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On the weakness of short-range interactions in Fermi gases

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Abstract
Ultracold quantum gases of equal-spin fermions with short-range interactions are often considered free even in the presence of strongly binding spin-up–spin-down pairs. We describe a large class of many-particle Schrödinger operators with short-range pair interactions, where this approximation can be justified rigorously.

Keywords Fermi gas · Short-range interactions · Zero-range limit of many-particle Schrödinger operators · Contact interactions for fermions · Ultracold quantum gases

1 Introduction

Short-range interactions among equal-spin fermions in ultracold quantum gases are often neglected, while at the same time the interaction between particles of opposite spin is modeled by zero-range (i.e., contact) interactions [6, 10, 20]. This can be justified by the fact that zero-range interactions among spinless (or equal spin) fermions are prohibited by the Pauli principle (see Theorem 5.1), and by the recent approximation results for zero-range interactions in terms of short-range potentials [2, 11, 12]. In the present paper, we give a more direct analysis of the weakness of short-range interactions among spinless fermions in terms of estimates for the resolvent difference of free and interacting Hamiltonians. Our main results hold for all space dimensions $d \leq 3$.

We consider fermionic N-particle systems in the Hilbert space

$$\mathscr{H}_f = \wedge^N L^2(\mathbb{R}^d)$$

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described by Schrödinger operators

$$H_\varepsilon = -\Delta - \lambda_\varepsilon \sum_{i<j} V_\varepsilon(x_i - x_j), \tag{1}$$

where $V_\varepsilon(r) = \varepsilon^{-2} V(r/\varepsilon)$, $V(-r) = V(r)$, $\lambda_\varepsilon > 0$ and $d \in \{1, 2, 3\}$. We are primarily interested in the case, where the interaction strength of two distinct particles, in their center-of-mass frame described by

$$-2\Delta - \lambda_\varepsilon V_\varepsilon, \quad \text{in } L^2(\mathbb{R}^d), \tag{2}$$

is independent of ε in the sense that (2) has a ground state energy E_ε that is fixed or convergent $E_\varepsilon \to E$ with limit $E < 0$. It is well known in the spectral theory of Schrödinger operators what this means for λ_ε [17, 22]. In fact, $\varepsilon \mapsto \lambda_\varepsilon$ and V can be chosen, depending on d, in such a way that the resolvent of (2)—and hence the spectrum of (2)—has a limit as $\varepsilon \to 0$. Since $E < 0$, the limit of (2) describes a non-trivial point interaction at the origin [1].

Our main result, Theorem 3.1, can be described in simplified form as follows. Assuming $V \in L^1 \cap L^2(\mathbb{R}^d)$ with $V \geq 0$, $C_V = \sup_{r \in \mathbb{R}^d} V(r)|r|^2 < \infty$, some further decay of V in the case $d = 1$, and

$$\limsup_{\varepsilon \to 0} \lambda_\varepsilon < \frac{d^2}{C_V N}, \tag{3}$$

we show that

$$H_\varepsilon \to -\Delta \qquad (\varepsilon \to 0) \tag{4}$$

in norm resolvent sense. The rate of convergence depends on the size of λ_ε and—to some extent—on the decay of $V(x)$ as $|x| \to \infty$. Given sufficient decay of V, the regularity of $H^2(\mathbb{R}^d)$-functions, and hence the dimension d, begins to play a role. If we choose λ_ε as described above, where $E_\varepsilon \to E < 0$, then condition (3) is satisfied for all N if $d \leq 2$, and for some $N \geq 3$ if $d = 3$. Surprisingly, this particular choice of λ_ε conspires with the regularity of Sobolev functions in such a way that

$$\|(H_\varepsilon + z)^{-1} - (-\Delta + z)^{-1}\| = O(\varepsilon^2) \qquad (\varepsilon \to 0) \tag{5}$$

with the bound $O(\varepsilon^2)$ independent of the space dimension d. Another choice for the coupling constant, consistent with (3) as well, is the one where λ_ε is a positive constant smaller than $d^2/C_V N$. Then, operator (2), upon a rescaling, is proportional to ε^{-2} and hence the (negative) binding energy E_ε diverges. In this case, the limit $\varepsilon \to 0$ amounts to a combined short-range and strong interaction limit, which is interesting and relevant physically [20].

In summary, we can say that fully spin-polarized Fermi gases in $d \leq 2$ with short-range interactions—the spin-up–spin-down interaction strength being fixed—are asymptotically free in the limit of zero-range interaction. This is true in dimension

$d = 3$ as well for suitable V and small $N \geq 3$, depending on V. The result remains correct even in a suitable combined limit of short-range and strong interaction.

We conclude with some remarks on the literature: For single particles, the approximation of point-like disturbances by short-range potentials is discussed at length and in rich detail in [1], see also [5]. For systems of $N \geq 3$ particles in $d \leq 2$ dimensions, it was recently shown in a series of papers that contact interactions (of TMS-type) can be approximated by rescaled two-body potentials in the norm resolvent sense of N-particle Hamiltonians [2, 11–14]. From these results, the mere convergence (4) can be derived by a reduction in the Hilbert space to antisymmetric wave functions. This works for the very special choice of λ_ε needed for the approximation of contact interactions, and for $d \leq 2$, only. For $d = 3$, systems of $N \geq 3$ distinct (or bosonic) particles with two-body short-range interactions are prone to suffer collapse, a phenomenon known as Thomas effect. See Proposition 4.1. In order to avoid this effect, suitable many-body forces are required [3, 8, 9].

This work is organized as follows. In Sect. 2, we present an explicit estimate of the norm of $(H_{\varepsilon=1}+z)^{-1} - (-\Delta+z)^{-1}$ in terms of the pair potential V. This estimate is then used in Sect. 3 to prove (4) and (5). The proofs benefit from the methods and tools developed in [11, 12]. Section 4 gives examples illustrating our results, in particular for $d = 3$ and $N = 3$. Finally, in Sect. 5, we prove the impossibility of fermionic contact interaction in $d \geq 2$. This improves, for fermions, a well-known result about the impossibility of contact interactions in dimensions $d \geq 4$ [23].

