\mathrm{d}^*_{\nabla^k} + \mathrm{d}^*_{\nabla^k}\mathrm{d}_{\nabla^k}$ and $\zeta_{\Delta_i^k}$, respectively. However, even knowing these further contributions from instanton vicinities we still cannot *a priori* conclude that the full partition function would be a sum of these terms only.

The paper is organized as follows. In Sect. 2, we recall the calculation of the quadratic Gaußian and certain quartic Gaußian integrals in finite dimensions. The computation of these latter integrals is due to Svensson [22]. The resulting formulata allow formal generalizations to infinite dimensions. Then, in Sect. 3 classical pure gauge theory with $\theta$-term is introduced in the standard way and its truncated partition function is computed by evaluating these infinite-dimensional formal integrals using $\zeta$-function and heat kernel techniques. Finally, Sect. A is an Appendix and consists a well-known no-go result from infinite-dimensional measure theory [9, 12]. This has been added to gain a more comprehensive picture.

## 2 Some quadratic and quartic Gaußian integrals

In this preliminary section, we recall the computation of the well-known quadratic Gaußian and a less-known quartic Gaußian integral in finite dimensions; these considerations then allow us to formally generalize these integrals to infinite dimensions which is the relevant case for quantum field theory.

*The Gaußian integral.* Let $(\mathbb{R}^m, \eta)$ be the $m$- dimensional Euclidean space and $S : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ a positive definite symmetric bilinear form on it given by $S(x, x) := \eta(x, Mx)$ where $M : \mathbb{R}^m \to \mathbb{R}^m$ is a positive symmetric matrix whose eigenvalues therefore are real and satisfy $0 < \lambda_i < +\infty$ for all $i = 1, \ldots, m$. Using a linear change of variables one can pass to a principal axis basis of $S$, i.e., in which it looks like $S(y, y) = \lambda_1 y_1^2 + \cdots + \lambda_m y_m^2$, and then, performing a further change of variables $u_i := \sqrt{\lambda_i}\, y_i$ we find that

$$\lim_{a_i \to +\infty} \int_{-a_i}^{+a_i} \mathrm{e}^{-\lambda_i y_i^2} \mathrm{d}y_i = \lim_{a_i \to +\infty} \frac{1}{\sqrt{\lambda_i}} \int_{-\sqrt{\lambda_i} a_i}^{+\sqrt{\lambda_i} a_i} \mathrm{e}^{-u_i^2} \mathrm{d}u_i = \frac{\sqrt{\pi}}{\sqrt{\lambda_i}};$$

hence, taking their product we come up with

$$\int\limits_{\mathbb{R}^m} \mathrm{e}^{-S(x,x)}\mathrm{d}x = \prod_{i=1}^{m} \frac{\sqrt{\pi}}{\sqrt{\lambda_i}} = \frac{\pi^{\frac{m}{2}}}{\sqrt{\det M}}$$

giving rise to the well-known result. This integral has a truncated version, too. Let $0 < \delta < +\infty$ be a fixed number, and using an orthonormal frame $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ adapted to $S$ let

$$C_\delta^m := \left\{ y \in \mathbb{R}^m \;\middle|\; y = \sum_{i=1}^{m} y_i \mathbf{e}_i \;,\; M\mathbf{e}_i = \lambda_i \mathbf{e}_i \;,\; -\frac{\delta}{\sqrt{\lambda_i}} < y_i < +\frac{\delta}{\sqrt{\lambda_i}} \right\} \tag{1}$$

denote the "principal axis hypercube" of $S$ more precisely an open rectangular parallelepiped whose edges are parallel with the principal axes labeled by the eigenvalues $\lambda_i$ of $S$ and having sizes $\frac{2\delta}{\sqrt{\lambda_i}}$, respectively. Then, introducing $a_i := \frac{\delta}{\sqrt{\lambda_i}}$ we can repeat the previous calculation as follows:

$$\int\limits_{-a_i}^{+a_i} \mathrm{e}^{-\lambda_i y_i^2}\mathrm{d}y_i = \frac{1}{\sqrt{\lambda_i}} \int\limits_{-\delta}^{+\delta} \mathrm{e}^{-u_i^2}\mathrm{d}u_i = \frac{\sqrt{K(\delta)}}{\sqrt{\lambda_i}}$$

where $K(\delta)$, the square of the classical error function, is defined as

$$\sqrt{K(\delta)} := \int\limits_{-\delta}^{+\delta} \mathrm{e}^{-u_i^2}\mathrm{d}u_i = 2\sum_{j=0}^{+\infty} (-1)^j \frac{\delta^{2j+1}}{j!(2j+1)} = 2\left(\delta - \frac{\delta^3}{3} + \frac{\delta^5}{10} - \frac{\delta^7}{42} + \ldots\right).$$

It is independent of $S$ and is monotonically increasing in $0 \leqq \delta \leqq +\infty$ such that $0 \leqq K(\delta) \leqq \pi$. Taking product again, we obtain an expression

$$\int\limits_{C_\delta^m} \mathrm{e}^{-S(x,x)}\mathrm{d}x = \prod_{i=1}^{m} \frac{\sqrt{K(\delta)}}{\sqrt{\lambda_i}} = \frac{K(\delta)^{\frac{m}{2}}}{\sqrt{\det M}} \leqq \frac{\pi^{\frac{m}{2}}}{\sqrt{\det M}}$$

for the integral over the principal axis hypercube, similar for the entire integral above.

*A Gaußian-like integral*. Now let us compute a more general integral following Svensson [22]. Namely, picking two positive definite bilinear forms $S_1$, $S_2$, we are interested in the quartic integral

$$\int\limits_{\mathbb{R}^m} \mathrm{e}^{-S_1(x,x)^2 - S_2(x,x)}\mathrm{d}x \;\;.$$

Consider $\gamma_s := \{u + \sqrt{-1}\, s \mid u \in \mathbb{R}\} \subset \mathbb{C}$, i.e., a straight line in the complex plane running parallel with the real axis $\mathbb{R} \subset \mathbb{C}$. Introducing $t := \gamma_s(u)$, it is easy to see that

$\int_{\gamma_s} \mathrm{e}^{-\left(\frac{t}{2}-\sqrt{-1}\,S_1(x,x)\right)^2}\mathrm{d}t$ exists such that its value is equal to $2\sqrt{\pi}$ hence independent of $s \in \mathbb{R}$. Referring to [22], we adjust our integral by carefully inserting the Gaußian integral $1 = \frac{1}{2\sqrt{\pi}} \int_{\gamma_s} \mathrm{e}^{-\left(\frac{t}{2}-\sqrt{-1}\,S_1(x,x)\right)^2}\mathrm{d}t$ as follows:

$$\begin{aligned}\int\limits_{\mathbb{R}^m} \mathrm{e}^{-S_1(x,x)^2-S_2(x,x)}\mathrm{d}x &= \int\limits_{\mathbb{R}^m} \mathrm{e}^{-S_1(x,x)^2-S_2(x,x)} \left(\frac{1}{2\sqrt{\pi}} \int\limits_{\gamma_s} \mathrm{e}^{-\left(\frac{t}{2}-\sqrt{-1}\,S_1(x,x)\right)^2}\mathrm{d}t\right)\mathrm{d}x \\ &= \frac{1}{2\sqrt{\pi}} \int\limits_{\mathbb{R}^m} \left( \int\limits_{\gamma_s} \mathrm{e}^{-S_2(x,x)+\sqrt{-1}\,tS_1(x,x)-\frac{t^2}{4}}\mathrm{d}t\right)\mathrm{d}x \ .\end{aligned}$$

If $s \geqq 0$, then $\left|\int_{\mathbb{R}^m} \mathrm{e}^{-S_2(x,x)+\sqrt{-1}\,tS_1(x,x)}\mathrm{d}x\right| \leqq \int_{\mathbb{R}^m} \mathrm{e}^{-S_2(x,x)-sS_1(x,x)}\mathrm{d}x < +\infty$ for every fixed $t$; hence, this integral exists. Moreover, since the corresponding matrix $M_2-\sqrt{-1}\,tM_1$ is symmetric therefore diagonalizable, we can proceed in the standard way as above to get

$$\begin{aligned}\int\limits_{\mathbb{R}^m} \mathrm{e}^{-S_2(x,x)+\sqrt{-1}\,tS_1(x,x)}\mathrm{d}x &= \pi^{\frac{m}{2}} \left(\det(M_2-\sqrt{-1}\,tM_1)\right)^{-\frac{1}{2}} \\ &= (\sqrt{-1}\pi)^{\frac{m}{2}} \left(\det M_1\right)^{-\frac{1}{2}} \left(\det(t\mathbf{1}+\sqrt{-1}M_2M_1^{-1})\right)^{-\frac{1}{2}} \\ &= (\sqrt{-1}\pi)^{\frac{m}{2}} \left(\det M_1\right)^{-\frac{1}{2}} \big((t-z_1)\dots(t-z_m)\big)^{-\frac{1}{2}}\end{aligned}$$

where $z_1, \dots, z_m \in \mathbb{C}$ are the (not necessarily different) eigenvalues of the matrix $-\sqrt{-1}M_2M_1^{-1}$. Consequently, if $\int_{\gamma_s} \mathrm{e}^{-\frac{t^2}{4}} \big((t-z_1)\dots(t-z_m)\big)^{-\frac{1}{2}}\mathrm{d}t$ also exists and is single valued, the two integrations are interchangeable via Fubini's theorem and we end up with

$$\int\limits_{\mathbb{R}^m} \mathrm{e}^{-S_1(x,x)^2-S_2(x,x)}\mathrm{d}x = \frac{(\sqrt{-1}\pi)^{\frac{m}{2}}}{2\sqrt{\pi}\sqrt{\det M_1}} \int\limits_{\gamma_s} \frac{\mathrm{e}^{-\frac{t^2}{4}}}{\sqrt{(t-z_1)\dots(t-z_m)}}\,\mathrm{d}t \ .$$

