Letters in Mathematical Physics (2023) 113:43
https://doi.org/10.1007/s11005-023-01658-y

On a matrix element representation of the GKZ hypergeometric functions

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Received: 12 December 2022 / Revised: 28 February 2023 / Accepted: 8 March 2023 /
Published online: 11 April 2023

Abstract

We develop a representation theory approach to the study of generalized hypergeometric functions of Gelfand, Kapranov and Zelevisnky (GKZ). We show that the GKZ hypergeometric functions may be identified with matrix elements of non-reductive Lie algebras $\mathfrak{L}_N$ of oscillator type. The Whittaker functions associated with principal series representations of $\mathfrak{gl}_{\ell+1}(\mathbb{R})$ being special cases of GKZ hypergeometric functions thus admit along with a standard matrix element representations associated with reductive Lie algebra $\mathfrak{gl}_{\ell+1}(\mathbb{R})$, another matrix element representation in terms of $\mathfrak{L}_{\ell(\ell+1)}$.

Keywords GKZ hypergeometric functions · Multidimensional Oscillator Lie algebra · Whittaker functions

Mathematics Subject Classification 33C70 · 33C80 · 81R12 · 37K10

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1 Introduction

One way of solving explicitly a quantum integrable system is to realize the system as a quantum reduction in a larger quantum integrable system for which explicit solution may be easily found. Choice of the larger system is obviously not unique and is a matter of convenience. To practically implement this approach, one should represent wave functions of quantum integrable system as particular matrix elements of suitable Lie algebra representations allowing central characters. In this representation, the reduction is realized via proper choice of the matrix element. Interesting examples arise by considering reductions with respect to non-abelian Lie group symmetries. Typical case (along with various kinds of the Calogero models [18]) is given by families of quantum open Toda chains solved via special matrix elements of principle series representations of non-abelian reductive Lie groups identified with Whittaker functions [13, 14, 19] and [15, 16].

Recall that another more traditional approach to integration of quantum integrable system employs large abelian Lie group of symmetries generated by mutually commuting integrable flows. The large abelian symmetry allows to find a proper set of quantum canonical variables and to realize wave functions of the quantum integrable systems as matrix elements of representations of the Heisenberg Lie algebras. However, this straightforward quantum generalization of the classical integration algorithm encounters various difficulties. The main reason is that quite non-trivial realization of the corresponding abelian symmetries does not allow to fix properly operator ordering ambiguities in a simple and explicit way.

It is natural to try to combine these two complimentary approaches by considering a larger explicitly integrable quantum theory completely defined in terms of its Lie symmetry given by extensions of the Heisenberg Lie algebras. As it was demonstrated in the previous short announcement [11], the theory of generalized hypergeometric functions developed in [4, 5] supplies us with a large class of quantum integrable systems which may be solved this way.

In this note, we show that general GKZ hypergeometric function may be identified with a matrix element in an irreducible representation of a suitable multidimensional oscillator Lie algebra. The explicit identification is most natural when is done in terms of Gelfand–Graev hypergeometric functions (referred to as GG-functions). The latter were introduced in [3] as a more symmetric formulation of the original GKZ hypergeometric functions.

The class of integrable systems solved in terms of GKZ hypergeometric functions includes open $\mathfrak{gl}_{\ell+1}$-Toda chains so that the corresponding solutions given by $\mathfrak{gl}_{\ell+1}$-Whittaker functions allow integral representations via GKZ integrals (see [1]). The matrix element representation of open $\mathfrak{gl}_{\ell+1}$-Toda chain wave functions introduced in this note should be contrasted with another one based on representation theory of reductive Lie algebra $\mathfrak{gl}_{\ell+1}(\mathbb{R})$ [15, 16]. To stress the difference, we provide a detailed description of both formulations in the case of Toda type models associated with maximal and minimal parabolic subalgebras of $\mathfrak{gl}_{\ell+1}$ [1, 10]. On a more fundamental level, the relation between two matrix element formulations will be considered elsewhere.

Our interest in various realizations of the Whittaker functions stems for the fact that the Whittaker functions play important role in the formulation of local Archimedean

Langlands correspondence (see [6, 7, 9] and references therein). Thus, the identification of the Whittaker functions with matrix elements of the non-reductive oscillator Lie algebra possibly leads to interesting implications for Archimedean Langlands correspondence related to various approaches based on abelianization/torification and mirror symmetry. This theme will be discussed in future publications.

The plan of the paper is as follows. In Sect. 2, we provide basics of GKZ hypergeometric structure and associated hypergeometric functions. Following mostly [3], we also define GG-system as a symmetric form of the GKZ hypergeometric structure. The corresponding GG-functions are the main objects of interest in the following Sections. In Sect. 3, matrix element realization of the general GG hypergeometric functions via representations theory of multidimensional oscillator algebras is presented. In Sect. 4, the set of defining equations for GG hypergeometric functions is rederived using matrix element representation introduced in Sect. 3. Finally, in Sect. 5 the matrix element representation in terms of oscillator Lie algebra and reductive Lie algebra $\mathfrak{gl}_{\ell+1}$ for $\mathfrak{gl}_{\ell+1}$-Whittaker functions associated with maximal and minimal parabolic subalgebras is considered in detail.

2 The GKZ and GG hypergeometric structures

The GKZ hypergeometric functions are defined as solutions to system of differential equations associated with the $(\mathcal{A}, c)$-hypergeometric structures, first introduced and studied in [2, 4, 5]. The GKZ hypergeometric structure is defined by the following data. We choose an N-element subset

$$\mathcal{A} = \{a_i : \quad i \in I\} \subset \mathbb{Z}^m, \qquad I = \{1, 2, \dots, N\}, \tag{2.1}$$

such that $\mathcal{A}$ generates $\mathbb{Z}^m$ as abelian group. In addition, we fix a complex vector,

$$c = (c^1, \dots, c^m) \in \mathbb{C}^m. \tag{2.2}$$

The collection $\mathcal{A}$ defines the $m \times N$-matrix of maximal possible rank:

$$A = \|a_i^s\| \in \mathrm{Mat}_{m\times N}(\mathbb{Z}), \qquad \mathrm{rank}(A) = m \tag{2.3}$$

where $a_i = (a_i^1, \cdots, a_i^m) \in \mathbb{Z}^m$ are elements of $\mathcal{A}$.

Let $\mathbb{L} \subset \mathcal{A}$ be the relations lattice of $\mathcal{A}$:

$$\mathbb{L} = \{(l_1, \dots, l_N) \in \mathbb{Z}^N : l_1 a_1 + \cdots + l_N a_N = 0\}. \tag{2.4}$$

Choosing a basis $\{l^\alpha,\ \alpha \in J\}$ of the lattice $\mathbb{L}$ indexed by $J = \{1, \dots, N - m\}$:

$$\mathbb{L} = \mathrm{span}\Big\{l^\alpha = (l_1^\alpha,\ \dots,\ l_N^\alpha) : \quad \alpha \in J\Big\}, \tag{2.5}$$

we consider the corresponding relation matrix

$$M = \|l_i^\alpha\| \in \mathrm{Mat}_{(N-m)\times N}(\mathbb{Z}), \qquad \mathrm{rank}(M) = N - m. \tag{2.6}$$

The matrices A and M enjoy the orthogonality property

$$AM^{\top} = 0 \in \mathrm{Mat}_{m\times(N-m)}(\mathbb{Z}), \tag{2.7}$$

where $\top$ denotes the standard matrix transposition. The GKZ hypergeometric function $f(u)$ is a solution to the GKZ-system introduced below.

Definition 2.1 Given a subset $\mathcal{A}\subset\mathbb{Z}^m$ from (2.1) and a vector $c\in\mathbb{C}^m$ from (2.2), the GKZ-system of differential equations associated with the data $(\mathcal{A},\, c)$ consists of the following set of equations in variables $u=(u_1,\cdots,u_N)\in(\mathbb{R}_+)^N$:

(1) For every $l\in\mathbb{L}$, one has

$$\prod_{\substack{i\in I\\ l_i<0}}\left(-\frac{\partial}{\partial u_i}\right)^{-l_i} f = \prod_{\substack{i\in I\\ l_i>0}}\left(-\frac{\partial}{\partial u_i}\right)^{l_i} f. \tag{2.8}$$

(2) The differential equations enumerated by $s\in\{1,\ldots,m\}$:

$$a_1^s u_1\frac{\partial f}{\partial u_1}+\cdots+a_N^s u_N\frac{\partial f}{\partial u_N}=c^s f. \tag{2.9}$$

Equation (2.9) may be easily integrated and thus allow a reduction in the solutions to the system of Eqs. (2.8), (2.9) to functions in $(N-m)$-variables indexed by $J=\{1,\ldots,N-m\}$. However, the resulting equations on the function $(N-m)$-variables have more complicated form.

Now we fix a special solution to the GKZ-system (2.8), (2.9).

Proposition 2.1 *Given the GKZ data $(\mathcal{A}, c)$, the GKZ-system (2.8), (2.9) allows the following solution:*

$$f_\gamma(u) = \int_{\mathbb{R}_+^N}\prod_{i=1}^{N}\frac{dt_i}{t_i}\,t_i^{\gamma_i}\,e^{-u_it_i}\prod_{\alpha\in J}\delta\left(\prod_{j\in I}t_j^{l_j^\alpha}-1\right),\qquad \mathrm{Re}(\gamma_i)>0. \tag{2.10}$$

Here $\{l^\alpha,\,\alpha\in J\}\subset\mathbb{L}$ is a basis (2.5) and $\gamma=(\gamma_1,\ldots,\gamma_N)\in\mathbb{C}^N$ is a vector subjected to

$$c^s+\sum_{j\in I}a_j^s\gamma_j = 0,\qquad s\in\{1,\ldots,m\}. \tag{2.11}$$

The function (2.10) is independent of the choice of the basis as well as of the choice of $\gamma\in\mathbb{C}^N$ satisfying (2.11).