2 The resolvent difference

Let $H_0 = -\Delta$ in $\mathscr{H}_f$ and let $R_0(z) = (H_0+z)^{-1}$. We consider N-body Hamiltonians $H = H_0 - W$ in $\mathscr{H}_f$ with

$$W = \sum_{i<j} V_{ij},$$

where V_{ij} denotes multiplication with $V(x_i - x_j)$ and $V \in L^1 \cap L^2(\mathbb{R}^d)$ is real-valued and even. There is no scaling parameter and no coupling constant. We assume $d \leq 3$ and hence H is self-adjoint on $D(H) = D(H_0)$ [21]. The result of this section, Proposition 2.1, is based on a suitable factorization $W = A^*B$ and an iterated (second) resolvent identity related to the Konno–Kuroda formula [18]. We start by constructing the factorization.

Let

$$\mathfrak{X}_\mathrm{f} := L^2_\mathrm{odd}(\mathbb{R}^d, \mathrm{d}r) \otimes L^2(\mathbb{R}^d, \mathrm{d}R) \otimes \bigwedge_{i=3}^{N} L^2(\mathbb{R}^d, \mathrm{d}x_i), \tag{6}$$

where L^2_odd denotes the subspace of odd functions from L^2. The integration variables r and R in (6) correspond to the relative and center-of-mass coordinates of the fermion positions x_1 and x_2. This change of coordinates is implemented isometrically by the operator $\mathscr{K} : \mathscr{H}_\mathrm{f} \to \mathfrak{X}_\mathrm{f}$ given by

$$\left(\mathscr{K}\psi\right)(r, R, x_3, ..., x_N) := \psi\left(R - \frac{r}{2}, R + \frac{r}{2}, x_3, ..., x_N\right). \tag{7}$$

From $\langle\varphi, V_{ij}\psi\rangle = \langle\varphi, V_{12}\psi\rangle$ and from $\langle\varphi, V_{12}\psi\rangle = \langle\mathscr{K}\varphi, \mathscr{K}V_{12}\psi\rangle = \langle\mathscr{K}\varphi, (V \otimes 1)\mathscr{K}\psi\rangle$, it follows that

$$W = \binom{N}{2}\mathscr{K}^*(V \otimes 1)\mathscr{K}. \tag{8}$$

We therefore write $V = vu$ with

$$\begin{aligned} v(r) &:= |V(r)|^{1/2}, \\ u(r) &:= J|V(r)|^{1/2}, \qquad J := \mathrm{sgn}(V), \end{aligned}$$

and we set

$$A := \binom{N}{2}^{1/2}(v \otimes 1)\mathscr{K}, \tag{9}$$

$$B := \binom{N}{2}^{1/2}(u \otimes 1)\mathscr{K} = JA. \tag{10}$$

Recall that $V \in L^1(\mathbb{R}^d)$ and hence $u, v \in L^2(\mathbb{R}^d)$. The domain $D(A)$ of $A : D(A) \subset \mathscr{H}_\mathrm{f} \to \mathfrak{X}_\mathrm{f}$ is determined by the domain of the multiplication operator $v \otimes 1$, so it follows that A and B are densely defined and closed on $D(A) \supset D(H_0)$. Hence, H can be rewritten in the form

$$H = H_0 - A^*B. \tag{11}$$

By an iteration of the resolvent identity, we find

$$(H + z)^{-1} = R_0(z) + R_0(z)WR_0(z) + R_0(z)W(H + z)^{-1}WR_0(z). \tag{12}$$

Upon setting $W = A^*B$, Identity (12) can be written in the form

$$(H + z)^{-1} = R_0(z) + (AR_0(\overline{z}))^*S(z)BR_0(z), \tag{13}$$

$$S(z) = 1 + B(H + z)^{-1}A^*. \tag{14}$$

The following proposition is our tool for proving norm resolvent convergence in the next section.

Proposition 2.1 *Suppose there exists $\delta > 0$ such that $H_0 - (1+\delta)W \geq 0$ and suppose that $V \geq 0$, $V \leq 0$, or that $\sum_{i<j}|V_{ij}| \leq CH_0$ with some $C > 0$. Then, $H \geq 0$ and,* *for all* $z > 0$,

$$\|(H+z)^{-1} - (H_0+z)^{-1}\| \leq C_\delta \binom{N}{2} \|v(-\Delta+z)^{-1}\|^2_{\mathrm{odd}},$$

where $v = |V|^{1/2}$*, and* C_δ *is a function of* C *and* δ*. Here,* $\|\cdot\|_{\mathrm{odd}}$ *denotes the operator norm in* $L^2_{\mathrm{odd}}(\mathbb{R}^d, \mathrm{d}r)$.

Proof Suppose, temporarily, that $H \geq 0$ and let $z > 0$. Then, from (13), the definition of A, and from $\|(-\Delta_r \otimes 1 + z)\mathscr{K}(H_0+z)^{-1}\| \leq 1$, it follows that

$$\|(H+z)^{-1} - R_0(z)\| \leq \binom{N}{2} \|v(-\Delta+z)^{-1}\|^2_{\mathrm{odd}} \|S(z)\|.$$

It remains to prove $H \geq 0$ and $\|S(z)\| \leq C_\delta$ under the various assumptions on V.

In the case $V \leq 0$, we have $J = -1$, $B = -A$, and hence, $H = H_0 + A^*A \geq A^*A \geq 0$. By Lemma 2.2, it follows that $\|S(z)\| \leq 2$.

In the case $V \geq 0$, we have $J = 1$, $B = A$, and hence the assumption $H_0 - (1+\delta)W \geq 0$ implies that

$$H \geq \delta W = \delta A^*A.$$

By Lemma 2.2, it follows that $A(H+\mu)^{-1}A^* \leq 1/\delta$, and hence $\|S(z)\| \leq 1 + \delta^{-1}$, by (14).