Therefore, our task is to arrange $\gamma_s$ with $s \geqq 0$ so that the corresponding complex integral exists and is single valued. Certainly existence is achieved if $\gamma_s \subset \mathbb{C}$ does not hit $z_1, \dots, z_m$ (because any of them might be a multiple eigenvalue hence might give a pole in the integrand). In order to make the integral single valued, we perform usual branch cutting. Firstly, $z_1, \dots, z_m$ are clearly branching points of the integral and if $m$ is even, then these are the only branching points; if $m$ is odd, then beyond them the infinitely remote point is also a branching point. Secondly, $M_1^{-\frac{1}{2}}$ exists and is positive symmetric; since the eigenvalues of $M_2M_1^{-1} = (M_2M_1^{-\frac{1}{2}})M_1^{-\frac{1}{2}}$ and $M_1^{-\frac{1}{2}}(M_2M_1^{-\frac{1}{2}})$ coincide and the latter operator is positive symmetric, the eigenvalues of $M_2M_1^{-1}$ continue to be positive real numbers. Thus, all the eigenvalues of $-\sqrt{-1}\,M_2M_1^{-1}$ are

in fact aligned along the negative imaginary axis according to their magnitude, i.e., we can suppose $0 > z_1 \geqq z_2 \geqq \cdots \geqq z_m > -\sqrt{-1}\,\infty$. Let us therefore do branch cutting in the standard way: cut up $\mathbb{C}$ along the at most $\left[\frac{m+1}{2}\right]$ segments of the negative imaginary axis connecting $z_1$ with $z_2$ (if $z_1 \neq z_2$), $z_3$ with $z_4$ (if $z_3 \neq z_4$) and finally $z_{m-1}$ with $z_m$ (if $z_{m-1} \neq z_m$) whenever $m$ is even; or $z_1$ with $z_2$ (if $z_1 \neq z_2$), $z_3$ with $z_4$ (if $z_3 \neq z_4$) and finally $z_m$ with $-\sqrt{-1}\,\infty$ whenever $m$ is odd. Thus, the complex integral will be single valued if $\gamma_s \subset \mathbb{C}$ avoids these cutting segments as well.[2] Thus, to summarize, $\int_{\gamma_s} \mathrm{e}^{-\frac{t^2}{4}} \big((t-z_1)\ldots(t-z_m)\big)^{-\frac{1}{2}} \mathrm{d}t$ both exists and is single valued if we take any $\gamma_s(u) = u + \sqrt{-1}\,s$ with $s \geqq 0$.

Let us specialize from now on to the case $S_1 := c_1 S$ and $S_2 := c_2 S$ with $c_1, c_2 > 0$ real constants; this yields $-\sqrt{-1}(c_2 M)(c_1 M)^{-1} = -\sqrt{-1}\frac{c_2}{c_1}\mathbf{1}$; hence, $z_1 = \cdots = z_m = -\sqrt{-1}\frac{c_2}{c_1} \notin \mathbb{R}$. Therefore, either there is no branch cutting if $m$ is even or there is a single branch cutting running from $-\sqrt{-1}\frac{c_2}{c_1}$ to $-\sqrt{-1}\,\infty$ if $m$ is odd. We eventually come up with

$$\int\limits_{\mathbb{R}^m} \mathrm{e}^{-c_1^2 S(x,x)^2 - c_2 S(x,x)} \mathrm{d}x = \frac{(\sqrt{-1}\pi)^{\frac{m}{2}}}{2\sqrt{\pi}\sqrt{\det(c_1 M)}} \int\limits_{\gamma_s} \left(t + \sqrt{-1}\,\frac{c_2}{c_1}\right)^{-\frac{m}{2}} \mathrm{e}^{-\frac{t^2}{4}}\,\mathrm{d}t$$

together with the truncated integral

$$\int\limits_{C_\delta^m} \mathrm{e}^{-c_1^2 S(x,x)^2 - c_2 S(x,x)} \mathrm{d}x = \frac{(\sqrt{-1}\,K(\delta))^{\frac{m}{2}}}{2\sqrt{\pi}\sqrt{\det(c_1 M)}} \int\limits_{\gamma_s} \left(t + \sqrt{-1}\,\frac{c_2}{c_1}\right)^{-\frac{m}{2}} \mathrm{e}^{-\frac{t^2}{4}}\,\mathrm{d}t \tag{2}$$

where $\gamma_s(u) = u + \sqrt{-1}\,s$ with any $s \geqq 0$ is the contour as before. It is easy to see that taking the limit $c_1 \to 0$ these integrals reduce to the corresponding (i.e., the full or the truncated, respectively) Gaußian ones. However, we shall be more interested in the limit $c_2 \to 0$ of the full (i.e., not-truncated) integral which readily looks like

$$\int\limits_{\mathbb{R}^m} \mathrm{e}^{-c_1^2 S(x,x)^2} \mathrm{d}x = \frac{(\sqrt{-1}\pi)^{\frac{m}{2}}}{2\sqrt{\pi}\sqrt{\det(c_1 M)}} \int\limits_{\gamma_s} t^{-\frac{m}{2}} \mathrm{e}^{-\frac{t^2}{4}}\,\mathrm{d}t \tag{3}$$

where now we allow $\gamma_s(u) = u + \sqrt{-1}\,s$ with $s > 0$ only to avoid the pole at the origin (if $m > 1$) as well as the single branch cutting along the whole non-positive imaginary axis (if $m$ is odd).

Having warmed up with these rigorous but only finite-dimensional results, let us generalize them to infinite dimensions at least formally. Let $(M, g)$ be a connected, compact, oriented Riemannian 4-manifold without boundary and consider the Laplacian $\Delta_k : C^\infty(M; \wedge^k M) \to C^\infty(M; \wedge^k M)$, i.e., the second order linear,

[2] Or equivalently we can lift any $\gamma_s$ not hitting the eigenvalues over the corresponding at most $\left[\frac{m+1}{2}\right]$-genus Riemann surface regarded as a branching cover of the Riemann sphere and then define the already single-valued integral there.

symmetric, elliptic partial differential operator $\Delta_k = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ naturally acting on the space of smooth $k$-forms. This space admits Hilbert space completions like $L^2_s(M;\wedge^k M)$ for any $s \in \mathbb{R}$, and one can demonstrate via elliptic regularity that $\Delta_k$ extends to a densely defined, self-adjoint, unbounded linear operator $\Delta_k : L^2(M;\wedge^k M) \to L^2(M;\wedge^k M)$. By elliptic regularity, the kernel of this map contains precisely the space $\mathscr{H}^k(M) \subset C^\infty(M;\wedge^k M) \subset L^2(M;\wedge^k M)$ of smooth harmonic $k$-forms; by the Hodge decomposition theorem this kernel is isomorphic to the de Rham cohomology group $H^k(M)$ and hence is finite-dimensional, i.e., a closed subspace. Therefore, $c\Delta_k$ with $c > 0$ a real constant gives rise to a positive self-adjoint operator on the orthogonal complement Hilbert space

$$\mathscr{H}^k(M)^\perp \subset L^2(M;\wedge^k M) \ .$$

By the finite-dimensional analogue (3), it is therefore convenient to define a non-truncated quartic integral involving the Laplacian as

$$\int\limits_{\mathscr{H}^k(M)^\perp} \mathrm{e}^{-(a\,,\,c\Delta_k a)^2_{L^2(M)}}\,\mathrm{D}a := \frac{(\sqrt{-1}\pi)^{\frac{1}{2}\mathrm{rk}'(c\Delta_k)}}{2\sqrt{\pi}\sqrt{\det{}'\,(c\Delta_k)}} \int\limits_{\gamma_s} t^{-\frac{1}{2}\mathrm{rk}'(c\Delta_k)}\mathrm{e}^{-\frac{t^2}{4}}\,\mathrm{d}t \qquad (4)$$

where the regularized rank $\mathrm{rk}'$ and determinant $\det'$ are yet to be defined somehow.

Likewise, let $C_\delta \subset L^2(M;\wedge^k M)$ be the "principal axis hypercube" for $\Delta_k$ defined as in the finite-dimensional case (1) more precisely as the corresponding *finite* linear combinations of the eigen-forms of $\Delta_k$. Note that by elliptic regularity these eigen-forms belong to $C^\infty(M;\wedge^k M) \subset L^2(M;\wedge^k M)$, but in spite of the fact that they span a dense subspace of $L^2(M;\wedge^k M)$ the subset $C_\delta$ is *not* open (unlike in finite dimensions). This is because the eigenvalues of the Laplacian form an unbounded sequence, i.e., $\lambda_i \to +\infty$; hence, the size of the edges of $C_\delta$ satisfies $2a_i \to 0$ as $i \to +\infty$. Keeping in mind this subtlety and taking into account (2), nevertheless we put

$$\int\limits_{C_\delta\cap\mathscr{H}^k(M)^\perp} \mathrm{e}^{-(a\,,\,c_1\Delta_k a)^2_{L^2(M)}-(a\,,\,c_2\Delta_k a)_{L^2(M)}}\,\mathrm{D}a := \frac{(\sqrt{-1}K(\delta))^{\frac{1}{2}\mathrm{rk}'(c_1\Delta_k)}}{2\sqrt{\pi}\sqrt{\det{}'(c_1\Delta_k)}} \times$$
$$\int\limits_{\gamma_s} \left(t+\sqrt{-1}\,\frac{c_2}{c_1}t\right)^{-\frac{1}{2}\mathrm{rk}'(c_1\Delta_k)} \mathrm{e}^{-\frac{t^2}{4}}\,\mathrm{d}t \ . \qquad (5)$$

We will be also assuming that the following "monotonicity principles" hold true for these infinite-dimensional formal integrals:

**Monotonicity principles**. If $\emptyset \subsetneqq A, B \subsetneqq L^2(S^4;\wedge^1 S^4)$ are two "measurable" subsets in the $L^2$ Hilbert space of 1-forms over the 4-sphere satisfying $A \subsetneqq B$ and $f : L^2(S^4;\wedge^1 S^4) \to \mathbb{R}$ is a non-negative "integrable" function then

$$0 \leqq \int_A f(a)\mathrm{D}a \leqq \int_B f(a)\mathrm{D}a \leqq +\infty \ .$$

Moreover, if $f, g : L^2(S^4; \wedge^1 S^4) \to \mathbb{R}$ are two "integrable" functions satisfying $0 \leqq f \leqq g$, then

$$0 \leqq \int_A f(a)\mathrm{D}a \leqq \int_A g(a)\mathrm{D}a \leqq +\infty$$

is valid.

***Remark*** 1. As we mentioned before, the "principal axis hypercube" $C_\delta \subset L^2(M; \wedge^k M)$ for the Laplacian is not open in infinite dimensions. If, nevertheless, the formal integral (5) happens to attain a nonzero value, then this would imply that infinite-dimensional integration over very small (i.e., which do not contain any open ball) subsets might yield non-trivial results.