Proof Let $\gamma=(\gamma_1,\ldots,\gamma_N)\in\mathbb{C}^N$ such that $\mathrm{Re}(\gamma_i)>0$, $i\in I$. For every $l\in\mathbb{L}$, let us verify the first assertion by substituting (2.10) into (2.8) and (2.9). For the first equation,

$$\prod_{\substack{i\in I\\ l_i>0}}\left(-\frac{\partial}{\partial u_i}\right)^{l_i}\cdot f_\gamma(u)-\prod_{\substack{i\in I\\ l_i<0}}\left(-\frac{\partial}{\partial u_i}\right)^{-l_i}\cdot f_\gamma(u)$$

$$=\int\limits_{\mathbb{R}_+^N}\prod_{i=1}^{N}\frac{dt_i}{t_i}\,t_i^{\gamma_i}\,e^{-u_it_i}\left(\prod_{\substack{i\in I\\ l_i>0}}t_i^{l_i}-\prod_{\substack{i\in I\\ l_i<0}}t_i^{-l_i}\right)\prod_{\alpha\in J}\delta\Big(\prod_{j\in I}t_j^{l_j^\alpha}-1\Big) \tag{2.12}$$

$$=\int\limits_{\mathbb{R}_+^N}\prod_{i=1}^{N}\frac{dt_i}{t_i}\,t_i^{\gamma_i}\,e^{-u_it_i}\prod_{\substack{i\in I\\ l_i<0}}t_i^{-l_i}\ \left(\prod_{i\in I}t_i^{l_i}-1\right)\prod_{\alpha\in J}\delta\Big(\prod_{j\in I}t_j^{l_j^\alpha}-1\Big)=0. \tag{2.13}$$

The last equality follows since for $l=\sum_{\alpha\in J}n_\alpha l^\alpha\in\mathbb{L}$, we have

$$\prod_{i\in I}t_i^{l_i}=\prod_{i\in I}t_i^{\sum\limits_{\alpha\in J}n_\alpha l_i^\alpha}=\prod_{\alpha\in J}\Big(\prod_{i\in I}t_i^{l_i^\alpha}\Big)^{n_\alpha}, \tag{2.14}$$

which equals 1 by taking into account the delta-factors in the integrand of (2.10).

To prove the second equation, it is useful to change the integration variables $t_i\to u_i^{-1}t_i$, so the integral (2.10) takes the form:

$$f_\gamma(u)=\prod_{i\in I}u_i^{-\gamma_i}\int\limits_{\mathbb{R}_+^N}\prod_{i=1}^{N}\frac{dt_i}{t_i}\,t_i^{\gamma_i}\,e^{-t_i}\prod_{\alpha\in J}\delta\left(\prod_{j\in I}u_j^{-l_j^\alpha}t_j^{l_j^\alpha}-1\right). \tag{2.15}$$

Then, Eq. (2.9) follows from the orthogonality relation (2.7) between A and M.

For the solution (2.10), independence of the choice of the basis $\{l^\alpha,\ \alpha\in J\}\subset\mathbb{L}$ can be verified as follows. Let $\{\tilde{l}^\alpha,\ \alpha\in J\}\subset\mathbb{L}$ be another basis, and let $\|g_\beta^\alpha\|\in GL_{N-m}(\mathbb{Z})$ be the transition matrix between the two bases:

$$\tilde{l}^\alpha=\sum_{\beta\in J}g_\beta^\alpha l^\beta,\qquad \det\|g_\beta^\alpha\|\in\mathbb{Z}^*=\{\pm1\}. \tag{2.16}$$

By change of integration variables $T_i=\ln t_i,\ i\in I$, (2.10) takes the following form:

$$f_\gamma(u)=\int\limits_{\mathbb{R}^N}\prod_{i=1}^{N}dT_i\,e^{\gamma_iT_i-u_ie^{T_i}}\prod_{\alpha\in J}\delta\left(\sum_{j\in I}l_j^\alpha T_j\right). \tag{2.17}$$

For $T=(T_1,\dots,T_N)\in\mathbb{R}^N$, consider vector $S\in\mathbb{R}^{N-m}$ with the coordinates $S^\alpha=\sum\limits_{j\in I}l_j^\alpha T_j$, and define

$$\delta(S)=\prod_{\alpha\in J}\delta(S^\alpha)=\prod_{\alpha\in J}\delta\left(\sum_{j\in I}l_j^\alpha T_j\right). \tag{2.18}$$

Then, applying standard rule for linear change of arguments in delta-functions

$$\delta(gS) = \frac{1}{|\det(g)|}\,\delta(S) = \delta(S), \qquad \det(g) \in \mathbb{Z}^* = \{\pm 1\}, \tag{2.19}$$

which verifies the independence of the choice of the basis $\{l^\alpha\} \subset \mathbb{L}$ for (2.10).

Finally, for $f_\gamma(u)$, independence of the choice of solution $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{C}^N$ to Eq. (2.11) may be checked as follows. Given a pair of solutions, γ and γ', they are related by

$$\gamma_j = \gamma'_j + \sum_{\alpha \in J} l^\alpha_j \xi_\alpha, \qquad \xi = \|\xi_\alpha\| \in \mathbb{C}^{N-m}. \tag{2.20}$$

Then, substituting this into the integral (2.10) results in $f_\gamma = f_{\gamma'}$ by taking into account the delta-factors in the integrand. □

Example 2.1 Consider the two elementary examples of the GKZ-hypergeometric functions associated with $\mathcal{A}$, corresponding to the cases $|I| = 1$, $|J| = 0$ and $|I| = |J| = 1$.

1. The GKZ-hypergeometric function corresponding to the GKZ data with $N = m = 1$ and $J = \emptyset$ according to Definition 2.1 is given by

$$f_\gamma(u) \;=\; \int\limits_{\mathbb{R}_+} \frac{dt}{t} t^\gamma\, e^{-ut} \;=\; u^{-\gamma}\,\Gamma(\gamma), \qquad \mathrm{Re}(\gamma) > 0, \tag{2.21}$$

   and satisfies the following equation (a special case of (2.9)):

$$u\frac{\partial}{\partial u} f_\gamma(u) \;=\; -\gamma\, f_\gamma(u). \tag{2.22}$$

   Note that Eq. (2.8) is absent in this case.
2. The GKZ-hypergeometric function associated with the data $|I| = |J| = 1$, $m = 0$ is given by the exponential function:

$$f_\gamma(u) \;=\; \int\limits_{\mathbb{R}_+} \frac{dt}{t}\, t^\gamma\, e^{-ut}\, \delta(t-1) \;=\; e^{-u}, \tag{2.23}$$

   and satisfies the following equation:

$$\left(-\frac{\partial}{\partial u}\right) f_\gamma(u) \;=\; f_\gamma(u), \tag{2.24}$$

   which is a special case of (2.8). Equation (2.9) is absent in this case.

Definition 2.1 of the GKZ system has an obvious asymmetry that we would like to get rid of. Namely, it is natural to consider a solution to (2.8),(2.9) as a function both in $u \in \mathbb{R}_+^N$ and $c \in \mathbb{C}^m$. Here the vector c plays the role of spectral variables. The difference between number of u-variables and c-variables is taken into account by additional symmetries generated by linear equations (2.9). Clearly u-variables and c-variables are treated quite differently in Definition 2.1. On the other hand, the explicit solution $f_\gamma(u)$ written down using spectral γ-variables of the same number as u-variables. As a compensation, we gain an additional symmetry over spectral parameters $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{C}^N$ reducing it effectively to the m parameters $\{c^s,\ 1 \leq s \leq m\}$ via (2.20). All this suggests a completely symmetric definition of the GKZ system of differential equations. Basic ingredients of this reformulation are already appeared in [3], and thus, we will call this GG-hypergeometric structure.

Let the set $\mathcal{A}$ be the same as in the definition of GKZ-data $(\mathcal{A}, c)$, and let $\mathbb{L}$ be the corresponding relation lattice (2.4). Let us introduce GG-hypergeometric system associated with $\mathcal{A}$. The GG-hypergeometric function $\Phi_\gamma(u)$ is a solution to the GG-system.

Definition 2.2 The GG-system associated with $\mathcal{A}$ consists of the following set of equations in variables $u = (u_1, \ldots, u_N) \in \mathbb{R}_+^N$ and $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{C}^N$:

(1) For every $l \in \mathbb{L}$, one has

$$\prod_{\substack{i \in I \\ l_i < 0}} \left(-\frac{\partial}{\partial u_i} + \frac{\gamma_i}{u_i} \right)^{-l_i} \Phi = \prod_{\substack{i \in I \\ l_i > 0}} \left(-\frac{\partial}{\partial u_i} + \frac{\gamma_i}{u_i} \right)^{l_i} \Phi. \tag{2.25}$$

(2) For every $l \in \mathbb{L}$, one has

$$\sum_{i \in I} l_i \left(\frac{\partial}{\partial \gamma_i} - \ln u_i \right) \Phi = 0, \tag{2.26}$$

or, equivalently,

$$\prod_{i \in I} e^{\theta l_i \partial_{\gamma_i}} \Phi = \prod_{i \in I} u_i^{\theta l_i} \Phi, \qquad \forall \theta \in \mathbb{R}. \tag{2.27}$$

(3) The system of m differential equations for $s \in \{1, \ldots, m\}$:

$$\left\{ a_1^s u_1 \frac{\partial}{\partial u_1} + \cdots + a_N^s u_N \frac{\partial}{\partial u_N} \right\} \Phi = 0. \tag{2.28}$$

(4) The system of m difference equations for $s \in \{1, \ldots, m\}$:

$$\left\{ a_1^s \left(e^{\frac{\partial}{\partial \gamma_1}} - \gamma_1 \right) + \ldots + a_N^s \left(e^{\frac{\partial}{\partial \gamma_N}} - \gamma_N \right) \right\} \Phi = 0. \tag{2.29}$$

Note that in comparison with Definition 2.1 we introduce additional dual difference equations over the spectral variables $\gamma \in \mathbb{C}^N$.

To contrast Definitions 2.1 and 2.2, it is useful to consider the two simple cases continuing Example 2.1.

Example 2.2 1. The GG-hypergeometric function corresponding to $\mathcal{A}$ with $N = m = 1$ and $J = \emptyset$ according to Definition 2.2 is given by the standard Gamma-function:

$$\Phi_\gamma(u) = \int\limits_{\mathbb{R}_+} \frac{dt}{t}\, t^\gamma\, e^{-t} = \Gamma(\gamma), \qquad \mathrm{Re}(\gamma) > 0, \tag{2.30}$$

satisfying the equations

$$\frac{\partial}{\partial u}\, \Phi_\gamma(u) \,=\, 0, \tag{2.31}$$

and

$$\left(e^{\partial_\gamma} - \gamma\right)\, \Phi_\gamma(u) \,=\, 0. \tag{2.32}$$

Here (2.31) is an instance of (2.28); meanwhile, (2.32) is an instance of (2.29). Note that Eqs. (2.25) and (2.26),(2.27) are absent in this case.

2. The GG-hypergeometric function associated with $\mathcal{A}$ such that $|I| = |J| = 1,\ m = 0$ is given by

$$\Phi_\gamma(u) \,=\, u^\gamma\, e^{-u}, \tag{2.33}$$

satisfying the differential equation:

$$\left\{u\frac{\partial}{\partial u} \,-\, \gamma\right\}\Phi_\gamma(u) \,=\, u\,\Phi_\gamma(u), \tag{2.34}$$

and the difference equation:

$$\left\{e^{\partial_\gamma} \,-\, u\right\}\Phi_\gamma(u) \,=\, 0. \tag{2.35}$$

Here (2.34) is an instance of (2.25); meanwhile, (2.35) is an instance of (2.27). Equations (2.28), (2.29) are absent in this case.

