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Segre products and Segre morphisms in a class of Yang–Baxter algebras

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Abstract
Let (X, r_X) and (Y, r_Y) be finite nondegenerate involutive set-theoretic solutions of the Yang–Baxter equation, and let $A_X = \mathcal{A}(\mathbf{k}, X, r_X)$ and $A_Y = \mathcal{A}(\mathbf{k}, Y, r_Y)$ be their quadratic Yang–Baxter algebras over a field $\mathbf{k}$. We find an explicit presentation of the Segre product $A_X \circ A_Y$ in terms of one-generators and quadratic relations. We introduce analogues of Segre maps in the class of Yang–Baxter algebras and find their images and their kernels. The results agree with their classical analogues in the commutative case.

Keywords Quadratic algebras · PBW algebras · Koszul algebras · Segre products · Segre maps · Yang–Baxter equation · Yang–Baxter algebras

Mathematics Subject Classification Primary 16S37 · 16T25 · 16S38 · 16S15 · 81R60

1 Introduction

It was established in the last three decades that solutions of the linear braid or Yang–Baxter equations (YBE)

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$

on a vector space of the form $V^{\otimes 3}$ lead to remarkable algebraic structures. Here, $r : V \otimes V \longrightarrow V \otimes V$, $r^{12} = r \otimes \mathrm{id}$, $r^{23} = id \otimes r$ is a notation and structures

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include coquasitriangular bialgebras $A(r)$, their quantum group (Hopf algebra) quotients, quantum planes and associated objects, at least in the case of specific standard solutions, see [21, 30]. On the other hand, the variety of all solutions on vector spaces of a given dimension has remained rather elusive in any degree of generality. It was proposed by Drinfeld [4], to consider the same equations in the category of sets, and in this setting, numerous results were found. It is clear that a set-theoretic solution extends to a linear one, but more important than this is that set-theoretic solutions lead to their own remarkable algebraic and combinatoric structures, only somewhat analogous to quantum group constructions. In the present paper, we continue our systematic study of set-theoretic solutions of YBE and the associated quadratic algebras that they generate.

The study of non-commutative algebras defined by quadratic relations as examples of quantum non-commutative spaces has undoubtedly received considerable impetus from the seminal work [5], where the authors considered general deformations of quantum groups and spaces arising from an R-matrix, and from Manin's program for non-commutative geometry [23]. The quadratic algebras related to set-theoretic solutions of the Yang–Baxter equation studied here are important for both noncommutative algebra and non-commutative algebraic geometry, as they provide a rich source of examples of interesting associative algebras and non-commutative spaces some of which are Artin-Schelter regular algebras. Our work is motivated by the relevance of those algebras for non-commutative geometry, especially in relation to the theory of quantum groups, and inspired by the interpretation of morphisms between non-commutative algebras as "maps between non-commutative spaces". In [14] and the present paper, we consider non-commutative analogues of the Veronese and Segre embeddings, two fundamental maps that play pivotal roles not only in classical algebraic geometry but also in applications to other fields of mathematics.

In this paper "a solution of YBE," or shortly, "a solution" means "a nondegenerate involutive set-theoretic solution of YBE," see Definition 2.5.

The Yang–Baxter algebras $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ related to solutions (X, r), of finite order $|X| = n$ will play a central role in the paper. It was proven in [17] that the quadratic algebra $\mathcal{A}_X$ of every finite solution (X, r) of YBE has remarkable algebraic, homological and combinatorial properties. In general, the algebra $\mathcal{A}_X$ is noncommutative and in most cases it is not even a PBW algebra, but it preserves various good properties of the commutative polynomial ring $\mathbf{k}[x_1, \dots, x_n]$: $\mathcal{A}_X$ has finite global dimension and polynomial growth, it is Cohen-Macaulay, Koszul, and a Noetherian domain.

It was shown through the years that there are close relations between various combinatorial properties of the solution (X, r) and the properties of the corresponding algebra $\mathcal{A}_X$, see [9, 10, 12, 13, 16, 29]. In the special case when (X, r) is *a finite nondegenerate involutive square-free quadratic set* whose quadratic algebra $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ has a $\mathbf{k}$-basis of Poincaré–Birkhoff–Witt type, the conditions "*$\mathcal{A}$ is an Artin-Schelter regular algebra*" and "*(X, r) is a solution of YBE*" *are equivalent*, see details in Sect. 2. The study of Artin-Schelter regular algebras is a central problem for noncommutative algebraic geometry.

A first stage of noncommutative geometry on $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ was proposed in [16], Sect. 6, where the quantum spaces under investigation are Yang–Baxter algebras

$\mathcal{A}(\mathbf{k}, X, r)$ associated with multipermutation (square-free) solutions of level two. In [2] a class of quadratic PBW algebras called "noncommutative projective spaces" were investigated and analogues of Veronese and Segre morphisms between noncommutative projective spaces were introduced and studied. It is natural to formulate similar problems for the class of Yang–Baxter algebras $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ related to finite solutions (X, r), but to find reasonable solutions of these problems is a nontrivial task. In contrast with [2], where the "noncommutative projective spaces" under investigation have almost commutative quadratic relations which form Gröbner bases, and the main results follow naturally from the theory of Noncommutative Gröbner bases, the Yang–Baxter algebras $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ have complicated quadratic relations, which in most cases do not form Gröbner bases. These relations remain complicated even when $\mathcal{A}$ is a PBW algebra, so we need more sophisticated arguments and techniques, see for example [14]. In the present paper, we consider the following problem.

Problem 1.1 Let (X, r_X) and (Y, r_Y) be finite (not necessarily square-free) solutions of YBE whose Yang–Baxter algebras are $A = \mathcal{A}(\mathbf{k}, X, r)$ and $B = \mathcal{A}(\mathbf{k}, Y, r_Y)$, respectively.

(1) Find a presentation of the Segre product $A \circ B$ in terms of one-generators and linearly independent quadratic relations.
(2) Introduce analogues of Segre maps for the class of Yang–Baxter algebras of finite solutions of YBE.
(3) Study separately Segre products and analogues of Segre maps in the special case when (X, r_X) and (Y, r_Y) are square-free solutions. (Note that only in this case the algebras A and B are binomial skew polynomial rings).

The special attention to Problem 1.1 (3) is motivated by Remarks 2.13 and 2.14.

Our main results are Theorems 3.10, 4.5 and 5.1 which solve completely the problem.

The paper is organized as follows. In Sect. 2, we recall some basic definitions, we fix the main settings and conventions we present useful facts about the Yang–Baxter algebras $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$, results adapted to our settings and needed for the proofs of the main theorems. In Sect. 3, we study the Segre product $A \circ B$ of the Yang–Baxter algebras $A = \mathcal{A}(\mathbf{k}, X, r_1)$ and $B = \mathcal{A}(\mathbf{k}, Y, r_2)$ of two finite solutions (X, r_1) and (Y, r_2), respectively. We prove Theorem 3.10 which gives an explicit finite presentation of the Segre product $A \circ B$ in terms of one-generators and linearly independent quadratic relations. In Sect. 4, we introduce analogues of Segre morphisms $s_{m,n}$ for quantum spaces $A = \mathcal{A}(\mathbf{k}, X, r_1)$ and $B = \mathcal{A}(\mathbf{k}, Y, r_2)$ related to finite solutions (X, r_1) and (Y, r_2) of orders m and n, respectively. We involve an abstract solution (Z, r_Z) of order mn which is isomorphic to the Cartesian product of solutions $(X \times Y, r_{X\times Y})$ and define the Segre map $s_{m,n}\ \mathcal{A}(\mathbf{k}, Z, r_Z) \longrightarrow A \otimes B$. Theorem 4.5 shows that the image of the map $s_{m,n}$ is the Segre product $A \circ B$ and describes explicitly *a minimal set of generators for its kernel*. Corollary 4.6 shows that the Segre product $A \circ B$ is left and right Noetherian. The results agree with their classical analogues in the commutative case, [18]. We end the section with open questions, see Questions 4.7. In Sect. 5, we pay special attention to the subclass of Yang–Baxter algebras of finite square-free solutions. It is known that all algebras in this subclass are binomial skew polynomial rings, see [29]. Theorem 5.1 shows that the Segre product $A \circ B$ of the YB algebras $A = \mathcal{A}(\mathbf{k}, X, r_1)$ and $B = \mathcal{A}(\mathbf{k}, Y, r_2)$ is a PBW algebra, whenever (X, r_1) and (Y, r_2) are finite square-free solutions and provides an explicit standard finite presentation of $A \circ B$ in terms of PBW generators and quadratic relations which form a Gröbner basis. The theorem implies that analogues of Segre maps are well-defined in the subclass of YB algebras of finite square-free solutions. In Sect. 6, we give an example which illustrates our results.

2 Preliminaries

Let X be a non-empty set, and let $\mathbf{k}$ be a field. We denote by $\langle X\rangle$ the free monoid generated by X, where the unit is the empty word denoted by 1, and by $\mathbf{k}\langle X\rangle$-the unital free associative $\mathbf{k}$-algebra generated by X. For a non-empty set $F \subseteq \mathbf{k}\langle X\rangle$, (F) denotes the two sided ideal of $\mathbf{k}\langle X\rangle$ generated by F. When the set X is finite, with $|X| = n$, and ordered, we write $X = \{x_1, \dots, x_n\}$ and fix the degree-lexicographic order $<$ on $\langle X\rangle$, where $x_1 < \cdots < x_n$. In what follows, $\mathbb{N}$ denotes the set of all positive integers, and $\mathbb{N}_0$ is the set of all non-negative integers.

We shall consider associative graded $\mathbf{k}$-algebras. Suppose $A = \bigoplus_{m\in\mathbb{N}_0} A_m$ is a graded $\mathbf{k}$-algebra such that $A_0 = \mathbf{k}$, $A_pA_q \subseteq A_{p+q}$, $p, q \in \mathbb{N}_0$, and such that A is finitely generated by elements of positive degree. Recall that its Hilbert function is $h_A(m) = \dim A_m$ and its Hilbert series is the formal series $H_A(t) = \sum_{m\in\mathbb{N}_0} h_A(m)t^m$. In particular, the algebra $\mathbf{k}[X] = \mathbf{k}[x_1, \dots, x_n]$ of commutative polynomials satisfies

$$h_{\mathbf{k}[X]}(d) = \binom{n+d-1}{d} = \binom{n+d-1}{n-1} \quad \text{and} \quad H_{\mathbf{k}[X]} = \frac{1}{(1-t)^n}. \tag{2.1}$$

We shall use the *natural grading by length* on the free associative algebra $\mathbf{k}\langle X\rangle$. For $m \geq 1$, X^m will denote the set of all words $u = x_{i_1} \dots x_{i_m}$ of length m in $\langle X\rangle$. Then,

$$\langle X\rangle = \bigsqcup_{m\in\mathbb{N}_0} X^m, \;\; X^0 = \{1\}, \;\text{and}\;\; X^kX^m \subseteq X^{k+m},$$

so the free monoid $\langle X\rangle$ is naturally *graded by length*.

Similarly, the free associative algebra $\mathbf{k}\langle X\rangle$ is also graded by length:

$$\mathbf{k}\langle X\rangle = \bigoplus_{m\in\mathbb{N}_0} \mathbf{k}\langle X\rangle_m, \quad \text{where}\;\; \mathbf{k}\langle X\rangle_m = \mathbf{k}X^m.$$

A polynomial $f \in \mathbf{k}\langle X\rangle$ is *homogeneous of degree m* if $f \in \mathbf{k}X^m$. We denote by

$$\mathcal{T} = \mathcal{T}(X) := \left\{x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \langle X\rangle \mid \alpha_i \in \mathbb{N}_0, i \in \{1, \dots, n\}\right\}$$

the set of ordered monomials (terms) in $\langle X\rangle$.

2.1 Gröbner bases for ideals in the free associative algebra

We shall briefly remind some basics on noncommutative Gröbner bases which will be used throughout the paper. In this subsection, $X = \{x_1, \dots, x_n\}$. Suppose $f \in \mathbf{k}\langle X\rangle$ is a nonzero polynomial. Its leading monomial with respect to the degree-lexicographic order $<$ on $\langle X\rangle$ will be denoted by $\mathbf{LM}(f)$. One has $\mathbf{LM}(f) = u$ if $f = cu + \sum_{1\leq i\leq m} c_i u_i$, where $c, c_i \in \mathbf{k}$, $c \neq 0$ and $u > u_i$ in $\langle X\rangle$, for every $i \in \{1, \dots, m\}$. Given a set $F \subseteq \mathbf{k}\langle X\rangle$ of non-commutative polynomials, $\mathbf{LM}(F)$ denotes the set

$$\mathbf{LM}(F) = \{\mathbf{LM}(f) \mid f \in F\}.$$

A monomial $u \in \langle X\rangle$ is *normal modulo* F if it does not contain any of the monomials $\mathbf{LM}(f)$, $f \in F$ as a subword. The set of all normal monomials modulo F is denoted by $N(F)$.

Let I be a two sided graded ideal in $\mathbf{k}\langle X\rangle$ and let $I_m = I \cap \mathbf{k}X^m$. We shall assume that *$I$ is generated by homogeneous polynomials of degree* ≥ 2 and $I = \bigoplus_{m\geq 2} I_m$. Then, the quotient algebra $A = \mathbf{k}\langle X\rangle/I$ is finitely generated and inherits its grading $A = \bigoplus_{m\in\mathbb{N}_0} A_m$ from $\mathbf{k}\langle X\rangle$. We shall work with the so-called *normal* $\mathbf{k}$*-basis of* A. We say that a monomial $u \in \langle X\rangle$ is *normal modulo I* if it is normal modulo $\mathbf{LM}(I)$. We set

$$N(I) := N(\mathbf{LM}(I)).$$

In particular, the free monoid $\langle X\rangle$ splits as a disjoint union

$$\langle X\rangle = N(I) \sqcup \mathbf{LM}(I). \tag{2.2}$$

The free associative algebra $\mathbf{k}\langle X\rangle$ splits as a direct sum of $\mathbf{k}$-vector subspaces

$$\mathbf{k}\langle X\rangle \simeq \mathrm{Span}_{\mathbf{k}} N(I) \oplus I,$$

and there is an isomorphism of vector spaces $A \simeq \mathrm{Span}_{\mathbf{k}} N(I)$.

It follows that every $f \in \mathbf{k}\langle X\rangle$ can be written uniquely as $f = f_0 + h$, where $f_0 \in \mathbf{k}N(I)$ and $h \in I$. The element f_0 is called *the normal form of f (modulo I)* and denoted by $\mathrm{Nor}(f)$. We define

$$N(I)_m = \{u \in N(I) \mid u \text{ has length } m\}.$$

In particular, $N(I)_1 = X$, and by definition $N(I)_0 = 1$. Then, $A_m \simeq \mathrm{Span}_{\mathbf{k}} N(I)_m$ for every $m \in \mathbb{N}_0$.

