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Hamiltonian reductions in matrix Painlevé systems

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Abstract

For certain finite groups G of Bäcklund transformations, we show that a dynamics of G-invariant configurations of $n|G|$ Calogero–Painlevé particles is equivalent to a certain n-particle Calogero–Painlevé system. We also show that the reduction of a dynamics on G-invariant subset of $n|G| \times n|G|$ matrix Painlevé system is equivalent to a certain $n \times n$ matrix Painlevé system. The groups G correspond to folding transformations of Painlevé equations. The proofs are based on Hamiltonian reductions.

Contents

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1 Introduction

In the paper we construct a relation between matrix Painlevé equations of different size and Calogero–Painlevé systems of different number of particles. Such relations are in a correspondence with folding transformations in the Painlevé theory. We recall these notions and illustrate our results by simple instructive examples.

Folding transformations   This notion was introduced in [13], but actual examples of such transformations were known for many years. By definition, the folding transformation is an algebraic (of degree greater than 1) map between solutions of Painlevé equations. Moreover, this map should go through a quotient of Okamoto-Sakai space of initial conditions (see e.g. [8] for the review). Probably, the simplest example is the folding transformation of Painlevé II to itself.

Example 1.1 The Painlevé II equation is a second-order differential equation with parameter θ

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = 2q^3 + tq + \theta. \tag{1.1}$$

This equation is equivalent to a Hamiltonian system with Hamiltonian

$$H_{\mathrm{II}}(p, q; t) = \frac{1}{2}p^2 - \frac{1}{2}\left(q^2 + \frac{t}{2}\right)^2 - \theta q. \tag{1.2}$$

This equation (system) has a natural symmetry r which transforms parameter $\theta \mapsto -\theta$ and maps $(p, q) \mapsto (-p, -q)$. Such symmetries are called Bäcklund transformations.

For the special value of the parameter $\theta = 0$ the equation is preserved by r so one can ask for an equation on functions invariant under this transformation. If we introduce new invariant coordinates P, Q and new time s

$$Q = -2^{-1/3}\frac{p}{q}, \quad P = 2^{1/3}\left(q^2 - \frac{p^2}{2q^2}\right) - \frac{s}{2}, \quad s = -2^{1/3}t, \qquad (1.3)$$

we get Painlevé II equation on Q with parameter $\theta = -1/2$

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}s^2} = 2Q^3 + sQ - \frac{1}{2}. \qquad (1.4)$$

This is a transformation of degree 2 between the spaces of initial conditions.

Note that in this example we start from the Painlevé equation with the special value of the parameter (namely $\theta = 0$) and come to the Painlevé equation with the special value of the parameter (namely $\theta = -1/2$). But, it appears that we can come to the equation with an arbitrary value of parameter if we start from Calogero–Painlevé system.

Calogero–Painlevé systems   These systems can be viewed as a certain N-particle generalization of the Painlevé equations. Let $q_1, \ldots, q_N$ be coordinates of these particles and $p_1, \ldots, p_N$ be the corresponding momenta with the standard Poisson bracket between them. The dynamics of a Calogero–Painlevé system is defined by the Hamiltonian of the form [12]

$$H_{\mathrm{CP}}(\{(p_i, q_i)\}; t) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N} V^{(1)}(q_i, t) + \sum_{1 \le i < j \le N} V^{(2)}(q_i, q_j). \quad (1.5)$$

Here $V^{(1)}(q)$ is a Painlevé potential (i.e. for $N = 1$ we get a Hamiltonian of the Painlevé equation) and $V^{(2)}(q_1, q_2)$ is a Calogero-type interaction. Note that the Hamiltonian (1.5) is symmetric under permutations. By definition the phase space of a Calogero–Painlevé system is a quotient by the action of permutation group S_N.

Note that the Hamiltonian (1.5) is non-autonomous, since Painlevé potential $V^{(1)}$ depends on time t. In the autonomous limit, in which t is a coupling constant, the Hamiltonian (1.5) belongs to Inozemtsev extension of Calogero integrable system [4, 5].

Another important feature of Calogero–Painlevé systems is that they describe isomonodromic deformations of certain natural $2N \times 2N$ systems [2, 6, 12].

In the paper we construct a natural analog of the folding transformations for the Calogero–Painlevé systems. Let us give an example of Calogero–Painlevé II for $N = 2$.

Example 1.2 Hamiltonian (1.5) in this case has the form

$$H_{\mathrm{CPII}}(\{(p_i, q_i)\}; t) = \sum_{i=1}^{2}\left(\frac{1}{2}p_i^2 - \frac{1}{2}\left(q_i^2 + \frac{t}{2}\right)^2 - \theta q_i\right) + \frac{g^2}{(q_1 - q_2)^2}. \quad (1.6)$$

This system has a symmetry $r_{CP} : \{(p_1, q_1), (p_2, q_2)\} \mapsto \{(-p_1, -q_1), (-p_2, -q_2)\}$ with $\theta \mapsto -\theta$. For $\theta = 0$ the subset of r_{CP}–invariant points is defined by the equations (recall that we take the quotient by permutation group S_2)

$$q_1 + q_2 = 0, \quad p_1 + p_2 = 0. \tag{1.7}$$

It is easy to check that these equations are preserved by the dynamics.

For $g = 0$ Calogero–Painlevé system is equivalent to the system of two non-interacting Painlevé particles q_1, q_2 up to the permutation. Hence, by Example 1.1, the dynamics on r_{CP}-invariant subset is equivalent to the Painlevé II equation with $\theta = -1/2$.

For $g \neq 0$ let us take the following coordinates P, Q on r_{CP}-invariant subset and rescale the time

$$Q = -2^{-1/3}\left(\frac{p_1}{q_1} - \frac{ig}{2q_1^2}\right), \quad P = 2^{1/3}q_1^2 - Q^2 - \frac{s}{2}, \quad s = -2^{1/3}t. \tag{1.8}$$

Note that these formulas are g-deformed version of formulas (1.3). It is straightforward to compute the dynamics in terms of p_1, q_1

$$\frac{dq_1}{dt} = p_1, \quad \frac{dp_1}{dt} = 2q_1^3 + tq_1 + \frac{g^2}{4q_1^3}, \tag{1.9}$$

and then get

$$\frac{dQ}{ds} = P, \quad \frac{dP}{ds} = 2Q^3 + sQ - ig - 1/2, \tag{1.10}$$

which is the Painlevé II equation with parameter $\theta = -\frac{1}{2} - ig$.

It appears that this example can be generalized to the Calogero–Painlevé system with more than 2 particles. Namely, for $2n$-particle Calogero–Painlevé II with $\theta = 0$ the dynamics on (dense open subset of) r_{CP}-invariant subset is equivalent to the dynamics of n-particle Calogero Painlevé II system with $\theta = -\frac{1}{2} - ig$. Moreover, similar statements hold for other folding transformations, see Theorem 4.1 and remarks after it. This is one of the main results of the paper.

Remark 1.1 Part of our motivation comes from the papers by Rumanov [10, 11], where certain solutions of N-particle Calogero–Painlevé II systems are related to a spectrum of β-ensemble with $\beta = 2N$. It was also observed in [11] that this solution for $N = 2$ is described by the scalar Painlevé II equation. It appears that this solution is invariant with respect to r_{CP}, hence this observation is a corollary of Example 1.2 above. It would be interesting to study subsets corresponding to Rumanov's solutions.

Matrix Painlevé systems　Calogero–Painlevé systems can be obtained by a Hamiltonian reduction à la Kazhdan–Kostant–Sternberg [7] from matrix Painlevé systems [2].[1] In the Calogero–Painlevé case we consider the Bäcklund invariant subset of the phase space. It appears that in the matrix case it is natural to consider a subset invariant under the Bäcklund transformation twisted with conjugation by a certain permutation matrix. In the matrix case we also perform an additional reduction.

Example 1.3　The phase space of matrix Painlevé II consists of pairs of $N \times N$ matrices p, q with symplectic form $\mathrm{Tr}(\mathrm{d}p \wedge \mathrm{d}q)$. The Hamiltonian has the form [cf. Hamiltonian in scalar case (1.2)]

$$H_{\mathrm{MPII}}(p, q; t) = \mathrm{Tr}\left(\frac{1}{2}p^2 - \frac{1}{2}\left(q^2 + \frac{t}{2}\right)^2 - \theta q\right), \qquad (1.11)$$

where parameter θ remains a scalar variable.

As before we have Bäcklund transformation $r : (p, q) \mapsto (-p, -q)$ with $\theta \mapsto -\theta$. Let us take $2n \times 2n$ matrix Painlevé II with $\theta = 0$ and consider the subset of the phase space invariant under $\mathrm{Ad}_{S_2} \circ r$, where $S_2 = (\mathbf{1}_{n \times n}, -\mathbf{1}_{n \times n})$. This invariant subset is given by block matrices with $n \times n$ blocks

$$p = \begin{pmatrix} 0 & \mathfrak{p}_{12} \\ \mathfrak{p}_{21} & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & \mathfrak{q}_{12} \\ \mathfrak{q}_{21} & 0 \end{pmatrix}. \qquad (1.12)$$

It is easy to see that the dynamics preserves this subset. It is given by

$$\dot{\mathfrak{q}}_{12} = \mathfrak{p}_{12}, \quad \dot{\mathfrak{q}}_{21} = \mathfrak{p}_{21}, \quad \dot{\mathfrak{p}}_{12} = 2\mathfrak{q}_{12}\mathfrak{q}_{21}\mathfrak{q}_{12} + t\mathfrak{q}_{12}, \quad \dot{\mathfrak{p}}_{21} = 2\mathfrak{q}_{21}\mathfrak{q}_{12}\mathfrak{q}_{21} + t\mathfrak{q}_{21}. \qquad (1.13)$$

One can check that $m_2 = \mathfrak{p}_{21}\mathfrak{q}_{12} - \mathfrak{q}_{21}\mathfrak{p}_{12}$ is an integral of motion. Let us fix its value as $m_2 = \mathrm{i}g\mathbf{1}_{n \times n}$. Then, taking $n \times n$ matrices (P, Q) and rescaling the time

$$Q = -2^{-1/3}\mathfrak{p}_{12}\mathfrak{q}_{12}^{-1}, \quad P = 2^{1/3}\mathfrak{q}_{12}\mathfrak{q}_{21} + Q^2 + s/2, \quad s = -2^{1/3}t, \quad (1.14)$$

we obtain matrix Painlevé II with $\theta = -\mathrm{i}g - 1/2$

$$\frac{\mathrm{d}Q}{\mathrm{d}s} = P, \qquad \frac{\mathrm{d}P}{\mathrm{d}s} = 2Q^3 + sQ - \mathrm{i}g - 1/2. \qquad (1.15)$$

Note that the parameter value for the resulting matrix Painlevé system coincides with one for the scalar Painlevé II obtained in Example 1.2 [cf. (1.15) with (1.10)].

Actually, the last step in the example is a Hamiltonian reduction with respect to the conjugation by a certain $\mathrm{GL}_n \subset \mathrm{GL}_{2n}$. In Theorem 3.1 we construct such Hamiltonian reductions of matrix Painlevé systems, corresponding to all folding transformations

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[1] In the paper we consider only matrix Painlevé systems that are Hamiltonian, there also exist other interesting matrix Painlevé analogs, see e.g. [1].

of the Painlevé equations. This is one of the main results of the paper. Moreover, our proof of Theorem 4.1 mentioned above is based on Theorem 3.1.

Plan of the paper In Sect. 2 we recall matrix Painlevé equations in their Hamiltonian forms. We also lift all Bäcklund transformations known in scalar case to the matrix case. This lift is not completely straightforward due to noncommutativity of variables, see e.g. Tables 1 and 2 below.

In Sect. 3 we construct block reductions for the matrix Painlevé systems. The construction works for Bäcklund transformations which preserve parameters of certain Painlevé systems but act on p, q nontrivially (similarly to the folding transformations). Let w be such a transformation, d denotes order of w and $\bar{w}$ denotes w twisted by adjoint action of a certain permutation matrix of order d. In Theorem 3.1 we consider a subset in the phase space of $nd \times nd$ matrix Painlevé system that is invariant under the action of $\bar{w}$, and show that the dynamics on its Hamiltonian reduction is equivalent to $n \times n$ matrix Painlevé system. The proof of this theorem is based on case by case considerations, which occupy the bulk of Sect. 3. In Sect. 3.4 we prove a similar statement for the non-cyclic subgroups.

In Sect. 4.1 we recall the definition of Calogero–Painlevé systems. In the following Sect. 4.2 we prove Theorem 4.1. Roughly speaking, this theorem states that the dynamics on w-invariant set of nd Calogero–Painlevé particles is equivalent to the dynamics of n Calogero–Painlevé particles. As we mentioned above, the proof is based on Theorem 3.1 and the construction of Calogero–Painlevé systems via the Hamiltonian reduction of matrix Painlevé systems. In particular, we do not need a case by case analysis here.

It appears that there are more relations between matrix Painlevé systems and Calogero–Painlevé systems, similar to ones found in Theorems 3.1 and 4.1. We do not intend to classify them and just give several examples in Sect. 5. In particular, in Sect. 5.1 we study another block reductions of matrix Painlevé equations. In Sect. 5.2 we study w-invariant configurations of Calogero–Painlevé particles in which several particles evolve by algebraic solutions of a Painlevé equation. Finally, in Sect. 5.3 we discuss spin generalization of the Calogero–Painlevé systems.

2 Matrix Painlevé systems and their Bäcklund transformations

We follow [2] in conventions on matrix Painlevé systems.

2.1 Painlevé II

Scalar case We will consider a parametrized family PII(θ) of ordinary differential equations

$$\ddot{q} = 2q^3 + tq + \theta, \tag{2.1}$$

where by dot we denote $\frac{\mathrm{d}}{\mathrm{d}t}$. These equations are Hamiltonian. Namely, one can take $\mathbb{C}^2$ with standard symplectic structure $\omega = \mathrm{d}p \wedge \mathrm{d}q$, then the Hamiltonian (1.2)

$$H(p, q; t) = \frac{1}{2}p^2 - \frac{1}{2}\left(q^2 + \frac{t}{2}\right)^2 - \theta q, \tag{2.2}$$

leads to dynamics (2.1).

This family of ordinary differential equations has discrete symmetries, called Bäcklund transformations. For example, it is easy to see that if $q(t)$ is a solution of PII(θ), then $-q(t)$ is a solution of PII$(-\theta)$.

Let us define Bäcklund transformations in general case. Let $\mathbb{D}(\alpha)$ be a system of ordinary differential equations with extended phase space M, which depends on the set of parameters $\alpha \in \mathbb{C}^k$.

Definition 2.1 A pair of maps $(\pi, \tilde{\pi}) : \text{M} \times \mathbb{C}^k \to \text{M} \times \mathbb{C}^k$ is called Bäcklund transformation if π maps solutions of $\mathbb{D}(\alpha)$ to solutions of $\mathbb{D}(\tilde{\pi}(\alpha))$.

Let us consider a hamiltonian dynamics of a system of ordinary differential equations $\mathbb{D}(\alpha)$ on a symplectic manifold (M, ω) with a time-dependent Hamiltonian $H_\alpha(x; t)$, $(x, t) \in \text{M} = M \times \mathbb{C}$. The equations of motion can be identified with the one dimensional distribution on TM defined as $\text{Ker}(\omega - dH_\alpha \wedge dt)$. Then the sufficient condition for $(\pi, \tilde{\pi})$ to be a Bäcklund transformation is

$$\pi^*\left(\omega - dH_{\tilde{\pi}(\alpha)} \wedge dt\right) = h_\alpha(x; t)\left(\omega - dH_\alpha \wedge dt\right), \tag{2.3}$$

for a certain function $h_\alpha(x; t)$. Note that it is important to consider extended phase space since we work with non-autonomous systems and non-autonomous symmetries.

Below we follow [13] in the description of Bäcklund transformations.

Matrix generalisation Hamiltonian system corresponding to PII(θ) can be generalized to the matrix case. Let us consider a phase space $\text{Mat}_N(\mathbb{C}) \times \text{Mat}_N(\mathbb{C})$ with coordinates (p, q) and symplectic structure $\omega = \text{Tr}(dp \wedge dq)$. Matrix PII$(\theta)$ can be defined by the Hamiltonian (1.11)

$$H_\theta(p, q; t) = \text{Tr}\left(\frac{p^2}{2} - \frac{1}{2}\left(q^2 + \frac{t}{2}\right)^2 - \theta q\right). \tag{2.4}$$

Note that the parameter θ remains scalar. In all cases below we will consider matrix Painlevé equations with scalar parameters only. Hamiltonian (2.4) leads to equations of motion

$$\dot{q} = p, \qquad \dot{p} = 2q^3 + tq + \theta. \tag{2.5}$$

In case $N = 1$ this system is equivalent to the ordinary differential equation PII(θ).

Now we generalize formulas for the Bäcklund transformations of scalar PII(θ) to the matrix case. Here and below we will use notation C_n for the cyclic group of order n.

Proposition 2.1 *Transformations in the table below are Bäcklund transformations of matrix* PII. *These transformations generate group* $C_2 \ltimes W\left(A_1^{(1)}\right)$.

Table 1 Bäcklund transformations for matrix PII(α_0, α_1)

	q	p	t
s_1	$q + \alpha_1 f^{-1}$	$p - \alpha_1 \left(qf^{-1} + f^{-1}q \right) - \alpha_1^2 f^{-2}$	t
r	$-q$	$-p$	t
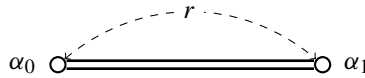
Let us explain the notation. Here $f = p + q^2 + \frac{t}{2}$, $\alpha_1 = \theta + \frac{1}{2}$, $\alpha_0 = 1 - \alpha_1$. Parameters α_0 and α_1 are called root variables. Action of s_1 on them is given by $s_1(\alpha_1) = -\alpha_1$, $s_1(\alpha_0) = \alpha_0 + \alpha_1$. The action of r on root variables is indicated on the diagram.

For any Painlevé equation group of Bäcklund transformations is isomorphic to an extended affine Weyl group. Thus it is convenient to encode generators and relations of these groups through diagrams. Solid lines and nodes define an affine Dynkin diagram. On the i-th node we write the corresponding root variable α_i. These root variables parametrize the family of equations. By s_i we denote the reflection corresponding to i'th simple root. A solid line between i'th and j'th nodes corresponds to the relation $s_i s_j s_i = s_j s_i s_j$. Absence of a solid line between i'th and j'th nodes corresponds to the relation $s_i s_j = s_j s_i$. The action of reflections on the root variables is given by

$$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i, \tag{2.6}$$

where $\{a_{ij}\}$ is the Cartan matrix corresponding to a given Dynkin diagram. Then we extend Coxeter group by a finite group, acting by automorphisms of the diagram. Dashed arrows show action of automorphisms on parameters and define adjoint action on $\{s_i\}$

$$\text{if } g(\alpha_i) = \alpha_j, \text{ then } g s_i g^{-1} = s_j. \tag{2.7}$$

The generators coming from the automorphisms of diagrams will be very important in Sect. 3.

Proof of proposition 2.1 Let us check that transformations from Table 1 are symmetries of matrix PII(α_0, α_1) using (2.3).

For transformation r it is easy to see that it maps Hamiltonian with parameter θ to the Hamiltonian with parameter $-\theta$. Also it preserves ω and t. Thus r satisfies equation (2.3) with $h_\theta(x; t) = 1$.

For transformation s_1 let us consider matrix coordinates (f, q). Then

$$\omega = \text{Tr} (df \wedge dq) + \frac{1}{2} d\text{Tr} (q) \wedge dt, \tag{2.8}$$

$$H_\theta(f, q; t) = \mathrm{Tr}\left(\frac{1}{2}f^2 - f\left(q^2 + \frac{t}{2}\right) - \theta q\right). \tag{2.9}$$

So we have

$$\omega - \mathrm{d}H_\theta \wedge \mathrm{d}t = \mathrm{Tr}\left(\mathrm{d}f \wedge \mathrm{d}q\right) - \mathrm{d}H_{\alpha_1} \wedge \mathrm{d}t, \quad \text{where } \alpha_1 = \theta + \frac{1}{2}. \tag{2.10}$$

It is easy to see that s_1 acts as $f \mapsto f, \alpha_1 \mapsto -\alpha_1$. Hence we get

$$s_1^*\left(\mathrm{Tr}\left(\mathrm{d}f \wedge \mathrm{d}q\right)\right) = \mathrm{Tr}\left(\mathrm{d}f \wedge \mathrm{d}\left(q + \alpha_1 f^{-1}\right)\right) = \mathrm{Tr}\left(\mathrm{d}f \wedge \mathrm{d}q\right),$$

$$s_1^*\left(H_{-\alpha_1}\right) = \mathrm{Tr}\left(\frac{1}{2}f^2 - f\left(\left(q + \alpha_1 f^{-1}\right)^2 + \frac{t}{2}\right) + \alpha_1\left(q + \alpha_1 f^{-1}\right)\right)$$

$$= H_{\alpha_1},$$

$$s_1^*(t) = t. \tag{2.11}$$

Then we have

$$s_1^*(\mathrm{Tr}\left(\mathrm{d}f \wedge \mathrm{d}q\right) - \mathrm{d}H_{-\alpha_1} \wedge \mathrm{d}t) = \mathrm{Tr}\left(\mathrm{d}f \wedge \mathrm{d}q\right) - \mathrm{d}H_{\alpha_1} \wedge \mathrm{d}t. \tag{2.12}$$

Hence s_1 is a symmetry of the matrix $\mathrm{PII}\left(-\theta + \frac{1}{2}, \theta + \frac{1}{2}\right)$.