A relation between the two formulations of the GKZ-hypergeometric structures is manifested by the following.

Proposition 2.2 *Given the GKZ-datum $\mathcal{A}$, the GG-system (2.25)–(2.29) allows the following solution:*

$$\Phi_\gamma(u) \,=\, \int\limits_{\mathbb{R}_+^N} \prod_{i=1}^N \frac{dt_i}{t_i}\, t_i^{\gamma_i}\, e^{-t_i} \prod_{\alpha\in J} \delta\left(\prod_{j\in I} u_j^{-l_j^\alpha}\, t_j^{l_j^\alpha} \,-\, 1\right), \qquad \mathrm{Re}(\gamma_i) > 0, \tag{2.36}$$

where $\{l^\alpha,\ \alpha\in J\}\subset\mathbb{L}$ *is the basis* (2.5). *The solution* (2.36) *is independent of the choice of basis* $\{l^\alpha,\ \alpha\in J\}\subset\mathbb{L}$ *and may be identified with* (2.10) *via*

$$\Phi_\gamma(u) \;=\; \left(\prod_{i\in I} u_i^{\gamma_i}\right) \times f_\gamma(u). \tag{2.37}$$

Proof Let $\gamma\in\mathbb{C}^N$ be such that $\mathrm{Re}(\gamma_i)>0,\ i\in I$. Similar to the proof of Proposition 2.1, one checks that (2.36) satisfies (2.25), (2.26) and (2.28) in straightforward way. For the integral (2.36), independence of the basis choice follows from (2.37) and from the independence for $f_\gamma(u)$ by Proposition 2.1.

One observes that Eqs. (2.25) and (2.28) are identified with (2.8) and (2.9) via (2.37) and the following identity for every $i\in I$:

$$\prod_{j\in I} u_j^{\gamma_j}\left(\frac{\partial}{\partial u_i}\right)^n \prod_{j\in I} u_j^{-\gamma_j} \;=\; \left(\frac{\partial}{\partial u_i} - \frac{\gamma_i}{u_i}\right)^n, \qquad n\in\mathbb{Z}_{\geq 0}. \tag{2.38}$$

To verify (2.29), consider the integrand of (2.36):

$$F_\gamma(t) \;=\; \prod_{i\in I} t_i^{\gamma_i} e^{-t_i} \prod_{\alpha\in J}\delta\left(\prod_{j\in I} u_j^{-l_j^\alpha} t_j^{l_j^\alpha} - 1\right). \tag{2.39}$$

Then, its differential is given by

$$\begin{aligned} dF_\gamma(t) = \sum_{i\in I}\frac{\partial F_\gamma(t)}{\partial t_i}\,dt_i \;&=\; \sum_{i\in I}(\gamma_i - t_i)F_\gamma(t)\frac{dt_i}{t_i} \\ &+ \sum_{i\in I}\sum_{\alpha\in J} l_i^\alpha \prod_{j\in I} u_j^{-l_j^\alpha} t_j^{l_j^\alpha}\,\delta'\left(\prod_{j\in I} u_j^{-l_j^\alpha} t_j^{l_j^\alpha} - 1\right)\prod_{\substack{\beta\in J\\ \beta\neq\alpha}}\delta\left(\prod_{j\in I} u_j^{-l_j^\beta} t_j^{l_j^\beta} - 1\right)\frac{dt_i}{t_i}. \end{aligned} \tag{2.40}$$

Therefore, observing that

$$\left(\gamma_i - t_i\right)F_\gamma(t) \;=\; \left(\gamma_i - e^{\partial_{\gamma_i}}\right) F_\gamma(t), \tag{2.41}$$

by the orthogonality relation (2.7), one deduces the following, for $1\leq s\leq k+1$:

$$\begin{aligned} \sum_{i\in I} a_i^s\left(\gamma_i - e^{\partial_{\gamma_i}}\right)\Phi_\gamma(u) \;&=\; \int_{\mathbb{R}_+^N}\prod_{j=1}^N \frac{dt_j}{t_j}\sum_{i\in I} a_i^s\left(\gamma_i - t_i\right)F_\gamma(t) \\ &= \sum_{i\in I} a_i^s \int_{\mathbb{R}_+^{N-1}}\prod_{\substack{j\in I\\ j\neq i}}\frac{dt_j}{t_j}\int_{\mathbb{R}_+} dt_i\,\frac{\partial F_\gamma(t)}{\partial t_i} \;=\; 0, \end{aligned} \tag{2.42}$$

since $F_\gamma(t)\,\big|_{t_i=0} = F_\gamma(t)\big|_{t_i=+\infty} = 0$ for each $i \in I$. □

As it is already mentioned in Introduction, the system of equations (2.25)–(2.29) is naturally related to the GG-system introduced and studied in [3]. In particular, the integral (2.36) can be identified with the GG-function as it is defined in [3]. Alternatively, in [3] the function (2.36) is described as a solution to (2.25), (2.26), (2.28), and the additional set of first-order differential difference equations (2.43) from the following proposition.

Proposition 2.3 *The GG-function (2.36) satisfies the following system of GG-equations:*

$$\left\{-u_i \frac{\partial}{\partial u_i} + \gamma_i\right\} \cdot \Phi_\gamma(u) \;=\; e^{\partial_{\gamma_i}} \cdot \Phi_\gamma(u), \qquad i \in I. \tag{2.43}$$

Proof To verify the assertion, for every $j \in I$, we substitute (2.36) (after changing $t_i \to u_i t_i,\ i \in I$) into (2.43):

$$\begin{aligned}
\left\{-u_j \frac{\partial}{\partial u_j} + \gamma_j\right\} \Phi_\gamma(u) &= \int\limits_{\mathbb{R}^N_+} \prod_{i=1}^N \frac{dt_i}{t_i} \prod_{\alpha \in J} \delta\left(\prod_{j \in I} t_j^{l_j^\alpha} - 1\right) \\
&\quad \times \left\{-u_j \frac{\partial}{\partial u_j} + \gamma_j\right\} \prod_{i=1}^N (u_i t_i)^{\gamma_i}\, e^{-u_i t_i} \\
&= \int\limits_{\mathbb{R}^N_+} \prod_{i=1}^N \frac{dt_i}{t_i} \prod_{\alpha \in J} \delta\left(\prod_{j \in I} t_j^{l_j^\alpha} - 1\right) (u_j t_j) \prod_{i=1}^N (u_i t_i)^{\gamma_i}\, e^{-u_i t_i} \\
&= \Phi_{\gamma_1,\dots,\gamma_j+1,\dots,\gamma_N}(u) \;=\; e^{\partial_{\gamma_j}} \cdot \Phi_\gamma(u).
\end{aligned} \tag{2.44}$$

□

One might note that (2.43) and (2.28) entail the dual difference equations (2.29).

3 Matrix element representation

In this section, we provide a representation of the GG-hypergeometric functions (2.36) as matrix elements in irreducible representation of a suitable class of non-reductive Lie algebras. Namely, for $I = \{1, \dots, N\}$, let $\mathfrak{L}_N$ be the Lie algebra generated by the central element $\mathcal{C}$ and by $\{\mathcal{E}_i, \mathcal{F}_i, \mathcal{H}_i,\ i \in I\}$ subjected to the following defining relations:

$$\begin{aligned}
&[\mathcal{E}_i, \mathcal{F}_j] = \delta_{ij}\mathcal{C}, \qquad [\mathcal{H}_i, \mathcal{E}_j] = -\delta_{ij}\mathcal{E}_j, \qquad [\mathcal{H}_i, \mathcal{F}_j] = \delta_{ij}\mathcal{F}_j, \\
&\qquad\qquad [\mathcal{E}_i, \mathcal{E}_j] = [\mathcal{F}_i, \mathcal{F}_j] = [\mathcal{H}_i, \mathcal{H}_j] = 0, \qquad i, j \in I.
\end{aligned} \tag{3.1}$$

The subalgebra in $\mathfrak{L}_N$ generated by $\mathcal{C}$ and by $\{\mathcal{E}_i,\ \mathcal{F}_i,\ i\in I\}$ is isomorphic to the $(2N+1)$-dimensional Heisenberg algebra. Thus, the Lie algebra $\mathfrak{L}_N$ may be considered as multidimensional version of the standard oscillator Lie algebra.

For $\gamma=(\gamma_1,\dots,\gamma_N)\in\mathbb{C}^N$ such that $\mathrm{Re}(\gamma_i)>0$, $i\in I$, let π_γ be the $\mathfrak{L}_N$-representation in the Schwartz space $\mathcal{V}_\gamma$ of smooth functions in $t=(t_1,\dots,t_N)\in(\mathbb{R}_+)^N$ decreasing rapidly with all its derivatives at infinity:

$$\mathcal{V}_\gamma=\Big\{f\in C^\infty(\mathbb{R}_+^N):\lim_{t_i\to\infty}\big(t_1^{n_1}\cdots t_N^{n_N}|\partial_{t_1}^{m_1}\cdots\partial_{t_N}^{m_N}f(t)|\big)=0,\ \forall n_i,m_i\in\mathbb{Z}_{\geq 0}\Big\}. \tag{3.2}$$

Namely, the representation $(\pi_\gamma,\mathcal{V}_\gamma)$ is defined by the following action of the generators (3.1):

$$\pi_\gamma(\mathcal{C})=1,\quad \pi_\gamma(\mathcal{E}_i)=-\partial_{t_i},\quad \pi_\gamma(\mathcal{F}_i)=-t_i,\quad \pi_\gamma(\mathcal{H}_i)=\gamma_i+t_i\partial_{t_i},\quad i\in I. \tag{3.3}$$

Let $\mathcal{U}(\mathfrak{L}_N)$ be the universal enveloping algebra of $\mathfrak{L}_N$, then $\mathcal{V}_\gamma$ affords a structure of $\mathcal{U}(\mathfrak{L}_N)$-module. Moreover, the action of generator $\mathcal{H}_i\in\mathfrak{L}_N$ in $\mathcal{V}_\gamma$ integrates to the action of the group $(\mathbb{R}_+)^N$. It will be also important in the following that the generators $\mathcal{F}_i$, $i\in I$ are invertible in the representation $\mathcal{V}_\gamma$; therefore, $(\pi_\gamma,\mathcal{V}_\gamma)$ extends to representation of a larger algebra containing negative powers of $\mathcal{F}_i$, $i\in I$. We fix a dual module $\mathcal{V}_\gamma^\vee$ realized in the space of generalized functions on $\mathbb{R}_+^N$ and denote the corresponding contragradient representation by $\pi_\gamma^\vee$. There is a non-degenerate $\mathfrak{L}_N$-invariant pairing $\langle,\rangle:\mathcal{V}_\gamma^\vee\times\mathcal{V}_\gamma\to\mathbb{C}$,

$$\begin{aligned}\langle\phi,\varphi\rangle&=\int_{\mathbb{R}_+^N}\prod_{i\in I}dt\,\phi(t)\,\varphi(t),\qquad \varphi\in\mathcal{V}_\gamma,\quad \phi\in\mathcal{V}_\gamma^\vee,\\ \langle\pi_\gamma^\vee(X)\cdot\phi,\varphi\rangle&=-\langle\phi,\pi_\gamma(X)\cdot\varphi\rangle,\qquad \forall X\in\mathfrak{L}_N.\end{aligned} \tag{3.4}$$