A subset $G \subseteq I$ of monic polynomials is a *Gröbner basis* of I (with respect to the ordering $<$ on $\langle X\rangle$) if

(1) G generates I as a two-sided ideal, and
(2) for every $f \in I$ there exists $g \in G$ such that $\mathbf{LM}(g)$ is a subword of $\mathbf{LM}(f)$, that is $\mathbf{LM}(f) = a\mathbf{LM}(g)b$, for some $a, b \in \langle X\rangle$.

A Gröbner basis G of I is *reduced* if (i) the set $G \setminus \{f\}$ is not a Gröbner basis of I, whenever $f \in G$, and (ii) each $f \in G$ is a linear combination of normal monomials modulo $G \setminus \{f\}$.

It is well-known that every ideal I of $\mathbf{k}\langle X\rangle$ has a unique reduced Gröbner basis $G_0 = G_0(I)$ with respect to $<$. However, G_0 may be infinite. For more details, we refer the reader to [20, 24, 25].

Bergman's Diamond lemma [3, Theorem 1.2] implies the following.

Remark 2.1 Let $G \subset \mathbf{k}\langle X\rangle$ be a set of noncommutative polynomials. Let $I = (G)$ and let $A = \mathbf{k}\langle X\rangle/I$. Then, the following conditions are equivalent. (1) The set G is a Gröbner basis of I. (2) Every element $f \in \mathbf{k}\langle X\rangle$ has a unique normal form, $\mathrm{Nor}(f)$, modulo G. (3) There is an equality $N(G) = N(I)$, so there is an isomorphism of vector spaces

$$\mathbf{k}\langle X\rangle \simeq I \oplus \mathbf{k}N(G).$$

(4) The image of $N(G)$ in A is a $\mathbf{k}$-basis of A. In this case A can be identified with the $\mathbf{k}$-vector space $\mathbf{k}N(G)$, made a $\mathbf{k}$-algebra by the multiplication $a \cdot b := \mathrm{Nor}(ab)$.

In this paper, we focus on a class of quadratic finitely presented algebras A associated with finite set-theoretic solutions (X, r) of the Yang–Baxter equation. Following Yuri Manin, [22], we call them *Yang–Baxter algebras*.

2.2 Quadratic algebras

A quadratic algebra is an associative graded algebra $A = \bigoplus_{i\geq 0} A_i$ over a ground field $\mathbf{k}$ determined by a vector space of generators $V = A_1$ and a subspace of homogeneous quadratic relations $R = R(A) \subset V \otimes V$. We assume that A is finitely generated, so $\dim A_1 < \infty$. Thus, $A = T(V)/(R)$ inherits its grading from the tensor algebra $T(V)$.

As usual, we take a combinatorial approach to study A. The properties of A will be read off a finite presentation $A = \mathbf{k}\langle X\rangle/(\Re)$, where by convention, X is a fixed finite set of generators of degree 1, (X is a basis of A_1), $|X| = n$, and $(\Re)$ is the two-sided ideal of relations, generated by a finite linearly independent set $\Re$ of homogeneous polynomials of degree two.

Definition 2.2 A quadratic algebra A is *a Poincarè–Birkhoff–Witt type algebra* or shortly *a PBW algebra* if there exists an enumeration $X = \{x_1, \dots, x_n\}$ of X, such that the quadratic relations $\Re$ form a (noncommutative) Gröbner basis with respect to the degree-lexicographic ordering $<$ on $\langle X\rangle$. In this case, the set of normal monomials (mod $\Re$) forms a $\mathbf{k}$-basis of A called a *PBW basis* and $x_1, \dots, x_n$ (taken exactly with this enumeration) are called *PBW-generators of* A.

The notion of a *PBW* algebra was introduced by Priddy, [27]. His *PBW basis* is a generalization of the classical Poincaré–Birkhoff–Witt basis for the universal enveloping of a finite dimensional Lie algebra. PBW algebras form an important class of Koszul algebras. The interested reader can find information on quadratic algebras and, in particular, on Koszul algebras and PBW algebras in [26]. A special class of PBW algebras important for this paper is the *binomial skew polynomial rings*, introduced and studied first in [7, 8].

Definition 2.3 A *binomial skew polynomial ring* is a quadratic algebra $A = \mathbf{k}\langle x_1, \dots, x_n\rangle/(\Re_0)$ with precisely $\binom{n}{2}$ defining relations

$$\Re_0 = \{f_{ji} = x_j x_i - c_{ij} x_{i'} x_{j'} \mid 1 \le i < j \le n\} \quad \text{such that} \tag{2.3}$$

(a) $c_{ij} \in \mathbf{k}^{\times}$; (b) For every pair $i, j,\ 1 \le i < j \le n$, the relation $x_j x_i - c_{ij} x_{i'} x_{j'} \in \Re_0$, satisfies $j > i', i' < j'$; (c) Every ordered monomial $x_i x_j$, with $1 \le i < j \le n$ occurs (as a second term) in some relation in $\Re_0$; (d) The set $\Re_0$ is the *reduced Gröbner basis* of the two-sided ideal $(\Re_0)$, with respect to the degree-lexicographic order $<$ on $\langle X\rangle$, or equivalently, (d′) The set of terms $\mathcal{T} = \left\{x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \langle X\rangle \mid \alpha_i \in \mathbb{N}_0, i \in \{0, \dots, n\}\right\}$ projects to a $\mathbf{k}$-basis of A.

The equivalence of (d) and (d′) follows from Remark 2.1.

Clearly, each binomial skew polynomial ring A is a PBW algebra with a set of PBW generators $x_1, \dots, x_n$. Moreover, A defines via its relations a square-free solution of the Yang–Baxter equation, see [17]. Conversely, if (X, r) is a finite square-free solution, then there exists an enumeration $X = \{x_1, x_2, \dots, x_n\}$ such that the Yang–Baxter algebra $\mathcal{A}(\mathbf{k}, X, r)$ is a binomial skew-polynomial ring, see [29].

Example 2.4 Let $A = \mathbf{k}\langle x_1, x_2, x_3, x_4\rangle/(\Re_0)$, where

$$\Re_0 = \{x_4x_2 - x_1x_3,\ x_4x_1 - x_2x_3,\ x_3x_2 - x_1x_4,\ x_3x_1 \\ -x_2x_4,\ x_4x_3 - x_3x_4,\ x_2x_1 - x_1x_2\}.$$

The algebra A is a binomial skew-polynomial ring. It is a PBW algebra with PBW generators $X = \{x_1, x_2, x_3, x_4\}$. The relations of A define in a natural way a solution of the Yang–Baxter equation.

2.3 Set-theoretic solutions of the Yang–Baxter equation and their Yang–Baxter algebras

Definition 2.5 [9, 15] Let X be a nonempty set, and let $r : X \times X \longrightarrow X \times X$ be a bijective map. Then, the pair (X, r) is called *a quadratic set*. (This is a set-theoretic analogue of "a quadratic algebra").

The image of (x, y) under r is presented as

$$r(x, y) = ({}^x y, x^y).$$

This formula defines a "left action" $\mathcal{L} : X \times X \longrightarrow X$, and a "right action" $\mathcal{R} : X \times X \longrightarrow X$, on X as: $\mathcal{L}_x(y) = {}^x y$, $\mathcal{R}_y(x) = x^y$, for all $x, y \in X$. (i) (X, r) is *non-degenerate*, if the maps $\mathcal{L}_x$ and $\mathcal{R}_x$ are bijective for each $x \in X$. (ii) (X, r) is *involutive* if $r^2 = id_{X\times X}$. (iii) (X, r) is *square-free* if $r(x, x) = (x, x)$ for all $x \in X$.

(iv) (X, r) is *a set-theoretic solution of the Yang–Baxter equation (YBE)* if the braid relation

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$

holds in $X \times X \times X$, where $r^{12} = r \times \mathrm{id}_X$, and $r^{23} = \mathrm{id}_X \times r$. In this case, (X, r) is also called *a braided set*. (v) A braided set (X, r) with r involutive is called *a symmetric set*. (vi) A nondegenerate symmetric set will be called *"a solution of YBE"*, or shortly, *"a solution"*.

Convention 2.6 In this paper, we shall always assume that (X, r) is nondegenerate. "*A solution of YBE*", or simply "*a solution*" means "*a non-degenerate symmetric set*" (X, r), where X is a set of arbitrary cardinality.

As a notational tool, we shall often identify the sets $X^{\times m}$ of ordered m-tuples, $m \geq 2$, and X^m, the set of all monomials of length m in the free monoid $\langle X \rangle$. Sometimes for simplicity, we shall write $r(xy)$ instead of $r(x, y)$.

Definition 2.7 [9, 15] To each quadratic set (X, r) we associate canonically algebraic objects generated by X and with quadratic relations $\Re = \Re(r)$ naturally determined as

$$xy = y'x' \in \Re(r) \text{ iff } r(x, y) = (y', x') \text{ and } (x, y) \neq (y', x') \text{ hold in } X \times X.$$

The monoid associated with (X, r) is defined as $S = S(X, r) = \langle X;\ \Re(r) \rangle$. It has a set of generators X and a set of defining relations $\Re(r)$. For an arbitrary fixed field $\mathbf{k}$, *the* $\mathbf{k}$*-algebra associated with* (X, r) is defined as

$$\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) = \mathbf{k}\langle X \rangle / (\Re_0), \text{ where } \Re_0 = \{xy - y'x' \mid r(xy) = y'x' \text{ and } r(xy) \neq xy \text{ holds in } X^2\}.$$

Clearly, $\mathcal{A}$ is a quadratic algebra generated by X and with defining relations $\Re_0$ (or equivalently, $\Re(r)$), which is isomorphic to the monoid algebra $\mathbf{k}S(X, r)$. When (X, r) is a solution of YBE, $\mathcal{A}$ is called *an Yang–Baxter algebra*, or shortly *an YB algebra*.

Suppose (X, r) is a finite quadratic set. Then, $A = A(\mathbf{k}, X, r)$ is *a connected graded* $\mathbf{k}$-algebra (naturally graded by length), $A = \bigoplus_{i \geq 0} A_i$, where $A_0 = \mathbf{k}$, and each graded component A_i is finite dimensional.

By [11, Proposition 2.3.] if (X, r) is a nondegenerate involutive quadratic set of finite order $|X| = n$ then the set $\Re(r)$ consists of precisely $\binom{n}{2}$ quadratic relations. In this case, the associated algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ satisfies

$$\dim \mathcal{A}_2 = \binom{n+1}{2}.$$

Definition-Notation 2.8 [13] Suppose (X, r) is an involutive quadratic set. Then, the cyclic group $\langle r \rangle = \{1, r\}$ acts on the set X^2 and splits it into disjoint r-orbits $\{xy, r(xy)\}$, where $xy \in X^2$. An r-orbit $\{xy, r(xy)\}$ is *non-trivial* if $xy \neq r(xy)$. The element $xy \in X^2$ is *an r-fixed point* if $r(xy) = xy$. *The set of r-fixed points* in X^2 will be denoted by $\mathcal{F}(X, r)$:

$$\mathcal{F}(X, r) = \{xy \in X^2 \mid r(xy) = xy\}. \tag{2.4}$$

The following useful corollary is a consequence from [13, Lemma 3.7].

Corollary 2.9 *Let (X, r) be a nondegenerate symmetric set of finite order $|X| = n$, $\mathcal{A} = \mathcal{A}(\boldsymbol{k}, X, r)$. (1) There are exactly n fixed points $\mathcal{F} = \mathcal{F}(X, r) = \{x_1y_1, \dots, x_ny_n\} \subset X^2$, so $|\mathcal{F}(X, r)| = |X| = n$. In the special case, when (X, r) is a square-free solution, one has $\mathcal{F}(X, r) = \Delta_2 = \{xx \mid x \in X\}$, the diagonal of X^2. (2) The number of non-trivial r-orbits is exactly $\binom{n}{2}$. Each such an orbit has two distinct elements: xy and $r(xy)$, where $xy, r(xy) \in X^2$. (3) The set X^2 splits into $\binom{n+1}{2}$ r-orbits. For $xy, zt \in X^2$ there is an equality $xy = zt$ in $\mathcal{A}$* iff *$zt \in \{xy, r(xy)\}$.*

The following lemma is involved in our interpretation of [17, Theorem 1.3]) as Facts 2.11 (1), which is used in our proofs.

Lemma 2.10 [14, Lemma 3.2] *Every nondegenerate involutive quadratic set (X, r) satisfy the following condition:*

$$\begin{array}{l}\textit{Given } a, b \in X \textit{ there exist unique } c, d \in X \textit{ such that } r(ca) = db.\\ \textit{Furthermore, if } a = b, \textit{ then } c = d.\end{array} \tag{2.5}$$

The following facts are a compilation of results from [17] and are true for every finite nondegenerate symmetric set (X, r).

Facts 2.11 Suppose (X, r) is a finite solution of YBE of order n, $X = \{x_1, \dots, x_n\}$. Let $S = S(X, r)$ be the associated monoid and let $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ be the associated Yang–Baxter algebra. Then, the following conditions hold.

(1) (A modified version of [17, Theorem 1.3])
S is a semigroup of I-type, that is there is a bijective map $v : \mathcal{U} \mapsto S$, where $\mathcal{U}$ is the free n-generated abelian monoid $\mathcal{U} = [u_1, \dots, u_n]$ such that $v(1) = 1$, and such that

$$\{v(u_1a), \dots, v(u_na)\} = \{x_1v(a), \dots, x_nv(a)\}, \text{ for all } a \in \mathcal{U}.$$

(2) The Hilbert series of A is $H_A(t) = 1/(1-t)^n$.
(3) [17, Theorem 1.4] (a) A has finite global dimension and polynomial growth; (b) A is Koszul; (c) A is left and right Noetherian; (d) A satisfies the Auslander condition and is Cohen-Macaulay.
(4) [17, Corollary 1.5] A is a domain, and in particular, the monoid S is cancellative.

For convenience of the reader, we shall make a brief observation. Note first that the hypothesis of Facts 2.11 is satisfied by arbitrary finite solution (a nondegenerate symmetric set) (X, r) which is not necessarily square free, possibly the algebra

$\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is not a binomial skew polynomial ring, or equivalently, $\mathcal{A}$ is not a PBW algebra.