It remains to consider the case where V changes sign. The assumption $H_0 - (1+\delta)W \geq 0$ implies that

$$H = \frac{\delta}{1+\delta} H_0 + \frac{1}{1+\delta}(H_0 - (1+\delta)W) \geq \frac{\delta}{1+\delta} H_0.$$

Combining this with the assumption on $|V|$, that is, with $CH_0 \geq \sum_{i<j} |V_{ij}| = A^*A$, we find that

$$H \geq \frac{\delta}{1+\delta} \frac{1}{C} A^*A.$$

This, by Lemma 2.2, implies $A(H+\mu)^{-1}A^* \leq C(1+\delta)/\delta$ and the desired bound on $\|S(z)\|$ follows from (14). □

Lemma 2.2 *Let* H*,* A *be any two closed operators in a Hilbert space, with* $H^* = H \geq 0$ *and* $D(A) \supset D(H)$*. If* $H \geq \lambda A^*A$ *for some* $\lambda > 0$*, then* $D(A) \supset D(H^{1/2})$ *and for all* $\mu > 0$,

$$A(H+\mu)^{-1}A^* \leq \frac{1}{\lambda}.$$

Proof The assumption means that $\|(H+\mu)^{1/2}\psi\|^2 \geq \lambda\|A\psi\|^2$ for $\mu > 0$ and all $\psi \in D(H)$. By an approximation argument, this inequality extends to all $\psi \in D(H^{1/2})$, and $D(A) \supset D(H^{1/2})$ follows from the closedness of A. Upon rewriting the inequality in terms of $\varphi = (H+\mu)^{1/2}\psi \in \mathscr{H}_f$, the assertion follows. □

3 The resolvent convergence

We now apply the results from the previous section to Schrödinger operators with rescaled two-body potentials, that is

$$H_\varepsilon = -\Delta - \lambda_\varepsilon \sum_{i<j} V_{\varepsilon,ij}, \tag{15}$$

where $\lambda_\varepsilon > 0$ and $V_\varepsilon(r) = \varepsilon^{-2}V(r/\varepsilon)$. The Schrödinger operator $\varepsilon^2 H_\varepsilon$ is unitarily equivalent to $S_\lambda := -\Delta - \lambda\sum_{i<j} V_{ij}$ with $\lambda = \lambda_\varepsilon$. We therefore define

$$\lambda_{\max} := \sup\{\lambda \geq 0 \mid S_\lambda \geq 0\}. \tag{16}$$

Note that $S_\lambda \geq 0$ for all $\lambda \leq \lambda_{\max}$. This follows from the fact that $E_\lambda := \inf\sigma(S_\lambda)$ as a function of λ is concave (hence continuous) and $E_0 = 0$.

For Theorem 3.1 to be non-void, we need that $\lambda_{\max} > 0$. This can be achieved, e.g., by assuming that

$$C_V := \sup_{x\in\mathbb{R}^d} V(x)|x|^2 < \infty.$$

Then, $\lambda_{\max} \geq d^2/(C_V N) > 0$, by the Hardy inequality for fermionic wave functions $\psi \in \mathscr{H}_f$ [15],

$$\sum_{i<j}\int \frac{|\psi(x_1,\ldots,x_N)|^2}{|x_i - x_j|^2}\,\mathrm{d}x \leq \frac{N}{d^2}\|\nabla\psi\|^2. \tag{17}$$

Statement as well as proof of Theorem 3.1 depends on the regularity of H^2-Sobolev functions. Explicitly, we use the embedding $H^2(\mathbb{R}^d) \hookrightarrow C^{0,s}(\mathbb{R}^d)$, valid for $s \in I_d$, where $I_1 = [0,1]$, $I_2 = [0,1)$ and $I_3 = [0,1/2]$, and we use Lemma 3.2, which improves the embedding in the case $d = 2$. Here, $C^{0,s}(\mathbb{R}^d)$ denotes the space of continuous functions that are uniformly Hölder continuous of exponent s.

Theorem 3.1 *Suppose that $V \geq 0$, $V \leq 0$, or that $\sup_{x\in\mathbb{R}^d}|V(x)||x|^2 < \infty$. If $\lambda_0 := \limsup_{\varepsilon\to 0}\lambda_\varepsilon < \lambda_{\max}$, then $H_\varepsilon \geq 0$ for ε small enough, and for all $z > 0$,*

$$\|(H_\varepsilon + z)^{-1} - (H_0 + z)^{-1}\| = o(\lambda_\varepsilon \varepsilon^{d-2}) \qquad (\varepsilon \to 0). \tag{18}$$

Moreover, the following is true:

(a) *If* $\int |V(r)||r|^{2s}\,dr < \infty$ *for some* $s \in I_d$, *then*

$$\|(H_\varepsilon + z)^{-1} - (H_0 + z)^{-1}\| = O(\lambda_\varepsilon \varepsilon^{d-2+2s}) \qquad (\varepsilon \to 0).$$

(b) *If* $d = 2$ *and* $\int |V(r)||r|^2 |\log |r||\,dr < \infty$, *then*

$$\|(H_\varepsilon + z)^{-1} - (H_0 + z)^{-1}\| = O(\lambda_\varepsilon \varepsilon^d |\log \varepsilon|) \qquad (\varepsilon \to 0).$$

In the situation described in the Introduction, where $\inf \sigma(-2\Delta - \lambda_\varepsilon V_\varepsilon)$ has a limit $E < 0$, the bounds (a) and (b) reveal a surprising interplay between λ_ε and the (optimal) regularity of $H^2(\mathbb{R}^d)$-functions: if, depending on d, we choose $\lambda_\varepsilon = O(\varepsilon)$, $\lambda_\varepsilon = O(1/|\log \varepsilon|)$, and $\lambda_\varepsilon = O(1)$ for $d = 1$, $d = 2$, and $d = 3$, respectively, then, for all $d \in \{1, 2, 3\}$,

$$\|(H_\varepsilon + z)^{-1} - (H_0 + z)^{-1}\| = O(\varepsilon^2), \qquad (\varepsilon \to 0)$$

provided V decays fast enough, e.g., as in the hypothesis of (b), and N is small if $d = 3$.

Remarks 1. Part (a) of the theorem shows that, for a large class of potentials,

$$H_\varepsilon \to H_0 \qquad (\varepsilon \to 0) \tag{19}$$

in the norm resolvent sense, provided that $\lambda_\varepsilon \varepsilon^{d-2+2s} \to 0$ as $\varepsilon \to 0$. This is not true for the Hamiltonians $\tilde{H}_\varepsilon$ defined by (15) on the enlarged Hilbert space $L^2(\mathbb{R}^{Nd})$, as shown in the next section.

2. For the convergence (19) in norm resolvent sense to hold, it is necessary that $\inf \sigma(H_\varepsilon) \to \inf \sigma(H_0) = 0$. Therefore, the assumption $\limsup_{\varepsilon\to 0} \lambda_\varepsilon < \lambda_{\max}$ in Theorem 3.1 cannot be relaxed significantly. Strong resolvent convergence, by contrast, has much weaker spectral implications and hence—given some decay of V—much less is needed of λ_ε, see Proposition 3.3.
3. A weaker result, similar to Theorem 3.1, could be derived from [11, 12]. Indeed, for suitable λ_ε we know from [11, 12] that $H_\varepsilon \to H$ in norm resolvent sense, where $H = -\Delta$ on $\mathscr{H}_\mathrm{f}$. Information on the rate of convergence can also be found in these papers.