Then, the $\mathfrak{L}_N$-action in the dual module $\mathcal{V}_\gamma^\vee$ can be computed explicitly:

$$\begin{aligned}\pi_\gamma^\vee(\mathcal{C})=-1,\qquad \pi_\gamma^\vee(\mathcal{E}_i)=-\partial_{t_i},\qquad \pi_\gamma^\vee(\mathcal{F}_i)=t_i,&\\ \pi_\gamma^\vee(\mathcal{H}_i)=1-\gamma_i+t_i\partial_{t_i},\qquad i\in I.&\end{aligned} \tag{3.5}$$

Given the GKZ-datum $\mathcal{A}$, let us fix a basis $\{l^\alpha,\ \alpha\in J\}$ in the relation lattice $\mathbb{L}$ (2.5). Let us introduce a vector $\phi_R\in\mathcal{V}_\gamma$ and a covector $\phi_L\in\mathcal{V}_\gamma^\vee$ defined by

$$\pi_\gamma(\mathcal{E}_i)\cdot\phi_R=\phi_R,\qquad i\in I, \tag{3.6}$$

and

$$\begin{aligned}\prod_{\substack{i\in I\\ l_i^\alpha<0}}\pi_\gamma^\vee(\mathcal{F}_i)^{-l_i^\alpha}\cdot\phi_L&=\prod_{\substack{i\in I\\ l_i^\alpha>0}}\pi_\gamma^\vee(\mathcal{F}_i)^{l_i^\alpha}\cdot\phi_L,\quad \alpha\in\{1,\dots,N-m\},\\ \sum_{i=1}^N a_i^s\,\pi_\gamma^\vee(\mathcal{H}_i)\cdot\phi_L&=0,\qquad s\in\{1,\dots,m\}.\end{aligned} \tag{3.7}$$

Note that invertibility of the images of the generators $\mathcal{F}_i$ in the considered representation allows to rewrite equivalently the first line of (3.7) as follows:

$$\prod_{i\in I}\pi_\gamma^\vee(\mathcal{F}_i)^{l_i^\alpha}\cdot\phi_L\,=\,\phi_L,\qquad \alpha\in\{1,\ldots,N-m\}. \tag{3.8}$$

Furthermore, for an arbitrary $l=\sum\limits_{\alpha\in J}n_\alpha l^\alpha=(l_1,\cdots,l_N)\in\mathbb{L}$, the following holds:

$$\begin{aligned}\prod_{i\in I}\pi_\gamma^\vee(\mathcal{F}_i)^{l_i}\cdot\phi_L\,&=\,\prod_{i\in I}\prod_{\alpha\in J}\pi_\gamma^\vee(\mathcal{F}_i)^{n_\alpha l_i^\alpha}\cdot\phi_L\\ &=\prod_{\alpha\in J}\left(\prod_{i\in I}\pi_\gamma^\vee(\mathcal{F}_i)^{l_i^\alpha}\right)^{n_\alpha}\cdot\phi_L\,=\,\phi_L.\end{aligned} \tag{3.9}$$

Lemma 3.1 *For $\phi_L\in\mathcal{V}_\gamma$ defined in (3.6) and $\phi_R\in\mathcal{V}_\gamma^\vee$ defined in (3.7), the following explicit expressions hold:*

$$\phi_R(t)\,=\,e^{-\sum\limits_{i=1}^N t_i},\qquad \phi_L(t)\,=\,\prod_{i\in I}t_i^{\gamma_i-1}\prod_{\alpha\in J}\delta\left(\prod_{j\in I}t_j^{l_j^\alpha}\,-\,1\right). \tag{3.10}$$

Proof The solution to (3.6) is given by the first expression in (3.10). As for ϕ_L, the first defining relation in (3.7) is a direct consequence of presence of the delta-factors in expression for ϕ_L in (3.10). For the second relation, substituting $\pi_\gamma^\vee(\mathcal{H}_j)$ from (3.5) into (3.10), for each $s\in\{1,\ldots,m\}$ implies

$$\begin{aligned}&\sum_{j=1}^N a_j^s\langle\pi_\gamma^\vee(\mathcal{H}_j)\cdot\phi_L,\varphi\rangle\\ &\;=\sum_{j=1}^N a_i^s\int\limits_{\mathbb{R}_+^N}\prod_{i\in I}dt_i\,\varphi(t)\,(1-\gamma_j+t_j\partial_{t_j})\prod_{i\in I}t_i^{\gamma_i-1}\prod_{\alpha\in J}\delta\left(\prod_{j\in I}t_j^{l_j^\alpha}\,-\,1\right)\\ &\;=\sum_{\alpha\in J}\sum_{j=1}^N a_j^s l_j^\alpha\int\limits_{\mathbb{R}_+^N}\prod_{i\in I}\frac{dt_i}{t_i}\;t_i^{\gamma_i+l_i^\alpha}\,\delta'\left(\prod_{j\in I}t_j^{l_j^\alpha}\,-\,1\right)\prod_{\beta\neq\alpha}\delta\left(\prod_{j\in I}t_j^{l_j^\beta}\,-\,1\right)\,\varphi(t),\end{aligned} \tag{3.11}$$

which vanishes due to the orthogonality relation $\sum_{i=1}^N a_i^s l_i^\alpha=0$ in (2.7). □

Let us note that the conditions (3.7) on the covector ϕ_L may be linearized as follows. Consider the subalgebra in $\mathfrak{L}_N$ generated by $\{\mathcal{H}_i,\,\mathcal{F}_i,\,i\in I\}$, subjected to

$$\mathcal{H}_i\mathcal{F}_j-\mathcal{F}_j\mathcal{H}_i=\delta_{ij}\mathcal{F}_j,\qquad i,j\in I. \tag{3.12}$$

Then, it may be embedded into the appropriately completed Heisenberg algebra $\mathfrak{H}_N$ generated by the $P_i,\,Q_i,\;i\in I$ subjected to

$$P_i Q_j - Q_j P_i = \delta_{ij}, \qquad i,j\in I. \tag{3.13}$$

Explicitly, we define the following embedding

$$\mathcal{H}_i \longmapsto P_i, \qquad \mathcal{F}_i \longmapsto e^{Q_i}. \tag{3.14}$$

Then, the defining relations (3.7) for covector ϕ_L are equivalent to the condition of annihilation of ϕ_L by the following $|I|=N$ operators:

$$A_s = \sum_{i=1}^{N} a_i^s P_i, \quad s\in\{1,\ldots,m\}; \qquad B_\alpha = \sum_{i=1}^{N} l_i^\alpha Q_i, \quad \alpha\in J. \tag{3.15}$$

These operators generate an N-dimensional commutative subalgebra in the Heisenberg algebra $\mathfrak{H}_N$ and define a linear polarization that in general differs from the one defined by maximal commutative subalgebra generated by $P_i,\;i\in I$.

Theorem 3.1 *The integral solution* (2.36) *to the GG-system* (2.25)–(2.29) *allows the following expression in terms of the matrix element in the* $\mathcal{U}(\mathfrak{L}_N)$*-representation* $(\pi_\gamma,\mathcal{V}_\gamma)$ *for* $\gamma=(\gamma_1,\ldots,\gamma_N)\in\mathbb{C}^N$, $\mathrm{Re}(\gamma_i)>0$*:*

$$\Phi_\gamma(e^{y_1},\ldots,e^{y_N}) = \Big\langle \phi_L,\,\pi_\gamma\left(e^{\sum\limits_{j=1}^{N} y_j\mathcal{H}_j}\right)\phi_R\Big\rangle, \tag{3.16}$$

where ϕ_R *and* ϕ_L *are defined in* (3.6) *and* (3.7)*, respectively.*

Proof Substituting (3.10) into the matrix element, we have

$$\begin{aligned}
\Phi_{\gamma_1,\ldots,\gamma_N}(e^{y_1},\ldots,e^{y_N}) &= \Big\langle \phi_L,\pi_\gamma\left(e^{\sum\limits_{j=1}^{N} y_j\mathcal{H}_j}\right)\phi_R\Big\rangle \\
&= \int\limits_{(\mathbb{R}_+)^N} \prod_{i=1}^{N}\frac{dt_i}{t_i}\,\phi_L(t)\,e^{\sum\limits_{j=1}^{N} y_j\{\gamma_i+t_j\partial_{t_j}\}}\,\phi_R(t) \\
&= \int\limits_{(\mathbb{R}_+)^N} \prod_{i=1}^{N}\frac{dt_i}{t_i}\,t_i^{\gamma_i}\,e^{\gamma_i y_i - t_i e^{y_i}} \prod_{\alpha\in J}\delta\left(\prod_{j\in I} t_j^{l_j^\alpha}-1\right),
\end{aligned} \tag{3.17}$$

which coincides with (2.36) for $u_i=e^{y_i}$ via substitution $t_i\mapsto u_i^{-1}t_i,\;i\in I$. □

4 Differential equations satisfied by the GKZ functions

In this section, we derive the defining Eqs. (2.25), (2.28) using matrix element representation (3.16). The way to derive these equations is similar to the way how equations on matrix elements arise from the action of Casimir elements. Note that the dual equations (2.26)–(2.27) and (2.29) include differentiation over spectral parameters and arise in a different way by considering intertwining operators acting between different representations of oscillator algebra $\mathfrak{L}_N$.