Next observe that part (1) of Facts 2.11 is a modification of the original second part of [17, Theorem 1.3] which states (in our terminology): "Suppose that (X, r) is a finite symmetric set of order n satisfying the condition (2.5). Then, the monoid $S(X, r)$ is of I type." However, under the hypothesis of Facts 2.11, Lemma 2.10 implies the necessary condition (2.5).

The following corollary is straightforward from Facts 2.11 (1) and will be used throughout the paper.

Corollary 2.12 *In notation and conventions as above, let* (X, r) *be a finite solution of YBE. Then for every integer* $d \geq 1$*, there are equalities*

$$\dim \mathcal{A}_d = \binom{n+d-1}{d} = |\mathcal{N}_d|. \tag{2.6}$$

The following remark observes the importance of finite square-free solutions and their close relations to Artin-Schelter regularity. The results are extracted from [10, 12, 17] and [29].

Remark 2.13 Suppose (X, r) is a square-free nondegenerate and involutive *quadratic set* of order n. Let $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ be the associated quadratic algebra. The following conditions are equivalent.

(1) $\mathcal{A}$ is an Artin-Schelter regular PBW algebra.
(2) (X, r) is a solution of YBE.
(3) There exists an enumeration $X = \{x_1, x_2, \dots, x_n\}$ such that $\mathcal{A}$ is a binomial skew-polynomial algebra.

The implication (1)$\Longrightarrow$ (3) follows from [12, Theorem 1.2]. (3) $\Longrightarrow$ (1) is proven in [10, Theorem B] (see also [17]). (3) $\Longrightarrow$ (2) is proven in [17, Theorem 1.1]. The implication (2) $\Longrightarrow$ (3) was conjectured by the author and proven by Rump, see [29, Theorem 1].

Remark 2.14 Note that among all Yang–Baxter algebras of finite solutions studied in this paper, the only PBW algebras $\mathcal{A} = \mathcal{A}(K, X, r)$ are those corresponding to square-free solutions. Indeed, our recent result [14, Theorem 3.8] shows that if (X, r) is a finite solution of YBE such that its Yang–Baxter algebra $\mathcal{A} = \mathcal{A}(K, X, r)$ is a PBW algebra with a set of PBW generators $X = \{x_1, x_2, \dots, x_n\}$, then (X, r) is a square-free solution.

Convention 2.15 Let (X, r) be a finite solution of YBE of order n, and let $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ be the associated Yang–Baxter algebra. (a) If (X, r) is square-free we fix an enumeration such that $X = \{x_1, \dots, x_n\}$ is a set of PBW generators of $\mathcal{A}$. In this case, $\mathcal{A}$ is a binomial skew polynomial ring, see Definition 2.3. (b) If (X, r) is not square-free, we fix an arbitrary enumeration $X = \{x_1, \dots, x_n\}$ on X.

In each of the cases (a) and (b), we extend the fixed enumeration on X to the degree-lexicographic ordering $<$ on $\langle X \rangle$. By convention the Yang–Baxter algebra,

$\mathcal{A} = \mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ is presented as

$$\begin{aligned} \mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) = \mathbf{k}\langle X \rangle / (\Re) \simeq \mathbf{k}\langle X;\ \Re(r)\rangle, \quad \text{where} \\ \Re = \Re_{\mathcal{A}} = \{xy - y'x' \mid xy > y'x', \text{ and } r(xy) = y'x'\}. \end{aligned} \tag{2.7}$$

Consider the two-sided ideal $I = (\Re)$ of $\mathbf{k}\langle X \rangle$, let $G = G(I)$ be the unique reduced Gröbner basis of I with respect to $<$. Here, we do not need an explicit description of the reduced Gröbner basis G of I, but we need some details. In the case (a), one has $G = \Re$. It follows from Remark 2.14 that in the case (b), the set of relations $\Re$ is not a Gröbner basis of I, but $\Re \subsetneqq G$. Moreover, the shape of the relations $\Re$ and standard techniques from noncommutative Gröbner bases theory imply that the Gröbner basis G is finite, or countably infinite, and consists of homogeneous binomials $f_j = u_j - v_j$, where $\mathbf{LM}(f_j) = u_j > v_j$, and $u_j, v_j \in X^m$, for some $m \geq 2$. The set of all normal monomials modulo I is denoted by $\mathcal{N}$. As we mentioned above, $\mathcal{N} = \mathcal{N}(I) = \mathcal{N}(G)$. An element $f \in \mathbf{k}\langle X \rangle$ is in normal form (modulo I), if $f \in \mathrm{Span}_{\mathbf{k}}\mathcal{N}$. The free monoid $\langle X \rangle$ splits as a disjoint union $\langle X \rangle = \mathcal{N} \sqcup \mathbf{LM}(I)$. The free associative algebra $\mathbf{k}\langle X \rangle$ splits as a direct sum of $\mathbf{k}$-vector subspaces $\mathbf{k}\langle X \rangle \simeq \mathrm{Span}_{\mathbf{k}}\mathcal{N} \oplus I$, and there is an isomorphism of vector spaces $\mathcal{A} \simeq \mathrm{Span}_{\mathbf{k}}\mathcal{N}$. As usual, we denote $\mathcal{N}_d = \{u \in \mathcal{N} \mid u \text{ has length } d\}$. Then, $\mathcal{A}_d \simeq \mathrm{Span}_{\mathbf{k}}\mathcal{N}_d$ for every $d \in \mathbb{N}_0$. One has $\dim \mathcal{A}_d = |\mathcal{N}_d| = \binom{n+d-1}{d}$, $\forall d \geq 0$. Note that since the set of relations $\Re$ is a finite set of homogeneous polynomials, the elements of the reduced Gröbner basis $G = G(I)$ of degree $\leq d$ can be found effectively, (using the standard strategy for constructing a Gröbner basis), and therefore, the set of normal monomials $\mathcal{N}_d$ can be found inductively for $d = 1, 2, 3, \ldots$. It follows from Bergman's Diamond lemma, [3, Theorem 1.2], that if we consider the space $\mathbf{k}\mathcal{N}$ endowed with multiplication defined by

$$f \cdot g := \mathrm{Nor}(fg), \quad \text{for every } f, g \in \mathbf{k}\mathcal{N},$$

then $(\mathbf{k}\mathcal{N}, \cdot)$ has a well-defined structure of a graded algebra, and there is an isomorphism of graded algebras

$$\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) \cong (\mathbf{k}\mathcal{N}, \cdot), \quad \text{so} \quad \mathcal{A} = \bigoplus_{d \in \mathbb{N}_0} A_d \cong \bigoplus_{d \in \mathbb{N}_0} \mathbf{k}\mathcal{N}_d.$$

By convention, we shall often identify the algebra $\mathcal{A}$ with $(\mathbf{k}\mathcal{N}, \cdot)$.

In the case (a) when (X, r) is square-free, the set of normal monomials is exactly $\mathcal{T}$ (the set of ordered terms in X), so $\mathcal{A}$ is identified with $(\mathbf{k}\mathcal{T}, \cdot)$ and $S(X, r)$ is identified with $(\mathcal{T}, \cdot)$.

3 Segre products of Yang–Baxter algebras

In this section, we investigate the Segre products of Yang–Baxter algebras. The main result of the section is Theorem 3.10.

3.1 Segre products of quadratic algebras

In [6], Fröberg and Backelin made a systematic account on Koszul algebras and showed that their properties are preserved under various constructions such as tensor products, Segre products, and Veronese subalgebras. Our main reference on Segre products of quadratic algebras and their properties is [26, Section 3.2]. An interested reader may find results on Segre product of specific Artin-Schelter regular algebras in [32], and on twisted Segre product of noetherian Koszul Artin-Schelter regular algebras in [19].

We first recall the notion of Segre product of graded algebras as follows [26, Ch 3 Sect 2, Def. 1].

Definition 3.1 Let

$$A = \mathbf{k} \oplus A_1 \oplus A_2 \oplus \dots \quad \text{and} \quad B = \mathbf{k} \oplus B_1 \oplus B_2 \oplus \cdots$$

be $\mathbb{N}_0-$ graded algebras over a field $\mathbf{k}$, where $\mathbf{k} = A_0 = B_0$. The *Segre product* of A and B is the $\mathbb{N}_0$-graded algebra

$$A \circ B := \bigoplus_{i \geq 0} (A \circ B)_i \text{ with } (A \circ B)_i = A_i \otimes_{\mathbf{k}} B_i.$$

The Segre product $A \circ B$ is a subalgebra of the tensor product algebra $A \otimes B$. Note that the embedding is not a graded algebra morphism, as it doubles grading. If A and B are locally finite then the Hilbert function of $A \circ B$ satisfies

$$\begin{aligned} h_{A \circ B}(t) &= \dim(A \circ B)_t = \dim(A_t \otimes B_t) \\ &= \dim(A_t) \cdot \dim(B_t) = h_A(t) \cdot h_B(t), \end{aligned} \tag{3.1}$$

and for the Hilbert series, one has

$$H_A(t) = \Sigma_{n \geq 0} (\dim A_n) t^n, \quad H_B(t) = \Sigma_{n \geq 0} (\dim B_n) t^n,$$
$$H_{A \circ B}(t) = \Sigma_{n \geq 0} (\dim A_n)(\dim B_n) t^n.$$

The Segre product, $A \circ B$, inherits various properties from the two algebras A and B. In particular, if both algebras are one-generated, quadratic, and Koszul, it follows from [26, Chap 3.2, Proposition 2.1] that the algebra $A \circ B$ is also one-generated, quadratic, and Koszul.

The following remark gives more concrete information about the space of quadratic relations of $A \circ B$, see for example, [32].

Remark 3.2 [32] Suppose that A and B are quadratic algebras generated in degree one by A_1 and B_1, respectively, written as:

$$A = T(A_1)/(\Re_A) \text{ with } \Re_A \subset A_1 \otimes A_1,$$
$$B = T(B_1)/(\Re_B) \text{ with } \Re_B \subset B_1 \otimes B_1,$$

where $T(-)$ is the tensor algebra and $(\Re_A)$, $(\Re_B)$ are the ideals of relations of A and B.

Then, $A \circ B$ is also a quadratic algebra generated in degree one by $A_1 \otimes B_1$ and presented as

$$A \circ B = T(A_1 \otimes B_1)/(\sigma^{23}(\Re_A \otimes B_1 \otimes B_1 + A_1 \otimes A_1 \otimes \Re_B)), \tag{3.2}$$

where

$$\sigma^{23}(a_1 \otimes a_2 \otimes b_1 \otimes b_2) = a_1 \otimes b_1 \otimes a_2 \otimes b_2.$$

As usual, we take a combinatorial approach to study quadratic algebras. The properties of A will be read off a presentation $A = \mathbf{k}\langle X \rangle/(\Re_A)$, where by convention X is a fixed finite set of generators of degree one, $|X| = n$, and $(\Re_A)$ is the two-sided ideal of relations, generated by a *finite* set $\Re_A$ of homogeneous polynomials of degree two.

3.2 Segre products of Yang–Baxter algebras, generators and relations

Suppose (X, r_1) and (Y, r_2) are finite solutions of YBE of orders $|X| = m$ and $|Y| = n$.

Let $A = \mathcal{A}(\mathbf{k}, X, r_1)$, and $B = \mathcal{A}(\mathbf{k}, Y, r_2)$ be the corresponding YB algebras. As in Convention 2.15, we fix enumerations

$$X = \{x_1, \dots, x_m\}, \qquad Y = \{y_1, \dots, y_n\},$$

and consider the degree-lexicographic orders on the free monoids $\langle X \rangle$, and $\langle Y \rangle$ extending these enumerations. Then,

$$\begin{array}{c} A = \mathbf{k}\langle X \rangle/(\Re_1) \text{ where } \Re_1 \text{ is a set of } \binom{m}{2} \text{ binomial relations :} \\ \Re_1 = \{x_j x_i - x_{i'} x_{j'} \mid x_j x_i > x_{i'} x_{j'} \text{ and } r_1(x_j x_i) = x_{i'} x_{j'}\}. \end{array} \tag{3.3}$$

$$\begin{array}{c} B = \mathbf{k}\langle Y \rangle/(\Re_2) \text{ where } \Re_2 \text{ is a set of } \binom{n}{2} \text{ binomial relations :} \\ \Re_2 = \{y_b y_a - y_{a'} y_{b'} \mid y_b y_a > y_{a'} y_{b'} \text{ and } r_2(y_b y_a) = y_{a'} y_{b'}\}. \end{array} \tag{3.4}$$

In general, (3.3) and (3.4) are not necessarily relations of binomial skew polynomial algebras. One has

$$\dim A_2 = \binom{m+1}{2}, \quad \dim B_2 = \binom{n+1}{2}, \quad \dim (A \circ B)_2 = \binom{m+1}{2}\binom{n+1}{2}. \tag{3.5}$$

Remark 3.3 Note that if (X, r) is a quadratic set, then $r(xy) = xy$ *iff* ${}^x y = x$ and $x^y = y, x, y \in X$. Moreover, if the monoid $S(X, r)$ is with cancellation, then $r(xy) = xy$ is equivalent to ${}^x y = x$.

Let $\mathcal{N}(A)$ be the set of normal monomials modulo the ideal $(\Re_1)$ in $\mathbf{k}\langle X \rangle$ and let $\mathcal{N}(B)$ be the set of normal monomials modulo the ideal $(\Re_2)$ in $\mathbf{k}\langle Y \rangle$.

Remark 3.4 (1) A monomial $xy \in \mathcal{N}(A)_2, x, y \in X$ *iff* either (a) ${}^x y > x$, in this case $f = {}^x y.x^y - xy \in \Re_1$, $HM(f) = {}^x y.x^y$, or (b) $r_1(xy) = xy$, which is equivalent to ${}^x y = x$, since the monoid $S(X, r_1)$ is cancellative, see Facts 2.11, (4).

(2) $zt \in \mathcal{N}(B)_2, z, t \in Y$, *iff* either (a) ${}^z t > z$, in this case $g = {}^z t z^t - zt \in \Re_2$, $HM(g) = {}^z t z^t$, or (b) $r_2(zt) = zt$, which is equivalent to ${}^z t = z$.

Thus,

$$\mathcal{N}(A)_2 = \{xy \in X^2 \mid {}^x y \geq x\}, \quad \mathcal{N}(B)_2 = \{zt \in Y^2 \mid {}^z t \geq z\}.$$

Definition 3.5 Let (X, r_X) and (Y, r_Y) be disjoint braided sets (set-theoretic solutions of YBE, we do not assume involutiveness, nor nondegeneracy). We define *the Cartesian product of the braided sets* (X, r_X) *and* (Y, r_Y) as $(X \times Y, \rho_{X \times Y})$, where the map $\rho_{X \times Y} = \rho$ is defined as

$$\rho : (X \times Y) \times (X \times Y) \longrightarrow (X \times Y) \times (X \times Y), \quad \rho = \sigma_{23} \circ (r_X \times r_Y) \circ \sigma_{23},$$

where σ_{23} is the flip of the second and the third component.