Proof We are going to apply Proposition 2.1 to the Hamiltonian (15), and we assume that V changes sign, the other cases being easier. Due to the unitary equivalence of $\varepsilon^2 H_\varepsilon$ and S_{λ_ε}, the hypotheses of Proposition 2.1 are equivalent to

$$H_0 - (1+\delta)\lambda_\varepsilon \sum_{i<j} V_{ij} \geq 0 \tag{20}$$

$$C H_0 - \lambda_\varepsilon \sum_{i<j} |V_{ij}| \geq 0 \tag{21}$$

for some $\delta, C > 0$. Both (20) and (21) are true for ε small enough. This follows from $\lambda_0 = \limsup_{\varepsilon \to 0} \lambda_\varepsilon < \lambda_{\max}$, from $\sup_{x\in\mathbb{R}^d} |V(x)||x|^2 < \infty$, and from the Hardy inequality for fermions (17). Hence, by Proposition 2.1, for $\varepsilon > 0$ small enough,

$$\|(H_\varepsilon + z)^{-1} - (H_0 + z)^{-1}\| \le C_\delta \binom{N}{2} \lambda_\varepsilon \varepsilon^{d-2} \|v_\varepsilon(-\Delta + z)^{-1}\|^2_{\text{odd}}, \tag{22}$$

where $v_\varepsilon(x) := \varepsilon^{-d/2}|V(x/\varepsilon)|^{1/2}$.

By the Sobolev embedding $H^2(\mathbb{R}^d) \hookrightarrow C^{0,s}(\mathbb{R}^d)$, valid for $s \in I_d$, the elements $\psi \in H^2(\mathbb{R}^d) \cap L^2_{\text{odd}}(\mathbb{R}^d)$ are Hölder continuous (of exponent s) odd functions. It follows that $\psi(0) = 0$ and that

$$|\psi(x)| = |\psi(x) - \psi(0)| \le C_s \|\psi\|_{H^2} |x|^s. \tag{23}$$

Therefore, for all $\psi \in H^2(\mathbb{R}^d) \cap L^2_{\text{odd}}(\mathbb{R}^d)$,

$$\begin{aligned} \|v_\varepsilon \psi\|^2 &\le C_s^2 \|\psi\|^2_{H^2} \int |v_\varepsilon(x)|^2 |x|^{2s}\, dx \\ &= \varepsilon^{2s} C_s^2 \|\psi\|^2_{H^2} \int |V(x)||x|^{2s}\, dx. \end{aligned} \tag{24}$$

This is true for all $s \in I_d$ and, combined with (22), it proves statement (a) of the theorem. To prove (b), we use Lemma 3.2 in (23) (rather than $H^2 \hookrightarrow C^{0,s}$) and then (24) becomes $C\varepsilon^2 |\log \varepsilon| \int |V(x)||x|^2 (1 + |\log|x||)\, dx$, where the integral is finite by the assumptions on V.

It remains to prove (18). Equation (24) with $s = 0$ implies that $\|v_\varepsilon(-\Delta + z)^{-1}\|_{\text{odd}} = O(1)$, which can be improved as follows: let χ_k denote the characteristic function of the ball $|x| \le k$ in $\mathbb{R}^d$ and let $(v\chi_k)_\varepsilon = v_\varepsilon \chi_{\varepsilon k}$. Then, by (24),

$$\|(v\chi_k)_\varepsilon(-\Delta + z)^{-1}\|^2_{\text{odd}} = O(\varepsilon^{2s}) = o(1) \qquad (\varepsilon \to 0) \tag{25}$$

for any $s > 0$ in I_d. On the other hand,

$$\|(v - v\chi_k)_\varepsilon(-\Delta + z)^{-1}\|_{HS} \le C\|v - v\chi_k\| = o(1) \qquad (k \to \infty) \tag{26}$$

uniformly in $\varepsilon > 0$, where HS refers to Hilbert–Schmidt norm. The combination of (22), (25) and (26) proves (18) and concludes the proof of the theorem. □

In the proof of Theorem 3.1, we have used the following lemma, which can probably be found in the literature, but we are not aware of suitable reference.

Lemma 3.2 *For all $u \in H^2(\mathbb{R}^2)$ and all $x, y \in \mathbb{R}^2$, $y \neq 0$, we have*

$$|u(x + y) - u(x)| \le \frac{1}{2\sqrt{\pi}} |y| \big(2 + |\log|y||\big)^{1/2} \left(\|\Delta u\|^2 + \|\nabla u\|^2\right)^{1/2}.$$

Remark By our method of proof, this inequality can be generalized to derivatives $\partial^\alpha u$, $|\alpha| \leq k$, of functions $u \in H^s(\mathbb{R}^n)$ with $s - (n/2) = k + 1$, $s \in \mathbb{R}$ and $k \in \mathbb{N}_0$.

Proof We first note that $u \in H^2(\mathbb{R}^2)$ implies that $\widehat{u} \in L^1(\mathbb{R}^2)$, and hence, for all $x \in \mathbb{R}^2$,

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \widehat{u}(p) \exp(ipx)\, \mathrm{d}p.$$

Therefore, by Cauchy–Schwarz,

$$\begin{aligned} 2\pi \frac{|u(x+y) - u(x)|}{|y|} &\leq \int_{\mathbb{R}^2} \frac{|\exp(ipy) - 1|}{|y|} |\widehat{u}(p)|\, \mathrm{d}p \\ &\leq I(y)^{1/2} \left(\|\Delta u\|^2 + \|\nabla u\|^2 \right)^{1/2}, \end{aligned} \tag{27}$$

where

$$I(y) := \int_{\mathbb{R}^2} \frac{|\exp(ipy) - 1|^2}{|y|^2} \frac{1}{|p|^4 + |p|^2}\, \mathrm{d}p.$$