Now for $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathbb{C}^N$, given the $\mathcal{U}(\mathfrak{L}_N)$-module $(\pi_\gamma, \mathcal{V}_\gamma)$ (3.3), let $\mathcal{I}_\gamma \subset \mathcal{U}(\mathfrak{L}_N)$ be the primitive annihilation ideal:

$$\mathcal{I}_\gamma = \big\{X \in \mathcal{U}(\mathfrak{L}_N) : \pi_\gamma(X) = 0\big\}. \tag{4.1}$$

For every $l = (l_1, \dots, l_N) \in \mathbb{Z}^N$, introduce the following element in $\mathcal{U}(\mathfrak{L}_N)$:

$$\mathcal{C}(l) = \prod_{\substack{i\in I\\ l_i<0}} \mathcal{F}_i^{-l_i}\mathcal{E}_i^{-l_i} \prod_{\substack{i\in I\\ l_i>0}} \prod_{k=0}^{l_i-1}(\mathcal{H}_i - \gamma_i - k) - \prod_{\substack{i\in I\\ l_i>0}} \mathcal{F}_i^{l_i}\mathcal{E}_i^{l_i} \prod_{\substack{i\in I\\ l_i<0}} \prod_{k=0}^{|l_i|-1}(\mathcal{H}_i - \gamma_i - k). \tag{4.2}$$

Lemma 4.1 *For an arbitrary $l \in \mathbb{Z}^N$, the element $\mathcal{C}(l)$ belongs to the ideal $\mathcal{I}_\gamma$.*

Proof Substitution of (3.3) into (2.2) reads

$$\begin{aligned}\pi_\gamma\big(\mathcal{C}(l)\big) &= \prod_{\substack{i\in I\\ l_i<0}} t_i^{-l_i}(\partial_{t_i})^{-l_i} \prod_{\substack{i\in I\\ l_i>0}} \prod_{k=0}^{l_i-1}(t_i\partial_{t_i} - k)\\ &\quad - \prod_{\substack{i\in I\\ l_i>0}} t_i^{l_i}(\partial_{t_i})^{l_i} \prod_{\substack{i\in I\\ l_i<0}} \prod_{k=0}^{-l_i-1}(t_i\partial_{t_i} - k).\end{aligned} \tag{4.3}$$

Then, by the following identity

$$\prod_{k=0}^{n-1}(t\partial_t - k) = t^n(\partial_t)^n, \qquad \forall n > 0, \tag{4.4}$$

each of the two terms in (4.3) equals $\prod_{i\in I}(t_i)^{|l_i|}(\partial_{t_i})^{|l_i|}$ and hence cancels each other. □

Now given the GKZ-datum $\mathcal{A}$, let $\mathbb{L}$ be the corresponding relation lattice.

Proposition 4.1 *For each $l \in \mathbb{L}$, the fact that $\pi_\gamma(\mathcal{C}(l)) = 0$ entails the GG-equations (2.25) satisfied by the matrix element (3.16):*

$$\left\{\prod_{\substack{i\in I\\ l_i<0}}\Big(-\frac{\partial}{\partial u_i} + \frac{\gamma_i}{u_i}\Big)^{-l_i} - \prod_{\substack{i\in I\\ l_i>0}}\Big(-\frac{\partial}{\partial u_i} + \frac{\gamma_i}{u_i}\Big)^{l_i}\right\}\Phi_\gamma(u) = 0. \tag{4.5}$$

Proof Using the invertibility of $\pi_\gamma(\mathcal{F}_i)$, $i \in I$ in the $\mathcal{U}(\mathfrak{L}_N)$-representation $(\pi_\gamma, \mathcal{V}_\gamma)$, we introduce the elements $\widetilde{\mathcal{C}}(l) \in \mathcal{U}(\mathfrak{L}_N)$ defined by

$$\pi_\gamma(\widetilde{\mathcal{C}}(l)) = \prod_{\substack{i\in I\\ l_i^\alpha<0}} \pi_\gamma(\mathcal{F}_i)^{l_i^\alpha} \times \pi_\gamma(\mathcal{C}(l)), \qquad l \in \mathbb{L}, \tag{4.6}$$

also acting by zero in $\mathcal{V}_\gamma$. Substituting the factors $\prod_{k=0}^{n-1}(\mathcal{H}_i - \gamma_i - k)$ from (4.2) into the matrix element (3.16) gives

$$\Big\langle \phi_L,\, \pi_\gamma\Big(\prod_{k=0}^{n-1}(\mathcal{H}_i - \gamma_i - k)\, e^{\sum_j y_j \mathcal{H}_j}\Big)\phi_R\Big\rangle = \prod_{k=0}^{n-1}(\partial_{y_i} - \gamma_i - k)\Phi_\gamma(e^y). \tag{4.7}$$

Next, we insert the expression (4.6) into the matrix element (3.16) and take into account the defining relations (3.6) and (3.7),(3.9) on the right and left vectors. Explicitly, for $u_i = e^{y_i}$, $i \in I$ we deduce the following:

$$\begin{aligned}
&\Big\langle \phi_L,\, \pi_\gamma\Big(\widetilde{\mathcal{C}}(l)\, e^{\sum_j y_j \mathcal{H}_j}\Big)\phi_R\Big\rangle \\
&= \prod_{\substack{i\in I\\ l_i>0}} \prod_{k=0}^{l_i-1}\Big(u_i\frac{\partial}{\partial u_i} - \gamma_i - k\Big)\Big\langle \phi_L,\, \pi_\gamma\Bigg(\prod_{\substack{i\in I\\ l_i<0}} \mathcal{E}_i^{-l_i}\, e^{\sum_j y_j \mathcal{H}_j}\Bigg)\phi_R\Big\rangle \\
&\quad - \prod_{\substack{i\in I\\ l_i<0}} \prod_{k=0}^{-l_i-1}\Big(u_i\frac{\partial}{\partial u_i} - \gamma_i - k\Big)\Big\langle \phi_L,\, \prod_{i\in I}\pi_\gamma(\mathcal{F}_i)^{l_i}\,\pi_\gamma\Bigg(\prod_{\substack{i\in I\\ l_i>0}} \mathcal{E}_i^{l_i}\, e^{\sum_j y_j \mathcal{H}_j}\Bigg)\phi_R\Big\rangle
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&= \prod_{\substack{i\in I\\ l_i>0}} \prod_{k=0}^{l_i-1}\Big(u_i\frac{\partial}{\partial u_i} - \gamma_i - k\Big)\prod_{\substack{i\in I\\ l_i<0}} u_i^{-l_i}\Big\langle \phi_L,\, \pi_\gamma\Big(e^{\sum_j y_j \mathcal{H}_j}\Big)\phi_R\Big\rangle \\
&\quad - \prod_{\substack{i\in I\\ l_i<0}} \prod_{k=0}^{-l_i-1}\Big(u_i\frac{\partial}{\partial u_i} - \gamma_i - k\Big)\prod_{\substack{i\in I\\ l_i>0}} u_i^{l_i}\Big\langle \prod_{i\in I}(-1)^{l_i}\pi_\gamma^\vee(\mathcal{F}_i)^{l_i}\phi_L,\, \pi_\gamma\Big(e^{\sum_j y_j \mathcal{H}_j}\Big)\phi_R\Big\rangle.
\end{aligned} \tag{4.9}$$

Then, applying the relations (3.9), this results in

$$\begin{aligned}
\Big\langle \phi_L,\, \pi_\gamma\Big(\widetilde{\mathcal{C}}(l)\, e^{\sum_j y_j \mathcal{H}_j}\Big)\phi_R\Big\rangle = &\Bigg\{\prod_{\substack{i\in I\\ l_i>0}} \prod_{k=0}^{l_i-1}\Big(u_i\frac{\partial}{\partial u_i} - \gamma_i - k\Big)\prod_{\substack{i\in I\\ l_i<0}} u_i^{-l_i} \\
&- \prod_{i\in I}(-1)^{l_i}\prod_{\substack{i\in I\\ l_i<0}} \prod_{k=0}^{-l_i-1}\Big(u_i\frac{\partial}{\partial u_i} - \gamma_i - k\Big)\prod_{\substack{i\in I\\ l_i>0}} u_i^{l_i}\Bigg\}\Phi_\gamma(u)
\end{aligned}$$

$$= \prod_{\substack{i\in I\\ l_i>0}}(-1)^{l_i}\prod_{i\in I}u_i^{|l_i|}\left\{\prod_{\substack{i\in I\\ l_i>0}}(-u_i)^{-l_i}\prod_{k=0}^{l_i-1}\left(u_i\frac{\partial}{\partial u_i}-\gamma_i-k\right)\right.$$

$$\left.-\prod_{\substack{i\in I\\ l_i<0}}(-u_i)^{l_i}\prod_{k=0}^{-l_i-1}\left(u_i\frac{\partial}{\partial u_i}-\gamma_i-k\right)\right\}\Phi_\gamma(u)$$

$$= \prod_{\substack{i\in I\\ l_i>0}}(-1)^{l_i}\prod_{i\in I}u_i^{|l_i|}\left\{\prod_{\substack{i\in I\\ l_i>0}}\left(-\frac{\partial}{\partial u_i}+\frac{\gamma_i}{u_i}\right)^{l_i}-\prod_{\substack{i\in I\\ l_i<0}}\left(-\frac{\partial}{\partial u_i}+\frac{\gamma_i}{u_i}\right)^{-l_i}\right\}\Phi_\gamma(u). \tag{4.10}$$

where in the last line (4.4) is used. Thus, equating the above expression to zero and dividing by the invertible function $\prod_{\substack{i\in I\\ l_i>0}}(-1)^{l_i}\prod_{i\in I}u_i^{|l_i|}$ yields (4.5). □

5 Whittaker functions via GKZ structures

Investigations of quantum cohomology of partial flag manifolds give rise to new families of quantum integrable systems of Toda type. The corresponding wave functions (generating functions) are given by generalized Whittaker functions associated with choice of parabolic subalgebras $\mathfrak{p}\subset\mathfrak{gl}_{\ell+1}$; for the motivated discussions of these see [1, 12] and [8, 10, 17]. As in the original construction of the standard Whittaker function (see [13, 14, 19] and [15, 16]), the generalized Whittaker functions can be presumably realized as particular matrix elements of the principal series $\mathcal{U}(\mathfrak{gl}_{\ell+1})$-representations. One also expects that the generalized Whittaker functions allow expressions in terms the appropriate GKZ hypergeometric functions. Below we consider two special instances of the $\mathfrak{gl}_{\ell+1}$-Whittaker functions associated with minimal and maximal parabolic subalgebras and provide their expressions in terms of GKZ hypergeometric functions. Various parts of these results may be found in references mentioned above and are included here for completeness.