In other words,

$$\rho((x_j, y_b), (x_i, y_a)) := (({}^{x_j} x_i, {}^{y_b} y_a), (x_j^{x_i}, y_b^{y_a})), \tag{3.6}$$

for all $i, j \in \{1, \ldots, m\}$ and all $a, b \in \{1, \ldots, n\}$. It is easy to see that the Cartesian product $(X \times Y, \rho)$ is a braided set of order mn.

Remark 3.6 It is not difficult to prove that the Cartesian product of braided sets $(X \times Y, \rho_{X \times Y})$ satisfies the following conditions.

(1) $(X \times Y, \rho_{X \times Y})$ is nondegenerate *iff* (X, r_X) and (Y, r_Y) are nondegenerate.
(2) $(X \times Y, \rho_{X \times Y})$ is involutive *iff* (X, r_X) and (Y, r_Y) are involutive.
(3) $(X \times Y, \rho_{X \times Y})$ is a solution of YBE *iff* (X, r_X) and (Y, r_Y) are solutions of YBE.
(4) $(X \times Y, \rho_{X \times Y})$ is a square-free solution *iff* (X, r_X) and (Y, r_Y) are square-free solutions.

To simplify notation when we work with elements of the Segre product $A \circ B$, we shall write "$x \circ y$" instead of "$x \otimes y$," whenever $x \in X, y \in Y$, or "$u \circ v$" instead of "$u \otimes v$", whenever $u \in A_d$, $v \in B_d$, $d \geq 2$.

Proposition-Notation 3.7 Let (X, r_1) and (Y, r_2) be solutions on the disjoint sets $X = \{x_1, \ldots, x_m\}$, and $Y = \{y_1, \ldots, y_n\}$. Let $A \circ B$ be the Segre product of the YB algebras $A = \mathcal{A}(\mathbf{k}, X, r_1)$ and $B = \mathcal{A}(\mathbf{k}.Y, r_2)$, and let

$$X \circ Y = \{x_i \circ y_a \mid 1 \leq i \leq m,\ 1 \leq a \leq n\}.$$

There is a natural structure of a solution $(X \circ Y, r_{X \circ Y})$ on the set $X \circ Y$, where the map $r_{X \circ Y}$ is defined as

$$r_{X \circ Y}((x_j \circ y_b), (x_i \circ y_a)) := ((({}^{x_j} x_i) \circ ({}^{y_b} y_a)), ((x_j{}^{x_i}) \circ (y_b^{y_a}))), \tag{3.7}$$

for all $1 \leq i, j \leq m$ and all $1 \leq a, b \leq n$. The solution $(X \circ Y, r_{X \circ Y})$ is isomorphic to the Cartesian product of solutions $(X \times Y, \rho_{X \times Y})$. In particular, the solution $(X \circ Y, r_{X \circ Y})$ has cardinality mn and $\binom{mn}{2}$ nontrivial $r_{X \circ Y}$-orbits.

Proof The set $X \circ Y$ consists of mn distinct elements and is a basis of $(A \circ B)_1 = A_1 \otimes B_1$. The map $r : (X \circ Y) \times (X \circ Y) \longrightarrow (X \circ Y) \times (X \circ Y)$ defined via (3.7) is a well-defined bijection. Consider the bijective map

$$F : X \circ Y \to X \times Y, \quad F(x \circ y) = (x, y).$$

It follows from the definitions of the maps $\rho_{X \times Y}$ and $r_{X \circ Y}$ that

$$(F \times F) \cdot r_{X \circ Y} = \rho_{X \times Y} \cdot (F \times F).$$

Therefore, $r_{X \circ Y}$ obeys the YBE, and $(X \circ Y, r_{X \circ Y})$ is a solution isomorphic to the Cartesian product of solutions $(X \times Y, \rho_{X \times Y})$. In particular, $(X \circ Y, r_{X \circ Y})$ is nondegenerate and involutive. It is clear that $|X \circ Y| = mn$ and, by Corollary 2.9 (2), the solution $(X \circ Y, r_{X \circ Y})$ has $\binom{mn}{2}$ nontrivial $r_{X \circ Y}$-orbits. □

We shall often identify the solutions $(X \circ Y, r_{X \circ Y})$ and $(X \times Y, \rho_{X \times Y})$ and refer to $(X \circ Y, r_{X \circ Y})$ as "the Cartesian product of the solutions (X, r_1) and (Y, r_2)".

Proposition 3.8 *Let (X, r_1) and (Y, r_2) be solutions on the disjoint sets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$. In notation as above, let $(X \circ Y, r = r_{X \circ Y})$ be the Cartesian product of the solutions (X, r_1) and (Y, r_2). We order the set $X \circ Y$ lexicographically $X \circ Y = \{x_1 \circ y_1, \dots, x_1 \circ y_n, \dots, x_m \circ y_n\}$. The Yang–Baxter algebra $\mathbb{A} = \mathbb{A}_{X \circ Y} = \mathcal{A}(\boldsymbol{k}, X \circ Y, r)$ is generated by the set $X \circ Y$ and has $\binom{mn}{2}$ quadratic defining relations described in the two lists (3.8) and (3.9).*

$$\begin{aligned} f_{ji,ba} = (x_j \circ y_b)(x_i \circ y_a) - ({}^{x_j}x_i \circ {}^{y_b}y_a)(x_j^{x_i} \circ y_b^{y_a}), \textit{for all } \ 1 \leq i, j \leq m \\ \textit{such that } \ x_j > {}^{x_j}x_i, \quad \textit{and all } \ 1 \leq a, b \leq n. \end{aligned} \tag{3.8}$$

Every relation $f_{ji,ba}$ has a leading monomial $\mathbf{LM}(f_{ji,ba}) = (x_j \circ y_b)(x_i \circ y_a)$.

$$\begin{aligned} f_{ij,ba} = (x_i \circ y_b)(x_j \circ y_a) - (x_i \circ {}^{y_b}y_a)(x_j \circ y_b^{y_a}), \textit{for all } \ 1 \leq i, j \leq m \ \textit{ with} \\ r_1(x_i x_j) = x_i x_j, \quad \textit{and all } \ 1 \leq a, b \leq n, \quad \textit{such that } \ y_b > {}^{y_b}y_a. \end{aligned} \tag{3.9}$$

Every relation $f_{ij,ba}$ has a leading monomial $\mathbf{LM}(f_{ij,ba}) = (x_i \circ y_b)(x_j \circ y_a)$.
The solution $(X \circ Y, r)$ has exactly mn fixed points, namely:

$$\begin{aligned} \mathcal{F} \ &= \{(x_p \circ y_a)(x_q \circ y_b) \mid r_1(x_p x_q) = x_p x_q, \ p, q \in \{1 \dots, m\}, \\ &\textit{and } r_2(y_a y_b) = y_a y_b, \ a, b \in \{1, \dots, n\}\}. \end{aligned}$$

In this case, $x_p x_q \in \mathcal{N}(A)_2$ and $y_a y_b \in \mathcal{N}(B)_2$.

Proof The solution $(X \circ Y, r)$ is nondegenerate, it has order $|X \circ Y| = mn$, and therefore, by Corollary 2.9, the number of its fixed points is mn. It is clear that $r((x_p \circ y_a)(x_q \circ y_b)) = (x_p \circ y_a)(x_q \circ y_b)$ if and only if $r_1(x_p x_q) = x_p x_q$ and $r_2(y_a y_b) = y_a y_b$. The defining relations of the Yang–Baxter algebra $\mathbb{A}$ correspond bijectively to the nontrivial r-orbits, there are exactly $\binom{mn}{2}$ nontrivial r-orbits. Observe that there are $\binom{m}{2}n^2$ distinct relations given in (3.8), each of them corresponds to a pair $(x_j \circ y_b, x_i \circ y_a)$, where $x_j x_i > r_1(x_j x_i)$, and $y_b y_a$ is an arbitrary word in Y^2. There are $m\binom{n}{2}$ distinct relations in (3.9), each of them is determined by a fixed point $x_i x_j$ in X^2 and some nontrivial r_2-orbit in Y^2. One has

$$\binom{m}{2}n^2 + m\binom{n}{2} = \binom{mn}{2},$$

as desired. □

The next corollary is a straightforward consequence from [26, Chap 3, Proposition 2.1] and Facts 2.11 (3).

Corollary 3.9 *Let (X, r_1) and (Y, r_2) be finite solutions and let $A = \mathcal{A}(\boldsymbol{k}, X, r_1)$ and $B = \mathcal{A}(\boldsymbol{k}, Y, r_2)$ be their Yang–Baxter algebras. Then, the Segre product, $A \circ B$ is a one-generated quadratic and Koszul algebra.*

We shall see in the next section that the Segre product $A \circ B$ is also *a left and a right Noetherian algebra with polynomial growt*h, see Corollary 4.6.

Theorem 3.10 *Let (X, r_1) and (Y, r_2) be finite solutions, where $X = \{x_1 \dots, x_m\}$ and $Y = \{y_1 \dots, y_n\}$ are disjoint sets. Let $A \circ B$ be the Segre product of the YB algebras $A = \mathcal{A}(\boldsymbol{k}, X, r_1)$ and $B = \mathcal{A}(\boldsymbol{k}, Y, r_2)$, and let $(X \circ Y, r_{X \circ Y})$ be the solution of YBE from Proposition 3.7.*

The algebra $A \circ B$ has a set of mn one-generators $W = X \circ Y$ ordered lexicographically:

$$\begin{aligned} W &= \{w_{11} = x_1 \circ y_1 < w_{12} = x_1 \circ y_2 < \cdots < w_{1n} = x_1 \circ y_n < w_{21} = x_2 \circ y_1 \\ &\quad < \cdots < w_{mn} = x_m \circ y_n\}, \end{aligned} \tag{3.10}$$

and a set of $\binom{mn}{2} + \binom{m}{2}\binom{n}{2}$ linearly independent quadratic relations $\Re$. The set of relations $\Re$ splits as a disjoint union $\Re = \Re_a \cup \Re_b$, where the sets $\Re_a$ and $\Re_b$ are described below.

(1) *The set $\Re_a$ is a disjoint union $\Re_a = \Re_{a1} \cup \Re_{a2}$ of two sets described as follows.*

$$\begin{aligned} \Re_{a1} = \{ & f_{ji,ba} = (x_j \circ y_b)(x_i \circ y_a) - (x_{i'} \circ y_{a'})(x_{j'} \circ y_{b'}),\ 1 \le i, j \le m, \\ & 1 \le a, b \le n, \text{ where } r_1(x_j x_i) = x_{i'} x_{j'}, \text{ with } j > i', \text{ and } r_2(y_b y_a) = y_{a'} y_{b'}\}. \end{aligned}$$

Every relation $f_{ji,ba}$ *has leading monomial* $\mathbf{LM}(f_{ji,ba}) = (x_j \circ y_b)(x_i \circ y_a)$*. The cardinality of* $\Re_{a1}$ *is* $|\Re_{a1}| = \binom{m}{2}n^2$*.*

$$\begin{aligned}\Re_{a2} = \{&f_{ij,ba} = (x_i \circ y_b)(x_j \circ y_a) - (x_i \circ y_{a'})(x_j \circ y_{b'}),\ 1 \le i, j \le m,\\ &1 \le a, b \le n, \text{ where } x_i x_j = r_1(x_i x_j) \text{ is a fixed point and } r_2(y_b y_a) = y_{a'} y_{b'},\\ &\text{with } b > a'\}.\end{aligned}$$

Every relation $f_{ij,ba}$ *has leading monomial* $\mathbf{LM}(f_{ij,ba}) = (x_i \circ y_b)(x_j \circ y_a)$*. The cardinality of* $\Re_{a2}$ *is* $|\Re_{a2}| = m\binom{n}{2}$*.*

(2) *The set* $\Re_b$ *consists of* $\binom{m}{2}\binom{n}{2}$ *relations given explicitly in* (3.11)

$$\begin{aligned}\Re_b = \{&g_{ij,ba} = (x_i \circ y_b)(x_j \circ y_a) - (x_i \circ y_{a'})(x_j \circ y_{b'}),\ 1 \le i, j \le m,\\ &1 \le a, b \le n, \text{ where } r_1(x_i x_j) > x_i x_j,\ r_2(y_b y_a) = y_{a'} y_{b'}\\ &\text{and } b > a'.\end{aligned} \tag{3.11}$$

Every relation $g_{ij,ba}$ *has leading monomial* $\mathbf{LM}(g_{ij,ba}) = (x_i \circ y_b)(x_j \circ y_a)$*.*

Proof Note that the relations in $\Re_a$ are the same as the defining relations of the Yang–Baxter algebra $\mathbb{A}_{X \circ Y} = \mathcal{A}(\mathbf{k}, X \circ Y, r_{X \circ Y})$ from Proposition 3.8. There is an obvious 1-1 correspondence between the set of nontrivial $r_{X \circ Y}$- orbits in $(X \circ Y) \times (X \circ Y)$ and the set of relations $\Re_a$. Being a nondegenerate symmetric set of order mn, $(X \circ Y, r_{X \circ Y})$ has exactly $\binom{mn}{2}$ nontrivial $r_{X \circ Y}$-orbits, see Corollary 2.9, and therefore the cardinality of $\Re_a$ must satisfy

$$|\Re_a| = \binom{mn}{2}. \tag{3.12}$$

It is clear that $\Re_{a1}$ and $\Re_{a2}$ are disjoint subsets of $\Re_a$. To be sure that the sets $\Re_{a1}$ and $\Re_{a2}$ exhaust $\Re_a$, we count their cardinalities.

Each of the relations $f_{ji,ba} \in \Re_{a1}$ corresponds to a pair $(x_j \circ y_b, x_i \circ y_a)$, where $x_j x_i > r_1(x_j x_i)$, and $y_b y_a$ is an arbitrary word in Y^2. There are exactly $\binom{m}{2}n^2$ distinct elements of this type.

Each of the relations $f_{ij,ba} \in \Re_{a2}$ is determined by a fixed point $x_i x_j$ in X^2 and some nontrivial r_2-orbit in Y^2, $\{y_b y_a, y_{a'} y_{b'} = r_2(y_b y_a)\}$ with $b > a'$. There are $m\binom{n}{2}$ distinct elements of this type. One has

$$|\Re_{a1}| + |\Re_{a2}| = \binom{m}{2}n^2 + m\binom{n}{2} = \binom{mn}{2} = |\Re_a|,$$

as desired.