Now it remains to check group relations between generators. In the case considered we have to check that s_1, r are involutions, which can be done by a straightforward computation. The group obtained is not smaller than $C_2 \ltimes W\left(A_1^{(1)}\right)$ since the action on parameters is the same as in the case of scalar $\mathrm{PII}\left(-\theta + \frac{1}{2}, \theta + \frac{1}{2}\right)$ and action on parameters determines an element of $C_2 \ltimes W\left(A_1^{(1)}\right)$ uniquely. □

2.2 Painlevé VI

The definition, Hamiltonian and Bäcklund transformations of scalar $\mathrm{PVI}(\theta)$ can be found in [13]. Let us start from the definition of the matrix PVI. Here and below to define a matrix Painlevé equation we will consider a symplectic structure $\omega = \mathrm{Tr}\left(\mathrm{d}p \wedge \mathrm{d}q\right)$ on the space of pairs of matrices $\mathrm{Mat}_n\left(\mathbb{C}\right) \times \mathrm{Mat}_n\left(\mathbb{C}\right) \ni (p, q)$. Then the dynamics can be defined by the Hamiltonian, which for matrix PVI is given by

$$\begin{aligned}
t(t-1)H(p, q; t) = \mathrm{Tr}(&pq(q-1)p(q-t) \\
&- (\alpha_4(q-1)(q-t) + \alpha_3 q(q-t) \\
&+ (\alpha_0 - 1)q(q-1))p + \alpha_2(\alpha_1 + \alpha_2)(q-t)).
\end{aligned} \tag{2.13}$$

Proposition 2.2 *Transformations in the table below are Bäcklund transformations of the matrix* PVI. *These transformations generate group* $S_4 \ltimes W\left(D_4^{(1)}\right)$.

Table 2 Bäcklund transformations for matrix PVI($\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$)

	q	p	t
s_0	q	$p - \alpha_0(q-t)^{-1}$	
s_1	q	p	
s_2	$q + \alpha_2 p^{-1}$	p	t
s_3	q	$p - \alpha_3(q-1)^{-1}$	
s_4	q	$p - \alpha_4 q^{-1}$	
σ_{34}	$1 - q$	$-p$	$1-t$
σ_{14}	$t^{[p,q]}q^{-1}t^{-[p,q]}$	$-t^{[p,q]}q(pq+\alpha_2)t^{-[p,q]}$	t^{-1}
σ_{03}	$t^{[p,q]}(t^{-1}q)t^{-[p,q]}$	$t^{[p,q]}(tp)t^{-[p,q]}$	t^{-1}
π_1	tq^{-1}	$-t^{-1}q(pq+\alpha_2)$	
			t
π_2	$t(q-1)(q-t)^{-1}$	$-\left(t(t-1)\right)^{-1}(q-t)\left(p(q-t)+\alpha_2\right)$	
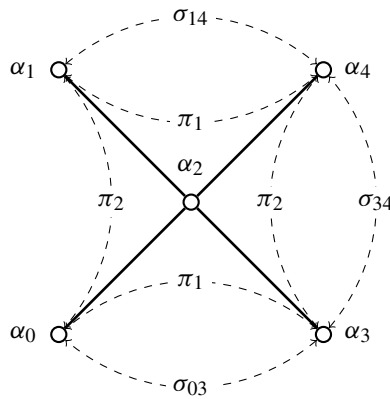
Automorphisms Aut $\left(D_4^{(1)}\right) = S_4$ are generated by permutations $\sigma_{34}, \sigma_{14}, \sigma_{03}$. In general this group changes the time variable t, the time preserving subgroup is C_2^2, generated by $\pi_1 = \sigma_{14}\sigma_{03}$, $\pi_2 = (\sigma_{34}\sigma_{03}\sigma_{14})^2$. This is just a group of automorphisms of the affine Weyl group.

Proof of Proposition 2.2 can be done by a tedious but straightforward calculation, similar to the above PII case.

Note that s_1, s_2 and σ's generate the rest of the transformations. Therefore it is sufficient to check the Bäcklund symmetry condition only for them. The following remarks simplify the proofs.

Remark 2.1 Let m_1, m_2 be monomials in p, q with coefficient 1 such that

$$\deg_p(m_1) = \deg_p(m_2), \qquad \deg_q(m_1) = \deg_q(m_2). \tag{2.14}$$

If $\deg_p (m_1) + \deg_q (m_1) \leq 3$, then

$$\mathrm{Tr}\, (m_1) = \mathrm{Tr}\, (m_2) . \tag{2.15}$$

If $\deg_p (m_1) + \deg_q (m_1) = 4$, then the same is true except for the case $\deg_p (m_1) = \deg_q (m_1) = 2$. In this case either $\mathrm{Tr}\, (m_1) = \mathrm{Tr}\, (p^2 q^2)$ or $\mathrm{Tr}\, (m_1) = \mathrm{Tr}\, (pqpq)$.

Remark 2.2 Let H_1 and H_2 be two Hamiltonians such that $H_1 - H_2 = f(t)\, (\mathrm{Tr}\, (pqpq) - \mathrm{Tr}\, (p^2 q^2))$. Then

$$\mathrm{Tr}\, (\mathrm{d}p \wedge \mathrm{d}q) - \mathrm{d}H_1 \wedge \mathrm{d}t = \mathrm{Tr}\, (\mathrm{d}\tilde{p} \wedge \mathrm{d}\tilde{q}) - \mathrm{d}H_2 \wedge \mathrm{d}t, \tag{2.16}$$

where

$$\tilde{p} = s^{[p,q]} p s^{-[p,q]}, \quad \tilde{q} = s^{[p,q]} q s^{-[p,q]}, \quad s = \exp\left(\int f(t) \mathrm{d}t \right). \tag{2.17}$$

In other words, the dynamics generated by H_1 can be mapped to the dynamics generated by H_2 by the certain non-autonomous change of variables.

One can see with the help of Remark 2.1 that for transformations s_0, s_2 form $\omega - \mathrm{d}H \wedge \mathrm{d}t$ changes in the same way as in the commutative case.

There is an additional conjugation in σ_{14}, σ_{03} which disappears in the commutative case. Let us explain its appearance. Consider $\tilde{\sigma}_{03}$ defined by the formula

$$\tilde{\sigma}_{03} : q \mapsto \frac{q}{t}, \quad p \mapsto tp, \quad t \mapsto t^{-1}. \tag{2.18}$$

In commutative case $\tilde{\sigma}_{03}$ is a Bäcklund transformation. In the matrix case almost all terms in the Hamiltonian transform in the same way as in the commutative case. The only difference appears in the transformation of the first term in the Hamiltonian

$$\tilde{\sigma}_{03}^{*} \left(\mathrm{d}t \wedge \mathrm{d} \left(\frac{1}{t(t-1)} \mathrm{Tr}\, (pq(q-1)p(q-t)) \right) \right)$$
$$= \mathrm{d}t \wedge \mathrm{d} \left(\frac{1}{t(t-1)} \mathrm{Tr}\, (pq(q-t)p(q-1)) \right). \tag{2.19}$$

Hence we have

$$\tilde{\sigma}_{03}^{*} \left(\omega - \mathrm{d}H_{\sigma_{03}(\alpha)} \wedge \mathrm{d}t \right)$$
$$= \omega - \mathrm{d}H_\alpha \wedge \mathrm{d}t - \mathrm{d} \left(\frac{\mathrm{Tr}\, (pq((q-t)p(q-1) - (q-1)p(q-t)))}{t(t-1)} \right) \wedge \mathrm{d}t$$
$$= \omega - \mathrm{d} \left(H_\alpha - \frac{1}{t} \mathrm{Tr}\, \left(pqpq - p^2 q^2 \right) \right) \wedge \mathrm{d}t. \tag{2.20}$$

We see that $\tilde{\sigma}_{03}$ fails to satisfy Eq. (2.3). But it follows from Remark 2.2 that for σ_{03} we have

$$\sigma_{03}^* \left(\omega - \mathrm{d}H_{\sigma_{03}(\alpha)} \wedge \mathrm{d}t \right) = \omega - \mathrm{d}H_\alpha \wedge \mathrm{d}t. \tag{2.21}$$

Transformations σ_{14}, σ_{34} can be treated similarly.

To finish the proof it remains to check group relations encoded in Table 2. It can be done directly.

2.3 Answers for the other Painlevé equations

In this part we present groups of Bäcklund transformations which generalize the Bäcklund groups of the scalar Painlevé equations for the matrix case. Namely, for each equation we list the following data:

1. A Hamiltonian which defines the equation.
2. A table and a diagram which encode action of Bäcklund transformations of the equation.
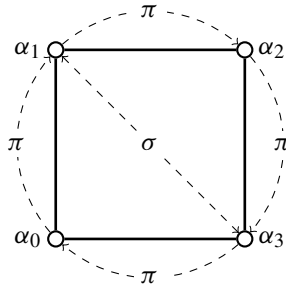3. Answer for the Bäcklund group.

Note that all transformations we list are symplectic. Also, they do preserve $[p, q]$.

Painlevé V The system is defined by Hamiltonian

$$tH(p, q; t) = \mathrm{Tr}\left(p(p + t)q(q - 1) - (\alpha_1 + \alpha_3)\, pq + \alpha_1 p + \alpha_2 tq \right). \tag{2.22}$$

Table 3 Bäcklund transformations for matrix PV($\alpha_0, \alpha_1, \alpha_2, \alpha_3$)

	q	p	t
s_0	$q + \alpha_0(p + t)^{-1}$	p	t
s_1	q	$p - \alpha_1 q^{-1}$	t
s_2	$q + \alpha_2 p^{-1}$	p	t
s_3	q	$p - \alpha_3(q - 1)^{-1}$	t
π	$-t^{-1}p$	$t(q - 1)$	t
σ	$1 - q$	$-p - t$	$-t$

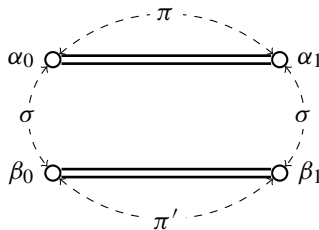These transformations generate group $(C_2 \ltimes C_4) \ltimes W(A_3^{(1)})$.

Painlevé III$\left(\mathrm{D}_6^{(1)}\right)$

The system is defined by Hamiltonian

$$tH(p, q; t) = \mathrm{Tr}\left(p^2q^2 - (q^2 - (\alpha_1 + \beta_1)q - t)p - \alpha_1 q\right). \qquad (2.23)$$

Table 4 Bäcklund transformations for matrix PIII$\left(\mathrm{D}_6^{(1)}\right)(\alpha_0, \alpha_1, \beta_0, \beta_1)$

	q	p	t
s_1	$q + \alpha_1 p^{-1}$	p	t
$\pi \circ \pi'$	$-q$	$1 - p + (\alpha_0 - \beta_1)q^{-1} - tq^{-2}$	t
s_1'	$q + \beta_1(p - 1)^{-1}$	p	t
π'	tq^{-1}	$-t^{-1}q\,(pq + \alpha_1)$	t
σ	$-q$	$1 - p$	$-t$

These transformations generate group $(C_2 \ltimes (C_2 \times C_2)) \ltimes W\left(A_1^{(1)}\right)^2$.

Painlevé III$\left(D_7^{(1)}\right)$

The system is defined by Hamiltonian

$$t H(p, q; t) = \text{Tr}\left(pqpq + \alpha_1 pq + tp + q\right). \tag{2.24}$$

Table 5 Bäcklund transformations for matrix PIII$\left(D_7^{(1)}\right)$ (α_0, α_1)

	q	p	t
s_0	q	$p - \alpha_0 q^{-1} + t q^{-2}$	$-t$
σ	tp	$-t^{-1}q$	$-t$
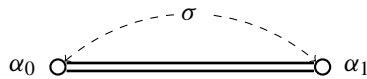
These transformations generate group $C_2 \ltimes W(A_1^{(1)})$.

Painlevé III$\left(D_8^{(1)}\right)$

The system is defined by Hamiltonian

$$t H(p, q; t) = \text{Tr}\left(pqpq + pq - q - t q^{-1}\right). \tag{2.25}$$

This transformation generates group C_2.

Table 6 Bäcklund transformations for matrix PIII$\left(D_8^{(1)}\right)$

	q	p	t
π	tq^{-1}	$-t^{-1}q\left(pq + \frac{1}{2}\right)$	t

Painlevé IV The system is defined by Hamiltonian

$$H(p, q; t) = \text{Tr}\left((p - q - 2t)pq - 2\alpha_1 p - 2\alpha_2 q\right). \tag{2.26}$$

Table 7 Bäcklund transformations for matrix PIV($\alpha_0, \alpha_1, \alpha_2$)

	q	p	t
s_0	$q + 2\alpha_0(p - q - 2t)^{-1}$	$p + 2\alpha_0(p - q - 2t)^{-1}$	t
s_1	q	$p - 2\alpha_1 q^{-1}$	t
s_2	$q + 2\alpha_2 p^{-1}$	p	t
π	$-p$	$-p + q + 2t$	t
σ_1	$-\mathrm{i}p$	$-\mathrm{i}q$	$\mathrm{i}t$
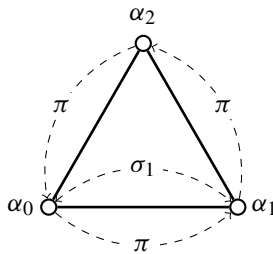
These transformations generate group $(C_4 \ltimes C_3) \ltimes W(A_2^{(1)})$. Structure of the semidirect product in $C_4 \ltimes C_3$ is defined by the relations

$$\pi^3 = e, \quad \sigma_1^4 = e, \quad \sigma_1 \pi \sigma_1^{-1} = \pi^{-1}. \tag{2.27}$$

Remark 2.3 There is central subgroup $\{\sigma_1^2, e\} \subset C_4 \ltimes C_3$ which acts trivially on the parameters. Quotient $C_4 \ltimes C_3/\{\sigma_1^2, e\}$ is isomorphic to the group of automorphisms of the diagram. This is only the case when the finite group we extend affine Weyl group by is not isomorphic to the group of automorphisms of the diagram.

Painlevé I   The system is defined by Hamiltonian

$$H(p, q; t) = \mathrm{Tr}\left(\frac{p^2}{2} - \frac{q^3}{2} - \frac{tq}{4}\right). \tag{2.28}$$

Table 8 Bäcklund transformations for matrix PI

	q	p	t
π	$\varpi_5^3 q$	$\varpi_5^2 p$	ϖt

Here ϖ_5 is a fifth root of unity: $\varpi_5^5 = 1$, $\varpi_5 \neq 1$. This transformation generates group C_5.

3 Block reduction of Matrix Painlevé systems

3.1 General construction

In this section we provide a construction which connects two matrix Painlevé systems of different sizes. In this construction input is a matrix Painlevé system and a Bäcklund transformation w of this system. Output is the matrix Painlevé system which we call the image. We specify input and output in the Table 9 below. Let us denote the extended phase space by $M = \mathrm{Mat}_N(\mathbb{C}) \times \mathrm{Mat}_N(\mathbb{C}) \times \mathbb{C} \times \mathbb{C}^k = \{(p, q, t, \alpha)\}$, where k is the number of parameters of the equation. For the phase space we will use notations $M_\alpha = \mathrm{Mat}_N(\mathbb{C}) \times \mathrm{Mat}_N(\mathbb{C}) \times \mathbb{C} = \{(p, q, t)\}$, $M_{\alpha,t} = \mathrm{Mat}_N(\mathbb{C}) \times \mathrm{Mat}_N(\mathbb{C}) = \{(p, q)\}$. By d we denote the order of the Bäcklund transformation w.

Remark 3.1 Transformations from Table 9 are specified by following conditions

- There exists $\alpha \in \mathbb{C}^k$ such that $M_{\alpha,t}$ is invariant under the action of w.
- w acts on $M_{\alpha,t}$ nontrivially.

Note that all of these transformations come from automorphisms of the corresponding Dynkin diagrams and preserve the time. They are known to be related to folding transformations and are classified in [13].

For each case from Table 9 we consider a matrix Painlevé of size $nd \times nd$. There is the adjoint action of $\mathrm{GL}_{nd}(\mathbb{C})$ on $M_{\alpha,t}$, namely $S : (p, q) \mapsto (SpS^{-1}, SqS^{-1})$. Consider a twisted Bäcklund transformation $\bar{w} = \mathrm{Ad}_{S_d} \circ w$, where $S_d = \mathrm{diag}\left(\mathbf{1}_{n \times n}, e^{\frac{2\pi i}{d}}\mathbf{1}_{n \times n}, ..., e^{\frac{2\pi(d-1)}{d}}\mathbf{1}_{n \times n}\right)$. Transformation $\bar{w}$ is an order d symmetry of the equation. Then for $M^{\bar{w}} = \{x \in M | \bar{w}(x) = x\}$ standard arguments imply

- $M_{\alpha,t}^{\bar{w}}$ is a symplectic submanifold of $M_{\alpha,t}$.
- $M_\alpha^{\bar{w}}$ is preserved by the dynamics of the equation.

Lemma 3.1 *Let w be a transformation from Table 9. Let $\alpha \in \mathbb{C}^k$ be preserved by w. Let M be the extended phase space of the corresponding matrix $nd \times nd$ Painlevé*

Table 9 Input and output

NO	Equation	Image	w	d	q	p	Section
1	PII	PII	r	2	$-q$	$-p$	3.2.1
2	PIII($\mathrm{D}_6^{(1)}$)	PIII($\mathrm{D}_8^{(1)}$)	$\pi \circ \pi'$	2	$-q$	$1 - p - tq^{-2}$	3.2.2
3	PV	PIII($\mathrm{D}_6^{(1)}$)	π^2	2	$1 - q$	$-t - p$	3.2.3
4	PIV	PIV	π	3	$-p$	$-p + q + 2t$	3.2.4
5	PV	PV	π	4	$-t^{-1}p$	$t(q - 1)$	3.2.5
6	PIII($\mathrm{D}_8^{(1)}$)	PIII($\mathrm{D}_6^{(1)}$)	π	2	tq^{-1}	$-t^{-1}q\left(pq + \frac{1}{2}\right)$	3.3.1
7	PIII($\mathrm{D}_6^{(1)}$)	PV	π'	2	tq^{-1}	$-t^{-1}q\,(pq + \alpha_1)$	3.3.2
8	PVI	PVI	π_1	2	tq^{-1}	$-t^{-1}q\,(pq + \alpha_2)$	3.3.3

system and H be the corresponding Hamiltonian. Then restriction of the dynamics, generated by H to $M_\alpha^{\bar{w}}$ is Hamiltonian.

Let us denote the corresponding Hamiltonian by $\tilde{H}$. Note that adjoint action of $\mathrm{GL}_{nd}(\mathbb{C})$ restricted to $\mathrm{GL}_n^d(\mathbb{C}) = \{\mathrm{diag}\,(h_1, h_2, \ldots, h_d)\,|h_i \in \mathrm{GL}_n(\mathbb{C})\}$ do commute with $\bar{w}$. Thus this action preserves $M_\alpha^{\bar{w}}$. Consider a Hamiltonian reduction on $M_\alpha^{\bar{w}}$ by a subgroup $\mathrm{GL}_n^{d-1}(\mathbb{C}) = \{\mathrm{diag}\,(1, h_2, \ldots, h_d)\,|\,h_i \in \mathrm{GL}_n(\mathbb{C})\} \subset \mathrm{GL}_n^d(\mathbb{C})$ with the moment map value fixed as $\mathbf{g} = (\mathrm{i}g_2\mathbf{1}_{n\times n}, \ldots, \mathrm{i}g_d\mathbf{1}_{n\times n})$. Let us denote the reduced space by $\mathbb{M}_\alpha := M_\alpha^{\bar{w}}//_{\mathbf{g}}\mathrm{GL}_n^{d-1}(\mathbb{C})$.

Theorem 3.1 *Under the conditions from Lemma 3.1 the dynamics on the manifold $\mathbb{M}_\alpha$ corresponding to $\tilde{H}$ is equivalent to the dynamics of a matrix $n \times n$ Painlevé system written as Image in Table 9.*

Proof of Theorem 3.1 and Lemma 3.1 is given below case by case. We perform the following steps for all cases.

Step 1. Solve fixed-point equations and obtain Darboux coordinates on $M_\alpha^{\bar{w}}$.
Step 2. Compute the action of $\mathrm{GL}_n^d(\mathbb{C})$ on $M^{\bar{w}}$ and the moment map of this action in these coordinates.
Step 3. Obtain Darboux coordinates on $\mathbb{M}_{\alpha,t}$.
Step 4. Calculate the Hamiltonian for the dynamics on $\mathbb{M}_\alpha$ and find coordinates on $\mathbb{M}_\alpha$, in which the Hamiltonian is a Painlevé's standard one.

Step 1, Step 2, Step 3 can be performed simultaneously for cases 1, 2, 3 and for cases 6, 7, 8 in Table 9. We call cases 1 – 5 linear, since for them to obtain a parametrization of $M^{\bar{w}}$ one has to solve only linear equations. Remaining cases 6, 7, 8 are called non-linear.

Remark 3.2 We use the following notation.

- Standard small letters (for example, (p, q)) to denote canonical coordinates on $M_{\alpha,t}$.
- Gothic letters (for example, $(\mathfrak{p}, \mathfrak{q})$) to denote coordinates on $M_{\alpha,t}^{\bar{w}}$.
- Standard capital letters (for example, (P, Q)) to denote canonical coordinates on $\mathbb{M}_{\alpha,t}$.

3.2 Linear cases

Step 1. Equations that determine the symplectic submanifold $M_{\alpha,t}^{\bar{w}}$ for cases 1, 2, 3

$$S_2 q S_2^{-1} = \xi - q, \quad S_2 p S_2^{-1} = -p + \eta(t) - \zeta t q^{-2}. \tag{3.1}$$

Here $\zeta, \xi \in \mathbb{C}$, $\eta \in \mathbb{C}[t]$ are specified for each case. For all cases which we consider $\zeta\xi = 0$ and for them we get

$$q = \begin{pmatrix} \frac{\xi}{2}\mathbf{1}_{n\times n} & \mathfrak{q}_{12} \\ \mathfrak{q}_{21} & \frac{\xi}{2}\mathbf{1}_{n\times n} \end{pmatrix},$$

$$p = \begin{pmatrix} \frac{\eta(t)}{2}\mathbf{1}_{n \times n} - t\frac{\zeta}{2}\mathfrak{q}_{21}^{-1}\mathfrak{q}_{12}^{-1} & \mathfrak{p}_{12} \\ \mathfrak{p}_{21} & \frac{\eta(t)}{2}\mathbf{1}_{n \times n} - t\frac{\zeta}{2}\mathfrak{q}_{12}^{-1}\mathfrak{q}_{21}^{-1} \end{pmatrix}. \tag{3.2}$$

Hence we get a symplectic submanifold of dimension $4n^2$ with symplectic form $\mathrm{Tr}(\mathrm{d}\mathfrak{p}_{12} \wedge \mathrm{d}\mathfrak{q}_{21}) + \mathrm{Tr}(\mathrm{d}\mathfrak{p}_{21} \wedge \mathrm{d}\mathfrak{q}_{12})$.