2. The monotonicity properties of integration are straightforward in finite dimensions, however, are not easily accessible in infinite dimensions. But more surprisingly, it seems these properties even may not hold over any 4-manifold. For instance, as discussed in Sect. 3, over the 4-sphere the regularized dimension of $L^2(S^4; \wedge^1 S^4)$ with respect to the Laplacian is *positive* (see Lemma 3.2); hence, the above monotonicity properties are expected to hold true. However, over the flat 4-torus for example, the regularized dimension of $L^2(T^4; \wedge^1 T^4)$ with respect to the Laplacian is *negative*; hence, one would expect that some sort of reversed form of the above monotonicity might work in this case.
   All of these oddities of integration in infinite dimensions likely are connected with the conflict between $\sigma$-additivity and infinite dimensionality (cf. Appendix here).

## 3 The partition function about the vacuum

After these preliminaries, we are ready to calculate the partition function. Let us begin with recalling and introducing 4- dimensional Euclidean non-supersymmetric SU(2) gauge theory with $\theta$ term in the usual way.

Consider $\mathbb{R}^4$ with its standard Euclidean metric $\eta$. Let $E \cong \mathbb{R}^4 \times \mathbb{C}^2$ be the unique trivial complex rank-two SU(2) vector bundle over $\mathbb{R}^4$ and take a compatible (i.e., SU(2)-valued) connection $\nabla$ on it. Denoting by $\wedge^k\mathbb{R}^4 \otimes \mathfrak{su}(2)$ the bundle of $\mathfrak{su}(2)$-valued $k$-forms over $\mathbb{R}^4$, by the global triviality of $E$ we can globally write $\nabla = \mathrm{d} + A$ where the gauge potential $A$ is a section of $\wedge^1\mathbb{R}^4 \otimes \mathfrak{su}(2)$ with the corresponding field strength $F_\nabla = \mathrm{d}A + A \wedge A$ giving rise to a section of $\wedge^2\mathbb{R}^4 \otimes \mathfrak{su}(2)$. Moreover, let $e \in \mathbb{R}$ and $\theta \in \mathbb{R}$ denote the coupling constant and the $\theta$-parameter of the theory, respectively. The non-supersymmetric 4-dimensional Euclidean SU(2) gauge theory is then defined by the action

$$S(\nabla, e, \theta) := -\frac{1}{2e^2}\int\limits_{\mathbb{R}^4} \mathrm{tr}(F_\nabla \wedge *F_\nabla) + \frac{\sqrt{-1}\,\theta}{16\pi^2}\int\limits_{\mathbb{R}^4} \mathrm{tr}(F_\nabla \wedge F_\nabla)\ .$$

The $\theta$-term is a characteristic class; hence, its variation is identically zero; consequently, the Euler–Lagrange equations (together with the Bianchi identity) of this theory are nothing but the usual vacuum Yang–Mills equations

$$\begin{cases} \mathrm{d}_\nabla F_\nabla = 0 \\ \mathrm{d}^*_\nabla F_\nabla = 0\ . \end{cases}$$

Introducing the complex coupling constant

$$\tau := \frac{\theta}{2\pi} + \frac{4\pi}{e^2}\sqrt{-1} \tag{6}$$

taking its values on the upper complex half-plane $\mathbb{C}^+$, and the *positive* definite $L^2$ scalar product $(\Phi, \Psi)_{L^2(\mathbb{R}^4)} := -\int_{\mathbb{R}^4} \mathrm{tr}(\Phi \wedge *\Psi)$ on the space of $\mathfrak{su}(2)$-valued 2-forms, with induced norm therefore satisfying $\|\Phi\|_{L^2(\mathbb{R}^4)} \geqq 0$, the action above can be re-written as

$$\begin{aligned} S(\nabla, \tau) = &-\frac{\sqrt{-1}\pi}{2}\tau\left(\frac{1}{8\pi^2}\|F_\nabla\|^2_{L^2(\mathbb{R}^4)} + \frac{1}{8\pi^2}(F_\nabla, *F_\nabla)_{L^2(\mathbb{R}^4)}\right) \\ &+\frac{\sqrt{-1}\pi}{2}\overline{\tau}\left(\frac{1}{8\pi^2}\|F_\nabla\|^2_{L^2(\mathbb{R}^4)} - \frac{1}{8\pi^2}(F_\nabla, *F_\nabla)_{L^2(\mathbb{R}^4)}\right) \end{aligned} \tag{7}$$

since $*^2 = \mathrm{Id}_{\wedge^2\mathbb{R}^4}$; hence, the topological term takes the shape $-\int_{\mathbb{R}^4} \mathrm{tr}(F_\nabla \wedge F_\nabla) = (F_\nabla, *F_\nabla)_{L^2(\mathbb{R}^4)}$ in this notation.

The orientation and the flat Euclidean metric $\eta$ on $\mathbb{R}^4$ are used to introduce various Sobolev spaces. Let $\nabla^0$ denote the trivial flat connection on $E$, i.e., the unique connection which satisfies $F_{\nabla^0} = 0$. Then, define

$$\mathscr{A}(\nabla^0) := \{\nabla \text{ is an SU}(2) \text{ connection on } E \mid \nabla - \nabla^0 \in L^2_1(\mathbb{R}^4\ ;\ \wedge^1\mathbb{R}^4 \otimes \mathfrak{su}(2))\}.$$

This is the $L^2_1$ Sobolev space of SU(2) connections on $E$ relative to $\nabla^0$. Notice that this is a vector space (not an affine space) and in fact $\mathscr{A}(\nabla^0) \ni \nabla \mapsto \nabla - \nabla^0 =: a \in L^2_1(\mathbb{R}^4\ ;\ \wedge^1\mathbb{R}^4 \otimes \mathfrak{su}(2))$ is a canonical isomorphism between $\mathscr{A}(\nabla^0)$ and $L^2_1(\mathbb{R}^4\ ;\ \wedge^1\mathbb{R}^4 \otimes \mathfrak{su}(2))$. Furthermore, write $\mathscr{SU}(2)$ for the $L^2_2$ completion of

$$\begin{aligned} &\{\gamma \text{ is an SU}(2) \text{ gauge transformation on } E \mid \gamma - \mathrm{Id}_E \in C^\infty_0(\mathbb{R}^4; \mathrm{End}E), \\ &\quad \gamma \in C^\infty(\mathbb{R}^4; \mathrm{Aut}E) \text{ a.e.}\} \end{aligned}$$

that is, the space of compactly supported smooth SU(2) gauge transformations. Therefore, $\gamma \in \mathscr{SU}(2)$ means that $\|\gamma - \mathrm{Id}_E\|_{L^2_2(\mathbb{R}^4)} < +\infty$. The space $\mathscr{A}(\nabla^0)$ is acted upon by $\mathscr{SU}(2)$ as $\nabla \mapsto \gamma^{-1}\nabla\gamma$ in the usual way and the corresponding gauge equivalence class of $\nabla \in \mathscr{A}(\nabla^0)$ is denoted by $[\nabla]$ and the orbit space $\mathscr{A}(\nabla^0)/\mathscr{SU}(2)$

of these equivalence classes with its quotient topology by $\mathscr{B}([\nabla^0])$ as usual. In the non-Abelian case, $\mathscr{B}([\nabla^0])$ is not a linear space; however, at least locally it can be modeled on various Banach spaces as we shall see shortly. Also note that $\nabla \in \mathscr{A}(\nabla^0)$ implies that if $\nabla = \nabla^0 + a$, then both the derivative term d$a$ and by the Sobolev multiplication theorem $L^2_1 \times L^2_1 \to L^2$ the interacting term $a \wedge a$ belong to $L^2$; therefore, $F_\nabla \in L^2(\mathbb{R}^4\,;\, \wedge^2\mathbb{R}^4 \otimes \mathfrak{su}(2))$ for any $\nabla \in [\nabla] \in \mathscr{B}([\nabla^0])$.

Having now the classical non-supersymmetric Euclidean gauge theory at our disposal, the *partition function* of the induced quantum theory is formally defined by the integral

$$Z(\mathbb{R}^4, \tau) := \frac{1}{\mathrm{Vol}\,(\mathscr{SU}(2))} \int\limits_{\nabla \in \mathscr{A}(\nabla^0)} \mathrm{e}^{-S(\nabla,\tau)} \mathrm{D}\nabla$$

or formally equivalently

$$Z(\mathbb{R}^4, \tau) := \int\limits_{[\nabla] \in \mathscr{B}([\nabla^0])} \mathrm{e}^{-S(\nabla,\tau)} \mathrm{D}[\nabla]$$

where D$\nabla$ is the formal (probably never definable) measure on $\mathscr{A}(\nabla^0)$ while D$[\nabla]$ is the induced formal measure (including the Faddeev–Popov determinant) on the orbit space $\mathscr{B}([\nabla^0])$. The ideal goal would be to calculate this integral in its full glory; however, it is an extraordinary difficult task because of the non-linearity of $\mathscr{B}([\nabla^0])$. Therefore, we will evaluate it in $\mathscr{B}_\varepsilon([\nabla^0])$ only, i.e., we are interested in an appropriately *truncated Feynman integral*

$$Z_\varepsilon(\mathbb{R}^4, \tau) := \int\limits_{[\nabla] \in \mathscr{B}_\varepsilon([\nabla^0])} \mathrm{e}^{-S(\nabla,\tau)} \mathrm{D}[\nabla]$$

where $\mathscr{B}_\varepsilon([\nabla^0])$ is a small open subset about $[\nabla^0]$ defined by $0 \leqq |S(\nabla, \tau)| < \frac{|\tau|}{8\pi}\varepsilon^2$ possessing the crucial property that, unlike the whole $\mathscr{B}([\nabla^0])$, it is well approximated by (a quotient of) a small open ball in an appropriate Hilbert space.