We shall prove next that the sets $\Re_a$ and $\Re_b$ described above are contained in the ideal of relations $(\Re(A \circ B))$ of $A \circ B$. Under the hypothesis of the theorem, we prove the following lemma.

Lemma 3.11 (1) *Suppose* $f = x_j x_i - x_{i'} x_{j'} \in \Re_1$*, with* $HM(f) = x_j x_i$*. Let* $y_b, y_a \in Y$*, and let* $r_2(y_b y_a) = y_{a'} y_{b'}$ *(it is possible that* $y_b y_a$ *is a fixed point, or* $y_b y_a <$

$y_{a'}y_{b'}$). Then,

$$f_{ji,ba} = (x_j \circ y_b)(x_i \circ y_a) - (x_{i'} \circ y_{a'})(x_{j'} \circ y_{b'}) \in (\Re(A \circ B)).$$

Moreover, the relation $f_{ji,ba}$ has leading monomial $\mathbf{LM}(f_{ji,ba}) = (x_j \circ y_b)(x_i \circ y_a)$.

(2) *Suppose $r_1(x_i x_j) = x_i x_j$ (that is $x_i x_j$ is a fixed point), for some $1 \leq i, j \leq m$, and let $r_2(y_b y_a) = y'_a y'_b$, with $y_b > y_{a'}$, (so $y_b y_a - y'_a y'_b \in \Re_2$). Then,*

$$f_{ij,ba} = (x_i \circ y_b)(x_j \circ y_a) - (x_i \circ y_{a'})(x_j \circ y_{b'}) \in (\Re(A \circ B)),$$

and $\mathbf{LM}(f_{ij,ba}) = (x_i \circ y_b)(x_j \circ y_a)$.

(3) *If $r_1(x_i x_j) > x_i x_j$, and $y_b y_a - y_{a'} y_{b'} \in \Re_2$, then*

$$g_{ij,ba} = (x_i \circ y_b)(x_j \circ y_a) - (x_i \circ y_{a'})(x_j \circ y_{b'}) \in (\Re(A \circ B)),$$

and $\mathbf{LM}(g_{ij,ba}) = (x_i \circ y_b)(x_j \circ y_a)$.

Proof (1) By hypothesis $x_j x_i - x_{i'} x_{j'} \in \Re_1$ and $y_b y_a - y_{a'} y_{b'}$ is in the ideal $(\Re_2)$. Then, by Remark 3.2

$$\begin{aligned} \varphi &= \sigma_{23}((x_j x_i - x_{i'} x_{j'}) \circ (y_b y_a)) \\ &= (x_j \circ y_b)(x_i \circ y_a) - (x_{i'} \circ y_b)(x_{j'} \circ y_a) \in (\Re(A \circ B)) \\ \psi &= \sigma_{23}((x_{i'} x_{j'}) \circ (y_b y_a - y_{a'} y_{b'})) \\ &= (x_{i'} \circ y_b)(x_{j'} \circ y_a) - (x_{i'} \circ y_{a'})(x_{j'} \circ y_{b'}) \in (\Re(A \circ B)). \end{aligned}$$

The elements φ and ψ are in the ideal of relations $(\Re(A \circ B))$, so the sum $\varphi + \psi$ is also in $(\Re(A \circ B))$. One has

$$\varphi + \psi = (x_j \circ y_b)(x_i \circ y_a) - (x_{i'} \circ y_{a'})(x_{j'} \circ y_{b'}) = f_{ji,ba} \in (\Re(A \circ B)).$$

By definition, $f = x_j x_i - x_{i'} x_{j'} \in \Re_1$ *iff* $r_1(x_j x_i) = x_{i'} x_{j'}$ and $x_j x_i > x_{i'} x_{j'}$. The cancellation low implies that $x_j > x_{i'}$. Thus $(x_{i'} \circ y_{a'})(x_{j'} \circ y_{b'}) < (x_j \circ y_b)(x_i \circ y_a)$, and $\mathbf{LM}(f_{ji,ba}) = (x_j \circ y_b)(x_i \circ y_a)$.

(2) By hypothesis $y_b y_a - y_{a'} y_{b'} \in \Re_2$, then by Remark 3.2 again,

$$\begin{aligned} f_{ij,ba} &= \sigma_{23}((x_i x_j) \circ (y_b y_a - y_{a'} y_{b'})) \\ &= (x_i \circ y_b)(x_j \circ y_a) - (x_i \circ y_{a'})(x_j \circ y_{b'}) \in (\Re(A \circ B)). \end{aligned}$$

It is clear that $HM(f_{ij,ba}) = (x_i \circ y_b)(x_j \circ y_a)$ which proves (2).

(3). Suppose $y_b y_a - y_{a'} y_{b'} \in \Re_2$, and $r_1(x_i x_j) > x_i x_j$. Then, $r_1(x_i x_j) = x_{j'} x_{i'}$ and $r_1(x_{j'} x_{i'}) = x_i x_j$ for some $1 \leq j', i' \leq m$, so $x_{j'} x_{i'} - x_i x_j \in \Re_1$. By Remark 3.2

$$\begin{aligned} \varphi_1 &= \sigma_{23}(x_{j'} x_{i'} - x_i x_j) \circ (y_b y_a)) \\ &= (x_{j'} \circ y_b)(x_{i'} \circ y_a) - (x_i \circ y_b)(x_j \circ y_a) \in (\Re(A \circ B)). \end{aligned}$$

By part (1)

$$f_{j'i',ba} = (x_{j'} \circ y_b)(x_{i'} \circ y_a) - (x_i \circ y_{a'})(x_j \circ y_{b'}) \in (\Re(A \circ B)).$$

It follows that $f_{j'i',ba} - \varphi_1 \in (\Re(A \circ B))$. The explicit computation gives

$$\begin{aligned} f_{j'i',ba} - \varphi_1 &= (x_{j'} \circ y_b)(x_{i'} \circ y_a) - (x_i \circ y_{a'})(x_j \circ y_{b'}) \\ &\quad -(x_{j'} \circ y_b)(x_{'} \circ y_a) + (x_i \circ y_b)(x_j \circ y_a) \\ &= (x_i \circ y_b)(x_j \circ y_a) - (x_i \circ y_{a'})(x_j \circ y_{b'}) = g_{ij,ba}. \end{aligned}$$

We have shown that $g_{ij,ba} \in (\Re(A \circ B))$. It is clear that $\mathbf{LM}(g_{ij,ba}) = (x_i \circ y_b)(x_j \circ y_a)$. □

Note that the sets $\Re_a$ and $\Re_b$ consist of quadratic polynomials in the set $X \circ Y$ of one-generators of $A \circ B$. It follows from Lemma 3.11 that every element of $\Re = \Re_a \cup \Re_b$ is a relation of $A \circ B$.

We have to show that the elements of $\Re$ form a basis of the ideal of relations $(\Re(A \circ B))$ of $A \circ B$. It will be convenient to use the description of $\Re_a$ and $\Re_b$ as sets of quadratic polynomials in the variables W, see (3.10), so we simply replace $x_i \circ y_a$ by w_{ia} in each of the relations in $\Re$.

Remark 3.12 Theorem 3.10 states that the Segre product $A \circ B$ has a finite presentation

$$A \circ B \simeq \mathbf{k}\langle w_{11}, \dots, w_{mn}\rangle/(\Re),$$

where $\Re$ is a set of $\binom{mn}{2} + \binom{m}{2}\binom{n}{2}$ quadratic polynomials in the free associative algebra $\mathbf{k}\langle w_{11}, \dots, w_{mn}\rangle$. More precisely, $\Re$ is a disjoint union $\Re = \Re_a \cup \Re_b$ of the sets $\Re_a$ and $\Re_b$ described below.

(1) The set $\Re_a$ consists of $\binom{mn}{2}$ relations given explicitly in (3.13) and (3.14):

$$\begin{gathered} f_{ji,ba} = w_{jb}w_{ia} - w_{i'a'}w_{j'b'}, 1 \le i, j \le m, 1 \le a, b \le n, \\ \text{where } r_1(x_j x_i) = x_{i'}x_{j'},\ \ j > i' \text{ and } r_2(y_b y_a) = y_{a'}y_{b'}. \end{gathered} \tag{3.13}$$

Every relation $f_{ji,ba}$ has leading monomial $\mathbf{LM}(f_{ji,ba}) = w_{jb}w_{ia}$.

$$\begin{gathered} f_{ij,ba} = w_{ib}w_{ja} - w_{ia'}w_{jb'}, 1 \le i, j \le m, 1 \le a, b \le n, \\ \text{where } r_1(x_i x_j) = x_i x_j, \text{ and } r_2(y_b y_a) = y_{a'}y_{b'} \text{ with } \ b > a'. \end{gathered} \tag{3.14}$$

Every relation $f_{ij,ba}$ has leading monomial $\mathbf{LM}(f_{ij,ba}) = w_{ib}w_{ja}$.

(2) The set $\Re_b$ consists of $\binom{m}{2}\binom{n}{2}$ relations given explicitly in (3.15)

$$\begin{gathered} g_{ij,ba} = w_{ib}w_{ja} - w_{ia'}w_{jb'}, 1 \le i, j \le m, 1 \le a, b \le n, \\ \text{where } r_1(x_i x_j) > x_i x_j, \ \text{ and } \ r_2(y_b y_a) = y_{a'}y_{b'} \ \text{ with } \ b > a'. \end{gathered} \tag{3.15}$$

Every relation $g_{ij,ba}$ has leading monomial $\mathbf{LM}(g_{ij,ba}) = w_{ib}w_{ja}$.

Next, we count the relations in $\Re_b$. The number of $x_i x_j$, $1 \leq i, j \leq m$, such that $r_1(x_i x_j) > x_i x_j$ is exactly the number of nontrivial r_1-orbits in $X \times X$ which is $\binom{m}{2}$. The number of pairs y_b, y_a with $y_b y_a > r_2(y_b y_a)$ equals the number of nontrivial r_2 -orbits, which is $\binom{n}{2}$, hence

$$|\Re_b| = \binom{m}{2}\binom{n}{2}. \tag{3.16}$$

The two sets $\Re_a$ and $\Re_b$ are disjoint. Indeed, the leading monomials of all elements in $\Re$ are pairwise distinct, and therefore the relations are pairwise distinct. So $\Re = \Re_a \cup \Re_b$ is a disjoint union of sets, and by (3.12) and (3.16), one has:

$$|\Re| = |\Re_a| + |\Re_b| = \binom{mn}{2} + \binom{m}{2}\binom{n}{2}. \tag{3.17}$$

It remains to show that $\Re$ is a linearly independent set.

Lemma 3.13 *Under the hypothesis of Theorem 3.10, the set of polynomials $\Re \subset \boldsymbol{k}\langle W\rangle$ is linearly independent.*

Proof This proof is routine. Note that the set of all words in $\langle W\rangle$ forms a basis of the free associative algebra $\mathbf{k}\langle W\rangle$ (considered as a vector space), in particular every finite set of distinct words in $\langle W\rangle$ is linearly independent. Consider the presentation of $\Re$ given in Remark 3.12. All words occurring in $\Re$ are monomials of length 2 in W^2, but some of them occur in more than one relation, e.g., the leading monomial $w_{ib}w_{ja}$, of g_{ijba} occurs as a second monomial in some f given in (3.13). Indeed, there is unique pair j_1, i_1 such that $r_1(x_{j_1}x_{i_1}) = x_i x_j$, $j_1 > i$. It is clear that $r_2(y_{a'}y_{b'}) = y_b y_a$ (since r_2 is involutive). Then, by definition

$$f_{j_1 i_1, a'b'} = (x_{j_1} \circ y_{a'})(x_{i_1} \circ y_{b'}) - (x_i \circ y_b)(x_j \circ y_a) = w_{j_1 a'}w_{i_1 b'} - w_{ib}w_{ja}.$$

We shall prove the lemma in three steps.

(1) The set of polynomials $\Re_a \subset \mathbf{k}\langle w_{11}, \ldots, w_{mn}\rangle$ is linearly independent. We have noticed that the polynomials in $\Re_a$ are in 1-to-1 correspondence with the nontrivial $r_{X\circ Y}$-oprbits in $(X \circ Y) \times (X \circ Y)$. The orbits are disjoint and therefore the relations $\Re_a$ involve exactly $m^2n^2 - mn$ distinct monomials in W^2. A linear relation

$$\sum_{f \in \Re_a} \alpha_f f = 0, \text{ where all } \alpha_f \in \mathbf{k},$$

involves only pairwise distinct monomials in W^2 and therefore it must be trivial: $\alpha_f = 0$, for all $f \in \Re_a$. It follows that $\Re_a$ is linearly independent.

(2) The set $\Re_b \subset \mathbf{k}\langle W\rangle$ is linearly independent. Assume the contrary. Then, there exists a nontrivial linear relation for the elements of $\Re_b$:

$$\sum_{g \in \Re_b} \beta_g g = 0, \text{ with all } \beta_g \in \mathbf{k}. \tag{3.18}$$

Let $g_{ij,ba}$ be the polynomial with $\beta_{g_{ij,ba}} \neq 0$ whose leading monomial is the highest among all leading monomials of polynomials $g \in \Re_b$, with $\beta_g \neq 0$. So we have

$$LM(g_{ij,ba}) = w_{ib}w_{ja} > \mathbf{LM}(g), \text{ for all } g \in \Re_b, g \neq g_{ij,ba} \text{ with } \beta_g \neq 0, . \tag{3.19}$$

We use (3.18) to find the following equality in $\mathbf{k}\langle W\rangle$:

$$w_{ib}w_{ja} = w_{ia'}w_{jb'} - \sum_{g\in\Re_b,\, \mathbf{LM}(g)<w_{ib}w_{ja}} \left(\frac{\beta_g}{\beta_{g_{ij,ba}}}\right) g.$$

It follows from (3.19) that the right-hand side of this equality is a linear combination of monomials strictly smaller than $w_{ib}w_{ja}$ (in the lexicographic order on $\langle W\rangle$), which is impossible. Therefore, the set $\Re_b \subset \mathbf{k}\langle W\rangle$ is linearly independent.