To estimate the integral $I(y)$, we may assume that $y = (|y|, 0)$. Then, upon the substitution $q = p|y|$, we find that, for any $Q > 0$,

$$I(y) = \int_{\mathbb{R}^2} \frac{|\exp(iq_1) - 1|^2}{|q|^4 + |y|^2|q|^2}\, \mathrm{d}q \leq \int_{|q| \leq Q} \frac{q_1^2}{|q|^4 + |y|^2|q|^2}\, \mathrm{d}q + \int_{|q| > Q} \frac{4}{|q|^4}\, \mathrm{d}q, \tag{28}$$

where $|\exp(iq_1) - 1| \leq |q_1|$ and $|\exp(iq_1) - 1| \leq 2$ was used, respectively. Both integrals on the right of (28) can be computed explicitly. For the first one, we obtain

$$\begin{aligned} \int_{|q| \leq Q} \frac{q_1^2}{|q|^4 + |y|^2|q|^2}\, \mathrm{d}q &= \frac{1}{2} \int_{|q| \leq Q} \frac{1}{|q|^2 + |y|^2}\, \mathrm{d}q \\ &= \frac{\pi}{2} \log\left(1 + \frac{Q^2}{|y|^2}\right) \leq \pi \left(|\log|y|| + |\log Q| + \frac{1}{2}\log(2) \right), \end{aligned} \tag{29}$$

where the inequality follows from $\log(1+t) \leq |\log t| + \log 2$, valid for all $t > 0$. The second integral on the right side of (28) equals $4\pi/Q^2$. Choosing $Q = 2\sqrt{2}$, we find from (28) and (29) that

$$I(y) \leq \pi(|\log|y|| + c)$$

with $c = \frac{1}{2} + \log 4 < 2$. This concludes the proof. $\square$

We conclude this section with the proposition announced in Remark 2. It is a consequence of Theorem 5.1 concerning the essential self-adjointness of the Laplacian. In all the following $\Omega = \mathbb{R}^{Nd} \setminus \Gamma$, where

$$\Gamma := \bigcup_{i<j} \{x = (x_1, \ldots, x_N) \in \mathbb{R}^{Nd} \mid x_i = x_j\}. \tag{30}$$

Furthermore, $C_0^\infty(\Omega) := \{\psi \in C_0^\infty(\mathbb{R}^{Nd}) \mid \operatorname{supp}\psi \subset \Omega\}$.

Proposition 3.3 *Let $d \geq 2$ and suppose that $V \in L^2(\mathbb{R}^d)$. Suppose there exists $s \geq 0$ such that $\int V(r)^2 |r|^{2s}\, \mathrm{d}r < \infty$ and $\limsup_{\varepsilon\to 0} \lambda_\varepsilon\, \varepsilon^{s+d/2-2} < \infty$. Then, $H_\varepsilon \to H_0$ in the strong resolvent sense as $\varepsilon \to 0$.*

Proof In view of $\|(H_\varepsilon+i)^{-1}\| = 1 = \|R_0(i)\|$ it suffices to prove that $(H_\varepsilon+i)^{-1}\psi \to R_0(i)\psi$ for ψ from a dense subset of $\mathscr{H}_f$. By Theorem 5.1, the set of all $\psi = (H_0+i)\varphi$ with $\varphi \in C_0^\infty(\Omega) \cap \mathscr{H}_\mathrm{f}$ is dense in $\mathscr{H}_\mathrm{f}$, and for such ψ,

$$\|(H_\varepsilon + i)^{-1}\psi - R_0(i)\psi\| \leq \left\| \lambda_\varepsilon \sum_{i<j} V_{\varepsilon,ij}\varphi \right\| \leq \lambda_\varepsilon \binom{N}{2} \|V_{\varepsilon,12}\varphi\|,$$

where the (anti-)symmetry of φ was used in the second inequality. Let $\widetilde{\varphi} := \mathscr{K}\varphi$, that is

$$\widetilde{\varphi}(r, R, x') = \varphi(R - r/2, R + r/2, x'),$$

where $x' = (x_3, \ldots, x_N)$. Like φ, $\tilde{\varphi}$ is compactly supported and hence $\operatorname{supp}\widetilde{\varphi} \subset \mathbb{R}^d \times B_{N-1}$ for some ball $B_{N-1} \subset \mathbb{R}^{d(N-1)}$. It follows that, for any $c > 0$,

$$\begin{aligned}\varepsilon^{4-d}\|V_{\varepsilon,12}\varphi\|^2 = &\int_{B_{N-1}} \mathrm{d}R\, \mathrm{d}x' \int_{|r|\leq c} |V(r)|^2 \left|\widetilde{\varphi}(\varepsilon r, R, x')\right|^2 \mathrm{d}r \\ &+ \int_{B_{N-1}} \mathrm{d}R\, \mathrm{d}x' \int_{|r|>c} |V(r)|^2 \left|\widetilde{\varphi}(\varepsilon r, R, x')\right|^2 \mathrm{d}r.\end{aligned} \tag{31}$$

By assumption on φ, $\tilde{\varphi}(r, R, x') = 0$ for $r < \operatorname{dist}(\operatorname{supp}\varphi, \Gamma)$. This means that the first summand vanishes for $\varepsilon c < \operatorname{dist}(\operatorname{supp}\varphi, \Gamma)$ and that

$$\left|\widetilde{\varphi}(r, R, x')\right| \leq C(\widetilde{\varphi}, s)|r|^s$$

for each $s \geq 0$. It follows that

$$\lambda_\varepsilon \|V_{\varepsilon,12}\varphi\| \leq \lambda_\varepsilon \varepsilon^{s+d/2-2} C(\widetilde{\varphi}, s)|B_{N-1}|^{1/2} \left(\int_{|r|>c} |V(r)|^2\, |r|^{2s} \mathrm{d}r\right)^{1/2},$$

where the integral can be made arbitrarily small by choosing c large. By assumption, $\limsup_{\varepsilon\to 0} \lambda_\varepsilon \varepsilon^{s+d/2-2} < \infty$; hence, it follows that $\lim_{\varepsilon\to 0} \lambda_\varepsilon \| V_{\varepsilon,12}\varphi\| = 0$, and the proof is complete. □

4 Examples and discussion

To put Theorem 3.1 into a broader perspective and to demonstrate its dependence on the Pauli principle, we now view H_ε as the restriction

$$H_\varepsilon = \tilde{H}_\varepsilon \restriction \mathscr{H}_f,$$

where $\tilde{H}_\varepsilon$ denotes the Schrödinger operator defined by expression (15) on the enlarged Hilbert space $L^2(\mathbb{R}^{Nd})$. We shall give choices for λ_ε and V, where $\tilde{H}_\varepsilon$, in contrast to H_ε, has a limit $\tilde{H}$ describing non-trivial contact interactions or no limit at all.