To make this picture more precise and in order to avoid several technical difficulties, we make a technical interlude and extend the SU(2) Yang–Mills theory from $(\mathbb{R}^4, \eta)$ to its one-point conformal compactification $(S^4, g_R)$ where $g_R$ denotes the standard round metric on $S^4 = \mathbb{R}^4 \cup \{\infty\}$ such that it has radius $0 < R < +\infty$. From the physical viewpoint, this conformal compactification is justified at least classically by the conformal invariance of classical gauge theory defined by (7) in four dimensions. From the mathematical or technical viewpoint a further support is Uhlenbeck's singularity removal theorem [23] or rather its generalization [24, Corollary 2.2] asserting that if $\nabla \in \mathscr{A}(\nabla^0)$ is *any* connection on $\mathbb{R}^4$ (which by definition means that there exists an $L^2_1$ gauge relative to $\nabla^0$ implying $\|F_\nabla\|_{L^2(\mathbb{R}^4)} < +\infty$ as we mentioned above) there exists an $L^2_2$ gauge transformation around the asymptotic region of $\mathbb{R}^4$ such that the gauge transformed connection $\nabla'$ extends over $\mathbb{R}^4 \cup \{\infty\} = S^4$. Therefore, from now

on, instead of $(\mathbb{R}^4, \eta)$ we consider the classical Yang–Mills theory (7) over $(S^4, g_R)$ and treat $R$ as a technical parameter of the original theory; correspondingly we are interested in calculating the formal truncated Feynman integral $Z_\varepsilon(\mathbb{R}^4, \tau)$ by working over $(S^4, g_R)$. It is therefore understood that the action $S$, the Sobolev space $\mathscr{A}(\nabla^0)$ consisting of our connections $\nabla$ and the various differential operators like $\mathrm{d}^*$, $\Delta_k$, etc. are defined over the round 4-sphere $(S^4, g_R)$ from now on. Uhlenbeck's gauge fixing theorem [25] can be formulated as follows (cf. [6, Proposition 2.3.13]): There exists a constant $0 < \varepsilon$ such that if a connection $\nabla \in \mathscr{A}(\nabla^0)$ on the trivial bundle $E \cong S^4 \times \mathbb{C}^2$ satisfies $\|F_\nabla\|_{L^2(S^4,g_R)} < \varepsilon$; then, there exists an $L^2_2$ gauge transformation $\gamma$ and a constant $0 < N(R) < +\infty$ such that the gauge transformed connection $\nabla' = \gamma^{-1}\nabla\gamma$ with corresponding decomposition $\nabla' = \mathrm{d} + A'$ satisfies the *Coulomb gauge condition* together with an estimate

$$\begin{cases} \mathrm{d}^* A' = 0 \\ \|A'\|_{L^2_1(S^4,g_R)} \leqq N(R)\|F_{\nabla'}\|_{L^2(S^4,g_R)} \end{cases} \tag{8}$$

implying $\|A'\|_{L^2_1(S^4,g_R)} \leqq N(R)\varepsilon$ in Coulomb gauge.

Now we are in a position to define the truncated partition function more carefully. Take a constant $0 < \varepsilon < \sqrt{8}\pi$ and consider those connections which satisfy $\|F_\nabla\|_{L^2(S^4,g_R)} < \varepsilon$. By conformal invariance of the norm, this is equivalent to consider those connections over the original space which satisfy $\|F_\nabla\|_{L^2(\mathbb{R}^4)} < \varepsilon$. The action takes a more clear shape in the compactified setting as follows. Regarding its topological term $\frac{1}{8\pi^2}\int_{S^4}\mathrm{tr}(F_\nabla \wedge F_\nabla)$, we know that it is proportional to the second Chern number of the extended SU(2) bundle over $S^4$; hence, it assumes integer values only; however, by the Cauchy–Schwarz inequality $0 \leqq \left|(F_\nabla\ ,\ *F_\nabla)_{L^2(S^4,g_R)}\right| \leqq \|F_\nabla\|^2_{L^2(S^4,g_R)} < \varepsilon^2 < 8\pi^2$ the $\theta$-term simply vanishes over $S^4$ in the small energy regime. This also implies that the connections we are interested in are realized in the extended gauge theory on the trivial bundle $E \cong S^4 \times \mathbb{C}^2$ alone and if $\varepsilon$ is small enough then Uhlenbeck's gauge fixing theorem applies. Consequently, the action (7) about $\nabla^0$ reduces to

$$S(\nabla, \tau) = S(\nabla^0 + a, \tau) = \frac{\mathrm{Im}\tau}{8\pi}\|F_{\nabla^0+a}\|^2_{L^2(S^4,g_R)} = \frac{\mathrm{Im}\tau}{8\pi}\|\mathrm{d}a + a\wedge a\|^2_{L^2(S^4,g_R)}$$

which also shows by conformal invariance of the action that $[\nabla] \in \mathscr{B}_\varepsilon([\nabla^0])$. The key technical observation now is [6, Proposition 4.2.9] saying that for a sufficiently small $\varepsilon$ there exists an $\eta > 0$ such that $\mathscr{B}_\varepsilon([\nabla^0])$ is homeomorphic to $(B_\eta(\nabla^0) \cap \ker \mathrm{d}^*)/G_0$ with $B_\eta(\nabla^0) \subset \mathscr{A}(\nabla^0)$ being a small open ball and $G_0 \cong \mathrm{SU}(2)$ the gauge isotropy subgroup of the flat hence reducible connection $\nabla^0$. Hence, put

$$\mathscr{A}_{\varepsilon,N(R)}(\nabla^0) := \left\{ a \in L^2_1(S^4; \wedge^1 S^4 \otimes \mathfrak{su}(2)) \;\middle|\; \|a\|_{L^2_1(S^4,g_R)} < \min(\eta, N(R)\varepsilon) \right\} \tag{9}$$

where $\varepsilon, N(R)$ are the same constants over $(S^4, g_R)$ as in (8). By the aid of the homeomorphism

$$\mathscr{B}_\varepsilon([\nabla^0]) \cong \frac{\mathscr{A}_{\varepsilon,N(R)}(\nabla^0) \cap \ker \mathrm{d}^*}{G_0}$$

we *suppose* that the "measure" $\mathrm{D}[\nabla]$ arises from a $G_0$-invariant "measure" on $\mathscr{A}_{\varepsilon,N(R)}(\nabla^0) \cap \ker \mathrm{d}^*$ what we denote $\mathrm{D}_R a$. The main advantage of this non-linear isomorphism is that it locally "straightens" the gauge orbits hence its effect is analogous to passing from a general curved coordinate system to the standard Descartes one. Consequently, the *Faddeev–Popov determinant* is locally transformed away, i.e., gives only a constant multiplier (cf. Footnote 3). Moreover, the *Gribov ambiguity* problem does not cause any headache here too; for this local quotient contains nearby gauge orbits precisely once only. Our truncated Feynman integral awaiting for computation is then defined more carefully as

$$Z_\varepsilon(\mathbb{R}^4, \tau) := \int\limits_{\mathscr{A}_{\varepsilon,N(R)}(\nabla^0)\cap\ker \mathrm{d}^*} \mathrm{e}^{-\frac{\mathrm{Im}\tau}{8\pi}\|\mathrm{d}a+a\wedge a\|^2_{L^2(S^4,g_R)}} \frac{\mathrm{D}_R a}{\mathrm{Vol}(G_0)} \tag{10}$$

having the following properties. In this formal integral, the integration domain $\mathscr{A}_{\varepsilon,N(R)}(\nabla^0) \cap \ker \mathrm{d}^*$ is a small open ball of radius $\min(\eta, N(R)\varepsilon)$ in the (by the compactness of $S^4$) closed hence Hilbert subspace $\ker \mathrm{d}^* \subset L^2_1(S^4; \wedge^1 S^4 \otimes \mathfrak{su}(2))$; consequently, the size of this ball depends on the radius $R$ through the Uhlenbeck constant $N(R)$ in (8). Moreover, in this formal integral the hypothetical integration "measure" $\mathrm{D}_R a$ may in principle depend on the radius $R$ of $S^4$ too. Consequently, in spite of the conformal invariance of the integrand $\mathrm{e}^{-\frac{\mathrm{Im}\tau}{8\pi}\|\mathrm{d}a+a\wedge a\|^2_{L^2(S^4,g_R)}}$, the formal integral itself may fail to be conformally invariant (cf. Lemma 3.3). For notational simplicity, we shall hide both the numerical factor $0 < \frac{1}{\mathrm{Vol}(G_0)} < +\infty$ and the $R$-dependence and denote $\frac{\mathrm{D}_R a}{\mathrm{Vol}(G_0)}$ simply as $\mathrm{D}a$ from now on.

Let us work out a two-sided estimate for the action appearing in (10) but along a perhaps resized integration domain as follows.

**Lemma 3.1** *For every fixed finite value $0 < \mathrm{Im}\tau < +\infty$ of the imaginary part of complex coupling constant (6), there exists a sufficiently small but yet finite value of the vicinity parameter $\varepsilon$ such that (8) is applicable and there exist constants $0 < N, c < +\infty$ where*

$$N := \lim_{R\to 0}\Big(\inf\big\{N(R) \;\big|\; \|a\|_{L^2_1(S^4,g_R)} \leqq N(R)\|\mathrm{d}a + a\wedge a\|_{L^2(S^4,g_R)}\,, \\ a \in \mathscr{A}_{\varepsilon,N(R)}(\nabla^0) \cap \ker \mathrm{d}^*\big\}\Big)$$

*such that for every $a \in \mathscr{A}_{\varepsilon,N}(\nabla^0) \cap \ker \mathrm{d}^*$ in the correspondingly resized ball a two-sided estimate*

$$\frac{2\,\mathrm{Im}\tau}{\pi N^4}\|\mathrm{d}a\|^4_{L^2(S^4)} \leqq \|\mathrm{d}a + a\wedge a\|^2_{L^2(S^4)} \leqq 2\|\mathrm{d}a\|^2_{L^2(S^4)} + 2c^2\|\mathrm{d}a\|^4_{L^2(S^4)} \tag{11}$$

*holds in Coulomb gauge.*

*Note that all norms in this inequality are conformally invariant. Accordingly, both* $0 < N, c < +\infty$ *are conformally invariant and* $1 \leqq N$ *such that* $N \to 1$ *as* $\varepsilon \to 0$.

***Proof*** We begin with the estimate from below in (11) which, as often happens, is much more difficult than obtaining an estimate from above.

Assume that $\varepsilon$ is small enough hence (8) is applicable; it readily follows that working over the unit sphere $(S^4, g_1)$ we have $\|a\|_{L^2_1(S^4,g_1)} = \|a\|_{L^2(S^4,g_1)} + \|\mathrm{d}a\|_{L^2(S^4,g_1)} \leqq N(1)\|\mathrm{d}a + a \wedge a\|_{L^2(S^4,g_1)}$. Observe that in this inequality both $\|\mathrm{d}a\|_{L^2(S^4,g_1)}$ and $\|\mathrm{d}a + a \wedge a\|_{L^2(S^4,g_1)}$ are conformally invariant, thus we shall denote them, respectively, as $\|\mathrm{d}a\|_{L^2(S^4)}$ and $\|\mathrm{d}a + a \wedge a\|_{L^2(S^4)}$ from now on, while $\|a\|_{L^2(S^4,g_1)}$ is not. More precisely, if we pass to $(S^4, g_R)$, then the latter norm scales as $\|a\|_{L^2(S^4,g_R)} = R\|a\|_{L^2(S^4,g_1)}$. Consequently, defining $N$ by taking the limit $R \to 0$ as above we obtain an inequality

$$\|\mathrm{d}a\|_{L^2(S^4)} \leqq N\|\mathrm{d}a + a \wedge a\|_{L^2(S^4)} \tag{12}$$

over the appropriately resized ball $\mathscr{A}_{\varepsilon,N}(\nabla^0) \cap \ker \mathrm{d}^*$ having the following properties. This $N$ is optimal and universal in the sense that it is the smallest available constant (at least in the Uhlenbeck setting) and hence satisfies $N \leqq N(R)$ for any Uhlenbeck constant from (8) over $(S^4, g_R)$; moreover, $N$ is conformally invariant.