Let us first fix the following notations for the general linear Lie algebra and its subalgebras. Let $V=\mathbb{C}^{\ell+1}$ be a $\mathbb{C}$-vector space and let $\mathfrak{gl}_{\ell+1}=\mathfrak{gl}(V)$ be the endomorphism Lie algebra spanned by the standard generators $\mathcal{E}_{ij}$, $1\le i,j\le\ell+1$ subjected to the relations:

$$[\mathcal{E}_{ij},\,\mathcal{E}_{kl}]=\delta_{jk}\mathcal{E}_{il}-\delta_{il}\mathcal{E}_{kj}. \tag{5.1}$$

Parabolic subalgebra $\mathfrak{p}$ in $\mathfrak{gl}_{\ell+1}$ is defined as a subalgebra satisfying $\mathfrak{b}_-\subseteq\mathfrak{p}\subset\mathfrak{gl}_{\ell+1}$, where $\mathfrak{b}_-\subset\mathfrak{gl}_{\ell+1}$ is the Borel subalgebra spanned by $\{\mathcal{E}_{ij},\ 1\le j\le i\le\ell+1\}$. Let $\mathfrak{n}_+\subset\mathfrak{gl}_{\ell+1}$ be the nilpotent subalgebra generated by $\{\mathcal{E}_{ij},\ 1\le i<j\le\ell+1\}$, so that the following decomposition holds:

$$\mathfrak{gl}_{\ell+1} = \mathfrak{b}_- \oplus \mathfrak{n}_+ \; = \; \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{5.2}$$

where $\mathfrak{h} \subset \mathfrak{b}$ is the Cartan subalgebra spanned by $\mathcal{E}_{nn}$, $1 \le n \le \ell+1$. Then, the Borel subalgebra $\mathfrak{b}_- \subset \mathfrak{gl}_{\ell+1}$ is the minimal parabolic subalgebra. On the other hand, the maximal parabolic subalgebra $\mathfrak{p}_{1,\ell+1} \subset \mathfrak{gl}_{\ell+1}$ is generated by $\mathfrak{b}_-$ and

$$\mathcal{E}_{i,\,i+1}, \quad 1 \le i < \ell. \tag{5.3}$$

5.1 Minimal parabolic $\mathfrak{gl}_{\ell+1}$-Whittaker function

The standard notion of the $\mathfrak{gl}_{\ell+1}$-Whittaker function defined for a reductive Lie algebra $\mathfrak{gl}_{\ell+1}$ is a special case a more general Whittaker function associated with a pair $\mathfrak{p} \subset \mathfrak{gl}_{\ell+1}$, for a parabolic subalgebra $\mathfrak{p}$. Precisely, the standard Whittaker function thus correspond to the case of minimal parabolic subalgebra (i.e., the Borel subalgebra $\mathfrak{b}_- \subset \mathfrak{gl}_{\ell+1}$).

For $\lambda = (\lambda_1, \ldots, \lambda_{\ell+1}) \in \mathbb{C}^{\ell+1}$, let $(\rho_\lambda,\, \mathcal{W}_\lambda)$ be the principal series representation of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_{\ell+1})$. Namely, for a generic character χ_λ of the Borel subgroup $B_- \subset GL_{\ell+1}$, $\mathrm{Lie}(B_-) = \mathfrak{b}_-$ and the modular character δ_{B_-}, let $\mathcal{W}_\lambda = \mathrm{Ind}_{B_-}^{GL_{\ell+1}}(\chi_\lambda \otimes \delta_{B_-}^{-1/2})$ the induced representation. We consider the infinitesimal form $\mathrm{Ind}_{\mathcal{U}(\mathfrak{b}_-)}^{\mathfrak{gl}_{\ell+1}}(\chi_\lambda \otimes \delta_{B_-}^{-1/2})$ of $\mathcal{W}_\lambda$, and we choose the appropriately defined dual $\mathcal{U}(\mathfrak{gl}_{\ell+1})$-module $(\rho_\lambda^\vee, \mathcal{W}_\lambda^\vee)$. Given the triangular decomposition (5.2), define after [15, 16] the pair of Whittaker vectors $\psi_R \in \mathcal{W}_\lambda$ and $\psi_L \in \mathcal{W}_\lambda^\vee$ to be the generic characters of $\mathfrak{n}_-,\ \mathfrak{n}_+$:

$$\rho_\lambda^\vee(\mathcal{E}_{i+1,\,i})\psi_L \;=\; -\psi_L, \quad \rho_\lambda(\mathcal{E}_{i,\,i+1})\psi_R \;=\; -\psi_R, \qquad 1 \le i \le \ell. \tag{5.4}$$

We assume that the action of the subalgebra $\mathfrak{h} \subset \mathfrak{gl}_{\ell+1}$ is integrated to the action of the maximal torus in $GL_{\ell+1}(\mathbb{R})$. We also imply the existence of $\mathfrak{gl}_{\ell+1}$-invariant non-degenerate pairing $\langle , \rangle$ between the submodules $\mathcal{U}(\mathfrak{gl}_{\ell+1})\psi_L \subseteq \mathcal{W}_\lambda^\vee$ and $\mathcal{U}(\mathfrak{gl}_{\ell+1})\psi_R \subseteq \mathcal{W}_\lambda$. Then, the $\mathfrak{gl}_{\ell+1}$-Whittaker function is given by the following matrix element:

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\ldots,\lambda_{\ell+1}}(e^{x_1}, \ldots, e^{x_{\ell+1}}) \;=\; e^{\sum\limits_{k=1}^{\ell+1}(\frac{\ell}{2}+1-k)x_k} \left\langle \psi_L,\, \rho_\lambda\left(e^{\sum\limits_{j=1}^{\ell+1} x_j\,\mathcal{E}_{jj}} \right) \psi_R \right\rangle. \tag{5.5}$$

This matrix element may be expressed as the following $\ell(\ell+1)$-fold integral [8] for $\mathrm{Re}(\lambda_i) > 0,\ 1 \le i \le \ell+1$:

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\ldots,\lambda_{\ell+1}}(e^{x_1}, \ldots, e^{x_{\ell+1}}) = e^{x_{\ell+1}\sum\limits_{i=1}^{\ell+1}\lambda_i} \int\limits_{\mathbb{R}_+^{2d}} \prod_{1 \le i \le k \le \ell} \frac{da_{k,i}}{a_{k,i}} \frac{db_{k,i}}{b_{k,i}}\; e^{-a_{k,i}-b_{k,i}}$$

$$\times \prod_{1\le i\le k\le \ell} (a_{k,i})^{\lambda_{k+1}} \prod_{k=1}^{\ell} (b_{k,k})^{\sum\limits_{i=1}^{k}\lambda_i}$$
$$\times \prod_{i=1}^{\ell} \delta\Big(e^{-x_i+x_{i+1}} a_{\ell,i} b_{\ell,i} - 1\Big) \prod_{1\le i\le k<\ell} \delta\Big(a_{k,i} b_{k,i} a^{-1}_{k+1,\,i+1} b^{-1}_{k+1,\,i} - 1\Big), \quad (5.6)$$

where $d = \frac{\ell(\ell+1)}{2}$. For $\lambda_1 = \ldots = \lambda_{\ell+1} = 0$, the integral (5.6) originally appeared in [12]. The structure of the integrand in (5.6) might be elucidated by invoking the type A_ℓ Gelfand–Zetlin graph:

$$\begin{tikzcd}
x_1 & & & \\
(\ell,1) \arrow[u, "a_{\ell,1}"] & x_2 \arrow[l, "b_{\ell,1}"'] & & \\
\vdots \arrow[u] & \ddots \arrow[u] & \ddots & \\
(1,1) \arrow[u, "a_{11}"] & \ldots \arrow[l, "b_{11}"] & (\ell,\ell) \arrow[l] \arrow[u, "a_{\ell\ell}"] & x_{\ell+1} \arrow[l, "b_{\ell\ell}"]
\end{tikzcd} \quad (5.7)$$

Namely, we attach the integration variables $a_{k,i}$, $b_{k,i}$ to the edges and add the arguments x_i, $1 \le i \le \ell+1$ at the boundary vertexes of the graph. The delta-factors in the integrand of (5.6) correspond to fixing products of variables $a_{k,i}$ and $b_{k,i}$ along various paths on the diagram (5.7). Namely, we consider the ℓ paths starting horizontally at x_{i+1} then turning upwards to the adjacent vertex x_i and the $\ell(\ell-1)/2$ elementary box-shaped paths.

On the other hand, the integral (5.6) can be identified with the GG-integral of the form (2.36) associated with $|I| = 2d = \ell(\ell+1)$ (corresponding to the arrows in (5.7)) and $|J| = d = \ell(\ell+1)/2$ (corresponding to the vertices in (5.7)). Let us introduce the integration variables $t_i \in \mathbb{R}_+$, $i \in I$:

$$\big\{t_i,\ 1 \le i \le \ell(\ell+1)\big\} = \{a_{k,i},\ b_{k,i}:\quad 1 \le i \le k \le \ell\}, \quad (5.8)$$

and the two sets of arguments:

$$\big\{y_{k,i},\ z_{k,i}:\quad 1 \le i \le k \le \ell\big\} \subset \mathbb{R}^{2d},$$
$$\big\{\gamma_{k,i},\ \nu_{k,i}:\quad 1 \le i \le k \le \ell\big\} \subset \mathbb{C}^{2d}, \qquad \mathrm{Re}(\gamma_{k,i}) > 0, \quad \mathrm{Re}(\nu_{k,i}) > 0. \quad (5.9)$$

Consider the following integral of GG-type:

$$\Phi_{\gamma,\nu}^{\mathfrak{gl}_{\ell+1}}(e^y,\,e^z) = \int_{\mathbb{R}_+^{2d}} \prod_{1\le k\le i\le \ell} \frac{da_{k,i}}{a_{k,i}}\frac{db_{k,i}}{b_{k,i}}\; a_{k,i}^{\gamma_{k,i}} b_{k,i}^{\nu_{k,i}}\, e^{-a_{k,i}-b_{k,i}} \times \prod_{i=1}^{\ell} \delta\Big(e^{-y_{\ell,i}-z_{\ell,i}} a_{\ell,i} b_{\ell,i} - 1\Big) \times \prod_{1\le i\le k<\ell} \delta\Big(e^{-y_{k,i}-z_{k,i}+y_{k+1,\,i+1}+z_{k+1,\,i}} a_{k,i} b_{k,i} a_{k+1,\,i+1}^{-1} b_{k+1,\,i}^{-1} - 1\Big), \tag{5.10}$$

associated with the following GKZ-data. Introduce the standard orthonormal basis in $\mathbb{Z}^{2d}$:

$$\left\{\epsilon^{k,i},\, \tilde{\epsilon}^{\,k,i} :\quad 1\le i\le k\le \ell\right\} \subset \mathbb{Z}^{2d}. \tag{5.11}$$

the relation lattice,

$$\mathbb{L} = \mathrm{span}\{l^{k,i},\; 1\le i\le k\le \ell\} \subseteq \mathbb{Z}^{2d}, \tag{5.12}$$

is generated by

$$l^{k,i} = \epsilon^{k,i} + \tilde{\epsilon}^{\,k,i} - \epsilon^{k+1,\,i+1} - \tilde{\epsilon}^{\,k+1,\,i} \in \mathbb{L}, \tag{5.13}$$

where $\epsilon^{\ell+1,\,i+1} = \tilde{\epsilon}^{\,\ell+1,\,i} = 0$ is assumed. Let $M \in \mathrm{Mat}_{d\times 2d}$ be the relation matrix with rows given by the generators $l^{k,i}$, $1\le i\le k\le \ell$. The defining matrix, $A \in \mathrm{Mat}_{d\times 2d}(\mathbb{Z})$, should satisfy the orthogonality relation (2.7), $AM^{\top} = 0 \in \mathrm{Mat}_{d\times d}(\mathbb{Z})$. One might choose the rows of A to be the following:

$$\alpha^{k,1} = \sum_{j=1}^{k}(\epsilon^{k,j} - \tilde{\epsilon}^{\,k,j}), \qquad 1\le k\le \ell,$$
$$\alpha^{k,i} = \sum_{j=1}^{i-1}\epsilon^{k-j,\,i-j} + \sum_{j=i}^{k}(\epsilon^{k,j} - \tilde{\epsilon}^{\,k,j}), \qquad 1< i\le k\le \ell. \tag{5.14}$$

Let us stress that GG-hypergeometric function (5.10) effectively depends only on the d independent variables,

$$R_i(y,z) := y_{\ell,i} + z_{\ell,i}, \qquad 1\le i\le \ell,$$
$$B_{k,i}(y,z) := y_{k,i} + z_{k,i} - y_{k+1,\,i+1} - z_{k+1,\,i}, \qquad 1\le i\le k<\ell, \tag{5.15}$$

which is obvious from the integral representation (5.10). This may be attributed to the set of linear Eq. (2.28) satisfied by the integral (5.10).