Recall that H is the Hamiltonian which defines a matrix Painlevé dynamics on M_α. Then the equations of motion can be identified with the one dimensional distribution on M_α, namely

$$\mathrm{Ker}\,(\omega - \mathrm{d}H \wedge \mathrm{d}t)\,. \tag{3.3}$$

Let us denote the embedding of the set of $\bar{w}$–invariant points by $\iota : M_\alpha^{\bar{w}} \to M_\alpha$. Since $\iota^*(\omega) = \mathrm{Tr}(\mathrm{d}\mathfrak{p}_{12} \wedge \mathrm{d}\mathfrak{q}_{21}) + \mathrm{Tr}(\mathrm{d}\mathfrak{p}_{21} \wedge \mathrm{d}\mathfrak{q}_{12})$ and the dynamics on $M_\alpha^{\bar{w}}$ is defined as $\mathrm{Ker}\,(\iota^*\,(\omega - \mathrm{d}H \wedge \mathrm{d}t))$ it follows that the dynamics on $M_\alpha^{\bar{w}}$ is also Hamiltonian and defined by Hamiltonian $\iota^*\,(H)$ and $(\mathfrak{p}_{12}, \mathfrak{p}_{21}, \mathfrak{q}_{21}, \mathfrak{q}_{12})$ are Darboux coordinates on $M_\alpha^{\bar{w}}$.

Step 2. The remaining gauge freedom consists of block diagonal matrices $h = \mathrm{diag}(h_1, h_2)$, $h_1, h_2 \in \mathrm{GL}_n(\mathbb{C})$ and the moment map is also block diagonal

$$[p, q] = m = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{p}_{12}\mathfrak{q}_{21} - \mathfrak{q}_{12}\mathfrak{p}_{21} & 0 \\ 0 & \mathfrak{p}_{21}\mathfrak{q}_{12} - \mathfrak{q}_{21}\mathfrak{p}_{12} \end{pmatrix}. \tag{3.4}$$

Remark 3.3 Block structure of $[p, q]$ appears not accidentally. Consider $(p, q) \in M_{\alpha,t}^{\bar{w}}$ with $\bar{w} = \mathrm{Ad}_{S_2} \circ w$. As we mentioned above $[p, q]$ is preserved by Bäcklund transformation w. Then we have

$$\bar{w}^*([p, q]) = [p, q] = w^*([p, q]),$$

which implies

$$\mathrm{Ad}_{S_2}^*([p, q]) = [p, q] \Rightarrow [S_2, [p, q]] = 0.$$

From the last equation it follows that $[p, q]$ is block diagonal.

Step 3. Let us perform Hamiltonian reduction with respect to $\mathrm{GL}_n(\mathbb{C}) = \{\mathrm{diag}(\mathbf{1}_{n \times n}, h_2)|h_2 \in \mathrm{GL}_n(\mathbb{C})\}$. We fix the moment map value as $m_2 = \mathfrak{p}_{21}\mathfrak{q}_{12} - \mathfrak{q}_{21}\mathfrak{p}_{12} = \mathrm{i}g_2\mathbf{1}_{n \times n}$. Hence we get $\mathfrak{p}_{21} = (\mathfrak{q}_{21}\mathfrak{p}_{12} + \mathrm{i}g_2)\mathfrak{q}_{12}^{-1}$. The functions $\tilde{P} = \mathfrak{p}_{12}\mathfrak{q}_{12}^{-1}$ and $\tilde{Q} = \mathfrak{q}_{12}\mathfrak{q}_{21}$ are invariant under the action of $\mathrm{diag}\,(1, h_2)$ and thus define functions on $\mathbb{M}_\alpha$. Moreover $(\tilde{P}, \tilde{Q})$ are Darboux coordinates on $\mathbb{M}_{\alpha,t}$.

3.2.1 PII to PII

Steps 1, 2, 3. We consider matrix $2n \times 2n$ PII with $\alpha_1 = \frac{1}{2}$. In this case $\xi = 0$, $\eta(t) = 0$, $\zeta = 0$.

Step 4. Substituting Darboux coordinates $\tilde{P}$, $\tilde{Q}$ into the Hamiltonian of matrix PII we get

$$H(\tilde{P}, \tilde{Q}; t) = \text{Tr}\left(\tilde{P}\tilde{Q}\tilde{P} + ig_2\tilde{P} - \left(\tilde{Q} + \frac{t}{2}\right)^2\right) = -\text{Tr}\left(\tilde{Q}(\tilde{Q} - \tilde{P}^2 + t) - ig_2\tilde{P}\right) - \frac{nt^2}{4}. \tag{3.5}$$

After the change of coordinates

$$\check{Q} = -\frac{1}{\sqrt[3]{2}}\tilde{P}, \quad \check{P} = \sqrt[3]{2}\tilde{Q}, \qquad s = -\sqrt[3]{2}t, \tag{3.6}$$

we obtain a system with matrix Darboux coordinates $(\check{P}, \check{Q})$ and Hamiltonian

$$H(\check{P}, \check{Q}; s) = \text{Tr}\left(\frac{1}{2}\check{P}(\check{P} - 2\check{Q}^2 - s) - (-ig_2)\check{Q}\right). \tag{3.7}$$

Then after the last change of coordinates

$$P = \check{P} - \check{Q}^2 - \frac{s}{2}, \quad Q = \check{Q} \tag{3.8}$$

we obtain a system with matrix Darboux coordinates (P, Q) and Hamiltonian

$$H(P, Q; s) = \text{Tr}\left(\frac{P^2}{2} - \frac{1}{2}\left(Q^2 + \frac{s}{2}\right)^2 - \left(-ig_2 - \frac{1}{2}\right)Q\right), \tag{3.9}$$

which is the Hamiltonian of matrix $n \times n$ $\text{PII}(1 + ig_2, -ig_2)$.

3.2.2 PIII$\left(D_6^{(1)}\right)$ to PIII$\left(D_8^{(1)}\right)$

Steps 1, 2, 3. We consider matrix $2n \times 2n$ $\text{PIII}\left(D_6^{(1)}\right)$ with $\alpha_0 = \alpha_1 = \frac{1}{2}$, $\beta_0 = \beta_1 = \frac{1}{2}$. In this case we have $\xi = 0$, $\eta(t) = 1$, $\zeta = 1$.

Step 4. Substituting Darboux coordinates $\tilde{P}$, $\tilde{Q}$ into the Hamiltonian of matrix PIII$\left(D_6^{(1)}\right)$ we get

$$tH(\tilde{P}, \tilde{Q}; t) = \text{Tr}\left(\tilde{P}^2\tilde{Q}^2 + \tilde{P}\tilde{Q}\tilde{P}\tilde{Q} + 2(1 + ig_2)\tilde{P}\tilde{Q} - \frac{1}{2}\tilde{Q} - \frac{1}{2}t^2\tilde{Q}^{-1}\right). \tag{3.10}$$

After the change of variables

$$P = 4t^{[\tilde{P},\tilde{Q}]}\left(\tilde{P} + \left(\frac{ig_2}{2}\right)\tilde{Q}^{-1}\right)t^{-[\tilde{P},\tilde{Q}]}, \quad Q = \frac{1}{4}t^{[\tilde{P},\tilde{Q}]}\tilde{Q}t^{-[\tilde{P},\tilde{Q}]}, \qquad s = \frac{t^2}{16},$$

$$(3.11)$$

we obtain a system with matrix Darboux coordinates (P, Q) and Hamiltonian

$$sH(P, Q; s) = \mathrm{Tr}\left(PQPQ + PQ - Q - sQ^{-1}\right),\qquad(3.12)$$

which is the Hamiltonian of matrix $n \times n$ PIII$(D_8^{(1)})$.

3.2.3 PV to PIII$(D_6^{(1)})$

Steps 1, 2, 3. We consider matrix $2n \times 2n$ PV with $\alpha_0 = \alpha_2 = \epsilon + \frac{1}{2}$, $\alpha_1 = \alpha_3 = -\epsilon$. In this case $\xi = 1$, $\eta(t) = -t$, $\zeta = 0$.

Step 4. Substituting Darboux coordinates $\tilde{P}$, $\tilde{Q}$ into the Hamiltonian of matrix PV we get

$$tH(\tilde{P}, \tilde{Q}; t) = \mathrm{Tr}\left(\tilde{P}^2\tilde{Q}^2 + \tilde{P}\tilde{Q}\tilde{P}\tilde{Q} - \frac{\tilde{P}^2\tilde{Q}}{4} + (4\epsilon + 2\mathrm{i}g_2)\tilde{P}\tilde{Q} - \frac{\mathrm{i}g_2}{2}\tilde{P} + \frac{t^2\tilde{Q}}{2}\right).$$
$$(3.13)$$

After the change of variables

$$P = 4t^{-[\tilde{P},\tilde{Q}]}\tilde{Q}t^{[\tilde{P},\tilde{Q}]},\quad Q = -\frac{1}{4}t^{-[\tilde{P},\tilde{Q}]}\tilde{P}t^{[\tilde{P},\tilde{Q}]},\quad s = -\frac{t^2}{16},\qquad(3.14)$$

we obtain a system with matrix Darboux coordinates (P, Q) and Hamiltonian

$$sH(P, Q; s) = \mathrm{Tr}\left(P^2Q^2 - (Q^2 + (2\epsilon + \mathrm{i}g_2)Q - s)P + \mathrm{i}g_2Q\right),\qquad(3.15)$$

which is the Hamiltonian of matrix $n \times n$ PIII$(D_6^{(1)})(1 + \mathrm{i}g_2, -\mathrm{i}g_2, 1 + 2\epsilon, -2\epsilon)$.

3.2.4 PIV to PIV

Step 1. This is case 4 in Table 9. We consider matrix $3n \times 3n$ PIV with $\alpha_0 = \alpha_1 = \alpha_2 = \frac{1}{3}$.

Equations that determine symplectic submanifold $M_{\alpha,t}^{\bar{w}}$ are

$$S_3qS_3^{-1} = -p,\quad S_3pS_3^{-1} = -p + q + 2t.\qquad(3.16)$$

Then

$$q = \begin{pmatrix} -\frac{2t}{3}\mathbf{1}_{n\times n} & q_{12} & q_{13} \\ q_{21} & -\frac{2t}{3}\mathbf{1}_{n\times n} & q_{23} \\ q_{31} & q_{32} & -\frac{2t}{3}\mathbf{1}_{n\times n} \end{pmatrix},\quad p = \begin{pmatrix} \frac{2t}{3}\mathbf{1}_{n\times n} & -\varpi^2 q_{12} & -\varpi q_{13} \\ -\varpi q_{21} & \frac{2t}{3}\mathbf{1}_{n\times n} & -\varpi^2 q_{23} \\ -\varpi^2 q_{31} & -\varpi q_{32} & \frac{2t}{3}\mathbf{1}_{n\times n} \end{pmatrix},\quad \varpi = e^{\frac{2\pi\mathrm{i}}{3}}.$$

$$\tag{3.17}$$

So we get a symplectic manifold of dimension $6n^2$, with symplectic form $\sqrt{3}i\,\mathrm{Tr}(dq_{12}\wedge dq_{21} + dq_{31} \wedge dq_{13} + dq_{23} \wedge dq_{32})$.

Similarly to cases above it follows that the restriction of the dynamics defined by Hamiltonian H on $M_\alpha^{\bar{w}}$ is Hamiltonian and defined by restriction of H on $M_\alpha^{\bar{w}}$.

Step 2. The remaining gauge freedom consists of block diagonal matrices $h = \mathrm{diag}(h_1, h_2, h_3)$, $h_1, h_2, h_3 \in \mathrm{GL}_n(\mathbb{C})$ and moment map is also block diagonal

$$[p, q] = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$$

$$= \sqrt{3}i \begin{pmatrix} q_{12}q_{21} - q_{13}q_{31} & 0 & 0 \\ 0 & q_{23}q_{32} - q_{21}q_{12} & 0 \\ 0 & 0 & q_{31}q_{13} - q_{32}q_{23} \end{pmatrix}. \tag{3.18}$$

Step 3. Now we can perform Hamiltonian reduction with respect to $\mathrm{GL}_n(\mathbb{C})^2 = \{\mathrm{diag}(1_{n\times n}, h_2, h_3)\,|\,h_2, h_3 \in \mathrm{GL}_n(\mathbb{C})\}$. We fix the moment map value as $m_2 = ig_2 1_{n\times n}$, $m_3 = ig_3 1_{n\times n}$.

The functions $\tilde{P} = (\varpi - \varpi^{-1})q_{12}q_{23}q_{13}^{-1}$ and $\tilde{Q} = q_{13}q_{23}^{-1}q_{21}$ are invariant under the action of $\mathrm{GL}_n^2(\mathbb{C})$, so they define functions on the manifold $\mathbb{M}_\alpha$. Moreover these functions are Darboux coordinates on $\mathbb{M}_{\alpha,t}$.

Step 4. Substitution $\tilde{P}$, $\tilde{Q}$ into the Hamiltonian gives

$$H(\tilde{P}, \tilde{Q}; t) = \mathrm{Tr}\left(\tilde{P}\tilde{Q}(\tilde{P} + \sqrt{3}i(\tilde{Q} + 2t)) + i(g_2 + g_3)\tilde{P} - \sqrt{3}g_2\tilde{Q}\right). \tag{3.19}$$

Then after the change of coordinates

$$P = \frac{1}{\kappa}\tilde{P}, \quad Q = \kappa\tilde{Q}, \qquad s = \kappa t, \quad \text{where } \kappa^2 = -\sqrt{3}i, \tag{3.20}$$

we get

$$H(P, Q; s) = \mathrm{Tr}\left(PQ(P - Q - 2s) + (ig_2 + ig_3)P - ig_2 Q\right), \tag{3.21}$$

which is the Hamiltonian of matrix $n \times n$ $\mathrm{PIV}\left(1 + \frac{ig_3}{2}, -\frac{ig_2 + ig_3}{2}, \frac{ig_2}{2}\right)$.

3.2.5 PV to PV

Step 1. This is case 5 from Table 9. We consider matrix $4n \times 4n$ PV with $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{4}$.

Equations that determine the symplectic submanifold $M_{\alpha,t}^{\bar{w}}$ in case 5 are

$$S_4 q S_4^{-1} = -\frac{p}{t}, \quad S_4 p S_4^{-1} = t(q - 1). \tag{3.22}$$

The solution is

$$
q = \begin{pmatrix}
\frac{1}{2}\mathbf{1}_{n\times n} & q_{12} & 0 & q_{14} \\
q_{21} & \frac{1}{2}\mathbf{1}_{n\times n} & q_{23} & 0 \\
0 & q_{32} & \frac{1}{2}\mathbf{1}_{n\times n} & q_{34} \\
q_{41} & 0 & q_{43} & \frac{1}{2}\mathbf{1}_{n\times n}
\end{pmatrix},
$$

$$
p = t \begin{pmatrix}
-\frac{1}{2}\mathbf{1}_{n\times n} & iq_{12} & 0 & -iq_{14} \\
-iq_{21} & -\frac{1}{2}\mathbf{1}_{n\times n} & iq_{23} & 0 \\
0 & -iq_{32} & -\frac{1}{2}\mathbf{1}_{n\times n} & iq_{34} \\
iq_{41} & 0 & -iq_{43} & -\frac{1}{2}\mathbf{1}_{n\times n}
\end{pmatrix}.
\tag{3.23}
$$

So we get symplectic manifold of dimension $8n^2$, with symplectic form

$$
2i\mathrm{Tr}\left(\mathrm{d}(t^{\frac{1}{2}}q_{12}) \wedge \mathrm{d}(t^{\frac{1}{2}}q_{21}) + \mathrm{d}(t^{\frac{1}{2}}q_{23}) \wedge \mathrm{d}(t^{\frac{1}{2}}q_{32})\right.
$$
$$
\left. + \mathrm{d}(t^{\frac{1}{2}}q_{34}) \wedge \mathrm{d}(t^{\frac{1}{2}}q_{43}) + \mathrm{d}(t^{\frac{1}{2}}q_{41}) \wedge \mathrm{d}(t^{\frac{1}{2}}q_{14})\right).
\tag{3.24}
$$

Similarly to cases 1–3 it follows that the restriction of the dynamics defined by the Hamiltonian H on $M_\alpha^{\bar{w}}$ is Hamiltonian and defined by restriction of H on $M_\alpha^{\bar{w}}$.

Step 2. The remaining gauge freedom consists of block diagonal matrices $h = \mathrm{diag}(h_1, h_2, h_3, h_4)$, $h_1, h_2, h_3, h_4 \in \mathrm{GL}_n(\mathbb{C})$ and moment map is also block diagonal

$$
[p, q] = \begin{pmatrix}
m_1 & 0 & 0 & 0 \\
0 & m_2 & 0 & 0 \\
0 & 0 & m_3 & 0 \\
0 & 0 & 0 & m_4
\end{pmatrix}
$$
$$
= 2it \begin{pmatrix}
q_{12}q_{21} - q_{14}q_{41} & 0 & 0 & 0 \\
0 & q_{23}q_{32} - q_{21}q_{12} & 0 & 0 \\
0 & 0 & q_{34}q_{43} - q_{32}q_{23} & 0 \\
0 & 0 & 0 & q_{41}q_{14} - q_{43}q_{34}
\end{pmatrix}.
\tag{3.25}
$$

Step 3. Now we can perform reduction with respect to $\mathrm{GL}_n^3(\mathbb{C}) = \mathrm{diag}(1_{n\times n}, h_2, h_3, h_4)$, $h_2, h_3, h_4 \in \mathbb{GL}_n(\mathbb{C})$). We fix the moment map value as $m_2 = ig_2\mathbf{1}_{n\times n}$, $m_3 = ig_3\mathbf{1}_{n\times n}$, $m_4 = ig_4\mathbf{1}_{n\times n}$.

The functions $\tilde{P} = 2it q_{12}q_{32}^{-1}q_{34}q_{41}$ and $\tilde{Q} = q_{14}q_{34}^{-1}q_{32}q_{12}^{-1}$ are invariant under the action of $\mathrm{GL}_n^3(\mathbb{C})$, so define functions on the manifold $\mathbb{M}_\alpha$. Moreover these functions are Darboux coordinates on $\mathbb{M}_{\alpha,t}$.

Step 4. Substituting $\tilde{P}$, $\tilde{Q}$ into the Hamiltonian we get

$$
tH(\tilde{P}, \tilde{Q}; t) = \mathrm{Tr}\left(\tilde{P}(\tilde{Q} - 1)^2 \tilde{P}\tilde{Q}\right.
$$
$$
+ \left(-(ig_2 + ig_3 + 2ig_4)\tilde{Q}^2 + (2it + 2ig_2 + ig_3 + 3ig_4)\tilde{Q}\right.
$$

$$-(ig_2 + ig_4))\tilde{P} + ig_4(ig_2 + ig_3 + ig_4)\tilde{Q}\Big). \tag{3.26}$$

Then after the change of coordinates

$$P = -\Big((\tilde{Q} - 1)\tilde{P} - ig_4\Big)(\tilde{Q} - 1), \quad Q = \tilde{Q}(\tilde{Q} - 1)^{-1}, \quad s = -2it, \tag{3.27}$$

we get the Hamiltonian

$$sH(P, Q; s) = \mathrm{Tr}\,(P(P + s)Q(Q - 1) - (ig_4 - ig_3)PQ + (ig_2 + ig_4)P - ig_4sQ), \tag{3.28}$$

which is the Hamiltonian of matrix $n \times n$ $\mathrm{PV}\big(1 + ig_3, ig_2 + ig_4, -ig_4, -ig_2 - ig_3\big)$.

3.3 Non-linear cases

Step 1. Equations that determine symplectic submanifold $M_{\alpha,t}^{\bar{w}}$ for cases 6–8 are

$$S_2 q S_2^{-1} = tq^{-1}, \quad S_2 p S_2^{-1} = -\frac{q(pq + v)}{t}. \tag{3.29}$$

On the open dense subset where the lower left $n \times n$ block of q is invertible the solution is given by

$$q = \begin{pmatrix} \tilde{q}_{11} & (\tilde{q}_{11}^2 - t)\tilde{q}_{21}^{-1} \\ \tilde{q}_{21} & \tilde{q}_{21}\tilde{q}_{11}\tilde{q}_{21}^{-1} \end{pmatrix}, \quad p = \begin{pmatrix} \tilde{p}_{11} & -\Big((\tilde{q}_{11}^2 - t)\tilde{q}_{21}^{-1}\tilde{p}_{21} + [\tilde{p}_{11}, \tilde{q}_{11}]_+ + v\Big)\tilde{q}_{21}^{-1} \\ \tilde{p}_{21} & [\tilde{q}_{21}\tilde{q}_{11}\tilde{q}_{21}^{-1}, \tilde{p}_{21}\tilde{q}_{21}^{-1}] + \tilde{q}_{21}\tilde{p}_{11}\tilde{q}_{21}^{-1} \end{pmatrix}. \tag{3.30}$$

So, $\tilde{p}_{11}, \tilde{q}_{11}, \tilde{p}_{21}, \tilde{q}_{21}$ are local coordinates on $M_{\alpha,t}^{\bar{w}}$. By ι denote the embedding $M_{\alpha}^{\bar{w}} \to M_{\alpha}$.