Taking Abelian 1-forms, i.e., $a \in \mathscr{A}_{\varepsilon,N}(\nabla^0) \cap \ker \mathrm{d}^*$ which satisfy $a \wedge a = 0$ a.e., then (12) shows that $\|\mathrm{d}a\|_{L^2(S^4)} \leqq N\|\mathrm{d}a\|_{L^2(S^4)}$; moreover, knowing that by the Coulomb gauge condition $\mathrm{d}a = 0$ if and only if $a = 0$ a.e. on the one hand $1 \leqq N$. In the generic non-Abelian case, $\|a \wedge a\|_{L^2(S^4)}$ is bounded by $\|a\|^2_{L^4(S^4)}$; but $\|a\|_{L^4(S^4)} \leqq c_1\|a\|_{L^2_1(S^4)}$ by the Sobolev embedding $L^2_1 \subset L^4$ which is sharp in 4 dimensions; moreover, elliptic regularity for $\mathrm{d} + \mathrm{d}^*$ gives $\|a\|_{L^2_1(S^4)} \leqq c_2\|(\mathrm{d} + \mathrm{d}^*)a\|_{L^2(S^4)} + c_3\|a\|_{L^2(S^4)} = c_2\|\mathrm{d}a\|_{L^2(S^4)}$ since $\mathrm{d}^*a = 0$ by the Coulomb gauge condition, and we can put $c_3 = 0$ because $H^1(S^4) = 0$; consequently, $\ker(\mathrm{d} + \mathrm{d}^*) = \ker \Delta_1 = \{0\}$. Combining these and introducing $c := (c_1c_2)^2 > 0$, we get

$$\begin{aligned}\|\mathrm{d}a\|_{L^2(S^4)} \leqq \|\mathrm{d}a + a \wedge a\|_{L^2(S^4)} + \|a \wedge a\|_{L^2(S^4)} \leqq \|\mathrm{d}a + a \wedge a\|_{L^2(S^4)} \\ + c\|\mathrm{d}a\|^2_{L^2(S^4)} \ .\end{aligned} \tag{13}$$

Regarding the constant $c$, note that it says $\|a\|_{L^4(S^4)} \leqq \sqrt{c}\ \|\mathrm{d}a\|_{L^2(S^4)}$ and both norms here are conformally invariant; hence, we can assume that $c$ is conformally invariant as well. Proceeding further, by the aid of (8) take any $N(R) \geqq 1$ satisfying $\|\mathrm{d}a\|_{L^2(S^4)} \leqq \|a\|_{L^2_1(S^4,g_R)} \leqq N(R)\|\mathrm{d}a + a \wedge a\|_{L^2(S^4)}$ over $(S^4, g_R)$. Adding the two estimates for $\|\mathrm{d}a\|_{L^2(S^4)}$ provided by (13) and this last inequality, we obtain $\|\mathrm{d}a\|_{L^2(S^4)}(2 - c\|\mathrm{d}a\|_{L^2(S^4)}) \leqq (N(R) + 1)\|\mathrm{d}a + a \wedge a\|_{L^2(S^4)}$. Moreover, we have $\|\mathrm{d}a\|_{L^2(S^4)} < N(R)\varepsilon$ in (9) thus $\|\mathrm{d}a\|_{L^2(S^4)}\,(2 - cN(R)\varepsilon) \leqq (N(R) + 1)\|\mathrm{d}a + a \wedge a\|_{L^2(S^4)}$. Provided $\varepsilon$ is small enough compared with the initial value of $N(R)$, more precisely if $\varepsilon < \frac{2}{cN(R)}$, then we can replace $N(R)$ with $\frac{N(R)+1}{2-cN(R)\varepsilon}$ and iterate this process; the general theory of iteration guarantees that $N(R)$ will converge to the

lower fixed point $N_*(R) = \frac{1}{2c\varepsilon}\big(1 - \sqrt{1-4c\varepsilon}\,\big)$ of this iteration. Since $N$ from (12) is the optimal constant, we have on the other hand $N \leqq N_*(R)$; consequently,

$$1 \leqq N \leqq N_*(R) = 1 + \frac{1}{4}c\varepsilon + \ldots$$

demonstrating that $N \to 1$ as $\varepsilon \to 0$. Assume that $0 < \varepsilon < \sqrt{\frac{\pi}{2\,\mathrm{Im}\tau}}$, then $\|\mathrm{d}a\|_{L^2(S^4)} < N\varepsilon < N\sqrt{\frac{\pi}{2\,\mathrm{Im}\tau}}$ within the ball $\mathscr{A}_{\varepsilon,N}(\nabla^0) \cap \ker \mathrm{d}^*$; consequently, multiplying the inequality (12) by $\|\mathrm{d}a\|_{L^2(S^4)}$ we get

$$\|\mathrm{d}a\|^2_{L^2(S^4)} \leqq N\|\mathrm{d}a + a\wedge a\|_{L^2(S^4)}\|\mathrm{d}a\|_{L^2(S^4)} \leqq \sqrt{\frac{\pi}{2\,\mathrm{Im}\tau}}\, N^2\|\mathrm{d}a + a\wedge a\|_{L^2(S^4)};$$

hence, squaring it we come up with the estimate from below in (11).

The estimate from above is simpler. We start with $\|\mathrm{d}a + a\wedge a\|^2_{L^2(S^4)} \leqq 2\|\mathrm{d}a\|^2_{L^2(S^4)} + 2\|a\wedge a\|^2_{L^2(S^4)}$ and then repeat the steps toward (13) to end up with

$$\|\mathrm{d}a + a\wedge a\|^2_{L^2(S^4)} \leqq 2\|\mathrm{d}a\|^2_{L^2(S^4)} + 2c^2\|\mathrm{d}a\|^4_{L^2(S^4)}$$

where $c$ is the conformally invariant constant used so far. Letting, for instance,

$$\varepsilon := \frac{1}{2}\min\left(\sqrt{8}\pi\ ,\ \text{the original Uhlenbeck condition in (8.)}\ ,\ \frac{2}{cN(R)}\ ,\ \sqrt{\frac{\pi}{2\,\mathrm{Im}\tau}}\ \right)$$

and then putting together the last two estimates, we obtain the desired two-sided inequality. ◊

Let us proceed further by multiplying each term in (11) with $-\frac{\mathrm{Im}\tau}{8\pi} < 0$ and then exponentiating:

$$\mathrm{e}^{-\left(\frac{\mathrm{Im}\tau}{2\pi N^2}\right)^2\|\mathrm{d}a\|^4_{L^2(S^4)}} \geqq \mathrm{e}^{-\frac{\mathrm{Im}\tau}{8\pi}\|\mathrm{d}a+a\wedge a\|^2_{L^2(S^4)}} \geqq \mathrm{e}^{-\frac{\mathrm{Im}\tau}{4\pi}\|\mathrm{d}a\|^2_{L^2(S^4)} - \frac{\mathrm{Im}\tau\, c^2}{4\pi}\|\mathrm{d}a\|^4_{L^2(S^4)}}$$

or equivalently, using $\mathrm{d}^*a = 0$ again

$$\begin{aligned}\mathrm{e}^{-\left(a\,,\,\frac{\mathrm{Im}\tau}{2\pi N^2}\Delta_1 a\right)^2_{L^2(S^4)}} &\geqq \mathrm{e}^{-\frac{\mathrm{Im}\tau}{8\pi}\|\mathrm{d}a+a\wedge a\|^2_{L^2(S^4)}}\\ &\geqq \mathrm{e}^{-\left(a\,,\,\sqrt{\frac{\mathrm{Im}\tau}{4\pi}}c\,\Delta_1 a\right)^2_{L^2(S^4)} - \left(a\,,\,\frac{\mathrm{Im}\tau}{4\pi}\Delta_1 a\right)_{L^2(S^4)}}\ .\end{aligned} \tag{14}$$

Having obtained these rigorous estimates, consider the vicinity of the vacuum in Coulomb gauge, i.e., the small ball about the flat connection $\mathscr{A}_{\varepsilon,N}(\nabla^0) \cap \ker \mathrm{d}^*$ as in (9), however, such that $N(R)$ in its radius has been replaced with the universal $N$ from (11). Take the Laplacian $\Delta_1$ and the corresponding $C_\delta \subset L^2_1(S^4; \wedge^1 S^4)$ introduced as its finite-dimensional analogue (1). If $0 < \lambda_{\min} < +\infty$ is the smallest

eigenvalue of $\Delta_1$, then picking any $0 < \delta < \sqrt{\lambda_{\min}}\ \min(\eta, N\varepsilon)$ we know by (9) that $\mathscr{A}_{\varepsilon,N}(\nabla^0) \supset C_\delta$ yielding two inclusions $\ker \mathrm{d}^* \supset \mathscr{A}_{\varepsilon,N}(\nabla^0) \cap \ker \mathrm{d}^* \supset C_\delta \cap \ker \mathrm{d}^*$ for these subsets in $L^2_1(S^4; \wedge^1 S^4)$. Now let us formally integrate the left term of (14) over $\ker \mathrm{d}^*$, the middle term of (14) over $\mathscr{A}_{\varepsilon,N}(\nabla^0) \cap \ker \mathrm{d}^*$, and finally, the right term of (14) over $C_\delta \cap \ker \mathrm{d}^*$. Referring at this step to our **Monotonicity principles**, this procedure obeys the ordering in (14); thus, formally

$$\int\limits_{\ker \mathrm{d}^*} \mathrm{e}^{-\left(a, \frac{\mathrm{Im}\tau}{2\pi N^2}\Delta_1 a\right)^2_{L^2(S^4)}} \mathrm{D}a \geqq \int\limits_{\mathscr{A}_{\varepsilon,N}(\nabla^0)\cap\ker \mathrm{d}^*} \mathrm{e}^{-\frac{\mathrm{Im}\tau}{8\pi}\|\mathrm{d}a + a\wedge a\|^2_{L^2(S^4)}} \mathrm{D}a$$
$$\geqq \int\limits_{C_\delta\cap\ker \mathrm{d}^*} \mathrm{e}^{-\left(a, \sqrt{\frac{\mathrm{Im}\tau}{4\pi}} c\,\Delta_1 a\right)^2_{L^2(S^4)} - \left(a,\ \frac{\mathrm{Im}\tau}{4\pi}\Delta_1 a\right)_{L^2(S^4)}} \mathrm{D}a$$

continues to hold. The time has come to apply our formal integral expressions from Sect. 2.