Now let us introduce the following restriction of the GG-hypergeometric function (5.10),

$$ {}^{\mathrm{res}}\Phi^{\mathfrak{gl}_{\ell+1}}_{\gamma,\,\nu}(e^x) := \Phi^{\mathfrak{gl}_{\ell+1}}_{\gamma,\,\nu}(e^y,\,e^z)\Big|_{\mathcal{L}}, \tag{5.16} $$

onto the ℓ-dimensional linear subspace $\mathcal{L}\subset\mathbb{R}^d$ with the coordinates x_i-x_{i+1}, $1\le i\le\ell$. Precisely, the subspace $\mathcal{L}$ is defined by the following constraints on the independent coordinates (5.15) in $\mathbb{R}^d$:

$$ \begin{aligned} R_i(y,z) &= x_i-x_{i+1}, \qquad 1\le i\le\ell,\\ B_{k,i}(y,z) &= 0, \qquad 1\le i\le k<\ell. \end{aligned} \tag{5.17} $$

Lemma 5.1 *For the following special values of spectral variables from* (5.9)*,*

$$ \gamma_{k,i}(\lambda)=\lambda_{k+1},\quad 1\le i\le k\le\ell;\qquad \nu_{k,k}(\lambda)=\sum_{i=1}^{k}\lambda_i,\quad 1\le k\le\ell, $$
$$ \nu_{k,i}(\lambda)=0,\quad 1\le i<k\le\ell, \tag{5.18} $$

the restricted GG-hypergeometric function (5.16) *is expressed through the* $\mathfrak{gl}_{\ell+1}$*-Whittaker function* (5.6) *as follows:*

$$ {}^{\mathrm{res}}\Phi^{\mathfrak{gl}_{\ell+1}}_{\gamma(\lambda),\,\nu(\lambda)}(e^x) = e^{-\sum\limits_{i=1}^{\ell+1}\lambda_i\,x_{\ell+1}}\,\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda}(e^x). \tag{5.19} $$

Proof The identification directly follows by comparing explicit integral representations (5.6) and (5.10). □

The difference in the exponential pre-factor may be taken into account by considering a slightly extended GKZ data and taking a limit of the resulting hypergeometric function. Let us extend the sets I, J defined by the graph (5.7) by adding one element to each J and I, and introduce the extended relation and defining matrices $\widehat{M},\ \widehat{A}\in\mathrm{Mat}_{(d+1)\times(2d+1)}(\mathbb{Z})$:

$$ \widehat{J}:=J\sqcup\{d+1\},\qquad \widehat{I}:=I\sqcup\{2d+1\},\qquad d=\frac{\ell(\ell+1)}{2}, $$
$$ \widehat{M}:=\begin{pmatrix} M & 0\\ 0 & 1\end{pmatrix},\quad \widehat{A}:=\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix},\qquad \widehat{A}\widehat{M}^{\top}=0\in\mathrm{Mat}_{(2d+1)\times(2d+1)}(\mathbb{Z}), \tag{5.20} $$

so that the extended relation lattice $\widehat{\mathbb{L}}$ is spanned by its rows $l^{k,i}$, $(k,i)\in J$ and $l^{d+1}=(0,\dots,0,1)$. Then, the matrix element (3.16) associated with the GKZ data (5.20) for $\gamma_*\in\mathbb{C}$, $\mathrm{Re}(\gamma_*)>0$ reads

$$ \widehat{\Phi}_{\gamma,\nu;\,\gamma_*}(e^y,e^z;\,e^{y_*}) = \int\limits_{\mathbb{R}_+^{2d+1}}\frac{dt_{2d+1}}{t_{2d+1}}\,\delta(e^{-y_*}t_{2d+1}-1)\,t_{2d+1}^{\gamma_*}\,e^{-t_{2d+1}} $$
$$ \times\prod_{1\le k\le i\le\ell}\frac{da_{k,i}}{a_{k,i}}\frac{db_{k,i}}{b_{k,i}}\;a_{k,i}^{\gamma_{k,i}}b_{k,i}^{\nu_{k,i}}\,e^{-a_{k,i}e^{y_{k,i}}-b_{k,i}e^{z_{k,i}}} $$

$$\times \prod_{i=1}^{\ell} \delta\Big(e^{-y_{\ell,i}-z_{\ell,i}} a_{\ell,i} b_{\ell,i} \;-\; 1\Big)$$
$$\times \prod_{1\le i\le k<\ell} \delta\Big(e^{-y_{k,i}-z_{k,i}+y_{k+1,\,i+1}+z_{k+1,\,i}} a_{k,i} b_{k,i} a^{-1}_{k+1,\,i+1} b^{-1}_{k+1,\,i} \;-\; 1\Big)$$
$$= e^{\gamma_* y_* - e^{y_*}} \times \Phi^{\mathfrak{gl}_{\ell+1}}_{\gamma,\,\nu}(e^y,\, e^z). \tag{5.21}$$

Therefore, the $\mathfrak{gl}_{\ell+1}$-Whittaker function (5.6) can be obtained as the following limit of the GG-hypergeometric function (5.21):

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\ldots,\lambda_{\ell+1}}(e^x) \;=\; \lim_{\epsilon\to-\infty}\Big({}^{\mathrm{res}}\widehat{\Phi}_{\gamma(\lambda),\nu(\lambda);\,\epsilon^{-1}\gamma_*}(e^x,\; e^{\epsilon y_*})\Big)\Big|_{\substack{y_*=x_{\ell+1}\\ \gamma_*=\lambda_1+\ldots+\lambda_{\ell+1}}}, \tag{5.22}$$

where ${}^{\mathrm{res}}\widehat{\Phi}$ is obtained by imposing the restrictions (5.17). Thus, $\mathfrak{gl}_{\ell+1}$-Whittaker function belongs to a partial compactification of the space of GG-hypergeometric functions.

5.2 Maximal parabolic $\mathfrak{gl}_{\ell+1}$-Whittaker function

Recall from [10] the definition of generalized Whittaker function associated with maximal parabolic subalgebra $\mathfrak{p}_{1,\ell+1} \subset \mathfrak{gl}_{\ell+1}$ (called the $(1;\,\ell+1)$-Whittaker function in [10]). To the maximal parabolic subalgebra $\mathfrak{p}_{(1,\ell+1)}$, we associate the following decomposition of $\mathfrak{gl}_{\ell+1}$

$$\mathfrak{gl}_{\ell+1} \;=\; \mathfrak{h}^{(1;\,\ell+1)} \,\oplus\, \mathfrak{n}_-^{(1;\,\ell+1)} \oplus \mathfrak{n}_+, \tag{5.23}$$

where $\mathfrak{n}_-^{(1;\,\ell+1)} \subset \mathfrak{b}_-$ is the $\ell(\ell+1)/2$-dimensional subalgebra generated by

$$\mathcal{E}_{\ell+1,\,1}; \qquad \mathcal{E}_{m+1,\,m}, \quad \mathcal{E}_{m+1,\,m+1}, \quad 1 \le m < \ell. \tag{5.24}$$

and $\mathfrak{h}^{(1,\ell+1)}$ is the $(\ell+1)$-dimensional commutative subalgebra spanned by

$$h_1^{(1;\,\ell+1)} = \mathcal{E}_{11}, \qquad h_2^{(1;\,\ell+1)} = \mathcal{E}_{22} + \ldots + \mathcal{E}_{\ell+1,\,\ell+1},$$
$$h_{m+1}^{(1;\,\ell+1)} = \mathcal{E}_{\ell+1,\,m}, \quad 1 < m \le \ell. \tag{5.25}$$

For $\lambda = (\lambda_1, \ldots, \lambda_{\ell+1}) \in \mathbb{C}^{\ell+1}$, let $(\rho_\lambda,\, \mathcal{W}_\lambda)$ be the principal series representation of $\mathcal{U}(\mathfrak{gl}_{\ell+1})$. Then, we choose the appropriately defined dual module $(\rho_\lambda^\vee,\, \mathcal{W}_\lambda^\vee)$. Given the decomposition (5.23), we introduce the pair of vectors $\psi_L^{(1;\,\ell+1)} \in \mathcal{W}_\lambda^\vee$ and $\psi_R \in \mathcal{W}_\lambda$ to be the generic characters of the subalgebras $\mathfrak{n}_-^{(1;\,\ell+1)}$ and $\mathfrak{n}_+$, where $\mathfrak{n}_-^{(1;\,\ell+1)}$ is generated by (5.24) (see [10, 17]):

$$\rho_\lambda^\vee(\mathcal{E}_{\ell+1,\,1})\psi_L^{(1;\,\ell+1)} \;=\; -(-1)^{\frac{\ell(\ell-1)}{2}}\,\psi_L^{(1;\,\ell+1)},$$

$$\rho_\lambda^\vee(\mathcal{E}_{21})\psi_L^{(1;\,\ell+1)} = \rho_\lambda^\vee(\mathcal{E}_{m+1,\,m})\psi_L^{(1;\,\ell+1)} = 0, \quad 1 < m < \ell; \tag{5.26}$$

and

$$\rho_\lambda(\mathcal{E}_{i,\,i+1})\psi_R \,=\, -\psi_R, \qquad 1 \leq i \leq \ell. \tag{5.27}$$