In the cases $d = 1$ and $d = 2$ we choose, for simplicity, a two-body potential $V \in L^\infty(\mathbb{R}^d)$ with compact support and $\int V(r)\,\mathrm{d}r = 1$. Suppose further that

$$\begin{aligned} \lambda_\varepsilon &= g\varepsilon > 0 && \text{if } d = 1,\\ \lambda_\varepsilon^{-1} &= \frac{|\log(\varepsilon)|}{4\pi} + a && \text{if } d = 2. \end{aligned}$$

Then, $\lambda_{\max} > 0$ and $\lambda_0 = \limsup_{\varepsilon\to 0} \lambda_\varepsilon = 0$. So, the hypotheses of Theorem 3.1 are satisfied and hence $H_\varepsilon \to H_0$ in norm resolvent sense. On the other hand, by [11, 12], $\tilde{H}_\varepsilon \to \tilde{H}$, where $\tilde{H}$ describes non-trivial contact interactions. That is, $\tilde{H}$ is a self-adjoint extension of $-\Delta\restriction C_0^\infty(\mathbb{R}^{Nd}\backslash\Gamma)$ distinct from $-\Delta$. See (30) for the definition of Γ.

We now turn to the more interesting case of N particles in $d = 3$ dimensions. In the following, N is exhibited in the notation: we write $H_{N,\varepsilon}$ for H_ε and $\tilde{H}_{N,\varepsilon}$ for $\tilde{H}_\varepsilon$. For the coupling constant and the two-body potential, we choose $\lambda_\varepsilon = 2$ and

$$V(r) := \begin{cases} \dfrac{2}{|r|} - 1 & \text{if } |r| \le 1\\ 0 & \text{if } |r| > 1. \end{cases} \tag{32}$$

Then, $0 \le V(r) \le |r|^{-2}$ and hence $C_V = \sup V(r)|r|^2 \le 1$. It follows that

$$\begin{aligned} \lambda_0 &= \limsup_{\varepsilon\to 0} \lambda_\varepsilon = 2\\ \lambda_{\max} &\ge \frac{d^2}{NC_V} \ge \frac{9}{N}. \end{aligned}$$

Thus, for $N \le 4$ we have $\lambda_0 < \lambda_{\max}$ and hence, by Theorem 3.1, $H_{N,\varepsilon} \to H_0$ in norm resolvent sense. On the other hand, concerning $\tilde{H}_{N,\varepsilon}$ the following can be said:

Proposition 4.1 *With the above notations, in the case $d = 3$ we have*

(a) *For* $N = 2$, $\tilde{H}_{2,\varepsilon} \to \tilde{H}_2$ *in norm resolvent sense, where* $\tilde{H}_2$ *is a non-trivial self-adjoint extension of* $-\Delta\lceil C_0^\infty(\mathbb{R}^6\backslash\Gamma)$.

(b) *For each* $N \geq 3$*, there exists a constant* $C_N < 0$ *such that*

$$\sigma(\tilde{H}_{N,\varepsilon}) = [C_N\varepsilon^{-2}, \infty).$$

Remark The divergence of the ground state energy established in Part (b) is known as Thomas effect [24].

Proof With respect to center-of-mass and relative coordinates $R = (x_1 + x_2)/2$ and $r = x_2 - x_1$, the Schrödinger operator for $N = 2$ takes the form

$$\begin{aligned}\tilde{H}_{2,\varepsilon} &= -\Delta_R/2 \otimes 1 + 1 \otimes h_\varepsilon \\ h_\varepsilon &= -2\Delta_r - \lambda_\varepsilon V_\varepsilon.\end{aligned}$$

By construction of V, $h = -\Delta - V \geq 0$ and $z = 0$ is not an eigenvalue but a resonance energy. This means that the Birman–Schwinger operator $V^{1/2}(-\Delta)^{-1}V^{1/2}$ has the (simple) eigenvalue 1, but the corresponding solution ψ of $(-\Delta - V)\psi = 0$ fails to be square-integrable. Explicitly, in the present case, $\psi(x) = e^{-|x|}$ for $|x| \leq 1$ and $\psi(x) = e^{-1}/|x|$ for $|x| > 1$. These properties of V imply that $h_\varepsilon \to -2\Delta_0$ in norm resolvent sense, as $\varepsilon \to 0$, where Δ_0 denotes a self-adjoint extension of $\Delta\lceil C_0^\infty(\mathbb{R}^3\backslash\{0\})$ that is distinct from the free Laplacian [1]. Since $-\Delta_R/2 \geq 0$, it follows that $\tilde{H}_{2,\varepsilon} \to -\Delta_R/2 \otimes 1 + 1 \otimes (-2\Delta_0)$ in norm resolvent sense, which proves assertion (a) (for details see [14]).

In the case (b), we use that the Schrödinger operator $\tilde{H}_{N,\varepsilon}$ is unitarily equivalent to $\varepsilon^{-2}\tilde{H}_{N,\varepsilon=1}$. For $N = 3$, the presence of a zero-energy resonance in the two-body Hamiltonian leads to non-empty (in fact, infinite) discrete spectrum in the three-particle Hamiltonian with center-of-mass motion removed [7, 16, 19]. This is the Efimov effect. It means, in particular, that $C_3 := \inf\sigma(\tilde{H}_{3,\varepsilon=1}) < 0$. By the HVZ theorem, $C_N := \inf\sigma(\tilde{H}_{N,\varepsilon=1}) \leq C_3$ for all $N \geq 3$. □

5 Absence of contact interactions for $d \geq 2$

In space dimensions $d \geq 2$, zero-range interactions among equal-spin fermions are prohibited by the Pauli principle. This is true in the very strong form of Theorem 5.1. For a related result in the physics literature concerning two fermions in $d = 2$, see [4].

Let $\Gamma_{ij} := \{x = (x_1, \ldots, x_N) \in \mathbb{R}^{Nd} \mid x_i = x_j\}$ and $\Omega_{ij} = \mathbb{R}^{Nd}\backslash\Gamma_{ij}$. Recall from Sect. 3 that $\Gamma = \cup_{i<j}\Gamma_{ij}$ and

$$\Omega = \mathbb{R}^{Nd}\backslash\Gamma = \bigcap_{i<j}\Omega_{ij}.$$

Theorem 5.1 *If $d \geq 2$, then $C_0^\infty(\Omega) \cap \mathscr{H}_f$ is dense in $H^2(\mathbb{R}^{Nd}) \cap \mathscr{H}_f$ with respect to the norm of H^2. This means that*

$$H_0^2(\Omega) \cap \mathscr{H}_f = H^2(\mathbb{R}^{Nd}) \cap \mathscr{H}_f,$$

and it implies that the Laplacian Δ in $\mathscr{H}_f$ is essentially self-adjoint on $C_0^\infty(\Omega) \cap \mathscr{H}_f$.