**Definition 3.1** (cf. [8, Definition 3.1]) Taking into account that $H^1(S^4) = \{0\}$ and $\dim_{\mathbb{R}} \mathfrak{su}(2) = 3$ substituting $c := \frac{\mathrm{Im}\tau}{2\pi N^2}$ in (4), we define a non-truncated quartic integral as

$$\int\limits_{\ker \mathrm{d}^*} \mathrm{e}^{-\left(a\ ,\ \frac{\mathrm{Im}\tau}{2\pi N^2}\Delta_1 a\right)^2_{L^2(S^4)}} \mathrm{D}a := \left(\frac{(\sqrt{-1}\pi)^{\frac{1}{2}\mathrm{rk}'\left(\frac{\mathrm{Im}\tau}{2\pi N^2}\Delta_1|_{\ker \mathrm{d}^*}\right)}}{\sqrt{\det{}'\left(\frac{\mathrm{Im}\tau}{2\pi N^2}\Delta_1|_{\ker \mathrm{d}^*}\right)}}\right)^3 \times$$
$$\frac{1}{2\sqrt{\pi}} \int\limits_{\gamma_s} t^{-\frac{3}{2}\mathrm{rk}'\left(\frac{\mathrm{Im}\tau}{2\pi N^2}\Delta_1|_{\ker \mathrm{d}^*}\right)} \mathrm{e}^{-\frac{t^2}{4}} \mathrm{d}t\ .$$

Likewise, substituting $c_2 := \frac{\mathrm{Im}\tau}{4\pi}$ and $c_1 := \sqrt{\frac{\mathrm{Im}\tau}{4\pi}}\, c$ in (5) we define a truncated quartic integral

$$\int\limits_{C_\delta\cap\ker \mathrm{d}^*} \mathrm{e}^{-\left(a\ ,\ \sqrt{\frac{\mathrm{Im}\tau}{4\pi}} c\,\Delta_1 a\right)^2_{L^2(S^4)} - \left(a\ ,\ \frac{\mathrm{Im}\tau}{4\pi}\Delta_1 a\right)_{L^2(S^4)}} \mathrm{D}a := \left(\frac{(\sqrt{-1}K(\delta))^{\frac{1}{2}\mathrm{rk}'\left(\frac{\mathrm{Im}\tau}{4\pi}\Delta_1|_{\ker \mathrm{d}^*}\right)}}{\sqrt{\det{}'\left(\frac{\mathrm{Im}\tau}{4\pi}\Delta_1|_{\ker \mathrm{d}^*}\right)}}\right)^3 \times$$
$$\frac{1}{2\sqrt{\pi}} \int\limits_{\gamma_s} \left(t + \sqrt{-1}\, c\sqrt{\frac{4\pi}{\mathrm{Im}\tau}}\right)^{-\frac{3}{2}\mathrm{rk}'\left(\frac{\mathrm{Im}\tau}{4\pi}\Delta_1|_{\ker \mathrm{d}^*}\right)} \mathrm{e}^{-\frac{t^2}{4}} \mathrm{d}t$$

where the common contour $\gamma_s$ is to be specified such that to meet all demands from avoiding possible poles and branch cuttings in both integrals.

A familiar way to make sense of $\mathrm{rk}'$ and $\det'$ in Definition 3.1, i.e., to regularize the dimension and the functional determinant in infinite dimensions, is an application of $\zeta$-function regularization.

**Lemma 3.2** (cf. [8, Lemma 3.1]) *Using $\zeta$-function regularization to define* rk′ *and* det′ *and then heat kernel techniques to calculate the zero values of various resulting $\zeta$-functions over $(S^4, g_R)$, we obtain from its definition above that the non-truncated quartic integral looks like*

$$\int\limits_{\ker \mathrm{d}^*} \mathrm{e}^{-\left(a\,,\,\frac{\mathrm{Im}\tau}{2\pi N^2}\Delta_1 a\right)^2_{L^2(S^4)}}\,\mathrm{D}a = \left(\frac{2\sqrt{-1}\,\pi^2 N^2}{\mathrm{Im}\tau}\right)^{\frac{11}{20}} \mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0)-3\zeta'_{\Delta_0}(0)}\,\times$$
$$\frac{1}{2\sqrt{\pi}}\int\limits_{-\infty}^{+\infty} t^{-\frac{11}{20}}\mathrm{e}^{-\frac{t^2}{4}}\,\mathrm{d}t$$

*over $(S^4, g_R)$. Likewise,*

$$\int\limits_{C_\delta\cap\ker \mathrm{d}^*} \mathrm{e}^{-\left(a\,,\,\sqrt{\frac{\mathrm{Im}\tau}{4\pi}}c\,\Delta_1 a\right)^2_{L^2(S^4)}-\left(a\,,\,\frac{\mathrm{Im}\tau}{4\pi}\Delta_1 a\right)_{L^2(S^4)}}\mathrm{D}a$$
$$= \left(\frac{4\sqrt{-1}\pi K(\delta)}{\mathrm{Im}\tau}\right)^{\frac{11}{20}} \mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0)-3\zeta'_{\Delta_0}(0)}\,\times$$
$$\frac{1}{2\sqrt{\pi}}\int\limits_{-\infty}^{+\infty}\left(t+\sqrt{-1}\,c\sqrt{\frac{4\pi}{\mathrm{Im}\tau}}\right)^{-\frac{11}{20}}\mathrm{e}^{-\frac{t^2}{4}}\,\mathrm{d}t$$

*is the shape of the truncated quartic integral over $(S^4, g_R)$.*

*Taking into account that the exponent in the complex integrals satisfies $-1 < -\frac{11}{20} < 0$, we know that there are no poles and there is a single branch cutting connecting 0 with $-\sqrt{-1}\,\infty$ along the non-positive imaginary axis in the first integral while connecting $-\sqrt{-1}\,c\sqrt{\frac{4\pi}{\mathrm{Im}\tau}}$ with $-\sqrt{-1}\,\infty$ along the negative imaginary axis in the second integral. Therefore, we can simply put $\gamma_s := \mathbb{R}$ in both integrals.*

***Remark*** Before embarking upon the proof, we note that the particular value $-1 < -\frac{11}{20} < 0$ of the exponents in these integral expressions is not important because it is just the consequence of one of the possible (namely $\zeta$-function combined with heat kernel) regularization procedures carried over one of the possible (namely $(S^4, g_R)$, i.e., the one-point conformal) compactifications of $(\mathbb{R}^4, \eta)$. Only its sign, namely that it is negative, bears relevance. Indeed, this exponent does not have to always assume a negative value because of some *a priori* reason. For example, in the case of the flat torus $T^4$ the corresponding exponent turns out to be $0 < \frac{9}{2} < 5$ leading to a completely different situation, e.g., the **Monotonicity principles** break down due to the opposite scaling of the integrals. These oddities are related with lacking a good measure in infinite dimensions, see Appendix.

***Proof*** Since the spectrum of the Laplacian over a compact Riemannian manifold $(M, g)$ is non-negative real and discrete, one sets

$$\zeta_{\Delta_k}(s) := \sum_{\lambda\in\mathrm{Spec}\,\Delta_k\setminus\{0\}} \lambda^{-s}, \quad \text{with } s\in\mathbb{C} \text{ and } \mathrm{Re}\, s > 0 \text{ sufficiently large}$$

and observes that this function can be meromorphically continued over the whole complex plane (cf., e.g., [19, Theorem 5.2]) having no pole at $s = 0 \in \mathbb{C}$. A formal calculation then convinces us that the regularized rank and the determinant of the Laplacian should be $\mathrm{rk}'\,\Delta_k := \zeta_{\Delta_k}(0)$ and $\det'\Delta_k := \mathrm{e}^{-\zeta'_{\Delta_k}(0)}$ yielding $\mathrm{rk}'(c\Delta_k) = \zeta_{\Delta_k}(0)$ and $\det'(c\Delta_k) = c^{\zeta_{\Delta_k}(0)}\mathrm{e}^{-\zeta'_{\Delta_k}(0)}$.

Because of the Coulomb gauge condition, we have to calculate restrictions of these $\zeta$-functions over the round 4-sphere $(S^4, g_R)$. Since $H^1(S^4) = \{0\}$, hence $\Delta_1$ has trivial kernel; the Hodge decomposition theorem says that $L^2(S^4;\wedge^1 S^4) \cong \mathrm{im}\,\mathrm{d}_0 \oplus \mathrm{im}\,\mathrm{d}_2^*$. Moreover, $\mathrm{im}\,\mathrm{d}_0 \cap \ker\mathrm{d}_1^* = \{0\}$ and $\mathrm{im}\,\mathrm{d}_2^* \subsetneqq \ker\mathrm{d}_1^*$; hence,

$$L^2(S^4;\wedge^1 S^4) \cong \mathrm{im}\,\mathrm{d}_0 \oplus \ker\mathrm{d}_1^* \ .$$