(3) The set $\Re \subset \mathbf{k}\langle W\rangle$ is linearly independent. Assume that the polynomials in $\Re$ satisfy a linear relation

$$\sum_{f\in\Re_a} \alpha_f f + \sum_{g\in\Re_b} \beta_g g = 0, \text{ where all } \alpha_f, \beta_g \in \mathbf{k}. \tag{3.20}$$

Every $f \in \Re_a$ can be written $f = u_f - u'_f$, where $u_f, u'_f \in W^2, u_f > u'_f$. Similarly, every $g \in \Re_b$ is $g = u_g - u'_g$, where $u_g, u'_g \in W^2, u_g > u'_g$. This gives the following equalities in the free associative algebra $\mathbf{k}\langle W\rangle$:

$$S_1 = \sum_{f\in\Re_a} \alpha_f u_f = \sum_{f\in\Re_a} \alpha_f u'_f - \sum_{g\in\Re_b} \beta_g g = S_2. \tag{3.21}$$

The element $S_1 = \sum_{f\in\Re_a} \alpha_f u_f$ on the left-hand side of (3.21) is in the space $V_1 = \text{Span} B_1$, where $B_1 = \mathbf{LM}(\Re_a) = \{u_f \mid f \in \Re_a\}$ is linearly independent since it consists of distinct monomials. The element S_2 on the right-hand side of the equality is in the space $V_2 = \text{Span} B$, where

$$B = \{u'_f \mid f \in \Re_a\} \cup \{\text{all monomials } u_g, u'_g \mid g \in \Re_b\}.$$

Take a subset $B_2 \subset B$ which forms a basis of V_2. Note that $B_1 \cap B = \emptyset$, hence $B_1 \cap B_2 = \emptyset$. Moreover each of the sets B_1, and B_2 consists of pairwise distinct monomials in W^2 and it is easy to show that $V_1 \cap V_2 = \{0\}$. Thus, the equality $S_1 = S_2 \in V_1 \cap V_2 = \{0\}$ implies a linear relation

$$S_1 = \sum_{f\in\Re_a} \alpha_f u_f = 0,$$

for the set B_1 of leading monomials of $\Re_a$. But B_1 consists of pairwise distinct monomials, and therefore it is linearly independent. It follows that all coefficients

α_f, $f \in \Re_a$ equal 0. This together with (3.20) implies the linear relation

$$\sum_{g \in \Re_b} \beta_g g = 0,$$

and since by (2), $\Re_b$ is linearly independent we get again $\beta_g = 0, \forall\ g \in \Re_b$. It follows that the linear relation (3.20) must be trivial, and therefore $\Re$ is a linearly independent set of quadratic polynomials in $\mathbf{k}\langle W\rangle$.

□

We claim that $\Re$ is a set of defining relations for $A \circ B$. We know that $A \circ B$ is a quadratic algebra, that is, its ideal of relations is generated by homogeneous polynomials of degree 2, see Corollary 3.9.

Consider the graded ideal $J = (\Re)$ of $\mathbf{k}\langle W\rangle$. To show that that $J = (\Re(A \circ B))$, it will be enowgh to verify that there is an isomorphism of vector spaces:

$$(\Re)_2 \oplus (A \circ B)_2 = (\mathbf{k}\langle W\rangle)_2,$$

or equivalently

$$\dim \mathrm{Span}_{\mathbf{k}} \Re + \dim_{\mathbf{k}} (A \circ B)_2 = \dim_{\mathbf{k}} (\mathbf{k}\langle W\rangle)_2. \tag{3.22}$$

We have shown that $\Re$ is linearly independent, so $\dim \mathrm{Span}_{\mathbf{k}} \Re = |\Re| = \binom{mn}{2} + \binom{m}{2}\binom{n}{2}$. It follows from (3.1) that $\dim_{\mathbf{k}}(A \circ B)_2 = \dim_{\mathbf{k}} A_2 \dim_{\mathbf{k}} B_2 = \binom{m+1}{2}\binom{n+1}{2}$. Therefore,

$$\begin{aligned} \dim Span_{\mathbf{k}} \Re + \dim_{\mathbf{k}} (A \circ B)_2 &= \binom{mn}{2} + \binom{m}{2}\binom{n}{2} + \binom{m+1}{2}\binom{n+1}{2} \\ &= m^2 n^2 = \dim_{\mathbf{k}} (\mathbf{k}\langle W\rangle)_2, \end{aligned}$$

as desired. It follows that $\Re$ is a set of defining relations for the Segre product $A \circ B$. □

4 Segre maps of Yang–Baxter algebras

In this section, we introduce and investigate non-commutative analogues of the Segre maps in the class of Yang–Baxter algebras of finite solutions. Our main result is Theorem 4.5. As a consequence, Corollary 4.6 shows that the Segre product $A \circ B$ of two Yang–Baxter algebras A and B is always left and right Noetherian. The results agree with their classical analogues in the commutative case, [18].

We keep the conventions and notation from the previous sections. As usual, (X, r_1) and (Y, r_2) are disjoint solutions of YBE of finite orders m and n, respectively, $A = \mathcal{A}(\mathbf{k}, X, r_1)$, and $B = \mathcal{A}(\mathbf{k}, Y, r_2)$ are the corresponding YB algebras. We fix enumerations

$$X = \{x_1, \dots, x_m\}, \quad Y = \{y_1, \dots, y_n\},$$

as in Convention 2.15 and consider the degree-lexicographic orders on the free monoids $\langle X\rangle$, and $\langle Y\rangle$ extending these enumerations. $A \circ B$ is the Segre product of A and B, its set of one-generators is

$$\begin{aligned} W = X \circ Y = \{&w_{11} = x_1 \circ y_1 < w_{12} = x_1 \circ y_2 < \cdots < w_{1n} = x_1 \circ y_n \\ &< w_{21} = x_2 \circ y_1 < \cdots < w_{mn} = x_m \circ y_n\}, \end{aligned}$$

ordered lexicographically, and $(X \circ Y, r_{X\circ Y})$ is the solution isomorphic to the Cartesian product $(X \times Y, \rho_{X\times Y})$, see Proposition-Notation 3.7.

Definition-Notation 4.1 Let $Z = \{z_{11}, z_{12}, \dots, z_{mn}\}$ be a set of order mn, disjoint with X and Y. Define a map

$$r = r_Z : Z \times Z \longrightarrow Z \times Z$$

induced canonically from the solution $(X \circ Y, r_{X\circ Y})$:

$$r(z_{jb}, z_{ia}) = (z_{i'a'}, z_{j'b'}) \textit{ iff } r_{X\circ Y}(x_j \circ y_b, x_i \circ y_a) = (x_{i'} \circ y_{a'}, x_{j'} \circ y_{b'}).$$

It is clear that (Z, r_Z) is a solution of YBE isomorphic to $(X \circ Y, r_{X\circ Y})$ (and isomorphic to the Cartesian product $(X \times Y, r_{X\times Y})$).

We consider the degree-lexicographic order on the free monoid $\langle Z\rangle$ induced by the enumeration of Z

$$Z = \{z_{11} < z_{12} < \cdots < z_{mn}\}.$$

Remark 4.2 Let $\mathbb{A}_Z = \mathcal{A}(\mathbf{k}, Z, r_Z)$ be the YB algebra of the solution (Z, r_Z). Then, $\mathbb{A}_Z = \mathbf{k}\langle Z\rangle/(\Re(\mathbb{A}_Z))$, where the ideal of relations of $\mathbb{A}_Z$ is generated by the set $\Re(\mathbb{A}_Z)$ consisting of $\binom{mn}{2}$ quadratic binomial relations given explicitly in (4.1) and (4.2):

$$\begin{gathered} \varphi_{ji,ba} = z_{jb}z_{ia} - z_{i'a'}z_{j'b'},\ 1 \le i, j \le m,\ 1 \le a, b \le n, \\ \text{where } r_Z(z_{jb}z_{ia}) = z_{i'a'}z_{j'b'},\ \text{ and } j > i', \text{or equivalently, } \ r_Z(z_{jb}z_{ia}) < z_{jb}z_{ia}. \end{gathered} \tag{4.1}$$

Every relation $\varphi_{ji,ba}$ has leading monomial $\mathbf{LM}(\varphi_{ji,ba}) = z_{jb}z_{ia}$.

$$\begin{gathered} \varphi_{ij,ba} = z_{ib}z_{ja} - z_{ia'}z_{jb'},\ \ 1 \le i, j \le m,\ 1 \le a, b \le n, \\ \text{where } r_Z(z_{ib}z_{ja}) = z_{ia'}z_{jb'} \text{ and } b > a'. \end{gathered} \tag{4.2}$$

Every relation $\varphi_{ij,ba}$ has leading monomial $\mathbf{LM}(\varphi_{ij,ba}) = z_{ib}z_{ja}$.

There is a bijective correspondence between the set of relations $\Re(\mathbb{A}_Z)$ and the set of nontrivial r_Z-orbits in $Z \times Z$.

By definition $A \circ B$ is a subalgebra of $A \otimes B$, so if $f = g$ holds in $A \circ B$, then it holds in $A \otimes B$.

Lemma 4.3 *In notation as above, let (X, r_1) and (Y, r_2) be solutions on the finite disjoint sets $X = \{x_1, \dots, x_m\}$, and $Y = \{y_1, \dots, y_n\}$, and let $A = \mathcal{A}(\boldsymbol{k}, X, r_1)$, and $B = \mathcal{A}(\boldsymbol{k}, Y, r_2)$ be the corresponding YB algebras. Let (Z, r_Z) be the solution of order mn from Definition-Notation 4.1, and let $\mathbb{A}_Z = \mathcal{A}(\boldsymbol{k}, Z, r_Z)$ be its YB algebra. Then, the assignment*

$$z_{11} \mapsto x_1 \otimes y_1, \ z_{12} \mapsto x_1 \otimes y_2, \ \dots, \ z_{mn} \mapsto x_m \otimes y_n$$

extents to an algebra homomorphism $s_{m,n} : \mathbb{A}_Z \longrightarrow A \otimes_{\boldsymbol{k}} B$.

Proof Naturally, we set $s_{m,n}(z_{i_1a_1} \dots z_{i_pa_p}) := (x_{i_1} \circ y_{a_1}) \dots (x_{i_p} \circ y_{a_p})$, for all words $z_{i_1a_1} \dots z_{i_pa_p} \in \langle Z \rangle$ and then extend this map linearly. Note that for each polynomial $\varphi_{ji,ba} \in \Re(\mathbb{A}_Z)$ given in (4.1), one has

$$s_{n,d}(\varphi_{ji,ba}) = f_{ji,ba} \in \Re_a,$$

where the set $\Re_a$ is a part of the relations of the Segre product $A \circ B$, given in Theorem 3.10 (1).

We have shown that $f_{ji,ba}$ equals identically zero in $A \circ B = \bigoplus_{i \geq 0} A_i \otimes_{\mathbf{k}} B_i$, which is a subalgebra of $A \otimes B$ and therefore $s_{n,d}(\varphi_{ji,ba}) = f_{ji,ba} = 0$ in $A \otimes B$.

Similarly for each $\varphi_{ij,ba}$ given in (4.2), one has

$$s_{n,d}(\varphi_{ij,ba}) = f_{ij,ba} \in \Re_a,$$

thus $s_{n,d}(\varphi_{ij,ba}) = 0$ holds in $A \circ B$, and therefore $s_{n,d}(\varphi_{ij,ba}) = 0$ in $A \otimes B$.

We have shown that the map $s_{m,n}$ agrees with the relations of the algebra $\mathbb{A}_Z$. It follows that the map $s_{m,n} : \mathbb{A}_Z \longrightarrow A \otimes_{\mathbf{k}} B$ is a well-defined homomorphism of algebras. □

Definition 4.4 We call the map $s_{m,n} : \mathbb{A}_Z \longrightarrow A \otimes_{\mathbf{k}} B$ from Lemma 4.3 *the (m, n)-Segre map.*

Theorem 4.5 *In notation as above. Let (X, r_1) and (Y, r_2) be solutions on the finite disjoint sets $X = \{x_1, \dots, x_m\}$, and $Y = \{y_1, \dots, y_n\}$, and let A and B be the corresponding Yang–Baxter algebras. Let (Z, r_Z) be the solution on the set $Z = \{z_{11}, \dots, z_{mn}\}$ defined in Definition-Notation 4.1, and let $\mathbb{A}_Z = \mathcal{A}(\boldsymbol{k}, Z, r_Z)$ be its Yang–Baxter algebra. Let $s_{m,n} : \mathbb{A}_Z \longrightarrow A \otimes_{\boldsymbol{k}} B$ be the Segre map extending the assignment*

$$z_{11} \mapsto x_1 \circ y_1, \ z_{12} \mapsto x_1 \circ y_2, \ \dots, \ z_{mn} \mapsto x_m \circ y_n.$$

1. *The image of the Segre map $s_{m,n}$ is the Segre product $A \circ B$. Moreover, $s_{m,n} : \mathbb{A}_Z \longrightarrow A \circ B$ is a homomorphism of graded algebras.*
2. *The kernel $\mathfrak{K} = \ker(s_{m,n})$ of the Segre map is generated by the set $\Re_s$ of $\binom{m}{2}\binom{n}{2}$ linearly independent quadratic binomials described below*

$$\begin{aligned} \Re_s = \{\gamma_{ij,ba} = z_{ib}z_{ja} - z_{ia'}z_{jb'}, 1 \leq i, j \leq m, 1 \leq a, b \leq n \mid \\ r_1(x_i x_j) > x_i x_j, \ and \ r_2(y_b y_a) = y_{a'} y_{b'} \ with \ b > a'\}. \end{aligned} \tag{4.3}$$

Proof (1) By definition the YB algebra $\mathbb{A}_Z$ is generated by the set Z, and therefore the image $s_{m,n}(\mathbb{A}_Z)$ of the Segre map is the subalgebra of $A \otimes B$ generated by the set $s_{m,n}(Z) = X \circ Y = W$ where

$$W = X \circ Y = \{w_{11} = x_1 \circ y_1 < w_{12} = x_1 \circ y_2 < \cdots < w_{mn} = x_m \circ y_n\}.$$

But the set $W = X \circ Y$ generates exactly the algebra $A \circ B$, see Theorem 3.10, and therefore $s_{m,n}(\mathbb{A}_Z) = A \circ B$.

(2) Observe that the elements of $\Re_s$ are considered both as elements of the free associative algebra $\mathbf{k}\langle Z\rangle$ and in the Yang–Baxter algebra $\mathbb{A}_Z$.