We can obtain restriction of the canonical 1-form Θ on $M_{\alpha}^{\bar{w}}$, namely

$$\iota^*\,(\Theta) = \mathrm{Tr}\,\Big(\mathfrak{p}_{11}d\mathfrak{q}_{11} + \mathfrak{p}_{12}d\mathfrak{q}_{21} - \tilde{\mathfrak{p}}_{21}\tilde{\mathfrak{q}}_{21}^{-1}dt\Big), \tag{3.31}$$

where Darboux coordinates on $M_{\alpha,t}^{\bar{w}}$ are

$$(\mathfrak{p}_{11}, \mathfrak{q}_{11}, \mathfrak{p}_{12}, \mathfrak{q}_{21}) = \Big(2(\tilde{\mathfrak{p}}_{11} + \tilde{\mathfrak{q}}_{11}\tilde{\mathfrak{q}}_{21}^{-1}\tilde{\mathfrak{p}}_{21}), \tilde{\mathfrak{q}}_{11},$$
$$2\Big(t\tilde{\mathfrak{q}}_{21}^{-1}\tilde{\mathfrak{p}}_{21} - \tilde{\mathfrak{p}}_{11}\tilde{\mathfrak{q}}_{11} - \tilde{\mathfrak{q}}_{11}\tilde{\mathfrak{q}}_{21}^{-1}\tilde{\mathfrak{p}}_{21}\tilde{\mathfrak{q}}_{11} - \frac{v}{2}\Big)\tilde{\mathfrak{q}}_{21}^{-1}, \tilde{\mathfrak{q}}_{21}\Big). \tag{3.32}$$

Recall that H is the Hamiltonian which defines a matrix Painlevé dynamics on M_{α}. Then the equations of motion can be identified with the one dimensional distribu-

tion on M_α, namely Ker $(\omega - \mathrm{d}H \wedge \mathrm{d}t)$. It follows that the dynamics on $M_\alpha^{\bar{w}}$ is also Hamiltonian and defined by the Hamiltonian $\iota^*(H) + \mathrm{Tr}\left(\tilde{\mathfrak{p}}_{21}\tilde{\mathfrak{q}}_{21}^{-1}\right)$.

Even though $(\mathfrak{p}_{11}, \mathfrak{q}_{11}, \mathfrak{p}_{12}, \mathfrak{q}_{21})$ are Darboux coordinates on $M_{\alpha,t}^{\bar{w}}$, it will be more convenient for us to use $(\tilde{\mathfrak{p}}_{11}, \tilde{\mathfrak{q}}_{11}, \tilde{\mathfrak{p}}_{12}, \tilde{\mathfrak{q}}_{21})$.

Step 2. As in the linear cases we have the action of block diagonal matrices

$$h = \mathrm{diag}\,(h_1, h_2) : \left(\{\tilde{\mathfrak{q}}_{ij}\}, \{\tilde{\mathfrak{p}}_{ij}\}\right) \mapsto \left(\{h_i\tilde{\mathfrak{q}}_{ij}h_j^{-1}\}, \{h_i\tilde{\mathfrak{p}}_{ij}h_j^{-1}\}\right). \qquad (3.33)$$

The moment map corresponding to this action is block diagonal, namely $[p, q] = \mathrm{diag}(m_1, m_2)$, where

$$\begin{aligned}
m_1 &= -2((\tilde{\mathfrak{q}}_{11}^2 - t)\tilde{\mathfrak{q}}_{21}^{-1}\tilde{\mathfrak{p}}_{21} + \tilde{\mathfrak{q}}_{11}\tilde{\mathfrak{p}}_{11}) - v, \\
m_2 &= 2\left(-t\tilde{\mathfrak{p}}_{21} + \tilde{\mathfrak{q}}_{21}\tilde{\mathfrak{p}}_{11}\tilde{\mathfrak{q}}_{11} + \tilde{\mathfrak{q}}_{21}\tilde{\mathfrak{q}}_{11}\tilde{\mathfrak{q}}_{21}^{-1}\tilde{\mathfrak{p}}_{21}\tilde{\mathfrak{q}}_{11}\right)\tilde{\mathfrak{q}}_{21}^{-1} + v.
\end{aligned} \qquad (3.34)$$

Step 3. Let us perform Hamiltonian reduction with respect to $\mathrm{GL}_n(\mathbb{C}) = \{\mathrm{diag}(1_{n\times n}, h_2)|h_2 \in \mathrm{GL}_n(\mathbb{C})\}$. Then, fixing $m_2 = ig_2\mathbf{1}_{n\times n}$ and resolving it with respect to $\tilde{\mathfrak{p}}_{11}$ on the open dense subset where $\tilde{\mathfrak{q}}_{11}$ is invertible, we get

$$\tilde{\mathfrak{p}}_{11} = t\tilde{\mathfrak{q}}_{21}^{-1}\tilde{\mathfrak{p}}_{21}\tilde{\mathfrak{q}}_{11}^{-1} - \tilde{\mathfrak{q}}_{11}\tilde{\mathfrak{q}}_{21}^{-1}\tilde{\mathfrak{p}}_{21} + \frac{1}{2}(ig_2 - v)\tilde{\mathfrak{q}}_{11}^{-1}. \qquad (3.35)$$

We can take the following coordinates on the reduction

$$\tilde{Q} = \tilde{\mathfrak{q}}_{11}, \quad \tilde{P} = 2t\tilde{\mathfrak{q}}_{21}^{-1}\tilde{\mathfrak{p}}_{21}\tilde{\mathfrak{q}}_{11}^{-1}. \qquad (3.36)$$

Then we have a section from $\mathbb{M}_\alpha$ to the intersection $M_\alpha^{\bar{w}} \cap \{m_2 = ig_2\mathbf{1}_{n\times n}\}$ which maps

$$\begin{aligned}
s : \left(\tilde{P}, \tilde{Q}, t\right) \mapsto \Bigg(\tilde{\mathfrak{p}}_{11} &= \frac{\tilde{P}}{2} - \frac{\tilde{Q}\tilde{P}\tilde{Q}}{2t} + \frac{1}{2}(ig_2 - v)\tilde{Q}^{-1}, \\
\tilde{\mathfrak{q}}_{11} &= \tilde{Q}, \tilde{\mathfrak{p}}_{21} = \frac{\tilde{P}\tilde{Q}}{2t}, \tilde{\mathfrak{q}}_{21} = 1, t\Bigg).
\end{aligned} \qquad (3.37)$$

Step 4. Hamiltonian on the reduction is

$$s^*\left(H + \mathrm{Tr}\left(\tilde{\mathfrak{p}}_{21}\tilde{\mathfrak{q}}_{21}^{-1}\right)\right) = s^*(H) + \mathrm{Tr}\left(\frac{\tilde{P}\tilde{Q}}{2t}\right). \qquad (3.38)$$

3.3.1 PIII$(D_8^{(1)})$ to PIII$(D_6^{(1)})$

Steps 1, 2, 3. We consider matrix $2n \times 2n$ PIII$(D_8^{(1)})$. In this case $\nu = \frac{1}{2}$.
Step 4. Here we start from coordinates

$$
\check{P} = -\frac{1}{2}\left(\frac{\tilde{Q}}{\sqrt{t}} - 1\right), \quad \check{Q} = 2\sqrt{t}\left(\tilde{P} + \left(ig_2 - \frac{1}{2}\right)\tilde{Q}^{-1}\right), \quad s = 16\sqrt{t},
$$
(3.39)

in which we get Hamiltonian

$$
sH(\check{P}, \check{Q}; s) = \text{Tr}\left(\check{P}\check{Q}\check{P}\check{Q} - \left(\check{Q}^2 - 2ig_2\check{Q} - s\right)\check{P} - ig_2\check{Q}\right).
$$
(3.40)

Then after change of variables

$$
P = s^{[\check{P},\check{Q}]}\check{P}s^{-[\check{P},\check{Q}]}, \quad Q = s^{[\check{P},\check{Q}]}\check{Q}s^{-[\check{P},\check{Q}]},
$$
(3.41)

we get

$$
sH(P, Q; s) = \text{Tr}\left(P^2Q^2 - \left(Q^2 - 2ig_2Q - s\right)P - ig_2Q\right),
$$
(3.42)

which is the Hamiltonian of matrix $n \times n$ PIII$(D_6^{(1)})(1 - ig_2, ig_2, 1 - ig_2, ig_2)$.

3.3.2 PIII$(D_6^{(1)})$ to PV

Steps 1, 2, 3. We consider matrix $2n \times 2n$ PIII$(D_6^{(1)})$ with $\beta_0 = \beta_1 = \frac{1}{2}, \alpha_1 = \epsilon$. In this case $\nu = \epsilon$.
Step 4. Here we start from coordinates

$$
\check{P} = 2\sqrt{t}\left(\tilde{P} + (ig_2 - \epsilon)\tilde{Q}^{-1}\right), \quad \check{Q} = \frac{1}{2}\left(\frac{\tilde{Q}}{\sqrt{t}} + 1\right), \quad s = -8\sqrt{t}, \quad (3.43)
$$

we get a system with Hamiltonian

$$
sH(\check{P}, \check{Q}; s) = \text{Tr}\left(\check{P}(\check{P}+s)\check{Q}(\check{Q}-1) + (1-2ig_2)\check{P}\check{Q}\right.
$$
$$
\left. + \left(\frac{2ig_2-1}{2}\right)\check{P} + (\epsilon-ig_2)s\check{Q} - \frac{[\check{P}, \check{Q}]^2}{4}\right).
$$
(3.44)

After change of coordinates

$$P = s^{-\frac{1}{2}[\check{P},\check{Q}]} \check{P} s^{\frac{1}{2}[\check{P},\check{Q}]}, \quad Q = s^{-\frac{1}{2}[\check{P},\check{Q}]} \check{Q} s^{\frac{1}{2}[\check{P},\check{Q}]}, \tag{3.45}$$

we get

$$sH(P, Q; s) = \mathrm{Tr}\left(P(P + s)Q(Q - 1) + (-2\mathrm{i}g_2)PQ + \mathrm{i}g_2 P + (\epsilon - \mathrm{i}g_2)s Q\right), \tag{3.46}$$

which is the Hamiltonian of matrix $n \times n$ PV$\left(1 - \mathrm{i}g_2 - \epsilon, \mathrm{i}g_2, \epsilon - \mathrm{i}g_2, \mathrm{i}g_2\right)$.

3.3.3 PVI to PVI

Steps 1, 2, 3. We consider matrix $2n \times 2n$ PVI with $\alpha_0 = \alpha_3 = \epsilon_0$, $\alpha_1 = \alpha_4 = \epsilon_1$. In this case $\nu = \alpha_2 = \frac{1}{2} - \epsilon_0 - \epsilon_1$.
Step 4. We start from coordinates

$$\check{P} = 2\sqrt{t}\left(\tilde{P} + \left(\epsilon_0 + \epsilon_1 + \mathrm{i}g_2 - \frac{1}{2}\right)\tilde{Q}^{-1}\right), \quad \check{Q} = \frac{1}{2}\left(\frac{\tilde{Q}}{\sqrt{t}} + 1\right),$$
$$s = \frac{1}{2} + \frac{1}{4}\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right), \tag{3.47}$$

in which we get Hamiltonian

$$s(s-1)H(\check{P}, \check{Q}; s) = \mathrm{Tr}\Bigg(\check{P}\check{Q}(\check{Q} - 1)\check{P}(\check{Q} - s)$$
$$+ \left(\mathrm{i}g_2(\check{Q}-1)(\check{Q}-s) + \mathrm{i}g_2\check{Q}(\check{Q}-s) + (2\epsilon_0-1)\check{Q}(\check{Q}-1)\right)\check{P}$$
$$+ \left(\left(\epsilon_0 - \mathrm{i}g_2 + \frac{1}{2}\right)^2 - \epsilon_1^2\right)\check{Q} - \frac{\sqrt{s(s-1)}}{2}[\check{P}, \check{Q}]^2\Bigg). \tag{3.48}$$

After the last substitution

$$P = e^{-2\mathrm{arccosh}(\sqrt{s})[\check{P},\check{Q}]}\check{P}e^{2\mathrm{arccosh}(\sqrt{s})[\check{P},\check{Q}]},$$
$$Q = e^{-2\mathrm{arccosh}(\sqrt{s})[\check{P},\check{Q}]}\check{Q}e^{2\mathrm{arccosh}(\sqrt{s})[\check{P},\check{Q}]}, \tag{3.49}$$

we get

$$s(s - 1)H(P, Q; s) = \mathrm{Tr}\Bigg(PQ(Q - 1)P(Q - s) + (\mathrm{i}g_2(Q - 1)(Q - s)$$
$$+ \mathrm{i}g_2 Q(Q - s) + (2\epsilon_0 - 1)Q(Q - 1))P + \left(\left(\epsilon_0 - \mathrm{i}g_2 + \frac{1}{2}\right)^2 - \epsilon_1^2\right)Q\Bigg), \tag{3.50}$$

which is the Hamiltonian of matrix $n \times n$ PVI$\left(2\epsilon_0, 2\epsilon_1, \frac{1}{2} - \mathrm{i}g_2 - \epsilon_0 - \epsilon_1, \mathrm{i}g_2, \mathrm{i}g_2\right)$.

3.4 Special $C_2 \times C_2$ cases

Let G be a group consisting of Bäcklund transformations which preserve a certain matrix Painlevé system and act trivially on the time variable. For the scalar Painlevé I-VI it was shown in [13] that such group G consists of transformations coming from automorphisms of diagram (up to overall conjugation of G). This is indeed true also for the described above Bäcklund transformation groups of matrix Painlevé equations. From the description of transformations corresponding to automorphisms in Sect. 2 it follows that possible G's are either cyclic, or isomorphic to $C_2 \times C_2$.

Twisting generators by $\mathrm{GL}_{n|G|}(\mathbb{C})$ action we get the group of symmetries $\bar{G}$, and the submanifold $M_\alpha^{\bar{G}}$ which is preserved by the dynamics. Condition of $\bar{G}$-invariance in case of cyclic G simplifies to invariance under the action of generator $\bar{w}$. These cases are described by Theorem 3.1.

There are only two cases of non-cyclic G. For them there is a construction of the reduction similar to one described by Theorem 3.1. The input and output are given in Table 10.

Let us use notation w_+, w_- for generators of G. We take twists $S_+ = \mathrm{diag}\,(\mathbf{1}_{n\times n}, \mathbf{1}_{n\times n}, -\mathbf{1}_{n\times n}, -\mathbf{1}_{n\times n})$, $S_- = \mathrm{diag}\,(\mathbf{1}_{n\times n}, -\mathbf{1}_{n\times n}, \mathbf{1}_{n\times n}, -\mathbf{1}_{n\times n})$. Then twisted generators are $\bar{w}_i = \mathrm{Ad}_{S_i} \circ w_i$.

3.4.1 $\mathrm{PIII}\big(\mathrm{D}_6^{(1)}\big)$ to $\mathrm{PIII}\big(\mathrm{D}_6^{(1)}\big)$

Step 1. Consider matrix $4n \times 4n$ $\mathrm{PIII}\big(\mathrm{D}_6^{(1)}\big)$ with parameters $\alpha_1 = \beta_1 = \frac{1}{2}$. In this case we take $w_+ = \pi'$ and $w_- = \pi \circ \pi'$.

Since $M_\alpha^{\bar{G}} = M_\alpha^{\bar{w}_+} \cap M_\alpha^{\bar{w}_-}$ let us start from $M_\alpha^{\bar{w}_+}$. Manifold $M_\alpha^{\bar{w}_+}$ is defined by equations (3.29) with $\nu = \frac{1}{2}$. So we have the solution on the dense open subset

$$
q = \begin{pmatrix} \check{q}_{11} & \check{q}_{12} & * & * \\ \check{q}_{21} & \check{q}_{22} & * & * \\ \check{q}_{31} & \check{q}_{32} & * & * \\ \check{q}_{41} & \check{q}_{42} & * & * \end{pmatrix}, \quad p = \begin{pmatrix} \check{p}_{11} & \check{p}_{12} & * & * \\ \check{p}_{21} & \check{p}_{22} & * & * \\ \check{p}_{31} & \check{p}_{32} & * & * \\ \check{p}_{41} & \check{p}_{42} & * & * \end{pmatrix}. \tag{3.51}
$$

Here $\check{q}_{ij}, \check{p}_{ij}$ are $n \times n$ blocks and $*$'s are defined in terms of $\check{q}_{ij}$ and $\check{p}_{ij}$ by (3.30). Note that for this case in equations (3.30) blocks have size $2n \times 2n$.

Table 10 Input and output for $C_2 \times C_2$ cases

No	Equation	Image	Generators	q	p	Section
1	$\mathrm{PIII}(\mathrm{D}_6^{(1)})$	$\mathrm{PIII}(\mathrm{D}_6^{(1)})$	$\pi \circ \pi'$	$-q$	$1 - p - tq^{-2}$	3.4.1
			π'	tq^{-1}	$-t^{-1}q\left(pq + \frac{1}{2}\right)$	
2	PVI	PVI	π_1	tq^{-1}	$-t^{-1}q(pq + \alpha_2)$	3.4.2
			π_2	$t(q-1)(q-t)^{-1}$	$\frac{(q-t)(p(q-t)+\alpha_2)}{t(1-t)}$	

The Darboux coordinates on $M_\alpha^{\bar{w}+}$ are given by (3.32). Note that one half of these coordinates are simply $\{\check{q}_{ij}\}_{1 \le i \le 4, \, 1 \le j \le 2}$, the other half are coordinates conjugate to them, defined by rather complicated formulas following from (3.32).

Now let us impose invariance under $\bar{w}_-$ to obtain $M_\alpha^{\bar{G}}$ as the submanifold of $M_\alpha^{\bar{w}+}$.

$$q = -S_2 q S_2^{-1}, \quad p = S_2 \left(-p + 1 - tq^{-2} \right) S_2^{-1}. \tag{3.52}$$

These equations are solved by

$$\check{q}_{11} = \check{q}_{22} = \check{q}_{31} = \check{q}_{42} = 0,$$

$$\check{p}_{11} = 1 - \frac{\check{q}_{12}\check{q}_{21}}{t}, \quad \check{p}_{22} = 1 - \frac{\check{q}_{21}\check{q}_{12}}{t}, \quad \check{p}_{31} = \frac{\check{q}_{32}\check{q}_{21}}{t}, \quad \check{p}_{42} = \frac{\check{q}_{41}\check{q}_{12}}{t}. \tag{3.53}$$

So matrix coordinates on $M_{\alpha,t}^{\bar{G}}$ are $(\check{q}_{12}, \check{q}_{21}, \check{q}_{32}, \check{q}_{41}, \check{p}_{12}, \check{p}_{21}, \check{p}_{32}, \check{p}_{41})$.

From the consideration above it follows that Darboux coordinates on $M_{\alpha,t}^{\bar{G}}$ are $\check{q}_{12}, \check{q}_{21}, \check{q}_{32}, \check{q}_{41}$ and conjugated to them which are restrictions of ones conjugated to them on $M_{\alpha,t}^{\bar{w}+}$ to $M_{\alpha,t}^{\bar{G}}$. For example, to obtain matrix coordinate conjugated to $\check{q}_{21}$ one should take upper right $n \times n$ block of $\mathfrak{p}_{11}$ from formula (3.32) and then restrict it to (3.53). Darboux coordinates on $M_{\alpha,t}^{\bar{G}}$ are

$$\left(\hat{p}_{21}, \hat{q}_{12}, \hat{p}_{12}, \hat{q}_{21}, \hat{p}_{23}, \hat{q}_{32}, \hat{p}_{14}, \hat{q}_{41} \right)$$

$$= \left(2(\check{p}_{21} + \check{q}_{21}\check{q}_{41}^{-1}\check{p}_{41}), \check{q}_{12}, 2(\check{p}_{12} + \check{q}_{12}\check{q}_{32}^{-1}\check{p}_{32}), \check{q}_{21}, \right.$$

$$2\left(t\check{q}_{32}^{-1}\check{p}_{32} - \check{p}_{21}\check{q}_{12} - \check{q}_{21}\check{q}_{41}^{-1}\check{p}_{41}\check{q}_{12} - \frac{1}{4} \right) \check{q}_{32}^{-1}, \check{q}_{32},$$

$$\left. 2\left(t\check{q}_{41}^{-1}\check{p}_{41} - \check{p}_{12}\check{q}_{21} - \check{q}_{12}\check{q}_{32}^{-1}\check{p}_{32}\check{q}_{21} - \frac{1}{4} \right) \check{q}_{41}^{-1}, \check{q}_{41} \right). \tag{3.54}$$

Step 2. We have the action of block diagonal matrices on $M_\alpha^{\bar{G}}$ which maps

$$h = \mathrm{diag}\,(h_1, h_2, h_3, h_4) : (\{\check{q}_{ij}\}, \{\check{p}_{ij}\}) \mapsto (\{h_i \check{q}_{ij} h_j^{-1}\}, \{h_i \check{p}_{ij} h_j^{-1}\}). \tag{3.55}$$

Moment map of this action is defined by blocks of block diagonal matrix $[p, q]$

$$m_1 = 2(t\check{q}_{41}^{-1}\check{p}_{41} - \check{q}_{12}\check{p}_{21} - \check{q}_{12}\check{q}_{21}\check{q}_{41}^{-1}\check{p}_{41}) - \frac{1}{2},$$

$$m_2 = 2(t\check{q}_{32}^{-1}\check{p}_{32} - \check{q}_{21}\check{p}_{12} - \check{q}_{21}\check{q}_{12}\check{q}_{32}^{-1}\check{p}_{32}) - \frac{1}{2},$$

$$m_3 = 2(-t\check{p}_{32} + \check{q}_{32}\check{p}_{21}\check{q}_{12} + \check{q}_{32}\check{q}_{21}\check{q}_{41}^{-1}\check{p}_{41}\check{q}_{12})\check{q}_{32}^{-1} + \frac{1}{2},$$

$$m_4 = 2(-t\check{p}_{41} + \check{q}_{41}\check{p}_{12}\check{q}_{21} + \check{q}_{41}\check{q}_{12}\check{q}_{32}^{-1}\check{p}_{32}\check{q}_{21})\check{q}_{41}^{-1} + \frac{1}{2}. \tag{3.56}$$

Step 3. Let us perform Hamiltonian reduction with respect to $\mathrm{GL}_n^3\,(\mathbb{C}) = \{\mathrm{diag}(1_{n \times n}, h_2, h_3, h_4) | h_2, h_3, h_4 \in \mathrm{GL}_n\,(\mathbb{C})\}$. We fix the moment map value as $m_2 = \mathrm{i}g_2 1_{n \times n}$, $m_3 = \mathrm{i}g_3 1_{n \times n}$, $m_4 = \mathrm{i}g_4 1_{n \times n}$.

Darboux coordinates on the reduction are $(\tilde{P}, \tilde{Q}) = \left(2(\breve{\mathfrak{q}}_{21}^{-1}\breve{\mathfrak{p}}_{21} + \breve{\mathfrak{q}}_{41}^{-1}\breve{\mathfrak{p}}_{41}), \breve{\mathfrak{q}}_{12}\breve{\mathfrak{q}}_{21}\right)$.