Applying this decomposition, we can write any element $a \in L^2(S^4;\wedge^1 S^4)$ uniquely as $a = \mathrm{d}_0 f + \alpha$ with $f \in L^2_1(S^4;\wedge^0 S^4)$ a function and $\alpha \in L^2(S^4;\wedge^1 S^4)$ satisfying $\mathrm{d}_1^*\alpha = 0$. A simple calculation ensures us that

$$\begin{aligned}(a\ ,\ \Delta_1 a)_{L^2(S^4)} &= (\mathrm{d}_0 f + \alpha\ ,\ \Delta_1(\mathrm{d}_0 f + \alpha))_{L^2(S^4)}\\ &= \left(f\ ,\ \Delta_0^2 f\right)_{L^2(S^4)} + (\alpha\ ,\ \Delta_1\alpha)_{L^2(S^4)}\end{aligned}$$

where $\Delta_0^2$ is the square of the scalar Laplacian on $(S^4, g_R)$. Taking into account these decompositions, then we obtain that $\mathrm{Spec}\,\Delta_1 = \mathrm{Spec}\,\Delta_0^2 \sqcup \mathrm{Spec}\,\Delta_1|_{\ker\mathrm{d}_1^*}$. This decomposition together with the proof of [19, Theorem 5.2] ensures us that $\zeta_{\Delta_1} = \zeta_{\Delta_0^2} + \zeta_{\Delta_1|\ker\mathrm{d}_1^*}$ consequently $\zeta_{\Delta_1|\ker\mathrm{d}^*} = \zeta_{\Delta_1} - \zeta_{\Delta_0^2}$. Moreover, $\zeta_{\Delta_0^2}(s) = \zeta_{\Delta_0}(2s)$; hence, $\mathrm{rk}'(c\Delta_0^2) = \zeta_{\Delta_0}(0)$ and $\det'(c\Delta_0^2) = c^{\zeta_{\Delta_0}(0)}\mathrm{e}^{-2\zeta'_{\Delta_0}(0)}$. Therefore, in the case of the first integral of Definition 3.1 putting $c := \frac{\mathrm{Im}\,\tau}{2\pi N^2}$ we find

$$\begin{cases}\mathrm{rk}'\left(\frac{\mathrm{Im}\,\tau}{2\pi N^2}\Delta_1|_{\ker\mathrm{d}^*}\right) &= \zeta_{\Delta_1}(0) - \zeta_{\Delta_0}(0)\\ \det'\left(\frac{\mathrm{Im}\,\tau}{2\pi N^2}\Delta_1|_{\ker\mathrm{d}^*}\right)) &= \left(\frac{\mathrm{Im}\,\tau}{2\pi N^2}\right)^{\zeta_{\Delta_1}(0)-\zeta_{\Delta_0}(0)}\mathrm{e}^{-\zeta'_{\Delta_1}(0)+2\zeta'_{\Delta_0}(0)}\end{cases} .$$

We can easily calculate at least $\zeta_{\Delta_1}(0) - \zeta_{\Delta_0}(0)$ explicitly applying standard heat kernel techniques. Over a compact 4-manifold $(M, g)$ without boundary, it is well known [19, Theorem 5.2] that

$$\zeta_{\Delta_k}(0) = -\dim_{\mathbb{R}}\ker\Delta_k + \frac{1}{16\pi^2}\int\limits_M \mathrm{tr}(u_k^4)\mathrm{d}V_g$$

where the sections $u_k^p \in C^\infty(M; \mathrm{End}(\wedge^k M))$ with $p = 0, 1, \ldots$ appear [19, Chapter 3] in the coefficients of the short time asymptotic expansion of the heat kernel for the $k$-Laplacian

$$\sum_{\lambda \in \mathrm{Spec}\,\Delta_k \setminus \{0\}} \mathrm{e}^{-\lambda t} \sim \frac{1}{(4\pi t)^2} \sum_{p=0}^{+\infty} \left( \int_M \mathrm{tr}(u_k^p) \mathrm{d}V_g \right) t^{\frac{p}{2}} \quad \text{as } t \to 0 \ .$$

These functions are expressible with the curvature of $(M, g)$ and one can demonstrate [10, p. 340] that

$$\begin{cases} u_0^4 & = \frac{1}{360} \left( 2|\mathrm{Riem}|_g^2 - 2|\mathrm{Ric}|_g^2 + 5\,\mathrm{Scal}^2 \right) \\ \mathrm{tr}(u_1^4) & = \frac{1}{360} \left( -22|\mathrm{Riem}|_g^2 + 172|\mathrm{Ric}|_g^2 - 40\,\mathrm{Scal}^2 \right) \end{cases}$$

yielding together with $\dim_{\mathbb{R}} \ker \Delta_0 = 1$ and $\dim_{\mathbb{R}} \ker \Delta_1 = 0$ over $(S^4, g_R)$ that

$$\zeta_{\Delta_1}(0) - \zeta_{\Delta_0}(0) = 1 - \frac{1}{\pi^2} \int_{S^4} \left( \frac{1}{120}|\mathrm{Riem}|_g^2 - \frac{87}{2880}|\mathrm{Ric}|_g^2 + \frac{1}{128}\mathrm{Scal}^2 \right) \mathrm{d}V_R \ .$$

In addition, we recall over $(S^4, g_R)$ the classical expressions

$$\begin{cases} |\mathrm{Riem}|_{g_R}^2 & = 2|\mathrm{Ric}|_{g_R}^2 - \frac{1}{3}\,\mathrm{Scal}^2 \\ |\mathrm{Ric}|_{g_R}^2 & = \frac{1}{4}\mathrm{Scal}^2 \\ \mathrm{Scal} & = \frac{12}{R^2} \end{cases}$$

and plug them into the integral and also perform $\int_{S^4} \mathrm{d}V_R = \frac{8\pi^2}{3} R^4$. We come up with

$$\frac{3}{2}\left(\zeta_{\Delta_1}(0) - \zeta_{\Delta_0}(0)\right) = \frac{3}{2}\left(1 - \frac{1}{\pi^2}\left(\frac{1}{120}\left(\frac{1}{2} - \frac{1}{3}\right) - \frac{87}{2880} \cdot \frac{1}{4} + \frac{1}{128}\right) \frac{144}{R^4} \cdot \frac{8\pi^2}{3} R^4\right) = \frac{11}{20}$$

and find in particular that $\frac{3}{2}\left(\zeta_{\Delta_1}(0) - \zeta_{\Delta_0}(0)\right)$ is independent of $R$ offering a sort of justification for using the conformal compactification $(S^4, g_R)$ in place of the original space $(\mathbb{R}^4, \eta)$. Inserting all of these formulata into the right hand side of the first integral of Definition 3.1, we obtain the first expression of the lemma.[3] Repeating the same with the truncated quartic integral, the corresponding result also follows.

[3] The Faddeev–Popov determinant is therefore formally equal to $\mathrm{e}^{3\zeta'_{\Delta_0}(0)}$ and hence is indeed constant in this picture.

The only remaining thing is to specify the common contour in the two complex integrals. Since $-1 < -\frac{11}{20} < 0$, there are no poles; however, branch cuttings required in these complex integrals as described; hence, for simplicity $\gamma_s$ can be taken to be the real line everywhere. $\Diamond$

By Lemma 3.2 and (10), we eventually arrive at the two-sided estimate

$$\left(\frac{2\sqrt{-1}\pi^2N^2}{\mathrm{Im}\tau}\right)^{\frac{11}{20}} \mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0)-3\zeta'_{\Delta_0}(0)} \frac{1}{2\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} t^{-\frac{11}{20}} \mathrm{e}^{-\frac{t^2}{4}} \mathrm{d}t$$

$$\geqq Z_\varepsilon(\mathbb{R}^4,\tau) \geqq \tag{15}$$

$$\left(\frac{4\sqrt{-1}\,\pi K(\delta)}{\mathrm{Im}\tau}\right)^{\frac{11}{20}} \mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0)-3\zeta'_{\Delta_0}(0)} \frac{1}{2\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} \left(t+\sqrt{-1}c\sqrt{\frac{4\pi}{\mathrm{Im}\tau}}\right)^{-\frac{11}{20}} \mathrm{e}^{-\frac{t^2}{4}}\,\mathrm{d}t\ .$$

**Lemma 3.3** *There exist constants $0 < t_0, \delta_0, R_\infty < +\infty$ with the following property. For any choice of the complex coupling constant (6) satisfying $t_0 < \mathrm{Im}\tau$ (with induced vicinity parameter $0 < \varepsilon < +\infty$ as in Lemma 3.1 such that (15) holds), the left- and right-hand sides of (15) get equal with some $\delta_0 < \delta(\tau)$ and with every $0 < R \leqq R(\tau)$, where $R(\tau) < R_\infty$. This yields that*

$$Z_\varepsilon(\mathbb{R}^4,\tau) = \left(\frac{\mathrm{Im}\tau}{2\pi^2N^2}\right)^{-\frac{11}{20}} \frac{2^{-\frac{11}{20}}}{\sqrt{\pi}} \cos\left(\tfrac{11\pi}{40}\right) \Gamma\left(\tfrac{9}{40}\right) \mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0)-3\zeta'_{\Delta_0}(0)}$$

*where $\Gamma$ is Euler's Gamma function.*

*Moreover, the partition function $Z_\varepsilon(\mathbb{R}^4,\tau)$ as calculated here depends on $R$, the radius of the conformal compactification $(S^4, g_R)$ of the original Euclidean space $(\mathbb{R}^4,\eta)$, only through its determinant term $\mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0)-3\zeta'_{\Delta_0}(0)}$. More precisely, if for a given $\tau \in \mathbb{C}^+$ two permitted conformal one-point compactifications $(S^4, g_{R_i})$ are taken, i.e., $0 < R_1 < R_2 \leqq R(\tau) < R_\infty$, then the corresponding partition functions are related by $Z^1_\varepsilon(\mathbb{R}^4,\tau) = \left(\frac{R_1}{R_2}\right)^{\frac{11}{10}} Z^2_\varepsilon(\mathbb{R}^4,\tau)$.*