We imply the existence of $\mathfrak{gl}_{\ell+1}$-invariant non-degenerate pairing $\langle,\,\rangle$ between the submodules $\mathcal{U}(\mathfrak{gl}_{\ell+1})\psi_L^{(1;\,\ell+1)} \subseteq \mathcal{W}_\lambda^\vee$ and $\mathcal{U}(\mathfrak{gl}_{\ell+1})\psi_R \subseteq \mathcal{W}_\lambda$. Then, the maximal parabolic Whittaker function is defined by the following matrix element:

$$\Psi_\lambda^{(1;\,\ell+1)}(e^{x_1}, e^{x_2}, \ldots, e^{x_{\ell+1}}) \,=\, e^{x_1 - x_{\ell+1}} \langle \psi_L^{(1;\,\ell+1)},\, \rho_\lambda\Big(e^{\sum_j x_j h_j^{(1;\,\ell+1)}}\Big)\psi_R\rangle. \tag{5.28}$$

The expression (5.28) implies that the action of the part of subalgebra $\mathfrak{b}_- \subset \mathfrak{gl}_{\ell+1}$ may be integrated to the action of the corresponding group. In the following, we consider the restricted maximal parabolic Whittaker function [10, 17], for $\mathrm{Re}(\lambda_i) > 0$, $1 \leq i \leq \ell+1$ given by

$$\begin{aligned}\Psi_\lambda^{(1;\,\ell+1)}(e^x) &:= \Psi_\lambda^{(1;\,\ell+1)}(e^x, 1\,\ldots, 1) \,=\, e^x \langle \psi_L^{(1;\,\ell+1)},\, \rho_\lambda(e^{x\,\mathcal{E}_{11}})\psi_R\rangle \\ &= e^{\lambda_{\ell+1}x} \int\limits_{\mathbb{R}_+^\ell} \prod_{i=1}^{\ell} \frac{dt_i}{t_i}\, t_i^{\lambda_i - \lambda_{\ell+1}}\, e^{-t_i}\, e^{-e^x \prod\limits_{i=1}^{\ell} t_i^{-1}} \\ &= \int\limits_{\mathbb{R}_+^{\ell+1}} \prod_{i=1}^{\ell+1} \frac{dt_i}{t_i}\, t_i^{\lambda_i}\, e^{-t_i}\, \delta\Big(e^{-x}\prod_{i=1}^{\ell+1} t_i \,-1\Big).\end{aligned} \tag{5.29}$$

Therefore, we only shall assume that the action of the generator $h_1^{(1;\,\ell+1)} = \mathcal{E}_{11} \in \mathfrak{h}^{(1;\,\ell+1)}$ integrates to the action of the corresponding one-parameter subgroup in $GL_{\ell+1}(\mathbb{R})$.

To put the restricted maximal parabolic Whittaker function (5.29) into the framework of GG-hyper- geometric functions, consider the following instance of GKZ-system with $|J| = 1$, $|I| = N = \ell+1$, $m = \ell$ and the following matrices $A \in \mathrm{Mat}_{\ell\times(\ell+1)}(\mathbb{Z})$ and $M \in \mathrm{Mat}_{1\times(\ell+1)}(\mathbb{Z})$

$$M = (1\ldots 1), \qquad A = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 & -1 \end{pmatrix},$$

$$A^\top = (\alpha_1, \ldots, \alpha_\ell), \qquad \alpha_j = \epsilon_j - \epsilon_{j+1} \in \mathbb{Z}^{\ell+1}, \qquad MA^\top = 0. \tag{5.30}$$

Here $\{\epsilon_j \,:\, j \in I\} \subset \mathbb{Z}^{\ell+1}$ is the standard basis. Let $\mathfrak{L}_{\ell+1}$ be the Lie algebra (3.1), and for $\lambda = (\lambda_1, \dots, \lambda_{\ell+1}) \in \mathbb{C}^{\ell+1}$, $\mathrm{Re}(\lambda_i) > 0$, let π_λ be the $\mathcal{U}(\mathfrak{L}_{\ell+1})$-module $\mathcal{V}_\lambda$ (3.3) modeled on $\mathcal{S}(\mathbb{R}_+^{\ell+1})$. Vector $\phi_R \in \mathcal{V}_\lambda$ and covector $\phi_L \in \mathcal{V}_\lambda^\vee$ read from (3.10):

$$\phi_R(t) = \prod_{i\in I} e^{-t_i}, \qquad \phi_L(t) = t_i^{\lambda_i - 1}\,\delta\Big(\prod_{j\in I} t_j - 1\Big). \tag{5.31}$$

Then, the matrix element (3.16) in the $\mathfrak{L}_{\ell+1}$-representation $\mathcal{V}_\lambda$ affords the following integral representation:

$$\begin{aligned}\Phi^{(1;\,\ell+1)}_{\lambda_1,\dots,\lambda_{\ell+1}}(e^{y_1}, \dots, e^{y_{\ell+1}}) &= \Big\langle \phi_L,\, \pi_\lambda\Big(e^{\sum_i y_i \mathcal{H}_i}\Big)\phi_R\Big\rangle \qquad (5.32)\\ &= \int\limits_{\mathbb{R}_+^{\ell+1}} \prod_{i\in I} \frac{dt_i}{t_i}\, t_i^{\lambda_i}\, e^{\lambda_i y_i - t_i e^{y_i}}\, \delta\Big(\prod_{j\in I} t_j - 1\Big)\\ &= \int\limits_{\mathbb{R}_+^{\ell+1}} \prod_{i\in I} \frac{dt_i}{t_i}\, t_i^{\lambda_i}\, e^{-t_i}\, \delta\Big(\prod_{j\in I} e^{-y_j} t_j - 1\Big). \qquad (5.33)\end{aligned}$$

Let us introduce variables $u_i := e^{y_i}$, $i \in I$ and write (5.32) as follows:

$$\Phi^{(1;\,\ell+1)}_{\lambda_1,\dots,\lambda_{\ell+1}}(u_1, \dots, u_{\ell+1}) = \int\limits_{\mathbb{R}_+^{\ell+1}} \prod_{i\in I} \frac{dt_i}{t_i}\, t_i^{\lambda_i}\, e^{-t_i}\, \delta\Big(\prod_{j\in I} u_j^{-1} t_j - 1\Big). \tag{5.34}$$

By the results of Sect. 2, (or via a simple direct computation), the GG-hypergeometric function $\Phi^{(1;\,\ell+1)}_\lambda(u)$ satisfies the following instances of Eqs. (2.25)–(2.29):

$$\begin{aligned}&\prod_{i\in I}\Big(-\frac{\partial}{\partial u_i} + \frac{\lambda_i}{u_i}\Big)\cdot \Phi^{(1;\,\ell+1)}_\lambda(u) = \Phi^{(1;\,\ell+1)}_\lambda(u);\\ &\prod_{i\in I} e^{\partial_{\lambda_i}} \cdot \Phi^{(1;\,\ell+1)}_\lambda(u) = \prod_{i\in I} u_i\, \Phi^{(1;\,\ell+1)}_\lambda(u);\\ &\Big(u_i\frac{\partial}{\partial u_i} - u_j\frac{\partial}{\partial u_j}\Big)\cdot \Phi^{(1;\,\ell+1)}_\lambda(u) = 0, \qquad i \neq j;\\ &\Big\{e^{\partial_{\lambda_i}} - e^{\partial_{\lambda_j}}\Big\}\cdot \Phi^{(1;\,\ell+1)}_\lambda(u) = 0, \qquad i \neq j,\end{aligned} \tag{5.35}$$

and the following instance of (2.43):

$$\Big\{-u_i\frac{\partial}{\partial u_i} + \lambda_i\Big\}\cdot \Phi^{(1;\,\ell+1)}_\lambda(u) = e^{\partial_{\lambda_i}} \cdot \Phi^{(1;\,\ell+1)}_\lambda(u). \tag{5.36}$$

The first line of equations in (5.35) can be written in the following form:

$$\left\{\prod_{i\in I}\Big(-u_i\frac{\partial}{\partial u_i}+\lambda_i\Big)-\prod_{i\in I}u_i\right\}\cdot\Phi_\lambda^{(1;\,\ell+1)}(u)\;=\;0, \tag{5.37}$$

As a consequence of the third line of equations in (5.35), the function $\Phi_\lambda^{(1;\,\ell+1)}(u)$ reduces to a function in one variable

$$\Phi^{(1;\,\ell+1)}_{\lambda_1,\ldots,\lambda_{\ell+1}}(e^{y_1},\ldots,e^{y_{\ell+1}})=\widetilde{\Phi}^{(1;\,\ell+1)}_{\lambda_1,\ldots,\lambda_{\ell+1}}(e^x),\qquad x=y_1+\ldots+y_{\ell+1}, \tag{5.38}$$

satisfying the reduced form of (5.37):

$$\left\{\prod_{i\in I}\Big(-\frac{\partial}{\partial x}+\lambda_i\Big)-e^x\right\}\cdot\widetilde{\Phi}_\lambda^{(1;\,\ell+1)}(e^x)\;=\;0. \tag{5.39}$$

Indeed, Eq. (5.37) is a simple consequence of the first line in (5.35). By the third equation in (5.35), the function (5.33) depends on a single variable $x=y_1+\ldots+y_{\ell+1}$:

$$\Phi^{(1;\,\ell+1)}_{\lambda_1,\ldots,\lambda_{\ell+1}}(e^{y_1},\ldots,e^{y_{\ell+1}})\;=\int\limits_{\mathbb{R}_+^{\ell+1}}\prod_{i\in I}\frac{dt_i}{t_i}\,t_i^{\lambda_i}\,e^{-t_i}\,\delta\Big(e^{-\sum\limits_{j=1}^{\ell+1}y_j}\prod_{j\in I}t_j\;-\;1\Big). \tag{5.40}$$

Thus, we may introduce the function $\widetilde{\Phi}_{\lambda_1,\ldots,\lambda_{\ell+1}}(e^x)$, which by (5.38) has the following integral representation:

$$\widetilde{\Phi}^{(1;\,\ell+1)}_{\lambda_1,\ldots,\lambda_{\ell+1}}(e^x):=\int\limits_{\mathbb{R}_+^{\ell+1}}\prod_{i\in I}\frac{dt_i}{t_i}\,t_i^{\lambda_i}\,e^{-t_i}\,\delta\Big(e^{-x}\prod_{j\in I}t_j\;-\;1\Big), \tag{5.41}$$

and satisfies Eq. (5.39) by construction.

Finally, we may identify the maximal parabolic Whittaker function $\Psi^{(1;\,\ell+1)}_{\lambda_1,\ldots,\lambda_{\ell+1}}(e^x)$ defined by (5.29) with the reduced GG-hypergeometric function (5.38) as follows:

$$\Psi^{(1;\,\ell+1)}_{\lambda_1,\ldots,\lambda_{\ell+1}}(e^x)\;=\;\widetilde{\Phi}^{(1;\,\ell+1)}_{\lambda_1,\ldots,\lambda_{\ell+1}}(e^x). \tag{5.42}$$

Acknowledgements The research of D.R.L. was supported by RSF grant 16-11-10075. The work of S.V.O. was partially supported by the EPSRC Grant EP/L000865/1.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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