Remarks 1. The main point of Theorem 5.1 is that elements of $C_0^\infty(\Omega)$ vanish in an entire neighborhood of the collision set Γ. The elements of $C_0^\infty(\mathbb{R}^{Nd}) \cap \mathscr{H}_f$ vanish on Γ too. But the weaker statement, that $C_0^\infty(\mathbb{R}^{Nd}) \cap \mathscr{H}_f$ is dense in $H^2(\mathbb{R}^{Nd}) \cap \mathscr{H}_f$, is true for all $d \geq 1$ and it easily follows from the fact that $C_0^\infty(\mathbb{R}^{Nd})$ is dense in $H^2(\mathbb{R}^{Nd})$.

2. For $d = 1$, the assertion of the theorem is false. To see this, consider a sequence (ψ_n) in $C_0^\infty(\Omega)$ with $\psi_n \to \psi$ in the norm of H^2. Then, $\nabla\psi_n \to \nabla\psi$ in the norm of H^1. Since the trace operators $T_{ij} : H^1(\mathbb{R}^N) \to L^2(\Gamma_{ij})$ are continuous, and since, clearly, $\nabla\psi_n = 0$ on all hyperplanes Γ_{ij}, it follows that

$$\nabla\psi = 0 \quad \text{on all } \Gamma_{ij}, \tag{33}$$

or, more precisely, $T_{ij}\nabla\psi = 0$ in $L^2(\Gamma_{ij})$. We now give an example of an antisymmetric wave function $\psi \in H^2(\mathbb{R}^N)$ without property (33), which proves that $C_0^\infty(\Omega) \cap \mathscr{H}_f$ is not dense in $H^2(\mathbb{R}^N) \cap \mathscr{H}_f$.

Let $|x|^2 := \sum_{i=1}^N x_i^2$ and

$$\psi(x_1, \ldots, x_N) := e^{-|x|^2} \prod_{i<j}(x_j - x_i).$$

Apart from the Gaussian, this is a Vandermonde determinant. This shows that ψ is antisymmetric. On the hyperplane Γ_{12}, we have

$$\frac{\partial\psi}{\partial x_1}\Big|_{x_1=x_2} = -e^{-|x|^2} \prod_{j=3}^N (x_j - x_1) \prod_{2\leq i<j} (x_j - x_i),$$

which shows that $\nabla\psi$ does not vanish on Γ_{12}.

The proof of Theorem 5.1 is based on the following lemmas, Lemma 5.2 being the heart of it.

Lemma 5.2 *If $d \geq 2$, then there exists a sequence $u_n \in C_0^\infty(\mathbb{R}^d, [0, 1])$ with $u_n(x) = 1$ if $|x| \leq 1/n$, and, in the limit $n \to \infty$, $\operatorname{diam}(\operatorname{supp} u_n) \to 0$ as well as*

$$\int |\nabla u_n(x)|^2\, dx \to 0 \tag{34}$$

$$\int |x|^2 |\Delta u_n(x)|^2\, dx \to 0. \tag{35}$$

Proof In the case $d \geq 3$, we may choose any function $u \in C_0^\infty(\mathbb{R}^d, [0, 1])$ with $u(x) = 1$ for $|x| \leq 1$ and define $u_n(x) = u(nx)$. Then, with the substitution $y = nx$, in the limit $n \to \infty$,

$$\int |\nabla u_n(x)|^2 \,\mathrm{d}x = \int |\nabla u(y)|^2 \,dy \cdot n^{2-d} \to 0,$$
$$\int |x|^2 |\Delta u_n(x)|^2 \,\mathrm{d}x = \int |y|^2 |\Delta u(y)|^2 \,dy \cdot n^{2-d} \to 0.$$

In the case $d = 2$, we define $u_n(0) := 1$ and for $|x| > 0$ we set

$$u_n(x) := g\left(\frac{\log(n|x|)}{\log\log n}\right),$$

where $g \in C^\infty(\mathbb{R}, [0, 1])$ with

$$g(s) := \begin{cases} 1 & s \leq 0 \\ 0 & s \geq 1. \end{cases}$$

It follows that $u_n(x) = 1$ for $|x| \leq 1/n$, $u_n(x) = 0$ for $|x| \geq (\log n)/n$ and hence that $u_n \in C_0^\infty(\mathbb{R}^d, [0, 1])$. Moreover,

$$\begin{aligned} \frac{1}{2\pi} \int |\nabla u_n(x)|^2 \,\mathrm{d}x &= \int_{1/n}^{(\log n)/n} g'\left(\frac{\log(nr)}{\log\log n}\right)^2 \frac{\mathrm{d}r}{r} \cdot \frac{1}{(\log\log n)^2} \\ &= \int_0^1 g'(s)^2 \,\mathrm{d}s \cdot \frac{1}{\log\log n}. \end{aligned}$$

On the other hand, using that on radially symmetric functions

$$r^2 \Delta = \left(r \frac{\partial}{\partial r}\right)^2,$$

we find

$$\begin{aligned} \frac{1}{2\pi} \int |x|^2 |\Delta u_n(x)|^2 \,\mathrm{d}x &= \int_{1/n}^{(\log n)/n} \left| \left(r \frac{\partial}{\partial r}\right)^2 g\left(\frac{\log(nr)}{\log\log n}\right) \right|^2 \frac{\mathrm{d}r}{r} \\ &= \int_{1/n}^{(\log n)/n} g''\left(\frac{\log(nr)}{\log\log n}\right)^2 \frac{\mathrm{d}r}{r} \cdot \frac{1}{(\log\log n)^4} \\ &= \int_0^1 g''(s)^2 \,\mathrm{d}s \cdot \frac{1}{(\log\log n)^3}. \end{aligned}$$