(i) We shall prove first that $\Re_s$ consists of nonzero elements of $\mathbb{A}_Z$. Assume the contrary: for some quadruple ij, ba, the element $\gamma_{ij,ba} \in \Re_s$ is zero in $\mathbb{A}_Z$. Then, there is an equality of monomials of degree two in the graded component $(\mathbb{A}_Z)_2$:

$$z_{ib}z_{ja} = z_{ia'}z_{jb'}.$$

By Corollary 2.9 (3), this is possible if and only if

$$z_{ia'}z_{jb'} \in \{z_{ib}z_{ja}, r_Z(z_{ib}z_{ja})\}. \tag{4.4}$$

It is clear, by the definition of R_s, that

$$z_{ia'}z_{jb'} \neq z_{ib}z_{ja} \text{ as words in } Z^2.$$

Moreover, by the definition of the element $\gamma_{ij,ba}$, one has

$$x_ix_j < r_1(x_ix_j) = x_{j'}x_{i'}, \text{ and } r_2(y_by_a) = y_{a'}y_{b'}, b > a'.$$

But the algebra $\mathbb{A}_Z$ is a domain (see Facts 2.11 (4)), so the inequality $x_ix_j < x_{j'}x_{i'}$ implies $i < j'$. At the same time, by the definition of the map r_Z, one has

$$r_Z(z_{ib}z_{ja}) = z_{j'a'}z_{i'b'} \neq z_{ia'}z_{jb'} \tag{4.5}$$

thus (4.4) is impossible. It follows that $z_{ib}z_{ja} \neq z_{ia'}z_{jb'}$ as elements of $\mathbb{A}_Z$, and therefore $\gamma_{ij,ba}$ is a nonzero element of $\mathbb{A}_Z$, which contradicts our assumption.

(ii) Next, we prove that $\Re_s \subset \mathfrak{K} = \ker(s_{m,n})$. Direct computation shows that

$$\begin{aligned} s_{m,n}(\gamma_{ij,ba}) = s_{m,n}(z_{ib}z_{ja} - z_{ia'}z_{jb'}) &= (x_i \circ y_b)(x_j \circ y_a) - (x_i \circ y_{a'})(x_j \circ y_{b'}) \\ &= g_{ij,ba} \in \Re_b, \end{aligned}$$

where $\Re_b$ is the subset of relations of $A \circ B$ described in (3.11). Therefore, $s_{m,n}(\gamma_{ij,ba}) = 0$ in $A \circ B$, for all elements of $\Re_s$, and

$$\Re_s \subset \mathfrak{K} = \ker(s_{m,n}).$$

(iii) We claim that $\Re_s$ is a minimal set of generators of the kernel $\mathfrak{K}$.

Note first that there is an equality of orders.

$$|\Re_s| = |\Re_b| = \binom{m}{2}\binom{n}{2}. \tag{4.6}$$

Indeed, it follows from the descriptions of $\Re_s$ and $\Re_b$, that there is a bijective correspondence between the two sets $\Re_s$ and $\Re_b$.

Moreover, the set $\Re_s$ is linearly independent, since $s_{m,n}(\Re_s) = \Re_b$, and $\Re_b$ is a linearly independent set in $A \otimes B$, by Lemma 3.13.

It is clear that the map $s_{m,n}$ agrees with the natural gradings by length of words in $\mathbb{A}_Z$ and the Segre product $A \circ B$ presented in terms of 1-generators and quadratic relations in Remark 3.12. By the First Isomorphism Theorem

$$\mathbb{A}_Z/\mathfrak{K} \simeq A \circ B, \text{ and } (\mathbb{A}_Z)_2/\mathfrak{K}_2 = (\mathbb{A}_Z/\mathfrak{K})_2 \simeq (A \circ B)_2.$$

Hence,

$$\dim(\mathbb{A}_Z)_2 = \dim(\mathfrak{K}_2) + \dim(A \circ B)_2, \text{ and } \dim(\mathfrak{K}_2) = \dim(\mathbb{A}_Z)_2 - \dim(A \circ B)_2. \tag{4.7}$$

But $\mathbb{A}_Z$ is the Yang–Baxter algebra of the solution (Z, r_Z) of order mn, so Corollary 2.12 implies

$$\dim(\mathbb{A}_Z)_2 = \binom{mn+1}{2}. \tag{4.8}$$

It follows from (3.1) that

$$\dim(A \circ B)_2 = \dim A_2 \dim B_2 = \binom{m+1}{2}\binom{n+1}{2}. \tag{4.9}$$

The second equality in (4.7) together with (4.8) and (4.9) imply

$$\dim(\mathfrak{K})_2 = \binom{mn+1}{2} - \binom{m+1}{2}\binom{n+1}{2} = \binom{m}{2}\binom{n}{2}.$$

This together with (4.6) imply the desired equality

$$\dim(\mathfrak{K})_2 = |\Re_s|.$$

We have shown that $\Re_s$ is a linearly independent subset of $\mathfrak{K}_2$, whose order equals the dimension of $\mathfrak{K}_2$ and therefore $\Re_s$ is a basis of the graded component $\mathfrak{K}_2$ of the ideal $\mathfrak{K}$. In particular, $\mathfrak{K}_2 = \mathbf{k}\Re_s$. The ideal $\mathfrak{K}$ is generated by homogeneous polynomials of degree 2, hence

$$\mathfrak{K} = (\mathfrak{K}_2) = (\Re_s).$$

It follows that $\mathfrak{R}_s$ is a minimal set of generators for the kernel $\mathfrak{K}$. □

Corollary 4.6 *In notation and assumption as above. Let $A = \mathcal{A}(\boldsymbol{k}, X, r_1)$, and $B = \mathcal{A}(\boldsymbol{k}, Y, r_2)$, be the Yang–Baxter algebras of the finite solutions (X, r_1) and (Y, r_2) of order m and n, respectively. Then, the Segre product $A \circ B$ is a left and a right Noetherian algebra. Moreover, $A \circ B$ has polynomial growth.*

Proof It follows from Theorem 4.5 that $A \circ B = s_{m,n}(\mathbb{A}_Z)$, the image of the Segre homomorphism $s_{m,n} : \mathbb{A}_Z \longrightarrow A \otimes_{\mathbf{k}} B$, where $\mathbb{A}_Z$ is the Yang–Baxter algebra of the solution (Z, r_Z) of order mn. By Facts 2.11 (3) (see also [17, Theorem 4.5]) the algebra $\mathbb{A}_Z$ is left and right Noetherian and has polynomial growth of degree mn, therefore its homomorphic image $A \circ B$ is left and right Noetherian and also has polynomial growth of degree $\leq mn$. □

We shall prove in the next section that in the special case, when A and B are binomial skew polynomial rings the Segre product $A \circ B$ has infinite global dimension, see Theorem 5.1 (4).

We end up the section with open questions, where we split the general case of arbitrary solutions (X, r_1) and (Y, r_2), and the particular case of square-free solutions.

Question 4.7 (1) Let $A = \mathcal{A}(\mathbf{k}, X, r_1)$, and $B = \mathcal{A}(\mathbf{k}, Y, r_2)$, be the Yang–Baxter algebras of the finite solutions (X, r_1) and (Y, r_2). Is it true that the Segre product $A \circ B$ is a domain?

(2) Let $A = \mathcal{A}(\mathbf{k}, X, r_1)$, and $B = \mathcal{A}(\mathbf{k}, Y, r_2)$, be the Yang–Baxter algebras of the finite square-free solutions (X, r_1) and (Y, r_2). Is it true that the Segre product $A \circ B$ is a domain?

(3) Let A and B be binomial skew polynomial algebras. Is it true that the Segre product $A \circ B$ is a domain?

Questions (2) and (3) are equivalent. Even in the general case we expect that due to the good algebraic and combinatorial properties of A and B, and the specific relations of $A \circ B$ the answer is affirmative. Moreover, in cases (2) and (3), the Segre product $A \circ B$ is a PBW algebra whose quadratic relations are explicitly given. Observe that A and B are Noetherian domains, and $A \circ B$ is a subalgebra of the tensor product $A \otimes B$. However, it is shown in [28] that the tensor product $D_1 \otimes_F D_2$ of two division algebras over an algebraically closed field contained in their centers may not be a domain.

5 Segre products and Segre maps for the class of square-free solutions

In this section (X, r_1) and (Y, r_2) are fixed disjoint square-free solutions of orders $|X| = m$ and $|Y| = n$. Let $A = \mathcal{A}(\mathbf{k}, X, r_1)$, and $B = \mathcal{A}(\mathbf{k}, Y, r_2)$ be the corresponding YB algebras. We keep Convention 2.15 (a) and choose enumerations $X = \{x_1, \dots, x_m\}$, and $Y = \{y_1, \dots, y_n\}$, such that A and B are binomial skew-polynomial algebras with respect to these enumerations, see Definition 2.3. In particular, A is a PBW algebra with PBW generators $x_1, \dots, x_m$ and B is a PBW algebra with PBW generators $y_1, \dots, y_n$. The following result collects various algebraic properties of the Segre product $A \circ B$.

Theorem 5.1 *Suppose (X, r_1) and (Y, r_2) are disjoint square-free solutions, where*

$$X = \{x_1, \dots, x_m\}, \quad and \quad Y = \{y_1, \dots, y_n\}$$

are enumerated so that the Yang–Baxter algebras $A = \mathcal{A}(\boldsymbol{k}, X, r_1)$, and $B = \mathcal{A}(\boldsymbol{k}, Y, r_2)$ are binomial skew polynomial rings with respect to these enumerations.

Then, the Segre product $A \circ B$ satisfies the following conditions.

(1) *$A \circ B$ is a PBW algebra with a set of mn PBW generators*

$$W = X \circ Y = \{w_{11} = x_1 \circ y_1,\ w_{12} = x_1 \circ x_2,\ \dots,\ w_{1n} = x_1 \circ y_n,\ \dots,\ w_{mn} = x_m \circ x_n\},$$

ordered lexicographically, and a standard finite presentation

$$A \circ B \simeq \boldsymbol{k}\langle w_{11}, \dots, w_{mn}\rangle / (\Re),$$

where the set of relations $\Re$ is a Gröbner basis of the ideal $I = (\Re)$ and consists of $\binom{mn}{2} + \binom{m}{2}\binom{n}{2}$ square-free quadratic polynomials. The set $\Re$ splits as a disjoint union $\Re = \Re_a \cup \Re_b$ of the sets $\Re_a$ and $\Re_b$ described below.

(a) *The set $\Re_a$ consists of $\binom{mn}{2}$ relations given explicitly in (5.1) and (5.2):*

$$\begin{gathered} f_{ji,ba} = w_{jb}w_{ia} - w_{i'a'}w_{j'b'}, 1 \le i < j \le m, 1 \le a, b \le n, \ where \\ r_1(x_j x_i) = x_{i'}x_{j'},\ j > i' \ and \ r_2(y_b y_a) = y_{a'}y_{b'}; \\ w_{jb} > w_{ia},\quad w_{jb} > w_{i'a'},\quad w_{i'a'} < w_{j'b'}. \end{gathered} \tag{5.1}$$

Every relation $f_{ji,ba}$ has leading monomial $\mathbf{LM}(f_{ji,ba}) = w_{jb}w_{ia}$.

$$\begin{gathered} f_{ii,ba} = w_{ib}w_{ia} - w_{ia'}w_{ib'}, 1 \le i \le m, 1 \le a < b \le n,\ where \\ r_2(y_b y_a) = y_{a'}y_{b'} \ with \ b > a'; \\ w_{ib} > w_{ia},\quad w_{ib} > w_{ia'},\quad w_{ia'} < w_{ib'}. \end{gathered} \tag{5.2}$$

Every relation $f_{ii,ba}$ has leading monomial $\mathbf{LM}(f_{ii,ba}) = w_{ib}w_{ia}$.

(b) *The set $\Re_b$ consists of $\binom{m}{2}\binom{n}{2}$ relations given explicitly in (5.3)*

$$\begin{gathered} g_{ij,ba} = w_{ib}w_{ja} - w_{ia'}w_{jb'}, 1 \le i < j \le m,\ 1 \le a < b \le n, \\ where \ r_2(y_b y_a) = y_{a'}y_{b'} \ with \ b > a'. \end{gathered} \tag{5.3}$$

One has $\mathbf{LM}(g_{ij,ba}) = w_{ib}w_{ja}$.

(2) *$A \circ B$ is a Koszul algebra.*
(3) *$A \circ B$ is left and right Noetherian.*
(4) *The algebra $A \circ B$ has polynomial growth and infinite global dimension.*

Proof (1). It follows from [26, Chap 4.4, Proposition 4.2] that the Segre product $A \circ B$ is a PBW algebra with a set of PBW one-generators $W = X \circ Y$, ordered lexicographically. The shape of the defining relations follows from our Theorem 3.10, and from the relations of the binomial skew-polynomial rings A and B which encode the properties of r_1 and r_2. To show that $\Re$ is a Gröbner basis of the ideal $I = (\Re)$ it will be enough to check that

$$\mathcal{N}(\Re)_3 = \mathcal{N}(I)_3.$$

Recall that $|\mathcal{N}(I)_3| = \dim(A \circ B)_3$ and by (3.1) one has

$$\dim(A \circ B)_3 = \dim A_3 \dim B_3 = \binom{m+2}{3}\binom{n+2}{3}.$$

In general, $\mathcal{N}(I)_3 \subseteq \mathcal{N}(\Re)_3$, so we have to show that

$$|\mathcal{N}(\Re)_3| = |\mathcal{N}(I)_3| = \dim(A \circ B)_3.$$

A monomial $u \in \langle W \rangle$ of length 3 is normal modulo $\Re$ *iff* it is normal modulo $\overline{\Re}$, where

$$\overline{\Re} = \{HM(f) \mid f \in \Re\},$$

or equivalently *iff* u is not divisible by any of the leading monomials $\mathbf{LM}(f)$, $f \in \Re$. Note that

$$w_{ia}w_{jb}w_{kc} \in \mathcal{N}(\Re)_3 \Longleftrightarrow w_{ia}w_{jb} \in \mathcal{N}(\Re)_2 \text{ and } w_{jb}w_{kc} \in \mathcal{N}(\Re)_2.$$

It follows from the shape of the leading monomials $\mathbf{LM}(f)$, $f \in \Re$, that $w_{ia}w_{jb} \in \mathcal{N}(\Re)_2$ if and only if $1 \le i \le j \le m$ and $1 \le a \le b \le n$. Therefore,

$$w_{ia}w_{jb}w_{kc} \in \mathcal{N}(\Re)_3 \Longleftrightarrow 1 \le i \le j \le k \le m \text{ and } 1 \le a \le b \le c \le n.$$

In other words,

$$\mathcal{N}(\Re)_3 = \{w_{ia}w_{jb}w_{kc} \mid 1 \le i \le j \le k \le m,\ 1 \le a \le b \le k \le m\}.$$

This implies that $|\mathcal{N}(\Re)_3| = \binom{m+2}{3}\binom{n+2}{3} = \dim(A \circ B)_3$, as desired. Therefore, the set of defining relations $\Re$ is a Gröbner basis.

(2). The Kosulity of $A \circ B$ follows from Corollary 3.9. Note that in this particular case, Kosulity also follows from the fact that every PBW algebra is Koszul, see [27].