Step 4. As in the non-linear cases restriction on the set of points invariant under π' shifts the Hamiltonian by $\mathrm{Tr}\left(\tilde{\mathfrak{p}}_{21}\tilde{\mathfrak{q}}_{21}^{-1}\right)$, which is in our case equal to $\mathrm{Tr}\left(\breve{\mathfrak{p}}_{32}\breve{\mathfrak{q}}_{32}^{-1} + \breve{\mathfrak{p}}_{41}\breve{\mathfrak{q}}_{41}^{-1}\right)$. Then it remains to restrict the Hamiltonian obtained to the set of points invariant under $\pi \circ \pi'$ and satisfying the moment map conditions and substitute the section from the reduction. Then we get

$$
\begin{aligned}
&tH(\tilde{P}, \tilde{Q}; t) \\
&= \mathrm{Tr}\left(\tilde{P}^2\tilde{Q}^2 - \left(\tilde{P}^2 + (\mathrm{i}g_4 + 2\mathrm{i}g_3 + \mathrm{i}g_2 - 1)\tilde{P} - 4\right)\tilde{Q} + (\mathrm{i}g_2 + \mathrm{i}g_3)\tilde{P}\right).
\end{aligned} \tag{3.57}
$$

Then after change of variables

$$
Q = -t\tilde{P}, \quad P = \frac{1}{t}\tilde{Q}, \quad s = -\frac{t}{4}, \tag{3.58}
$$

we get

$$
\begin{aligned}
&sH(P, Q; s) \\
&= \mathrm{Tr}\left(P^2Q^2 - \left(Q^2 - ((\mathrm{i}g_2 + \mathrm{i}g_3) + (\mathrm{i}g_3 + \mathrm{i}g_4))Q - s\right)P - (\mathrm{i}g_2 + \mathrm{i}g_3)Q\right),
\end{aligned} \tag{3.59}
$$

which is the Hamiltonian of matrix $n \times n$ $\mathrm{PIII}\big(\mathrm{D}_6^{(1)}\big)\big(1 - \mathrm{i}g_2 + \mathrm{i}g_3, \mathrm{i}g_2 + \mathrm{i}g_3, 1 - \mathrm{i}g_3 - \mathrm{i}g_4, \mathrm{i}g_3 + \mathrm{i}g_4\big)$.

3.4.2 PVI to PVI

Step 1. Consider matrix $4n \times 4n$ PVI with parameters $\alpha_0 = \alpha_1 = \alpha_3 = \alpha_4 = \epsilon$. In this case we take $w_+ = \pi_1$ and $w_- = \pi_2$.

Let us obtain $M_\alpha^{\bar{w}_+}$. This manifold is defined by equations (3.29) with $\nu = \alpha_2$. So we have the solution on the dense open subset

$$
q = \begin{pmatrix} \breve{\mathfrak{q}}_{11} & \breve{\mathfrak{q}}_{12} & * & * \\ \breve{\mathfrak{q}}_{21} & \breve{\mathfrak{q}}_{22} & * & * \\ \breve{\mathfrak{q}}_{31} & \breve{\mathfrak{q}}_{32} & * & * \\ \breve{\mathfrak{q}}_{41} & \breve{\mathfrak{q}}_{42} & * & * \end{pmatrix}, \quad p = \begin{pmatrix} \breve{\mathfrak{p}}_{11} & \breve{\mathfrak{p}}_{12} & * & * \\ \breve{\mathfrak{p}}_{21} & \breve{\mathfrak{p}}_{22} & * & * \\ \breve{\mathfrak{p}}_{31} & \breve{\mathfrak{p}}_{32} & * & * \\ \breve{\mathfrak{p}}_{41} & \breve{\mathfrak{p}}_{42} & * & * \end{pmatrix}. \tag{3.60}
$$

Here $\breve{\mathfrak{q}}_{ij}$, $\breve{\mathfrak{p}}_{ij}$ are $n \times n$ blocks and $*$'s are defined in terms of $\breve{\mathfrak{q}}_{ij}$ and $\breve{\mathfrak{p}}_{ij}$ by (3.30). Note that for this case in equations (3.30) blocks $\tilde{\mathfrak{p}}_{ij}$, $\tilde{\mathfrak{q}}_{ij}$ have size $2n \times 2n$.

Let us obtain $M_\alpha^{\bar{G}} = M_\alpha^{\bar{w}+} \cap M_\alpha^{\bar{w}-} \subset M_\alpha^{\bar{w}+}$. Then on $M_\alpha^{\bar{w}+}$ we have to solve

$$q - t = t(t-1)S_2(q-t)^{-1}S_2^{-1}, \quad p = -\frac{1}{t(t-1)} S_2\left((q-t)(p(q-t)+\alpha_2)\right)S_2^{-1}. \tag{3.61}$$

On the open dense subset, where $\breve{q}_{21}, \breve{q}_{31}, \breve{q}_{41}$ are invertible, these equations can be solved by

$$\begin{aligned}
\breve{q}_{12} &= (\breve{q}_{11} - t)(\breve{q}_{11} - 1)\breve{q}_{21}^{-1}, \quad \breve{q}_{22} = \breve{q}_{21}\breve{q}_{11}\breve{q}_{21}^{-1}, \\
\breve{q}_{32} &= \breve{q}_{31}(\breve{q}_{11} - t)\breve{q}_{21}^{-1}, \quad \breve{q}_{42} = \breve{q}_{41}(\breve{q}_{11} - 1)\breve{q}_{21}^{-1}, \\
\breve{p}_{11} &= \breve{p}_{22} = 0, \quad \breve{p}_{31} = -\breve{q}_{31}(\breve{q}_{21}^{-1}\breve{p}_{21} + \breve{q}_{41}^{-1}\breve{p}_{41}), \\
\breve{p}_{12} &= -\left((\breve{q}_{11} - 1)\breve{q}_{31}^{-1}\breve{p}_{32} + (\breve{q}_{11} - t)\breve{q}_{41}^{-1}\breve{p}_{42}\right).
\end{aligned} \tag{3.62}$$

So matrix coordinates on $M_{\alpha,t}^{\bar{G}}$ are $\breve{q}_{11}, \breve{q}_{21}, \breve{q}_{31}, \breve{q}_{41}, \breve{p}_{21}, \breve{p}_{41}, \breve{p}_{32}, \breve{p}_{42}$. Let us denote embedding by $\iota : M_\alpha^{\bar{G}} \to M_\alpha$. Note that on $M_\alpha^{\bar{G}}$ blocks of q depend only on $\breve{q}_{11}, \breve{q}_{21}, \breve{q}_{31}, \breve{q}_{41}, t$, hence

$$\iota^*\Theta = \text{Tr}\left(\hat{p}_{11}d\breve{q}_{11} + \hat{p}_{12}d\breve{q}_{21} + \hat{p}_{13}d\breve{q}_{31} + \hat{p}_{14}d\breve{q}_{41}\right) - Fdt, \tag{3.63}$$

for certain $\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{13}, \hat{p}_{14}, F$. One can calculate all of them, but we will use only $\hat{p}_{11}, F$, given by

$$\hat{p}_{11} = -2\left(t\breve{q}_{21}^{-1}\breve{p}_{21} + (t-1)\breve{q}_{41}^{-1}\breve{p}_{41}\right), \tag{3.64}$$

$$F = \frac{1}{t-1}\text{Tr}\left((1 - \breve{q}_{11})\breve{q}_{21}^{-1}\breve{p}_{21} + (t-1)\breve{q}_{41}^{-1}\breve{p}_{41} + \breve{q}_{41}^{-1}\breve{p}_{42}\breve{q}_{21} + \breve{q}_{31}^{-1}\breve{p}_{32}\breve{q}_{21} + \alpha_2\right). \tag{3.65}$$

Step 2. We have a Hamiltonian action of $\text{GL}_n^4(\mathbb{C})$ on $M_{\alpha,t}^{\bar{G}}$

$$h = \text{diag}(h_1, h_2, h_3, h_4) : (\{\breve{q}_{ij}\}, \{\breve{p}_{ij}\}) \mapsto (\{h_i\breve{q}_{ij}h_j^{-1}\}, \{h_i\breve{p}_{ij}h_j^{-1}\}). \tag{3.66}$$

The moment map of this action is given by blocks of block diagonal matrix $[p, q]$

$$\begin{aligned}
m_1 &= 2t(\breve{q}_{11} - 1)\breve{q}_{21}^{-1}\breve{p}_{21} + 2(t-1)\breve{q}_{11}\breve{q}_{41}^{-1}\breve{p}_{41} - \alpha_2, \\
m_2 &= 2t\breve{q}_{21}\breve{q}_{41}^{-1}\breve{p}_{42} - 2\breve{q}_{21}\breve{q}_{31}^{-1}\breve{p}_{32} - \alpha_2, \\
m_3 &= 2\breve{p}_{32}\breve{q}_{21}\breve{q}_{31}^{-1} - 2t\breve{q}_{31}(\breve{q}_{21}^{-1}\breve{p}_{21} + \breve{q}_{41}^{-1}\breve{p}_{41})(\breve{q}_{11} - 1)\breve{q}_{31}^{-1} + \alpha_2, \\
m_4 &= 2\breve{p}_{41}(\breve{q}_{11} - t)\breve{q}_{41}^{-1} + 2t\breve{p}_{42}\breve{q}_{21}\breve{q}_{41}^{-1} + \alpha_2.
\end{aligned} \tag{3.67}$$

Step 3. Let us perform Hamiltonian reduction with respect to $\text{GL}_n^3(\mathbb{C}) = \{\text{diag}(1_{n\times n}, h_2, h_3, h_4)|h_2, h_3, h_4 \in \text{GL}_n(\mathbb{C})\}$. We fix the moment map value as $m_2 = ig_2\mathbf{1}_{n\times n}$, $m_3 = ig_3\mathbf{1}_{n\times n}$, $m_4 = ig_4\mathbf{1}_{n\times n}$.

Darboux coordinates on the reduction are

$$(\tilde{P}, \tilde{Q}) = (\hat{\mathfrak{p}}_{11}, \check{\mathfrak{q}}_{11}) = \left(-2 \left(t \check{\mathfrak{q}}_{21}^{-1} \check{\mathfrak{p}}_{21} + (t-1) \check{\mathfrak{q}}_{41}^{-1} \check{\mathfrak{p}}_{41}\right), \check{\mathfrak{q}}_{11}\right). \qquad (3.68)$$

Step 4. From (3.63) it follows that the dynamics on $M_\alpha^{\bar{G}}$ is given by the Hamiltonian $\iota^*(H) + F$. The Hamiltonian on the reduction in the coordinates

$$\check{P} = \frac{\tilde{P}}{t-1}, \quad \check{Q} = (t-1)\tilde{Q} + 1, \qquad s = \frac{1}{1-t}, \qquad (3.69)$$

is given by

$$
\begin{aligned}
s(s-1)H(\check{P}, \check{Q}; s) = \mathrm{Tr}\,\Big(&\check{P}\check{Q}(\check{Q}-s)\check{P}(\check{Q}-1) \\
&- \Big((ig_3 + ig_4 - 1)\check{Q}(\check{Q}-1) + (ig_2 + ig_4)\check{Q}(\check{Q}-s) + \\
&+ (ig_2 + ig_3)(\check{Q}-1)(\check{Q}-s)\Big)\check{P} \\
&+ \frac{1}{4}(1 - 2ig_2 - 2ig_3 - 2ig_4 - 4\epsilon)(1 - 2ig_2 - 2ig_3 - 2ig_4 + 4\epsilon)\,\check{Q}\Big). \quad (3.70)
\end{aligned}
$$

After change of coordinates

$$P = s^{-[\check{P},\check{Q}]} \check{P} s^{[\check{P},\check{Q}]}, \quad Q = s^{-[\check{P},\check{Q}]} \check{Q} s^{[\check{P},\check{Q}]}, \qquad (3.71)$$

we get

$$
\begin{aligned}
s(s-1)H(P, Q; s) = \mathrm{Tr}\Big(&PQ(Q-1)P(Q-s) \\
&- ((ig_3 + ig_4 - 1)Q(Q-1) + (ig_2 + ig_4)Q(Q-s) + \\
&+ (ig_2 + ig_3)(Q-1)(Q-s))P \\
&+ \frac{1}{4}(1 - 2ig_2 - 2ig_3 - 2ig_4 - 4\epsilon)(1 - 2ig_2 - 2ig_3 - 2ig_4 + 4\epsilon)\,Q\Big), \quad (3.72)
\end{aligned}
$$

which is the Hamiltonian of matrix $n \times n$ $\mathrm{PVI}\big(ig_3 + ig_4, 4\epsilon, \frac{1}{2}(1 - 4\epsilon - 2(ig_2 + ig_3 + ig_4)), ig_2 + ig_4, ig_2 + ig_3\big)$.

4 Application to Calogero–Painlevé systems

4.1 From Matrix Painlevé to Calogero–Painlevé

Every matrix $N \times N$ Painlevé system defined above corresponds to a Calogero–Painlevé system. We briefly recall its construction following [2].

Consider phase space of matrix Painlevé system, which is $M_{\alpha,t} = \{(p, q) \in \mathrm{Mat}_{N \times N}^2(\mathbb{C})\}$. There is an action of $\mathrm{GL}_N(\mathbb{C})$ by overall conjugation of p and q.

This action is Hamiltonian and the moment map equals $\mu_N(p,q) = [p,q]$. Let us define phase space of the corresponding Calogero–Painlevé system as a Hamiltonian reduction

$$\mathsf{M}_{\alpha,t} = M_{\alpha,t} /\!/_{\mathbf{O}_{N,g}} \mathrm{GL}_N(\mathbb{C}). \tag{4.1}$$

Here $\mathbf{O}_{N,g}$ is a coadjoint orbit $\mathbf{O}_{N,g} = \{ig(\mathbf{1}_{N\times N} - \xi \otimes \eta) | \xi \in \mathrm{Mat}_{N\times 1}(\mathbb{C}), \eta \in \mathrm{Mat}_{1\times N}(\mathbb{C}), \eta\xi = N\}, g \in \mathbb{C}$.

Let H be the Hamiltonian of matrix $N \times N$ Painlevé system. The Hamiltonian H is invariant with respect to the action of $\mathrm{GL}_N(\mathbb{C})$, thus H defines a Hamiltonian dynamics on M_α, which is called dynamics of the corresponding Calogero–Painlevé system.

The open dense subset of $\mathsf{M}_{\alpha,t}$ can be described as the phase space of system of particles considered up to permutations. This space is $\mathrm{T}^*\left((\mathbb{C}^N \backslash \mathrm{diags})/S_N\right)$, its points are sets $\{(p_j, q_j)\}_{j=1,\dots,N}$ where q_j's are distinct. Let us consider a map $\zeta_N : \mathrm{T}^*\left((\mathbb{C}^N\backslash\mathrm{diags})/S_N\right) \to M_{\alpha,t}$

$$\zeta_N : \{(p_j,q_j)\}_{j=1,\dots,N} \mapsto \left(\begin{pmatrix} p_1 & \frac{ig}{q_1-q_2} & \cdots & \cdots & \frac{ig}{q_1-q_N} \\ \frac{ig}{q_2-q_1} & p_2 & \frac{ig}{q_2-q_3} & \ddots & \frac{ig}{q_2-q_N} \\ \vdots & \frac{ig}{q_3-q_2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{ig}{q_{N-1}-q_N} \\ \frac{ig}{q_N-q_1} & \cdots & \cdots & \frac{ig}{q_N-q_{N-1}} & p_N \end{pmatrix}, \begin{pmatrix} q_1 & 0 & \cdots & \cdots & 0 \\ 0 & q_2 & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & q_N \end{pmatrix} \right). \tag{4.2}$$

Note that $\mathrm{Im}(\zeta_N) \subset \mu_N^{-1}\left(\{ig\left(\mathbf{1}_{N\times N} - v_N \otimes v_N^t\right)\}\right)$, where $v_N = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

Let us denote by $[(p,q)]$ the orbit of (p,q) with respect to the action of $\mathrm{GL}_N(\mathbb{C})$. Consider a map

$$\tilde{\zeta}_N : \mathrm{T}^*\left((\mathbb{C}^N\backslash\mathrm{diags})/S_N\right) \to \mathsf{M}_{\alpha,t}$$
$$\{(p_j,q_j)\}_{j=1,\dots,N} \mapsto [\zeta_N(\{(p_j,q_j)\})]. \tag{4.3}$$

The map $\tilde{\zeta}_N$ is injective and the image of $\tilde{\zeta}_N$ consists of classes $[(p,q)]$ such that q is diagonalisable with different eigenvalues. Let us use the notations $\mathrm{Im}(\tilde{\zeta}_N) =$

$\mathsf{M}_{\alpha,t}^{\mathrm{reg}}, \quad M_{\alpha,t}^{\mathrm{reg}} = \{(p,q) \in \mathrm{Mat}_{N \times N}^2(\mathbb{C}) : [(p,q)] \in \mathsf{M}_{\alpha,t}^{\mathrm{reg}}\}$. We have the inverse map

$$\pi_N : M_{\alpha,t}^{\mathrm{reg}} \cap \mu_N^{-1}(\mathbf{O}_{N,g}) \to \mathrm{T}^*\left(\left(\mathbb{C}^N \backslash \mathrm{diags}\right)/\mathsf{S}_N\right)$$
$$(p,q) \to \{(p_j, q_j)\}_{j=1,\dots,N}. \tag{4.4}$$

Here q_j's are eigenvalues of q and p_j is the j-th element on the diagonal of p in a basis where $q = \mathrm{diag}(q_1, \dots, q_N)$.

Bäcklund transformations of matrix Painlevé systems obtained in Sect. 2 are rational in p, q, t and thus do commute with the action of $\mathrm{GL}_N(\mathbb{C})$. Also, Bäcklund transformations preserve $[p, q]$. Thus we have

Proposition 4.1 *Bäcklund transformation of a matrix Painlevé system defines Bäcklund transformation for the corresponding Calogero–Painlevé system.*

Indeed, let w be a Bäcklund transformation of a matrix Painlevé system. Then the corresponding transformation of the Calogero–Painlevé system can be written as $\pi_N \circ w \circ \zeta_N = \mathsf{w}$. As a result, we have a homomorphism from the described above Bäcklund transformation groups of matrix Painlevé equations to Bäcklund transformations of the corresponding Calogero–Painlevé systems.

Remark 4.1 One can lift the condition of w-invariance of $[(p,q)]$ to the level of matrix representatives (p,q), namely

$$[(p,q)] \in \mathsf{M}_{\alpha,t}^{\pi_N \circ w \circ \zeta_N} \Leftrightarrow \pi_N((p,q)) = \pi_N \circ w((p,q)) \Leftrightarrow \exists S \in \mathrm{GL}_N(\mathbb{C}) : w((p,q))$$
$$= (SpS^{-1}, SqS^{-1}). \tag{4.5}$$

Let us illustrate constructions above by an example (in addition to Example 1.2)

Example 4.1 To obtain the Calogero–Painlevé $\mathrm{III}(\mathrm{D}_6^{(1)})$ Hamiltonian one should restrict the corresponding matrix Painlevé $\mathrm{III}(\mathrm{D}_6^{(1)})$ Hamiltonian (2.23) to the image of the map (4.2). In this way we obtain

$$t H_N(\{(p_i, q_i)\}; t) = \sum_{i=1}^N (p_i^2 q_i^2 + (-q_i^2 + (\alpha_1 + \beta_1)q_i + t)p_i - \alpha_1 q_i)$$
$$+ g^2 \sum_{1 \le j < i \le N} \frac{q_i^2 + q_j^2}{(q_i - q_j)^2}. \tag{4.6}$$

Note that this Hamiltonian is not of the physical form (1.5), to obtain such form, one should make logarithmic change of variables

$$\mathsf{t} = \log t, \quad \mathsf{q}_i = \log q_i + \frac{\mathsf{t}}{2}, \quad \mathsf{p}_i = p_i q_i - \frac{q_i}{2} + \frac{t}{2q_i} + \frac{\alpha_1 + \beta_1}{2}, \tag{4.7}$$

which gives

$$
\begin{aligned}
H_{phys} &= \sum_{i=1}^{N} \left(\mathsf{p}_i^2 - e^{\mathsf{t}} \sinh^2 \mathsf{q}_i + e^{\mathsf{t}/2} \left(\left(\beta_1 + \frac{1}{2} \right) \cosh \mathsf{q}_i - \left(\alpha_1 + \frac{1}{2} \right) \sinh \mathsf{q}_i \right) \right) \\
&+ \sum_{1 \le j < i \le N} \frac{g^2}{2 \sinh^2 \left(\frac{\mathsf{q}_i - \mathsf{q}_j}{2} \right)}.
\end{aligned}
\tag{4.8}
$$

So this is a system of trigonometric Calogero type, in difference with rational Calogero–Painlevé II (1.6). However, Hamiltonian (4.6) is more convenient for us than (4.8) because it is given in terms of rational functions.

4.2 Reduction at Calogero–Painlevé level

Let G be a group of symmetries of a certain matrix Painlevé system and let G be the group of corresponding symmetries of the Calogero–Painlevé system. Then $\mathsf{M}_\alpha^{\mathsf{G}}$ is preserved by the Calogero–Painlevé dynamics. We aim to study the dynamics on this subset.

Consider a certain matrix $n|G| \times n|G|$ Painlevé system with the finite group G of its symmetries from Table 9 or from Table 10.

Theorem 4.1 *Let $G \cong C_2$ or $G \cong C_2 \times C_2$. Then there is an open subset $U \subset \left(M_\alpha^{\mathrm{reg}} \right)^G$ such that*

- *The dynamics on U is equivalent to the dynamics of the n–particle Calogero– Painlevé system. The type of this Calogero–Painlevé system is given at Image column in Tables 9 and 10. The coupling constant of this system is $|G|g$.*
- *U is open and dense on the connected component of the largest dimension in $\left(M_\alpha^{\mathrm{reg}} \right)^G$.*

In general $\left(\mathsf{M}_\alpha^{\mathrm{reg}} \right)^{\mathsf{G}}$ is not connected. We will see in the proof that the component of the largest dimension in $\left(\mathsf{M}_\alpha^{\mathrm{reg}} \right)^{\mathsf{G}}$ is unique.