This concludes the proof. □

Lemma 5.3 *Suppose that $d \geq 2$ and let $\psi \in C_0^\infty(\mathbb{R}^{Nd})$ with $\psi = 0$ on Γ. Then, for each pair $i, j \in \{1, \dots, N\}$, $i \neq j$ and for each $\varepsilon > 0$, there exists $\psi_\varepsilon \in C_0^\infty(\Omega_{ij})$ with* $\operatorname{supp} \psi_\varepsilon \subset \operatorname{supp} \psi$, *$\psi_\varepsilon = 0$ on Γ, and*

$$\|(-\Delta + 1)(\psi - \psi_\varepsilon)\| < \varepsilon.$$

Proof We may assume that $(i, j) = (1, 2)$ and we introduce the relative and center-of-mass coordinates

$$r := x_2 - x_1, \qquad R := \frac{1}{2}(x_1 + x_2).$$

Then,

$$\Delta = 2\Delta_r + \frac{1}{2}\Delta_R + \Delta_{x'}, \tag{36}$$

where $x' := (x_3, \dots, x_N)$. Let $\psi \in C_0^\infty(\mathbb{R}^{Nd})$ with $\psi = 0$ on Γ and let

$$\psi_n(x_1, \dots, x_N) = \psi(x_1, \dots, x_N) \cdot (1 - u_n(x_2 - x_1))$$

where u_n is given by Lemma 5.2. In the following u_n also denotes the function $(x_1, \dots, x_N) \mapsto u_n(x_2 - x_1)$. Then, $\psi - \psi_n = \psi u_n$ and hence

$$\|(-\Delta + 1)(\psi - \psi_n)\| \leq \|\psi u_n\| + \|\Delta(\psi u_n)\|.$$

Clearly $\|\psi u_n\| \to 0$ because $|\psi u_n| \leq |\psi|$ and because $\psi u_n \to 0$ pointwise as $n \to \infty$. On the other hand, using (36) and the fact that u_n depends on r only,

$$\Delta(\psi u_n) = 2\Delta_r(\psi u_n) + \frac{1}{2}(\Delta_R \psi)u_n + (\Delta_{x'}\psi)u_n$$

where the first term equals

$$2\Delta_r(\psi u_n) = 2(\Delta_r \psi)u_n + 4(\nabla_r \psi)(\nabla_r u_n) + 2\psi \Delta_r u_n.$$

By the pointwise convergence $u_n \to 0$, as explained above, $(\Delta_r \psi)u_n$, $(\Delta_R \psi)u_n$ and $(\Delta_{x'}\psi)u_n$ have vanishing L^2-norm in the limit $n \to \infty$. It remains to show that

$$\begin{aligned} \|\nabla_r \psi \cdot \nabla_r u_n\| &\to 0 \qquad (n \to \infty) \\ \|\psi \cdot \Delta_r u_n\| &\to 0 \qquad (n \to \infty). \end{aligned}$$

From Lemma 5.2, we know that

$$\|\nabla_r \psi \cdot \nabla_r u_n\|^2 \leq \int |\nabla_r \psi(r, R, x')|^2 |\nabla u_n(r)|^2 \, \mathrm{d}r \mathrm{d}R \mathrm{d}x'$$

$$\leq \sup_{r\in\mathbb{R}^d} \int |\nabla_r \psi(r, R, x')|^2 \, \mathrm{d}R\mathrm{d}x' \cdot \|\nabla u_n\|^2 \quad \to 0 \qquad (n \to \infty).$$

For $\psi \Delta u_n$, we use that $\psi(0, R, x') = 0$ and hence that

$$\psi(r, R, x') = \int_0^1 (\nabla_r \psi)(tr, R, x') \cdot r \, \mathrm{d}t.$$

It follows that

$$\begin{aligned} &\int |\psi \Delta u_n|^2 \, \mathrm{d}r\mathrm{d}R\mathrm{d}x' \\ &\leq \int \mathrm{d}r\mathrm{d}R\mathrm{d}x' \left(\int_0^1 |\nabla_r \psi(tr, R, x')|^2 \, \mathrm{d}t \right) |r|^2 |\Delta u_n(r)|^2 \\ &\leq C \int |r|^2 |\Delta u_n(r)|^2 \, \mathrm{d}r \to 0, \qquad (n \to \infty) \end{aligned}$$

by Lemma 5.2, because

$$C := \sup_{r\in\mathbb{R}^d} \int \mathrm{d}R\mathrm{d}x' \int_0^1 |\nabla_r \psi(tr, R, x')|^2 \, \mathrm{d}t < \infty.$$

□

Proof of Theorem 5.1 For given $\psi \in H^2(\mathbb{R}^{Nd}) \cap \mathscr{H}_f$ and $\varepsilon > 0$ it suffices to find $\phi_\varepsilon \in C_0^\infty(\Omega)$ with $\|\psi - \phi_\varepsilon\|_{H^2} < \varepsilon$. Let $P_f : L^2(\mathbb{R}^{Nd}) \to L^2(\mathbb{R}^{Nd})$ denote the orthogonal projection onto $\mathscr{H}_f$. Then, $\psi_\varepsilon := P_f \phi_\varepsilon$ belongs to $C_0^\infty(\Omega) \cap \mathscr{H}_f$ and

$$\|\psi - \psi_\varepsilon\|_{H^2} = \|P_f(\psi - \phi_\varepsilon)\|_{H^2} \leq \|\psi - \phi_\varepsilon\|_{H^2} < \varepsilon$$

because P_f is an orthogonal projection in H^2 (if suitably normed). To find ϕ_ε, we may assume that $\psi \in C_0^\infty(\mathbb{R}^{Nd}) \cap \mathscr{H}_f$, which is dense in $H^2(\mathbb{R}^{Nd}) \cap \mathscr{H}_f$, and we use Lemma 5.3 repeatedly. That is, we use $\{\sigma_k \mid k = 0, \ldots, n\}$ to denote the set of $n := \binom{N}{2}$ pairs (i, j), we define $\Omega_0 := \mathbb{R}^{Nd}$ and

$$\Omega_k := \cap_{j=1}^k \Omega_{\sigma_j}, \qquad k = 1 \ldots n.$$

Then, we construct smooth functions $(\gamma_k)_{k=0}^n$ recursively with $\gamma_0 := \psi$, $\mathrm{supp}(\gamma_k) \subset \mathrm{supp}(\gamma_{k-1}) \cap \Omega_k$, $\gamma_k = 0$ on Γ, and $\|(-\Delta + 1)(\gamma_k - \gamma_{k-1})\| < \varepsilon/n$. This is achieved with the help of Lemma 5.3. The function $\phi_\varepsilon := \gamma_n$ has the desired properties. □

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Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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