(3). Corollary 4.6 implies that $A \circ B$ is left and right Noetherian.

(4). By Corollary 4.6 again, $A \circ B$ has polynomial growth. It follows from our result [12, Theorem 1.1] that if a graded PBW algebra has mn one-generators, polynomial growth and finite global dimension, then the number of its defining relation must be $\binom{mn}{2}$. We have shown in part (1) that the algebra $A \circ B$ is a quadratic PBW algebra with mn PBW generators, and $\binom{mn}{2} + \binom{m}{2}\binom{n}{2}$ defining relations, therefore $A \circ B$ has infinite global dimension. □

As we mentioned before, we do not know if $A \circ B$ is a domain even in the square-free case, see Questions 4.7.

Let $(X \circ Y, r_{X \circ Y})$ be the solution on the set $X \circ Y$, defined in Proposition-Notation 3.7. Then, $(X \circ Y, r_{X \circ Y})$ is a square-free solution.

Consider now the solution (Z, r_Z) on the set $Z = \{z_{11}, z_{12}, \dots, z_{mn}\}$ defined in Definition-Notation 4.1. By construction (Z, r_Z) is isomorphic to the solution $(X \circ Y, r_{X \circ Y})$, and therefore it is a square-free solution of order mn.

Proposition 5.2 *Let $\mathbb{A}_Z = \mathcal{A}(\boldsymbol{k}, Z, r_Z)$ be the YB algebra of (Z, r_Z). Then $\mathbb{A}_Z$ is a binomial skew-polynomial ring with a standard finite presentation $\mathbb{A}_Z = \boldsymbol{k}\langle z_{11}, z_{12}, \dots, z_{mn}\rangle/(\Re(\mathbb{A}_Z))$, where the set of defining relations $\Re(\mathbb{A}_Z)$ consists of $\binom{mn}{2}$ binomial relations given explicitly in (5.4) and (5.5).*

$$\begin{array}{l} \varphi_{ji,ba} = z_{jb}z_{ia} - z_{i'a'}z_{j'b'},\ 1 \le i < j \le m,\ 1 \le a, b \le n,\ \textit{where} \\ r_Z(z_{jb}z_{ia}) = z_{i'a'}z_{j'b'},\ \textit{and}\ z_{jb} > z_{ia},\ z_{jb} > z_{i'a'},\ z_{i'a'} < z_{j'b'}. \end{array} \tag{5.4}$$

Every relation $\varphi_{ji,ba}$ has leading monomial $\mathbf{LM}(\varphi_{ji,ba}) = z_{jb}z_{ia}$.

$$\begin{array}{l} \varphi_{ii,ba} = z_{ib}z_{ia} - z_{ia'}z_{ib'},\ 1 \le i \le m,\ 1 \le a < b \le n,\ \textit{where} \\ r_Z(z_{ib}z_{ia}) = z_{ia'}z_{ib'}\ \textit{and}\ z_{ib} > z_{ia},\ z_{ib} > z_{ia'},\ z_{ia'} < z_{ib'}. \end{array} \tag{5.5}$$

Every relation $\varphi_{ij,ba}$ has leading monomial $\mathbf{LM}(\varphi_{ii,ba}) = z_{ib}z_{ia}$. Moreover, the set $\Re(\mathbb{A}_Z)$ forms a Gröbner basis of the ideal $I = (\Re(\mathbb{A}_Z))$ of the free associative algebra $\boldsymbol{k}\langle z_{11}, z_{12}, \dots, z_{mn}\rangle$ with respect to the degree-lexicographic order.

Proof The relations $\Re(\mathbb{A}_Z)$ described with details in (5.4) and (5.5) have the shape of the typical relations of a binomial skew-polynomial ring, see Definition 2.3, conditions (a), (b), (c). We have to show that the set $\Re(\mathbb{A}_Z)$ is a Gröbner basis of the ideal I with respect to the degree-lexicographic order on $\langle z_{11}, z_{12}, \dots, z_{mn}\rangle$. It follows from the shape of relations that the set $\mathcal{N}(I)$ of normal words modulo I is a subset of the set of terms (ordered monomials) in the alphabet Z:

$$\mathcal{N}(I) \subseteq \mathcal{T}(Z) = \{z_{11}^{k_{11}} \dots z_{mn}^{k_{mn}} \mid k_{ia} \ge 0,\ 1 \le i \le m,\ 1 \le a \le n\}.$$

By Facts 2.11 the Yang–Baxter algebra $\mathbb{A}_Z$ has Hilbert series $H_{\mathbb{A}}(t) = \frac{1}{(1-t)^{mn}}$ which implies that $\mathcal{N}(I) = \mathcal{T}(Z)$. In other words, $\mathcal{T}(Z)$ is the normal $\mathbf{k}$-basis of $\mathbb{A}_Z$, so condition (d′) in Definition 2.3 is satisfied, and therefore $\Re(\mathbb{A}_Z)$ is a Gröbner basis of the ideal I. □

The following corollary shows that our (noncommutative) analogue of Segre morphisms for Yang–Baxter algebras of finite solutions (the general case) can be defined also for the subclass of Yang–Baxter algebras related to square-free solutions. This is in contrast with our recent results [14, Corollary 6.5], which imply that the noncommutative analogue of Veronese morphisms for the class of Yang–Baxter algebras related to (arbitrary) finite solutions of YBE, introduced in [14] can not be restricted to the subclass of YB algebras of square-free solutions.

Corollary 5.3 *In notation as above. Let (X, r_1) and (Y, r_2) be disjoint square-free solutions of finite orders, m and n, respectively, let A, and B be the corresponding YB algebras. Let (Z, r_Z) be the square-free solution on the set $Z = \{z_{11}, \dots, z_{mn}\}$ defined in Definition-Notation 4.1, and let $\mathbb{A} = \mathcal{A}(\boldsymbol{k}, Z, r_Z)$ be its YB algebra. Let $s_{m,n} : \mathbb{A} \longrightarrow A \otimes_{\boldsymbol{k}} B$ be the Segre map extending the assignment*

$$z_{11} \mapsto x_1 \circ y_1, \quad z_{12} \mapsto x_1 \circ y_2, \quad \dots, \quad z_{mn} \mapsto x_m \circ y_n.$$

(1) *The image of the Segre map $s_{m,n}$ is the Segre product $A \circ B$.*
(2) *The kernel $\mathfrak{K} = \ker(s_{m,n})$ of the Segre map is generated by the set $\Re_s$ of $\binom{m}{2}\binom{n}{2}$ linearly independent quadratic binomials described below*

$$\begin{aligned} \Re_s = \{&\gamma_{ij,ba} = z_{ib}z_{ja} - z_{ia'}z_{jb'} \mid 1 \le i < j \le m, 1 \le a < b \le n \\ &\text{and } r_2(y_b y_a) = y_{a'} y_{b'} \text{ with } b > a', a' < b'\}. \end{aligned} \tag{5.6}$$

6 An Example

We shall present an example which illustrates the results of the paper. We use the notation of the previous sections.

Example 6.1 Let (X, r_1) be the square-free solution on the set $X = \{x_1, x_2, x_3\}$, where

$$\begin{aligned} r_1 : x_3x_2 &\leftrightarrow x_1x_3 \quad x_3x_1 \leftrightarrow x_2x_3 \quad x_2x_1 \leftrightarrow x_1x_2 \\ x_ix_i &\leftrightarrow x_ix_i \quad 1 \le i \le 3. \end{aligned}$$

and let (Y, r_2) be the solution on $Y = \{y_1, y_2\}$, where

$$r_2 : y_2y_2 \leftrightarrow y_1y_1 \;\; y_1y_2 \leftrightarrow y_1y_2 \;\; y_2y_1 \leftrightarrow y_2y_1.$$

Then,

$$\begin{aligned} A &= \mathcal{A}(\mathbf{k}, X, r_1) = \mathbf{k}\langle x_1, x_2, x_3\rangle/(x_3x_2 - x_1x_3, \; x_3x_1 - x_2x_3, \; x_2x_1 - x_1x_2); \\ B &= \mathcal{A}(\mathbf{k}, Y, r_2) = \mathbf{k}\langle y_1, y_2\rangle/(y_2^2 - y_1^2). \end{aligned}$$

The algebra A is a binomial skew-polynomial ring, its relation form a Gröbner basis of the ideal they generate. In contrast with the relations of A, the relations of the algebra B do not form a Gröbner basis of the ideal $J = (y_2^2 - y_1^2)$. The reduced Gröbner basis of the ideal J is the set $G = \{y_2^2 - y_1^2, y_2y_1y_1 - y_1y_1y_2\}$, see more details in [14].

Let $A \circ B$ be the Segre product of A and B, and let $(X \circ Y, r_{X \circ Y})$ be the solution from Proposition-Notation 3.7 isomorphic to the Cartesian product of solutions $(X \times Y, \rho_{X \times Y})$. Then, $A \circ B$ is a quadratic algebra with a set of one-generators

$$\begin{aligned} W = \{&w_{11} = x_1 \circ y_1, w_{12} = x_1 \circ y_2, w_{21} = x_2 \circ y_1, w_{22} = x_2 \circ y_2, \\ &w_{31} = x_3 \circ y_1, w_{32} = x_3 \circ y_2\} \end{aligned}$$

and 18 defining quadratic relations. More precisely,

$$A \circ B \simeq \mathbf{k}\langle w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}\rangle/(\Re),$$

where $\Re = \Re_a \cup \Re_b$ is a disjoint union of quadratic relations $\Re_a$ and $\Re_b$ given below.

(1) The set $\Re_a$ with $|\Re_a| = 15$ is a disjoint union $\Re_a = \Re_{a1} \cup \Re_{a2}$, where

$$\begin{aligned} \Re_{a1} = \{\ & f_{32,22} = w_{32}w_{22} - w_{11}w_{31}, \quad f_{32,11} = w_{31}w_{21} - w_{12}w_{32},\\ & f_{32,21} = w_{32}w_{21} - w_{12}w_{31}, \quad f_{32,12} = w_{31}w_{22} - w_{11}w_{32},\\ & f_{31,22} = w_{32}w_{12} - w_{21}w_{31}, \quad f_{31,11} = w_{31}w_{11} - w_{22}w_{32},\\ & f_{31,21} = w_{32}w_{11} - w_{22}w_{31}, \quad f_{31,12} = w_{31}w_{12} - w_{21}w_{32},\\ & f_{21,22} = w_{22}w_{12} - w_{11}w_{21}, \quad f_{21,11} = w_{21}w_{11} - w_{12}w_{22},\\ & f_{21,21} = w_{22}w_{11} - w_{12}w_{21}, \quad f_{21,12} = w_{21}w_{12} - w_{11}w_{22}\ \}. \end{aligned}$$

$$\begin{aligned} \Re_{a2} = \{\ & f_{33,22} = w_{32}w_{32} - w_{31}w_{31}, \quad f_{22,22} = w_{22}w_{22} - w_{21}w_{21},\\ & f_{11,22} = w_{12}w_{12} - w_{11}w_{11}\ \}. \end{aligned}$$

In fact the relations $\Re_a$ are exactly the defining relations of the YB algebra $\mathbb{A}_{X\circ Y} = \mathcal{A}(\mathbf{k}, X\circ Y, r_{X\circ Y})$, there is a one-to one correspondence between the set of relations $\Re_a$ and the set of nontrivial $r_{X\circ Y}$-orbits in $(X\circ Y)\times(X\circ Y)$. Note that each relation in $\Re_{a2}$ involves squares of generators.

(2) The set $\Re_b$ consists of 3 quadratic relations given below

$$\begin{aligned} \Re_b\ &= \{g_{23,22} = w_{22}w_{32} - w_{21}w_{31},\ g_{13,22} = w_{12}w_{32} - w_{11}w_{31},\\ & g_{12,22} = w_{12}w_{22} - w_{11}w_{21}\}. \end{aligned}$$

Let $Z = \{z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32}\}$, and let (Z, r_Z) be the solution defined in Definition-Notation 4.1. By construction, (Z, r_Z) is isomorphic to the solution $(X\circ Y, r_{X\circ Y})$.

The Yang–Baxter algebra $\mathbb{A}_Z = \mathcal{A}(\mathbf{k}, Z, r_Z)$ has a finite presentation

$$\mathbb{A}_Z = \mathbf{k}\langle z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32}\rangle/(\Re(\mathbb{A}_Z)),$$

where the set $\Re(\mathbb{A}_Z))$ of 15 defining relations is:

$$\begin{aligned} \Re(\mathbb{A}_Z) = \{\ & \varphi_{32,22} = z_{32}z_{22} - z_{11}z_{31}, \quad \varphi_{32,11} = z_{31}z_{21} - z_{12}z_{32},\\ & \varphi_{32,21} = z_{32}z_{21} - z_{12}z_{31}, \quad \varphi_{32,12} = z_{31}z_{22} - z_{11}z_{32},\\ & \varphi_{31,22} = z_{32}z_{12} - z_{21}z_{31}, \quad \varphi_{31,11} = z_{31}z_{11} - z_{22}z_{32},\\ & \varphi_{31,21} = z_{32}z_{11} - z_{22}z_{31}, \quad \varphi_{31,12} = z_{31}z_{12} - z_{21}z_{32},\\ & \varphi_{21,22} = z_{22}z_{12} - z_{11}z_{21}, \quad \varphi_{21,11} = z_{21}z_{11} - z_{12}z_{22},\\ & \varphi_{21,21} = z_{22}z_{11} - z_{12}z_{21}, \quad \varphi_{21,12} = z_{21}z_{12} - z_{11}z_{22},\\ & \varphi_{33,22} = z_{32}z_{32} - z_{31}z_{31}, \quad \varphi_{22,22} = z_{22}z_{22} - z_{21}z_{21},\\ & \varphi_{11,22} = z_{12}z_{12} - z_{11}z_{11}\ \}. \end{aligned}$$

Clearly, (Z, r_Z) is not a square-free solution, and therefore, by [14, Theorem 3.8], the defining relations $\Re(\mathbb{A}_Z)$ of $\mathbb{A}_Z$ do not form a Gröbner basis. In particular, $\mathbb{A}_Z$

is neither a binomial skew polynomial ring, nor a PBW algebra. The Segre map $s_{3,2} : \mathbb{A}_Z \longrightarrow A \otimes B$ has image $A \circ B$. The kernel $\ker(s_{3,2})$ is the ideal of $\mathbb{A}_Z$ generated by the following three polynomials

$$\gamma_{23,22} = z_{22}z_{32} - z_{21}z_{31}, \ \gamma_{13,22} = z_{12}z_{32} - z_{11}z_{31}, \ \gamma_{12,22} = z_{12}z_{22} - z_{11}z_{21}.$$

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