Proof In the proof we combine two different Hamiltonian reductions. First, recall that above we defined reduction (4.1), with the corresponding moment map $\mu_N : M_\alpha^{\mathrm{reg}} \to \mathfrak{gl}_N(\mathbb{C})$ and the projection $\pi_N : \mu_N^{-1}(\mathbf{O}_{N,g}) \cap M_{\alpha,t}^{\mathrm{reg}} \to \mathsf{M}_{\alpha,t}^{\mathrm{reg}}$. Second, in the setting of Theorem 3.1 we have the corresponding moment map $\mathbf{m} : M_\alpha^{\bar{G}} \to \left(\mathfrak{gl}_n(\mathbb{C}) \right)^{|\bar{G}|-1}$ and the projection $\mathrm{pr} : \mathbf{m}^{-1}(\mathfrak{g}) \to \mathbb{M}_\alpha$. Recall that $\mathbf{m}$ is given just by the diagonal $n \times n$ blocks of $[p, q]$ from the second to the last one.

The main idea of the proof is to construct a map φ such that diagram on Fig. 1 is commutative.

Let us consider the cases $G \cong C_2$ only (for the cases $G \cong C_2 \times C_2$ the proof is similar). Group $G \cong C_2$ is generated by the transformation w. Let w be the corresponding transformation of the Calogero–Painlevé system.

$$M_\alpha^{\bar{G}} \supset \mu_{|G|n}^{-1}\left(\mathbf{O}_{|G|n,g}\right) \cap (M_\alpha^{\text{reg}})^{\bar{G}} \cap \mathbf{m}^{-1}\left(\mathbf{g}\right) \xrightarrow{\ \pi_{|G|n}\ } U \subset (M_\alpha^{\text{reg}})^{\mathsf{G}}$$

$$\text{pr} \downarrow \qquad\qquad\qquad\qquad\qquad \vdots\varphi$$

$$\mathbb{M}_\alpha \supset \mu_n^{-1}\left(\mathbf{O}_{n,|G|g}\right) \xrightarrow{\ \pi_n\ } \mathsf{M}_\beta$$

Fig. 1 Description of the map φ

Step 1. Using explicit formulas for w we see that w has special form.

$$\mathsf{w}: \{(p_j, q_j)\}_{j=1,\dots,2n} \mapsto \{(a_\mathsf{w}(q_j, t)p_j + b_\mathsf{w}(q_j, t), c_\mathsf{w}(q_j, t))\}_{j=1,\dots,2n}. \quad (4.9)$$

Here $a_\mathsf{w}, b_\mathsf{w}, c_\mathsf{w}$ are certain rational functions. Invariance condition then can be written as follows.

$$\exists \sigma \in S_{2n} : a_\mathsf{w}(q_i, t)p_i + b_\mathsf{w}(q_i, t) = p_{\sigma(i)}, \quad c_\mathsf{w}(q_i, t) = q_{\sigma(i)}. \quad (4.10)$$

The permutation σ is unique since q_i's are pairwise distinct.

Only conjugacy class of σ is well–defined since coordinates $((p_1, q_1), (p_2, q_2), \dots, (p_{2n}, q_{2n}))$ are defined up to permutation. Hence for any point $x \in \left(M_\alpha^{\text{reg}}\right)^\mathsf{w}$ we have a conjugacy class which we denote by $[\sigma_x]$. In notation $U_{[\sigma]} = \{x \in \left(M_\alpha^{\text{reg}}\right)^\mathsf{w} | [\sigma_x] = [\sigma]\}$, we get $\left(M_\alpha^{\text{reg}}\right)^\mathsf{w} = \sqcup_{[\sigma]} U_{[\sigma]}$.

Step 2. Let us express σ as the product of independent cycles. Recall that w is an involution, thus, taking square of w, we get $q_{\sigma^2(i)} = q_i$, hence $\sigma^2 = \text{Id}$. Then σ is the product of independent transpositions. We denote the cyclic type of σ by $[2^k 1^{2n-2k}]$. Let us compute $\dim\left(U_{[2^k 1^{2n-2k}]}\right)$. It is easy to see that each cycle in σ imply system of equations (4.10) and systems for different cycles are independent.

Let $\sigma(i) = i$, then

$$a_\mathsf{w}(q_i, t)p_i + b_\mathsf{w}(q_i, t) = p_i, \quad c_\mathsf{w}(q_i, t) = q_i. \quad (4.11)$$

Hence $(p_i(t), q_i(t))$ is an algebraic solution of the corresponding Painlevé equation.

If we have cycle (l_1, l_2) in σ, then we get

$$a_\mathsf{w}(q_{l_1}, t)p_{l_1} + b_\mathsf{w}(q_{l_1}, t) = p_{l_2}, \quad c_\mathsf{w}(q_{l_1}, t) = q_{l_2}, \quad (4.12\text{a})$$
$$a_\mathsf{w}(q_{l_2}, t)p_{l_2} + b_\mathsf{w}(q_{l_2}, t) = p_{l_1}, \quad c_\mathsf{w}(q_{l_2}, t) = q_{l_1} \quad (4.12\text{b})$$

Note that since w is an involution (4.12b) follows from (4.12a). Thus the cycle (l_1, l_2) implies two independent equations.

As the result we get (for fixed t)

$$\dim\left(U_{[2^k 1^{2n-2k}], t}\right) = 4n - (2k + 2(2n - 2k)) = 2k. \quad (4.13)$$

So $U_t := U_{[2^n], t}$ has the maximal dimension equal to $2n$.

Step 3a. Map π_{2n} should be surjective for the existence of φ. In other words we have to check $\forall x \in U : \pi_{2n}^{-1}(\{x\}) \cap \left(M_\alpha^{\mathrm{reg}}\right)^{\bar{w}} \cap m_2^{-1}\left(\mathrm{i}g\mathbf{1}_{n \times n}\right) \neq \varnothing$.

For $\sigma \in S_{2n}$ let us denote by S_σ the matrix corresponding to σ. Then for $x \in U_{[\sigma]}$ consider $(\breve{p}, \breve{q}) = \zeta_{2n}(x) \in \pi_{2n}^{-1}(\{x\})$. Then we have

$$(S_\sigma \breve{p} S_\sigma^{-1}, S_\sigma \breve{q} S_\sigma^{-1}) = w(\breve{p}, \breve{q}). \tag{4.14}$$

Since $x \in U_{[2^n]}$ we get $\exists A \in \mathrm{GL}_{2n}(\mathbb{C}) : A S_\sigma A^{-1} = S_2$. Recall that $S_2 = \mathrm{diag}\left(\mathbf{1}_{n \times n}, -\mathbf{1}_{n \times n}\right)$ was defined in Sect. 3.1. Then we get

$$(S_2 A \breve{p} A^{-1} S_2^{-1}, S_2 A \breve{q} A^{-1} S_2^{-1}) = w(A \breve{p} A^{-1}, A \breve{q} A^{-1}). \tag{4.15}$$

Therefore $\exists (\hat{p}, \hat{q}) = (A \breve{p} A^{-1}, A \breve{q} A^{-1}) \in \pi_{2n}^{-1}(\{x\}) \cap M_{\alpha,t}^{\bar{w}}$.

From Remark 3.3 it follows that $[\hat{p}, \hat{q}]$ is block diagonal. Let us use the notation $[\hat{p}, \hat{q}] = \mathrm{diag}(m_1, m_2)$.

Since $(\hat{p}, \hat{q}) \in \mu_{2n}^{-1}\left(\mathbf{O}_{2n,g}\right)$ we get either $m_1 = \mathrm{i}g\mathbf{1}_{n \times n}$ or $m_2 = \mathrm{i}g\mathbf{1}_{n \times n}$. In the second case let us take $(p, q) := (\hat{p}, \hat{q}) \in m_2^{-1}(\mathrm{i}g\mathbf{1}_{n \times n})$. In the first case we take $(p, q) := (R \hat{p} R^{-1}, R \hat{q} R^{-1})$, where $R = \begin{pmatrix} 0 & \mathbf{1}_{n \times n} \\ \mathbf{1}_{n \times n} & 0 \end{pmatrix}$ is a matrix, which preserves $\mu_{2n}^{-1}\left(\mathbf{O}_{2n,g}\right) \cap \left(M_\alpha^{\mathrm{reg}}\right)^{\bar{w}}$. In both cases we get

$$(p, q) \in \pi_{2n}^{-1}(\{x\}) \cap \left(M_{\alpha,t}^{\mathrm{reg}}\right)^{\bar{w}} \cap m_2^{-1}\left(\mathrm{i}g\mathbf{1}_{n \times n}\right). \tag{4.16}$$

Step 3b. Let $(p, q) \in \pi_{2n}^{-1}(\{x\}) \cap \left(M_{\alpha,t}^{\mathrm{reg}}\right)^{\bar{w}} \cap m_2^{-1}(\mathrm{i}g\mathbf{1}_{n \times n})$. Then we have to check that $\pi((p, q)) \in \mu_n^{-1}\left(\mathbf{O}_{n,|G|g}\right)$.

We introduced coordinates $(\tilde{P}, \tilde{Q})$ on $\mathbb{M}_{\alpha,t}$ on Steps 3 in Sects. 3.2, 3.3 and 3.4. For them we get

$$m_1 = [\tilde{P}, \tilde{Q}] - \mathrm{i}g\mathbf{1}_{n \times n}. \tag{4.17}$$

For the final coordinates (P, Q) in each case one can check that

$$[\tilde{P}, \tilde{Q}] = [P, Q]. \tag{4.18}$$

Since $[p, q] = \mathrm{diag}(m_1, \mathrm{i}g\mathbf{1}_{n \times n}) \in \mu_{2n}^{-1}\left(\mathbf{O}_{2n,g}\right)$ we get

$$[P, Q] - \mathrm{i}g\mathbf{1}_{n \times n} = \mathrm{i}g\left(\mathbf{1}_{n \times n} - \xi \otimes \eta\right), \quad \eta\xi = 2n, \tag{4.19}$$

which means $(P, Q) \in \mu_n^{-1}\left(\mathbf{O}_{n,2g}\right)$.

Step 3c. Let us define $\varphi(x) = \pi_n \circ \mathrm{pr}((p, q))$. We have to check that the right side does not depend on the choice of $(p, q) \in \pi_{2n}^{-1}(\{x\}) \cap \left(M_{\alpha,t}^{\mathrm{reg}}\right)^{\bar{w}} \cap m_2^{-1}(\mathrm{i}g\mathbf{1}_{n \times n})$.

Consider $(p', q') \in \pi_{2n}^{-1}(\{x\}) \cap \left(M_{\alpha,t}^{\text{reg}}\right)^{\bar{w}} \cap m_2^{-1}(ig\mathbf{1}_{n \times n})$, then

$$\exists B \in \text{GL}_{2n}(\mathbb{C}) : (p', q') = (BpB^{-1}, BqB^{-1}). \tag{4.20}$$

Acting on both sides by $\bar{w}$ we get

$$(p', q') = ((S_2 B S_2^{-1})p(S_2 B S_2^{-1})^{-1}, (S_2 B S_2^{-1})q(S_2 B S_2^{-1})^{-1}), \tag{4.21}$$

which implies

$$[B^{-1}S_2 B S_2^{-1}, p] = [B^{-1}S_2 B S_2^{-1}, q] = 0. \tag{4.22}$$

Since $(p, q) \in \mu_{2n}^{-1}(\mathbf{O}_{2n,g}) \cap M_{\alpha,t}^{\text{reg}}$ we can consider these equalities in a gauge, where q is diagonal and eigenvalues of q are different. Then we get

$$B^{-1}S_2 B S_2^{-1} = \lambda \mathbf{1}_{2n \times 2n} \text{ for some } \lambda \in \mathbb{C}^*. \tag{4.23}$$

Rewriting this as $B^{-1}S_2 B = \lambda S_2$ and taking square of both sides we get $\lambda^2 = 1$. Then

$$B = \begin{cases} \text{diag}(b_1, b_2), & \lambda = 1, \\ R \cdot \text{diag}(b_1, b_2), & \lambda = -1. \end{cases} \tag{4.24}$$

Here $b_1, b_2 \in \text{GL}_n(\mathbb{C})$. Note that conjugation R does not preserve $\mu_n^{-1}(\mathbf{O}_{n,2g}) \cap m_2^{-1}(ig\mathbf{1}_{n \times n})$, while conjugation by $\text{diag}(b_1, b_2)$ does. Then the only possible case is $\lambda = 1$.

Conjugation by $\text{diag}(\mathbf{1}_{n \times n}, b_2)$ preserves pr by the definition. Conjugation by $\text{diag}(b_1, b_2)$ descends to the overall conjugation of (P, Q) by b_1. But this conjugation preserves π_n by the definition. Hence $\pi_n \circ \text{pr}((p, q)) = \pi_n \circ \text{pr}((p', q'))$.

Step 4. One can inverse φ on M_β^{reg} taking $\varphi^{-1} = \pi_{2n} \circ s \circ \zeta_n$. Here $s : \mathbb{M}_{\alpha,t} \to M_{\alpha,t}^{\bar{w}}$ is a section for the reduction from Theorem 3.1 (for an example in non-linear cases see (3.37)).

Step 5. It remains to check that φ maps the dynamics on U to the dynamics of the n–particle Calogero–Painlevé system called by Image in Table 9.

Let $\gamma \subset U$ be an integral curve of $2n$–particle Calogero–Painlevé system. Then for $(x_0, t_0) \in \gamma$ let $(\tilde{x}_0, t_0) \in \mu_{2n}^{-1}(\mathbf{O}_{2n,g}) \cap \left(M_\alpha^{\text{reg}}\right)^{\bar{w}} \cap \mathbf{m}^{-1}(\mathbf{g})$ be a lift of (x_0, t_0) i.e. $\pi_{2n}(\tilde{x}_0) = x_0$. Let us denote by Γ the integral curve of the corresponding matrix $2n \times 2n$ Painlevé system through $(\tilde{x}_0, t_0)$. Note that $\mu_{2n}^{-1}(\mathbf{O}_{2n,g}) \cap \left(M_\alpha^{\text{reg}}\right)^{\bar{w}} \cap \mathbf{m}^{-1}(\mathbf{g})$ is locally preserved by the dynamics of $2n \times 2n$ matrix Painlevé system, so $\Gamma \subset \mu_{2n}^{-1}(\mathbf{O}_{2n,g}) \cap \left(M_\alpha^{\text{reg}}\right)^{\bar{w}} \cap \mathbf{m}^{-1}(\mathbf{g})$. Then Γ is a lift of γ to $\mu_{2n}^{-1}(\mathbf{O}_{2n,g}) \cap \left(M_\alpha^{\text{reg}}\right)^{\bar{w}} \cap \mathbf{m}^{-1}(\mathbf{g})$. By Theorem 3.1, pr maps Γ to an integral curve of the $n \times n$ matrix Painlevé system, called by Image in Table 9. Then, by the definition π_n maps $\text{pr}(\Gamma)$ to an integral curve of the corresponding n–particle Calogero–Painlevé system.

$\square$

Remark 4.2 In cases 4, 5 from Table 9 we have open subset

$$U = \pi_{|G|n} \left(\mu_{|G|n}^{-1} \left(\mathbf{O}_{|G|n,g} \right) \cap \left(M_\alpha^{\text{reg}} \right)^{\bar{G}} \cap \mathbf{m}^{-1} \left(\mathbf{g} \right) \right) \subset \mathsf{M}_{\alpha,t}^{\mathsf{w}}. \tag{4.25}$$

For this subset one can also construct φ as in Theorem 4.1. Namely surjectivity of $\pi_{|G|n}$ follows from definition of U, while steps 3b–5 of the proof can be performed with the slight modification. Note that for cases 4, 5 Bäcklund transformations are not of the form (4.9), then steps 1, 2 of the proof have no sense. So for this U we do not have the second statement of Theorem 4.1.

Example 4.2 Let us consider Calogero–Painlevé $\text{III}(D_6^{(1)})$ and corresponding matrix Painlevé $\text{III}(D_6^{(1)})$. Hamiltonian (4.6) gives equations of motion

$$t\dot{q}_i = 2q_i^2 p_i - q_i^2 + (\alpha_1 + \beta_1)q_i + t,$$

$$t\dot{p}_i = -2p_i^2 q_i + 2p_i q_i - (\alpha_1 + \beta_1)p_i + \alpha_1 + 2g^2 \sum_{j=1, j\neq i}^{N} \frac{q_j(q_i + q_j)}{(q_i - q_j)^3}. \tag{4.26}$$

Let us at first illustrate Theorem 4.1 for G generated by $\pi \circ \pi'$ (case 2 from Table 9). We take $N = 2n$ particles and consider open subset $U_{[2^n]} \subset \left(\mathsf{M}_\alpha^{\text{reg}} \right)^{\mathsf{w}}$ with $\mathsf{w} = \pi_N \circ (\pi \circ \pi') \circ \zeta_N$, choosing $\sigma = (1, n+1) \ldots (n, 2n)$. Then $\alpha_0 = \alpha_1 = \beta_0 = \beta_1 = 1/2$ and on $U_{[2^n]}$ we have

$$q_{i+n} = -q_i, \quad p_{i+n} = 1 - p_i - t/q_i^2. \tag{4.27}$$

Then equations of motion (4.26) reduce to equations on $\{(p_i, q_i)\}_{i=1,\ldots n}$

$$t\dot{q}_i = 2q_i^2 \tilde{p}_i + q_i, \quad t\dot{\tilde{p}}_i = -2\tilde{p}_i^2 q_i - \tilde{p}_i + \frac{q_i}{2} - \frac{t^2}{2q_i^3} + 16g^2 \sum_{j=1, j\neq i}^{n} \frac{q_i q_j^2 (q_i^2 + q_j^2)}{(q_i^2 - q_j^2)^3}. \tag{4.28}$$

Here for convenience we introduce $\tilde{p}_i = p_i - 1/2 + t/(2q_i^2)$, such that $\mathsf{w}^*(\tilde{p}_i) = -\tilde{p}_i$. Now we find coordinates in which this dynamics has Calogero–Painlevé type. In other words, we have to find map φ from Theorem 4.1 using the Hamiltonian reduction on the matrix level (with $g_2 = g$). We take $\{(p_i, q_i)\}_{i=1,\ldots 2n} \in U_{[2^n]}$, namely under condition (4.27). Choosing the following matrix A from the proof of Theorem 4.1 we have the following $(p, q) \in M_\alpha^{\tilde{w}}$

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} \end{pmatrix}, \tag{4.29}$$

$$\text{Ad}_A \circ \zeta_{2n}(\{(p_i, q_i)\})$$
$$= \left(\begin{pmatrix} \text{diag}(\frac{1}{2} - \frac{t}{2q_i^2}) & \mathfrak{p}_{12} \\ \mathfrak{p}_{21} & \text{diag}(\frac{1}{2} - \frac{t}{2q_i^2}) \end{pmatrix}, \begin{pmatrix} 0 & -\text{diag}(q_i) \\ -\text{diag}(q_i) & 0 \end{pmatrix} \right),$$

$$(\mathfrak{p}_{12})_{ii} = -\tilde{p}_i + \frac{ig}{2q_i}, \ (\mathfrak{p}_{21})_{ii} = -\tilde{p}_i - \frac{ig}{2q_i}, \quad i \neq j : (\mathfrak{p}_{12})_{ij} = \frac{2igq_j}{q_j^2 - q_i^2},$$

$$(\mathfrak{p}_{21})_{ij} = \frac{2igq_i}{q_j^2 - q_i^2}. \tag{4.30}$$

After the conjugation by matrix A the moment map condition becomes

$$[p, q] = ig \begin{pmatrix} \mathbf{1}_{n \times n} - 2v_n v_n^t & 0 \\ 0 & \mathbf{1}_{n \times n} \end{pmatrix} \text{ due to } Av_{2n} = \sqrt{2}(\underbrace{1, \dots 1}_{n}, \underbrace{0, \dots 0}_{n})^t. \tag{4.31}$$

So, using coordinates $(\tilde{P}, \tilde{Q}) = (\mathfrak{p}_{12}\mathfrak{q}_{12}^{-1}, \mathfrak{q}_{12}\mathfrak{q}_{21})$ from Sect. 3.2 we obtain on $\mathrm{pr}\left(\mathrm{Ad}_A \circ \zeta_{2n}(U_{[2^n]})\right)$

$$\tilde{P}_{ii} = \tilde{p}_i/q_i - \frac{ig}{2q_i^2}, \quad i \neq j : \tilde{P}_{ij} = \frac{2ig}{q_i^2 - q_j^2}, \qquad \tilde{Q}_{ij} = \delta_{ij}q_i^2. \tag{4.32}$$

In coordinates (3.11) we obtain following coordinates $\{(P_i, Q_i)\}_{i=1,\dots n}$ and time s on $\pi_n \circ \mathrm{pr}\left(\mathrm{Ad}_A \circ \zeta_{2n}(U_{[2^n]})\right)$

$$P_i = 4\tilde{p}_i/q_i, \quad Q_i = q_i^2/4, \quad s = t^2/16. \tag{4.33}$$

Finally, one can check from (4.28) that these $\{(P_i, Q_i)\}_{i=1,\dots n}$ satisfy Calogero–Painlevé III$\left(\mathrm{D}_8^{(1)}\right)$ with coupling constant $2g$

$$s\frac{dQ_i}{ds} = 2Q_i^2 P_i + Q_i,$$

$$s\frac{dP_i}{ds} = -2P_i^2 Q_i - P_i + 1 - s/Q_i^2 + 8g^2 \sum_{j=1, j \neq i}^{n} \frac{Q_j(Q_i + Q_j)}{(Q_i - Q_j)^3}. \tag{4.34}$$

Example 4.3 Let us then illustrate Theorem 4.1 also for Calogero–Painlevé III$\left(\mathrm{D}_6^{(1)}\right)$ system, but for G generated by π' (case 7 from Table 9). Note that this case is non-linear in difference with the previous example. We take $N = 2n$ particles and consider $U_{[2^n]} \subset \left(\mathsf{M}_\alpha^{\mathrm{reg}}\right)^{\mathsf{w}}$ with $\mathsf{w} = \pi_N \circ \pi' \circ \zeta_N$. Then $\beta_0 = \beta_1 = 1/2$ and on $U_{[2^n]}$ we have

$$q_{i+n} = \frac{t}{q_i}, \quad p_{i+n} = -\frac{q_i(p_iq_i + \alpha_1)}{t}. \tag{4.35}$$