***Proof*** It is clear that the scissor (15) around the partition function closes up if the equation

$$\left(\frac{\pi N^2}{2K(\delta)}\right)^{\frac{11}{20}} = \frac{\int\limits_{-\infty}^{+\infty} \left(t+\sqrt{-1}c\sqrt{\frac{4\pi}{\mathrm{Im}\tau}}\right)^{-\frac{11}{20}} \mathrm{e}^{-\frac{t^2}{4}}\,\mathrm{d}t}{\int\limits_{-\infty}^{+\infty} t^{-\frac{11}{20}} \mathrm{e}^{-\frac{t^2}{4}}\,\mathrm{d}t} \tag{16}$$

can be solved for some $\delta$ without breaking the inclusion $C_\delta \subset \mathscr{A}_{\varepsilon,N}(\nabla^0)$ over some $(S^4, g_R)$. The right hand side of (16) monotonically grows from 0 to 1 as $0 \leqq \mathrm{Im}\tau \leqq +\infty$. Likewise via $0 \leqq K(\delta) \leqq \pi$ the left-hand side of (16) monotonically decays

from $+\infty$ to $\left(\frac{N^2}{2}\right)^{\frac{11}{20}}$ as $0 \leqq \delta \leqq +\infty$. Assume now that $N < \sqrt{2}$ hence $\left(\frac{N^2}{2}\right)^{\frac{11}{20}} < 1$. These together imply that we can find a constant $0 < t_0 < +\infty$ such that the right hand side of (16), when evaluated at $\mathrm{Im}\tau = t_0$, is equal to $\left(\frac{N^2}{2}\right)^{\frac{11}{20}}$. Likewise we can find another constant $0 < \delta_0 < +\infty$ such that the left hand side of (16), when evaluated at the constant $\delta_0$, is equal to 1. It then readily follows that for every $\tau \in \mathbb{C}^+$ satisfying $t_0 < \mathrm{Im}\tau$ there exists $\delta_0 < \delta(\tau)$ such that (16) can be solved. Note that as $\mathrm{Im}\tau \to +\infty$, then $\delta_0 \leftarrow \delta(\tau)$; however, as $t_0 \leftarrow \mathrm{Im}\tau$, then $\delta(\tau) \to +\infty$. Proceeding further, by shrinking $R$, i.e., conformally rescaling $(S^4, g_R)$ with a constant if necessary, we can scale up $\lambda_{\min}$, the smallest eigenvalue of $\Delta_1$, to be arbitrary large without affecting the other conformally invariant parameters $\varepsilon, N, c$ of the theory. Thus, for any permitted choice of $\tau \in \mathbb{C}^+$ there exists a radius $R(\tau)$ such that working over any $(S^4, g_R)$ obeying $0 < R \leqq R(\tau)$ we can take $\delta(\tau)$ without breaking $0 < \delta(\tau) < \sqrt{\lambda_{\min}}\ \min(\eta, N\varepsilon)$, i.e., the inclusion $C_{\delta(\tau)} \subset \mathscr{A}_{\varepsilon,N}(\nabla^0)$ which has been used in (15). Again note that as $\mathrm{Im}\tau \to +\infty$ then $R(\tau) \to R_\infty := \sup\left\{R \mid C_{\delta_0} \subset \mathscr{A}_{\varepsilon,N}(\nabla^0) \text{ is valid}\right\} < +\infty$, but as $t_0 \leftarrow \mathrm{Im}\tau$, then $0 \leftarrow R(\tau)$. Summarizing, we can consistently solve (16) whenever $N < \sqrt{2}$. However, this latter condition—*which is therefore the only but crucial condition*[4] *for our whole method to work here*—is already satisfied for small $\varepsilon$'s because Lemma 3.1 makes sure that $N \to 1$ as $\varepsilon \to 0$.

Therefore, (15) in fact provides us with an equality

$$Z_\varepsilon(\mathbb{R}^4, \tau) = \left(\frac{2\sqrt{-1}\pi^2 N^2}{\mathrm{Im}\tau}\right)^{\frac{11}{20}} \mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0) - 3\zeta'_{\Delta_0}(0)} \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} t^{-\frac{11}{20}} \mathrm{e}^{-\frac{t^2}{4}} \mathrm{d}t$$

and our last task is to evaluate the complex integral here. We can do this by executing a counterclockwise rotation of the negative part of the integration contour $\mathbb{R} \subset \mathbb{C}$ (together with the branch cutting along the negative imaginary axis) about the origin toward its positive part; this shows that

$$\sqrt{-1}^{\frac{11}{20}} \int_{-\infty}^{+\infty} t^{-\frac{11}{20}} \mathrm{e}^{-\frac{t^2}{4}} \mathrm{d}t = \sqrt{-1}^{\frac{11}{20}} \left(1 + (-1)^{-\frac{11}{20}}\right) \int_0^{+\infty} t^{-\frac{11}{20}} \mathrm{e}^{-\frac{t^2}{4}} \mathrm{d}t$$

with a real integral on the right. Firstly, $\sqrt{-1}^{\frac{11}{20}}\left(1 + \left(\frac{1}{(\sqrt{-1})^2}\right)^{\frac{11}{20}}\right) = \sqrt{-1}^{\frac{11}{20}} + \left(\frac{1}{\sqrt{-1}}\right)^{\frac{11}{20}} = 2\cos\left(\frac{11\pi}{40}\right)$. Secondly, the substitution $u := \frac{t^2}{4}$ yields $\int_0^{+\infty} t^{-\frac{11}{20}} \mathrm{e}^{-\frac{t^2}{4}} \mathrm{d}t = 2^{-\frac{11}{20}} \Gamma(\frac{9}{40})$ hence the result.

Concerning the role of the compactification radius, recall that $0 \leftarrow R(\tau)$ as $t_0 \leftarrow \mathrm{Im}\tau$; consequently, there exists no overall finite choice for $R$ which could work for every permitted value of $\tau$ thus the $R$ dependence of $Z_\varepsilon(\mathbb{R}^4, \tau)$, as has been computed here, is unavoidable. Nevertheless, since $N$ is conformally invariant,

[4] Honestly speaking, we also assume the validity of the **Monotonicity principles** as formulated above. However, the (in)validity of these assumptions is rather related with the more general problem of the existence of a satisfactory measure theory in infinite dimensions, cf. Appendix.

$Z_\varepsilon(\mathbb{R}^4,\tau)$ as it stands can depend on $R$ only through the functional determinant. If $(S^4, g_{R_i})$ are two conformal one-point compactifications of $(\mathbb{R}^4,\eta)$, then obviously $g_{R_1}=\left(\frac{R_1}{R_2}\right)^2 g_{R_2}$ which can be regarded as a homothety applied on $g_{R_1}$. Therefore, the eigenvalues of $\Delta_1$ under this re-sizing simply change as $\lambda_k\mapsto\left(\frac{R_2}{R_1}\right)^2\lambda_k$, i.e., coincide with that of the scaled Laplacian $\left(\frac{R_2}{R_1}\right)^2\Delta_1$; hence, $\zeta_{\Delta_1}\mapsto\zeta_{\left(\frac{R_2}{R_1}\right)^2\Delta_1}$. Consequently, $\mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0)-3\zeta'_{\Delta_0}(0)}\mapsto\left(\frac{R_2}{R_1}\right)^{2\left(-\frac{3}{2}\left(\zeta_{\Delta_1}(0)-\zeta_{\Delta_0}(0)\right)\right)}\mathrm{e}^{\frac{3}{2}\zeta'_{\Delta_1}(0)-3\zeta'_{\Delta_0}(0)}$, but we already know that $\frac{3}{2}\left(\zeta_{\Delta_1}(0)-\zeta_{\Delta_0}(0)\right)=\frac{11}{20}$; hence, the asserted scaling of $Z_\varepsilon(\mathbb{R}^4,\tau)$ follows. $\Diamond$

***Proof of Theorem 1.1*** Putting together the contents of Lemmata 3.1, 3.2, and 3.3, the result follows. $\Diamond$

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## Appendix A: There is no good measure in infinite dimensions

For completeness, we recall the following simple but important general fact about measures in infinite dimensions. Perhaps, this no-go result demonstrates in the sharpest way the existence of a deep chasm between finite- and infinite-dimensional integration. We also refer to the excellent survey book [12] to gain a broader picture.

Let $(X,\mu)$ be any measure space. As a very basic demand in measure theory the measure $\mu$ is always assumed to be *$\sigma$-additive*, i.e., $\mu(\sqcup_i A_i)=\sum_i\mu(A_i)$ to hold for all countable collection of pairwise disjoint measurable subsets $A_1, A_2, \ldots\subset X$. If $X$ admits further structures, further natural assumptions can be imposed on a measure. If $X$ can be given the structure of a Banach space, for instance, then mimicking the properties of the Lebesgue measure in finite dimensions, one can further demand $\mu$ to be (i) *positive*, i.e., $0\leqq\mu(U)\leqq+\infty$ for every open subset $\emptyset\subsetneqq U\subsetneqq X$; (ii) *locally finite*, i.e., every point $x\in X$ has an open neighborhood $N_x\subsetneqq X$ such that $-\infty<\mu(N_x)<+\infty$; (iii) and finally *translation invariant*, that is, for every

measurable subset $\emptyset \subsetneqq A \subsetneqq X$ and every vector $x \in X$, the translated set $x + A$ is measurable and $\mu(x + A) = \mu(A)$ holds.

However, as it is well known, these natural demands conflict each other in infinite dimensions:

**Theorem 3.1** (cf., e.g., [9, Theorem 4, p. 359], or [12, Theorem 3.1.5]) *Let* $(X, \|\cdot\|)$ *be an infinite-dimensional, separable Banach space. Then, the only locally finite and translation invariant Borel measure* $\mu$ *on* $X$ *is the trivial measure, with* $\mu(A) = 0$ *for every measurable subset* $A$*. Equivalently, every translation invariant measure that is not identically zero assigns infinite measure to all open subsets of* $X$*.*

***Proof*** Take a locally finite, translation invariant measure $\mu$ on an infinite-dimensional, separable Banach space $(X, \|\cdot\|)$. Using local finiteness, suppose that, for some $\varepsilon > 0$, the open ball $B_\varepsilon(0) \subset X$ of radius $\varepsilon$ and centered at the origin has a finite $\mu$-measure. Since $X$ is infinite-dimensional, there is a countable infinite sequence of pairwise disjoint open balls $B_{\frac{\varepsilon}{4}}(x_i)$ of radius for instance $\frac{\varepsilon}{4}$ and centers $x_i \in X$, with all the smaller balls $B_{\frac{\varepsilon}{4}}(x_i)$ with $i = 1, 2, \ldots$ contained within the larger ball $B_\varepsilon(0)$. By translation invariance, all of the smaller balls have the same measure; since by $\sigma$-additivity, the absolute value of the sum of these measures is estimated from above by $\mu(B_\varepsilon(0)) < +\infty$ and hence is finite, the smaller balls must all have $\mu$-measure zero. Now, since $X$ is separable, it can be covered by a countable collection of balls of radius $\frac{\varepsilon}{4}$; since each such ball has $\mu$-measure zero, by $\sigma$-additivity again so must the whole space $X$. Therefore, $\mu$ is the trivial measure. $\Diamond$

This means that our *ad hoc* "measure" D$a$ used for integration in a Hilbert space throughout Sects. 3 and 1 lacks at least one of the standard properties listed above. We already observed in the *Remark* after the **Monotonicity principles** that our hypothetical D$a$ assigns finite measure to certain subsets which do not contain open balls at all (like the "principal axis hypercube" $C_\delta \cap \ker \mathrm{d}^*$ which is not open in infinite dimensions). This oddity might be related with another one too, namely that it is locally finite for certain open subsets ( like the ball $\mathscr{A}_{\varepsilon,N}(\nabla^0) \cap \ker \mathrm{d}^*$ or the full Hilbert space $\ker \mathrm{d}^*$ itself).

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