Then equations of motion (4.26) reduce to the equations on $\{(p_i, q_i)\}_{i=1,\ldots n}$, namely

$$
\begin{aligned}
t\dot{q}_i &= 2q_i^2 p_i - q_i^2 + (\alpha_1 + 1/2)q_i + t \\
t\dot{p}_i &= -2p_i^2 q_i + 2p_i q_i - \left(\alpha_1 + \frac{1}{2}\right)p_i + \alpha_1 \\
&\quad + 2g^2 \frac{tq_i(q_i^2 + t)}{(q_i^2 - t)^3} + 2g^2 \sum_{j=1, j\neq i}^{n} \left(\frac{q_j(q_i + q_j)}{(q_i - q_j)^3} + \frac{tq_j^{-1}(q_i + tq_j^{-1})}{(q_i - tq_j^{-1})^3} \right).
\end{aligned}
\tag{4.36}
$$

We want to find φ from Theorem 4.1 using the Hamiltonian reduction at matrix level (with $g_2 = g$). We choose A as in the previous example, and take $\{(p_i, q_i)\}_{i=1,\ldots 2n} \in U_{[2^n]}$, namely under condition (4.35). Then we have following $(p, q) \in M_\alpha^{\tilde{w}}$

$$
\begin{aligned}
\mathrm{Ad}_A \circ \zeta_{2n}(\{(p_i, q_i)\}) &= \left(\begin{pmatrix} * & * \\ \tilde{\mathfrak{p}}_{21} & * \end{pmatrix}, \frac{1}{2} \begin{pmatrix} \mathrm{diag}(q_i + t/q_i) & \mathrm{diag}(-q_i + t/q_i) \\ \mathrm{diag}(-q_i + t/q_i) & \mathrm{diag}(q_i + t/q_i) \end{pmatrix} \right) \\
(\tilde{\mathfrak{p}}_{21})_{ii} &= -\frac{p_i}{2} - \frac{q_i(p_i q_i + \alpha_1)}{2t} + \frac{ig}{2} \frac{q_i + q_j}{t - q_i q_j}, \\
i \neq j : (\tilde{\mathfrak{p}}_{21})_{ij} &= \frac{ig}{2} \left(\frac{q_i q_j + t}{t(q_j - q_i)} + \frac{q_i + q_j}{t - q_i q_j} \right),
\end{aligned}
\tag{4.37}
$$

where we calculate only necessary matrix block for the momentum. So, using formulas from Sect. 3.3 we obtain following $(\tilde{P}, \tilde{Q})$ on $\mathrm{pr}\left(\mathrm{Ad}_A \circ \zeta_{2n}(U_{[2^n]})\right)$ (only diagonal elements of $\tilde{P}$ are calculated)

$$
\begin{aligned}
\tilde{P}_{ii} &= (2t\tilde{q}_{21}^{-1}\tilde{\mathfrak{p}}_{21}\tilde{q}_{11}^{-1})_{ii} = \frac{4}{q_i - tq_i^{-1}} \left(\left(p_i + \frac{\alpha_1}{q_i + tq_i^{-1}} \right) q_i + \frac{2tig}{q_i^2 - t^2 q_i^{-2}} \right), \\
\tilde{Q}_{ij} &= (\tilde{q}_{11})_{ij} = \frac{1}{2}\delta_{ij}(q_i + tq_i^{-1}).
\end{aligned}
\tag{4.38}
$$

In coordinates (3.43) we obtain following coordinates $\{(P_i, Q_i)\}_{i=1,\ldots n}$ and time s on $\pi_n \circ \mathrm{pr}\left(\mathrm{Ad}_A \circ \zeta_{2n}(U_{[2^n]})\right)$

$$
\begin{aligned}
P_i &= 8\frac{p_i q_i + \alpha_1/2}{q_i/\sqrt{t} - \sqrt{t}/q_i} + \frac{4ig\left(q_i/\sqrt{t} + \sqrt{t}/q_i\right)}{\left(q_i/\sqrt{t} - \sqrt{t}/q_i\right)^2}, \\
Q_i &= \frac{1}{2} + \frac{1}{4}\left(\frac{q_i}{\sqrt{t}} + \frac{\sqrt{t}}{q_i}\right), \qquad s = -8\sqrt{t},
\end{aligned}
\tag{4.39}
$$

Finally, these $\{(P_i, Q_i)\}_{i=1,...n}$ satisfy Calogero–Painlevé V$(1-\alpha_1-ig, ig, \alpha_1-ig, ig)$ with coupling constant $2g$, which one can check from (4.36)

$$
\begin{aligned}
s\frac{dQ_i}{ds} &= (2P_i + s)Q_i(Q_i - 1) + ig(1 - 2Q_i), \\
s\frac{dP_i}{ds} &= (P_i + s)P_i(1 - 2Q_i) + ig(2P_i + s) - \alpha_1 s \\
&\quad + 4g^2 \sum_{j=1, j\neq i}^{n} \frac{2Q_i Q_j + 2Q_j^2 - Q_i - 3Q_j}{(Q_i - Q_j)^3}.
\end{aligned}
\tag{4.40}
$$

Example 4.4 Let us now consider Calogero–Painlevé II system, as in Example 1.2 from Introduction, but with an arbitrary number of particles. Then, instead of Hamiltonian (1.6), we have

$$
\begin{aligned}
H_N(\{(p_i, q_i)\}; t) &= \sum_{i=1}^{N} \left(\frac{1}{2} p_i^2 - \frac{1}{2}\left(q_i^2 + \frac{t}{2} \right)^2 - (\alpha_1 - 1/2)q_i \right) \\
&\quad + \sum_{j,i=1, j<i}^{N} \frac{g^2}{(q_i - q_j)^2}.
\end{aligned}
\tag{4.41}
$$

The corresponding equations of motion are

$$
\dot{q}_i = p_i, \qquad \dot{p}_i = 2q_i^3 + tq_i + (\alpha_1 - 1/2) + 2g^2 \sum_{j=1, j\neq i}^{N} \frac{1}{(q_i - q_j)^3}.
\tag{4.42}
$$

Let us illustrate Theorem 4.1 for G generated by r (case 1 from Table 9). We take $N = 2n$ particles and consider $U_{[2^n]} \subset \left(M_\alpha^{\text{reg}} \right)^{\text{w}}$ with $\text{w} = \pi_N \circ r \circ \zeta_N$. Then $\alpha_1 = 1/2$ and on $U_{[2^n]}$ we have

$$
q_{i+n} = -q_i, \; p_{i+n} = -p_i.
\tag{4.43}
$$

Then equations of motion (4.42) reduce to equations on $\{(p_i, q_i)\}_{i=1,...n}$

$$
\dot{q}_i = p_i, \qquad \dot{p}_i = 2q_i^3 + tq_i + \frac{g^2}{4q_i^3} + 4g^2 \sum_{j=1, j\neq i}^{n} \frac{q_i(q_i^2 + 3q_j^2)}{(q_i^2 - q_j^2)^3}.
\tag{4.44}
$$

We want to find φ from Theorem 4.1 using the Hamiltonian reduction at matrix level (with $g_2 = g$). The calculations just resemble those from Example 4.2. As a result, we have (4.32) for $(\tilde{P}, \tilde{Q})$ with p_i instead of $\tilde{p}_i$. Then, in coordinates (3.6) we obtain momentum $\check{P}$ in a diagonal form

$$\check{Q}_i = -2^{-1/3}\tilde{P}_i = -2^{-1/3}\left(\frac{p_i}{q_i} - \frac{ig}{2q_i^2}\right), \quad \check{P}_i = 2^{1/3}\tilde{Q}_i = 2^{1/3}q_i^2, \quad s = -2^{1/3}t,$$

$$(4.45)$$

where $\check{Q}_i$ and $\check{P}_i$ are diagonal elements of $\check{Q}$ and $\check{P}$ respectively. So the natural projection π_N from $\mathrm{pr}(\mathrm{Ad}_A \circ \zeta_{2n}(U_{[2^n]}))$ gives us Calogero–Painlevé system in a gauge, where $\check{P}$ is diagonal.

It follows from (4.44) that

$$\frac{\mathrm{d}\check{P}_i}{\mathrm{d}s} = 2\check{P}_i\check{Q}_i - ig, \qquad \frac{\mathrm{d}\check{Q}_i}{\mathrm{d}s} = \check{P}_i - \check{Q}_i^2 - \frac{s}{2} + 4g^2 \sum_{j=1, j\neq i}^{n} \frac{\check{P}_i + 3\check{P}_j}{(\check{P}_i - \check{P}_j)^3}.$$

$$(4.46)$$

This dynamics is Hamiltonian with

$$H_n(\{(\tilde{P}_i, \check{Q}_i)\}; t) = \sum_{i=1}^{n}\left(\frac{\check{P}_i^2}{2} - \left(\check{Q}_i^2 + \frac{s}{2}\right)\check{P}_i + ig\check{Q}_i\right) - (2g)^2 \sum_{j,i=1, j<i}^{n} \frac{\check{P}_i + \check{P}_j}{(\check{P}_i - \check{P}_j)^2}.$$

$$(4.47)$$

It is easy to see that this Hamiltonian can be obtained from the matrix Painlevé Hamiltonian (2.9) in the gauge, where $\check{P}$ (f in loc. cit.) is diagonal. So we get Calogero–Painlevé II$(-ig - 1/2)$ with coupling constant $2g$.

In order to obtain a transformation of Calogero–Painlevé system to the standard gauge (as in r.h.s. of (4.2)), one should diagonalize matrix Q. This cannot give an algebraic formula for the general size n.

Remark 4.3 Note that for $N = 2$ and $g = 0$ coordinate transformations from above Examples 4.2, 4.3, 4.4 reproduce formulas [13, 5.10–5.12, 5.5–5.7, 9.5–9.7] for the folding transformations of the corresponding Painlevé equations.

5 Further examples

5.1 Matrix reduction generalization

For the construction in Sect. 3 we choose for a certain Bäcklund transformation the additional twist $S_d = \mathrm{diag}\left(\mathbf{1}_{n\times n}, e^{\frac{2\pi i}{d}}\mathbf{1}_{n\times n}, \ldots, e^{\frac{2\pi(d-1)}{d}}\mathbf{1}_{n\times n}\right)$. At the Calogero level this twist corresponds to permutation class $[d^n]$. We also choose moment map value $\mathbf{g} = (ig_2\mathbf{1}_{n\times n}, \ldots, ig_d\mathbf{1}_{n\times n})$. It is natural to try to weaken such restrictions. Below we present two examples for such generalizations.

Example 5.1 Let us consider $3n \times 3n$ matrix Painlevé II. We take $M_\alpha^{\bar{r}}$ with $\bar{r} = \mathrm{Ad}_S \circ r$, where $S = \mathrm{diag}\left(\mathbf{1}_{n\times n}, -\mathbf{1}_{2n\times 2n}\right)$. Recall that $r : (p, q) \mapsto (-p, -q)$ and $\theta \mapsto -\theta$, so we set $\theta = 0$.

Step 1. The matrices $(p, q) \in M_\alpha^{\bar{r}}$ are given by

$$
p = \begin{pmatrix} 0 & \mathfrak{p}_{12} & \mathfrak{p}_{13} \\ \mathfrak{p}_{21} & 0 & 0 \\ \mathfrak{p}_{31} & 0 & 0 \end{pmatrix}, \qquad q = \begin{pmatrix} 0 & \mathfrak{q}_{12} & \mathfrak{q}_{13} \\ \mathfrak{q}_{21} & 0 & 0 \\ \mathfrak{q}_{31} & 0 & 0 \end{pmatrix} \tag{5.1}
$$

with a symplectic form $\mathrm{Tr}(d\mathfrak{p}_{12} \wedge d\mathfrak{q}_{21}) + \mathrm{Tr}(d\mathfrak{p}_{21} \wedge d\mathfrak{q}_{12}) + \mathrm{Tr}(d\mathfrak{p}_{13} \wedge d\mathfrak{q}_{31}) + \mathrm{Tr}(d\mathfrak{p}_{31} \wedge d\mathfrak{q}_{13})$.

Step 2. The remaining gauge freedom is $\mathrm{diag}(\mathrm{GL}_n(\mathbb{C}))$, $\mathrm{GL}_{2n}(\mathbb{C})$ and the moment map is

$$
\begin{aligned}
[p, q] &= \begin{pmatrix} (m_1)_{n \times n} & 0 \\ 0 & (m_2)_{2n \times 2n} \end{pmatrix} \\
&= \begin{pmatrix} \mathfrak{p}_{12}\mathfrak{q}_{21}+\mathfrak{p}_{13}\mathfrak{q}_{31}-\mathfrak{q}_{12}\mathfrak{p}_{21}-\mathfrak{q}_{13}\mathfrak{p}_{31} & 0 & 0 \\ 0 & \mathfrak{p}_{21}\mathfrak{q}_{12}-\mathfrak{q}_{21}\mathfrak{p}_{12} & \mathfrak{p}_{21}\mathfrak{q}_{13}-\mathfrak{q}_{21}\mathfrak{p}_{13} \\ 0 & \mathfrak{p}_{31}\mathfrak{q}_{12}-\mathfrak{q}_{31}\mathfrak{p}_{12} & \mathfrak{p}_{31}\mathfrak{q}_{13}-\mathfrak{q}_{31}\mathfrak{p}_{13} \end{pmatrix}
\end{aligned} \tag{5.2}
$$

Step 3. We perform the Hamiltonian reduction with respect to $\mathrm{GL}_{2n}(\mathbb{C}) = \{\mathrm{diag}(\mathbf{1}_{n \times n}, h_2)|h_2 \in \mathrm{GL}_{2n}(\mathbb{C})\}$. We take the moment map value $m_2 = \mathrm{diag}(ig_2\mathbf{1}_{n \times n}, ig_3\mathbf{1}_{n \times n})$, where $g_2 \neq g_3$. Note that its stabilizer is $\mathrm{GL}_n^2(\mathbb{C}) = \{\mathrm{diag}(\mathbf{1}_{n \times n}, h_2, h_3)|h_2, h_3 \in \mathrm{GL}_n(\mathbb{C})\}$. On the reduction $\mathbb{M}_{\alpha,t}$ we can choose the following Darboux coordinates

$$
(\tilde{P}, \tilde{Q}) = (\mathfrak{p}_{12}\mathfrak{q}_{12}^{-1}, (1 - g_3/g_2)\mathfrak{q}_{12}\mathfrak{q}_{21}). \tag{5.3}
$$

Step 4. After such Hamiltonian reduction we obtain $n \times n$ matrix system with Hamiltonian

$$
H(\tilde{P}, \tilde{Q}; t) + 3t^2/8 = \mathrm{Tr}((ig_2 - ig_3)\tilde{P} - t\tilde{Q} - \tilde{Q}^2 + \tilde{P}\tilde{Q}\tilde{P}). \tag{5.4}
$$

Omitting $3t^2/8$ and making standard substitution

$$
\tilde{P} = 2^{1/3}Q, \quad \tilde{Q} = 2^{-1/3}(P + Q^2) - t/2, \qquad s = -2^{1/3}t, \tag{5.5}
$$

we obtain the standard Hamiltonian of PII$(-i(g_2 - g_3) - 1/2)$.

The block sizes and the moment map value in above example, as well as in Sect. 3 are quite special. At least after the Hamiltonian reduction the phase space dimension should correspond to a matrix system, namely it should be equal to a doubled square of an integer. For the standard situation from Sect. 3 we have

$$
\underbrace{2(dn)^2}_{\dim M_{\alpha,t}} \rightarrow \underbrace{2dn^2}_{\dim M_{\alpha,t}^{\bar{w}}} = \underbrace{2n^2}_{\dim \mathbb{M}_{\alpha,t}} + \underbrace{(d-1)n^2}_{\dim \mathrm{Mat}_n^{d-1}(\mathbb{C})} + \underbrace{(d-1)n^2}_{\dim \mathrm{GL}_n^{d-1}(\mathbb{C})}, \tag{5.6}
$$

where the last two terms correspond to the value of moment map and its stabilizer correspondingly. For the situation from Example 5.1 an analogous calculation gives

$$\underbrace{2(3n)^2}_{\dim M_{\alpha,t}} \to \underbrace{8n^2}_{\dim M^{\tilde{w}}_{\alpha,t}} = \underbrace{2n^2}_{\dim \mathbb{M}_{\alpha,t}} + \underbrace{4n^2}_{\dim \mathrm{Mat}_{2n}(\mathbb{C})} + \underbrace{2n^2}_{\dim \mathrm{GL}_n^2(\mathbb{C})} . \tag{5.7}$$

However, even if the dimension is not a doubled square of integer, sometimes we can make a certain additional reduction to obtain a matrix Painlevé. We illustrate this by the following example.

Example 5.2 Let us again consider a Hamiltonian reduction of $M_\alpha^{\bar{r}}$ for $3n \times 3n$ matrix Painlevé II, but, in difference with Example 5.1, with respect to $\mathrm{GL}_n(\mathbb{C}) = \mathrm{diag}(h_1, \mathbf{1}_{2n \times 2n} | h_1 \in \mathrm{GL}_n(\mathbb{C}))$. We take for the moment map (see (5.2)) the value

$$\mathfrak{p}_{12}\mathfrak{q}_{21} + \mathfrak{p}_{13}\mathfrak{q}_{31} - \mathfrak{q}_{12}\mathfrak{p}_{21} - \mathfrak{q}_{13}\mathfrak{p}_{31} = ig_1 \mathbf{1}_{n \times n}, \tag{5.8}$$

so after the reduction we obtain $6n^2$-dimensional system

$$\underbrace{2(3n)^2}_{\dim M_{\alpha,t}} \to \underbrace{8n^2}_{\dim M^{\tilde{w}}_{\alpha,t}} = \underbrace{6n^2}_{\dim \mathbb{M}_{\alpha,t}} + \underbrace{n^2}_{\dim \mathrm{Mat}_n(\mathbb{C})} + \underbrace{n^2}_{\dim \mathrm{GL}_n(\mathbb{C})} . \tag{5.9}$$

On the reduction we introduce Darboux coordinates

$$\begin{pmatrix} \tilde{P}_1 & \tilde{P}_2 & \tilde{P}_3 \\ \tilde{Q}_1 & \tilde{Q}_2 & \tilde{Q}_3 \end{pmatrix}$$
$$= \begin{pmatrix} \mathfrak{q}_{21}\mathfrak{p}_{13} - \mathfrak{p}_{21}\mathfrak{q}_{13} & (\mathfrak{p}_{31} - \mathfrak{q}_{31}\mathfrak{q}_{21}^{-1}\mathfrak{p}_{21})\mathfrak{q}_{21}^{-1} & \mathfrak{q}_{21}\mathfrak{q}_{12} - \frac{1}{2}(\mathfrak{p}_{21}\mathfrak{q}_{21}^{-1})^2 + \mathfrak{q}_{21}\mathfrak{q}_{13}\mathfrak{q}_{31}\mathfrak{q}_{21}^{-1} + \frac{t}{2} \\ \mathfrak{q}_{31}\mathfrak{q}_{21}^{-1} & \mathfrak{q}_{21}\mathfrak{q}_{13} & -\mathfrak{p}_{21}\mathfrak{q}_{21}^{-1} \end{pmatrix}. \tag{5.10}$$

This matrix system has Hamiltonian

$$H(\{\tilde{P}_i, \tilde{Q}_i\}; t) + 3t^2/8$$
$$= \mathrm{Tr}\left(-\tilde{P}_3^2 + \frac{1}{4}(\tilde{Q}_3^2 - t)^2 - (ig + 1/2)\tilde{Q}_3 + \tilde{P}_1\tilde{P}_2 - 2\tilde{Q}_2\tilde{P}_2\tilde{Q}_3 \right). \tag{5.11}$$

On coordinates $(\tilde{P}_3, \tilde{Q}_3)$ we have almost the matrix Painlevé II Hamiltonian. It appears that we can perform two successive Hamiltonian reductions to obtain a matrix system only on $(\tilde{P}_3, \tilde{Q}_3)$. For the first one, with respect to translations of $\tilde{Q}_1$, we fix moment map value $\tilde{P}_1 = 0$. Then the Hamiltonian becomes invariant with respect to $(\tilde{P}_2, \tilde{Q}_2) \mapsto (h\tilde{P}_2, \tilde{Q}_2 h^{-1})$, $h \in \mathrm{GL}_n(\mathbb{C})$. We fix the value of the corresponding moment map $\tilde{P}_2\tilde{Q}_2$ to be a scalar matrix, namely $\tilde{P}_2\tilde{Q}_2 = \theta \mathbf{1}_{n \times n} = \tilde{Q}_2\tilde{P}_2$. Then after an additional coordinate and time rescaling

$$P = 2^{1/3}\tilde{P}, \quad Q = 2^{-1/3}\tilde{Q}, \quad s = -2^{1/3}t, \tag{5.12}$$

we see that Hamiltonian (5.11) becomes the standard Hamiltonian of matrix $n \times n$ PII$\left(-\mathrm{i}g - \frac{1}{2} - 2\theta \right)$.

5.2 Adding algebraic solutions to Calogero–Painlevé systems

Let us consider a Calogero–Painlevé system together with order 2 Bäcklund transformation w. Let us suppose that w leads to the permutations of cyclic type $[2^n 1^m]$, in difference with Step 2 from the proof of Theorem 4.1. Then we obtain that w$((p_i, q_i)) = (p_i, q_i)$ for $2n < i \leq 2n + m$, so the last m particles evolve as certain algebraic functions. Below we give two examples for the dynamics on the Calogero–Painlevé invariant subset involving such additional algebraic solutions.

Example 5.3 We modify Example 4.4 for Calogero–Painlevé II. The w-invariant algebraic solution is

$$\mathsf{w}((p_i, q_i)) = (-p_i, -q_i) = (p_i, q_i) \Rightarrow (p_i, q_i) = (0, 0). \tag{5.13}$$

We can add only one such particle, due to condition $q_i \neq q_j$ for $i \neq j$ on $\mathsf{M}_\alpha^{\mathrm{reg}}$. Adding such particle $q_{2n+1} = p_{2n+1} = 0$, for the rest of the particles we have equations of motion on (p_i, q_i), $1 \leq i \leq n$

$$\dot{q}_i = p_i, \qquad \dot{p}_i = 2q_i^3 + tq_i + \frac{9g^2}{4q_i^3} + 4g^2 \sum_{j=1, j\neq i}^{n} \frac{q_i(q_i^2 + 3q_j^2)}{(q_i^2 - q_j^2)^3}. \tag{5.14}$$

which differ from (4.44) only in coefficient of term q_i^{-3}. Then it is easy to see that we can modify Example 4.4 by hand. Namely, it is enough to modify Q_i from (4.45) by $g \to 3g$

$$\check{Q}_i = -2^{-1/3} \left(\frac{p_i}{q_i} - \frac{3\mathrm{i}g}{2q_i^2} \right). \tag{5.15}$$

Finally we obtain Calogero–Painlevé II$(-3\mathrm{i}g - 1/2)$ instead of $(-\mathrm{i}g - 1/2)$.

Example 5.4 We modify Example 4.3 for Calogero–Painlevé III$\left(D_6^{(1)} \right)$. The w-invariant algebraic solutions are

$$\mathsf{w}((p_i, q_i)) = (-t^{-1}q_i(p_i q_i + \alpha_1), t/q_i) = (p_i, q_i) \Rightarrow (p_i, q_i) = \left(\mp \frac{\alpha_1}{2\sqrt{t}}, \pm\sqrt{t} \right). \tag{5.16}$$

We can add one of them or both to our system. So let us add to the Calogero–Painlevé III$\left(D_6^{(1)} \right)$ from Example 4.3 $n_1 = 0, 1$ particles $\sqrt{t}$ and $n_2 = 0, 1$ particles $(-\sqrt{t})$.

Modifying momenta P_i from (4.39) by an additional term as follows

$$P_i = 8 \frac{p_i q_i + \alpha_1/2}{q_i/\sqrt{t} - \sqrt{t}/q_i} + \frac{4(1 + 2n_1 n_2) \mathrm{i} g \left(q_i/\sqrt{t} + \sqrt{t}/q_i + 4(n_1 - n_2) \right)}{\left(q_i/\sqrt{t} - \sqrt{t}/q_i \right)^2}, \quad (5.17)$$

we obtain Calogero–Painlevé $\mathrm{V}(1 - \alpha_1 - (1 + 2n_1 n_2)\mathrm{i} g, (1 - 2(n_1 - n_2) + 2n_1 n_2)\mathrm{i} g, \alpha_1 - (1 + 2n_1 n_2)\mathrm{i} g, (1 - 2(n_2 - n_1) + 2n_1 n_2)\mathrm{i} g)$.

Remark 5.1 It would be interesting to obtain such Calogero–Painlevé relations from matrix Painlevé ones. Note that the permutation matrix for $[2^n 1^m]$ is conjugated to $S = \mathrm{diag}(\mathbf{1}_{(m+n) \times (m+n)}, -\mathbf{1}_{n \times n})$, cf. Section 5.1.

5.3 Spin Calogero–Painlevé systems

One can consider a more general case of Calogero-type systems taking in (4.1) general orbit $\mathbf{O}$ instead of $\mathbf{O}_{N,g}$. Let us denote the corresponding reduction map by $\pi_{\mathbf{O}}$ instead of π_N.

The regular part of the phase space of a spin Calogero–Painlevé system can be defined as

$$\mathsf{M}_{\alpha,t}^{\mathrm{reg}} = M_{\alpha,t}^{\mathrm{reg}} //_{\mathbf{O}} \mathrm{GL}_N(\mathbb{C}). \quad (5.18)$$

It will be more convenient for us to use the identification (for the details see [9], Theorem 3).

$$M_{\alpha,t}^{\mathrm{reg}} //_{\mathbf{O}} \mathrm{GL}_N(\mathbb{C}) = \left(\mathrm{T}^* \left(\mathbb{C}^N \backslash \mathrm{diags} \right) \times \mathbf{O} //_0 \mathrm{GL}_1^N(\mathbb{C}) \right) / S_N. \quad (5.19)$$

Let us recall the construction of the reduction $\mathbf{O} //_0 \mathrm{GL}_1^N(\mathbb{C})$ in the right hand side of (5.19).

Let $X \in \mathfrak{gl}_N(\mathbb{C})$, then the fundamental vector field, corresponding to the coadjoint action of X on $\mathbf{O}$ is $v_X(m) = -\mathrm{ad}_X^*(m)$. Then Kirillov–Kostant–Souriau form can be written as $\omega_{\mathrm{KKS}}|_m(v_X, v_Y) = \langle m, [X, Y] \rangle$. Then we have $\iota_{v_X} \omega_{\mathrm{KKS}}|_m(v_Y) = -\langle \mathrm{ad}_Y^*(m), X \rangle = \langle \mathrm{d}m(v_Y), X \rangle$, which means that the coadjoint action on $\mathbf{O}$ is Hamiltonian with the moment map $\mu_{KKS} : m \mapsto m$. Below we will identify points of $\mathbf{O} \subset \mathfrak{gl}_N(\mathbb{C})^*$ with matrices using Killing form.

Next, we restrict coadjoint action to the subgroup of diagonal matrices, so the moment map becomes the projection, which maps $(m_{ij})_{i,j=1,\dots,N} \mapsto (m_{ii})_{i=1,\dots,N}$. Then the element of $\mathbf{O} //_0 \mathrm{GL}_1^N(\mathbb{C})$ is a matrix $m \in \mathbf{O}$ such that $m_{ii} = 0$, $i = 1, \dots N$, considered up to the conjugations by diagonal matrices, namely $(m_{ij})_{i,j=1,\dots,N} \sim (a_i a_j^{-1} m_{ij})_{i,j=1,\dots,N}$. We will denote this class by $[m]$. It will be useful to denote the coadjoint orbit of m by $\mathbf{O}(m)$.

Points of $\mathrm{T}^* \left(\mathbb{C}^N \backslash \mathrm{diags} \right)$ are ordered sets of pairs $((p_j, q_j))_{j=1,\dots,N}$, where q_j's are distinct. The action of S_N on $\left(\mathrm{T}^* \left(\mathbb{C}^N \backslash \mathrm{diags} \right) \times \mathbf{O} //_0 \mathrm{GL}_1^N(\mathbb{C}) \right) / S_N$ is defined as

follows

$$\sigma : (((p_j, q_j))_{j=1,\ldots,N}, [m]) \mapsto (((p_{\sigma(j)}, q_{\sigma(j)}))_{j=1,\ldots,N}, [S_\sigma m S_\sigma^{-1}]), \quad (5.20)$$

where S_σ is the matrix corresponding to $\sigma \in S_N$.

Finally the identification 5.19 can be done as follows (cf. $\tilde{\zeta}_N$ in Sect. 4.1)

$$[(((p_j, q_j))_{j=1,\ldots,N}, [m])] \mapsto \left[\left(\left(\begin{pmatrix} p_1 & \frac{m_{12}}{q_1-q_2} & \cdots & \cdots & \frac{m_{1N}}{q_1-q_N} \\ \frac{m_{21}}{q_2-q_1} & p_2 & \frac{m_{23}}{q_2-q_3} & \ddots & \frac{m_{2N}}{q_2-q_N} \\ \vdots & \frac{m_{32}}{q_3-q_2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{m_{N-1,N}}{q_{N-1}-q_N} \\ \frac{m_{N1}}{q_N-q_1} & \cdots & \cdots & \frac{m_{N,N-1}}{q_N-q_{N-1}} & p_N \end{pmatrix} \right., \right. \right.$$
$$\left. \left. \left. \begin{pmatrix} q_1 & 0 & \cdots & \cdots & 0 \\ 0 & q_2 & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & q_N \end{pmatrix} \right) \right) \right]. \quad (5.21)$$

The dynamics of a spin Calogero–Painlevé system is defined as descent of the matrix Painlevé dynamics with respect to the reduction (5.18). Let us consider an analogue of Theorem 4.1 for the spin Calogero–Painlevé III$\left(D_6^{(1)}\right)$.

Example 5.5 The spin Calogero–Painlevé III$\left(D_6^{(1)}\right)$ is defined by the Hamiltonian (cf. (4.6)).

$$t H([(((p_j, q_j))_{j=1,\ldots,N}, [m])]; t) = \sum_{i=1}^{N} (p_i^2 q_i^2 + (-q_i^2 + (\alpha_1 + \beta_1)q_i + t)p_i - \alpha_1 q_i)$$
$$- \sum_{1 \leq i < j \leq N} \frac{m_{ij} m_{ji}(q_i^2 + q_j^2)}{(q_i - q_j)^2}. \quad (5.22)$$

We will consider a spin generalization of Example 4.2. So, we take $N = 2n$ and $\alpha_1 = \beta_1 = \frac{1}{2}$.

From the Diagram 1 it can be seen that the natural candidate for U from Theorem 4.1 is $\pi_{\mathbf{O}}\left(\tilde{U}\right)$, where $\tilde{U} = \left(\left(M_{\alpha,t}^{\text{reg}}\right)^{\overline{\pi \circ \pi'}} \cap \mu_{2n}^{-1}(\mathbf{O}) \cap \mathbf{m}^{-1}(ig_2 \mathbf{1}_{n \times n}) \right)$. We want to obtain coordinates on $\pi_{\mathbf{O}}\left(\tilde{U}\right)$ in which the dynamics corresponds to spin Calogero–Painlevé III$\left(D_8^{(1)}\right)$ i.e. we want to find an analogue of the map φ from the Diagram 1.

Step 1. Let us obtain coordinates on $\pi_{\mathbf{O}}\left(\tilde{U}\right)$. Consider $[(((p_j, q_j))_{j=1,\ldots,N}, [m])] = \pi_{\mathbf{O}}((p, q))$, where $(p, q) \in \tilde{U}$. We have that $[p, q] = \text{diag}(m_1, ig_2 \mathbf{1}_{n \times n})$. Let A be a

matrix such that $\tilde{q} := A^{-1}qA = \mathrm{diag}\,(q_1, \ldots, q_{2n})$. Then we get

$$\begin{aligned}
\tilde{S}_2 \tilde{q} \tilde{S}_2^{-1} &= -\tilde{q}, \\
\tilde{S}_2 \tilde{p} \tilde{S}_2^{-1} &= -\tilde{p} + 1 - t\tilde{q}^{-2}.
\end{aligned} \tag{5.23}$$

where $\tilde{S}_2 = A^{-1}S_2 A$, $\tilde{p} = A^{-1}pA$. Then without loss of generality we get

$$q_{i+n} = -q_i, \quad 1 \le i \le n. \tag{5.24}$$

It can be seen that after additional multiplication of A by a certain diagonal matrix from the right (which preserves $\tilde{q}$) we get $\tilde{S}_2 = \begin{pmatrix} 0 & \mathbf{1}_{n \times n} \\ \mathbf{1}_{n \times n} & 0 \end{pmatrix}$. So, for diagonal entries of $\tilde{p}$ from (5.23) we get

$$p_{i+n} = 1 - p_i - \frac{t}{q_i^2} \ where \ 1 \le i \le n. \tag{5.25}$$

Now let us obtain conditions on spin variables m_{ij}. Note that since $A^{-1}S_2 A = \tilde{S}_2$, we have $A^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_{n \times n} & -\mathbf{1}_{n \times n} \\ \mathbf{1}_{n \times n} & \mathbf{1}_{n \times n} \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$, where $\alpha_1, \alpha_2 \in \mathrm{GL}_n\,(\mathbb{C})$. Then we get

$$m = [\tilde{p}, \tilde{q}] = A^{-1}[p, q]A = \begin{pmatrix} \frac{1}{2}M & \frac{1}{2}M - \mathrm{i}g_2\mathbf{1}_{n \times n} \\ \frac{1}{2}M - \mathrm{i}g_2\mathbf{1}_{n \times n} & \frac{1}{2}M \end{pmatrix}, \tag{5.26}$$

where $M = \alpha_1 m_1 \alpha_1^{-1} + \mathrm{i}g_2\mathbf{1}_{n \times n}$. Then we have $M \in \tilde{\mathbf{O}}$, where $\tilde{\mathbf{O}} = \mathbf{O}\,(m_1) + \mathrm{i}g_2\mathbf{1}_{n \times n}$. Note that M has zero diagonal entries and is defined up to conjugations by diagonal matrices, which means that we have $[M] \in \tilde{\mathbf{O}}//_0\mathrm{GL}_n\,(\mathbb{C})$.

Taking into account remaining symmetry corresponding to permutations of (p_i, q_i)'s we get that $\pi_{\mathbf{O}}\left(\tilde{U}\right)$ is parametrized by points $[(((P_i, Q_i))_{i=1,\ldots,n}, [M])] \in \left(\tilde{\mathbf{O}}//_0\mathrm{GL}_1^n\,(\mathbb{C}) \times \mathrm{T}^*(\mathbb{C}^n \backslash \mathrm{diags})\right)/\mathrm{S}_n$, where

$$Q_i = \frac{1}{4}q_i^2, \quad P_i = \frac{4}{q_i}\left(p_i - \frac{1}{2} + \frac{t}{2q_i^2}\right), \quad 1 \le i \le n. \tag{5.27}$$

We will explain meaning of formulas (5.27) below.

Step 2. Let us explain why $[(((P_i, Q_i))_{i=1,\ldots,n}, [M])]$ are desired coordinates corresponding to the dynamics of spin Calogero–Painlevé $\mathrm{III}(\mathrm{D}_8^{(1)})$. Following the arguments similar to Step 5 from the proof of Theorem 4.1 we get that the coordinates given by the map $\pi_{\mathbf{O}} \circ \mathrm{pr}$ match the dynamics of spin Calogero–Painlevé $\mathrm{III}(\mathrm{D}_8^{(1)})$.

We have $\mathrm{pr}((p, q)) = (P, Q)$, where (P, Q) is given by (3.11). One can check that the pair (P, Q) is conjugated to $\left(\hat{P}, \hat{Q}\right) := \left(2\left(\mathfrak{p}_{12}\mathfrak{q}_{12}^{-1} + \mathfrak{q}_{21}^{-1}\mathfrak{p}_{21}\right), \frac{1}{4}\mathfrak{q}_{12}\mathfrak{q}_{21}\right)$, where $\mathfrak{p}_{ij}$, $\mathfrak{q}_{ij}$'s are given by (3.2) properly specialized for case 3.2.2.

To obtain $\pi_{\mathbf{O}}((P, Q)) = \pi_{\mathbf{O}}\left(\left(\hat{P}, \hat{Q}\right)\right)$ following the identification (5.19) one should consider the pair $\left(\hat{P}, \hat{Q}\right)$ in the basis, where $\hat{Q} = \frac{1}{4}\mathfrak{q}_{12}\mathfrak{q}_{21}$ is diagonal. Since $A^{-1}qA = \tilde{q}$ it is easy to see that

$$\alpha_1\mathfrak{q}_{12}\alpha_2^{-1} = \alpha_2\mathfrak{q}_{21}\alpha_1^{-1} = -\mathrm{diag}\,(q_1, \ldots, q_n)\,.$$

This proves that the formula $Q_i = \frac{1}{4}q_i^2$ gives part of proper coordinates. As well it implies that α_1 diagonalizes $\frac{1}{4}\mathfrak{q}_{12}\mathfrak{q}_{21}$, so momenta conjugated to Q_i's are diagonal entries of $\mathrm{Ad}_{\alpha_1}\left(\hat{P}\right)$. At first let us compute

$$p_i = (A^{-1}pA)_{ii} = -\frac{1}{2}\left(\alpha_1\mathfrak{p}_{12}\alpha_2^{-1} + \alpha_2\mathfrak{p}_{21}\alpha_1^{-1}\right)_{ii} + \frac{1}{2} - \frac{t}{2q_i^2}, \quad 1 \le i \le n.$$

Then we obtain

$$\left(\mathrm{Ad}_{\alpha_1}\left(\hat{P}\right)\right)_{ii} = \left(2\left(\alpha_1\mathfrak{p}_{12}\alpha_2^{-1}\alpha_2\mathfrak{q}_{12}^{-1}\alpha_1^{-1} + \alpha_1\mathfrak{q}_{21}^{-1}\alpha_2^{-1}\alpha_2\mathfrak{p}_{21}^{-1}\alpha_1^{-1}\right)\right)_{ii}$$

$$= -\frac{2}{q_i}\left(\alpha_1\mathfrak{p}_{12}\alpha_2^{-1} + \alpha_2\mathfrak{p}_{21}\alpha_1^{-1}\right)_{ii} = \frac{4}{q_i}\left(p_i - \frac{1}{2} + \frac{t}{2q_i^2}\right). \quad (5.28)$$

So, we have proved that the formula (5.27) gives part of coordinates corresponding to the dynamics of spin Calogero–Painlevé III$(\mathrm{D}_8^{(1)})$. The rest of coordinates is given by $[\mathrm{Ad}_{\alpha_1}\left([\hat{P}, \hat{Q}]\right)]$. By straightforward computation we obtain that $[\hat{P}, \hat{Q}] = m_1 + \mathrm{i}g_2\mathbf{1}_{n\times n}$, so

$$\mathrm{Ad}_{\alpha_1}\left([\hat{P}, \hat{Q}]\right) = M. \quad (5.29)$$

Additionally we can check that $[((P_j, Q_j))_{j=1,\ldots,n}, [M]]$ are proper coordinates at the level of dynamics. Since coordinates $[M]$ on $\pi_{\mathbf{O}}(\tilde{U})$ are tricky it is more convenient to use the analogue of φ^{-1} which is defined as follows

$$\tilde{\varphi}^{-1} : \left(\tilde{\mathbf{O}}//_0\mathrm{GL}_1^n\,(\mathbb{C}) \times \mathrm{T}^*\left(\mathbb{C}^n\backslash\mathrm{diags}\right)\right)/\mathrm{S}_n \to \pi_{\mathbf{O}}(\tilde{U})\begin{bmatrix}((P_j, Q_j))_{j=1,\ldots,n}, \\ [M])\end{bmatrix}$$

$$\mapsto \begin{bmatrix}(((\frac{\sqrt{Q_j}P_j+1}{2} + \frac{t}{8Q_j}, 2\sqrt{Q_j}))_{j=1,\ldots,n} \cup ((\frac{-\sqrt{Q_j}P_j+1}{2} + \frac{t}{8Q_j}, -2\sqrt{Q_j}))_{j=1,\ldots,n}, \\ [\begin{pmatrix}\frac{1}{2}M & \frac{1}{2}M - \mathrm{i}g_2\mathbf{1}_{n\times n} \\ \frac{1}{2}M - \mathrm{i}g_2\mathbf{1}_{n\times n} & \frac{1}{2}M\end{pmatrix}])\end{bmatrix}.$$

$$(5.30)$$

One can check that this map is symplectic. Let us compute the Hamiltonian with respect to variables $[(((P_j, Q_j))_{j=1,\dots,n}, [M])]$. Taking $s = \frac{t^2}{16}$ as the new time variable we get

$$
sH([((P_j, Q_j))_{j=1,\dots,n}, [M]], s) = \sum_{i=1}^{n} \left(P_i^2 Q_i^2 + P_i Q_i - Q_i - s Q_i^{-1} \right)
$$
$$
- \sum_{1 \leq i < j \leq n} M_{ij} M_{ji} \frac{2 Q_i Q_j}{(Q_i - Q_j)^2}
$$
$$
- \frac{1}{2} \sum_{1 \leq i < j \leq n} M_{ij} M_{ji} + \frac{n g^2}{4}. \qquad (5.31)
$$

Note that on $\tilde{\mathbf{O}}$, the term $-\frac{1}{2} \sum_{1 \leq i < j \leq n} M_{ij} M_{ji} + \frac{n g^2}{4}$ is just a constant since $\sum_{1 \leq i < j \leq n} M_{ij} M_{ji}$ is a Casimir function. So we get the system with the Hamiltonian

$$
sH([((P_j, Q_j))_{j=1,\dots,n}, [M]], s) = \sum_{i=1}^{n} \left(P_i^2 Q_i^2 + P_i Q_i - Q_i - s Q_i^{-1} \right)
$$
$$
- \sum_{1 \leq i < j \leq n} M_{ij} M_{ji} \frac{2 Q_i Q_j}{(Q_i - Q_j)^2}, \qquad (5.32)
$$

which is the Hamiltonian of spin Calogero–Painlevé $\mathrm{III}\big(\mathrm{D}_8^{(1)}\big)$.

References

1. Adler, V.E., Sokolov, V.V.: Matrix Painlevé II equations. Teoret. Mat. Fiz. **207**(2), 188–201 (2021). arXiv:2012.05639
2. Bertola, M., Cafasso, M., Roubtsov, V.: Noncommutative Painlevé equations and systems of Calogero type. Commun. Math. Phys. **363**(2), 503–530 (2018). arXiv:1710.00736

3. Helton, W.J., de Oliveira, M.C., Miller, B., Stankus, M.: NCAlgebra package. https://mathweb.ucsd.edu/~ncalg/
4. Inozemtsev, V.I., Meshcheryakov, D.V.: Extension of the class of integrable dynamical systems connected with semisimple Lie algebras. Lett. Math. Phys. **9**(1), 13–18 (1985)
5. Inozemtsev, V.I.: Lax representation with spectral parameter on a torus for integrable particle systems. Lett. Math. Phys. **17**(1), 11–17 (1989)
6. Kawakami, H.: Matrix Painlevé systems. J. Math. Phys. **56**(3), 033503–27 (2015)
7. Kazhdan, D., Kostant, B., Sternberg, S.: Hamiltonian group actions and dynamical systems of Calogero type. Commun. Pure Appl. Math. **31**(4), 481–507 (1978)
8. Kajiwara, K., Noumi, M., Yamada, Y.: Geometric aspects of Painlevé equations. J. Phys. A: Math. Theor. **50**(7), 073001 (2017). arXiv:1509.08186
9. Reshetikhin, N.: Degenerate integrability of spin Calogero–Moser systems and the duality with the spin Ruijsenaars systems. Lett. Math. Phys. **63**, 03 (2002)
10. Rumanov, I.: Classical integrability for beta-ensembles and general Fokker–Planck equations. J. Math. Phys. **56**(1), 013508 (2015). arXiv:1306.2117
11. Rumanov, I.: Painlevé representation of Tracy–Widom$_\beta$ distribution for $\beta = 6$. Commun. Math. Phys. **342**(3), 843–868 (2016). arXiv:1408.3779
12. Takasaki, K.: Painlevé–Calogero correspondence revisited. J. Math. Phys. **42**(3), 1443–1473 (2001). arXiv:math/0004118
13. Tsuda, T., Okamoto, K., Sakai, H.: Folding transformations of the Painlevé equations. Math. Ann. **331**(4), 713–738 (2005)