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Regularising transformations for the $(n,\ n+1)$-Liénard equations

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Abstract
We discuss regularising transformations for the Liénard equations of the form $y'' = F(z, y)y' + G(z, y)$, where F and G are polynomials of degrees n and $n+1$ respectively using the geometric approach. As a particular case we find a transformation for the Duffing–van der Pol equation which leads to the regularisation.

Keywords Algebraic singularities · Movable singularities · Liénard equations

Mathematics Subject Classification 34M35 · 34A34 · 34A12

1 Introduction

It is well-known that nonlinear differential equations have movable singularities, the locations of which depend on initial conditions. One can find series representations of solutions of a differential equation in the neighborhood of a singularity. However, it is in general difficult to show the convergence of such a series or to find a complete list of singular behaviours of all solutions.

In [3] movable singularities of solutions of equations of the form

$$y'' = F(z, y)y' + G(z, y) \tag{1}$$

were studied, where F and G are polynomials in y. It was shown that if $\deg_y G \leq \deg_y F + 1$, and certain resonance condition is satisfied, then any movable singularity of y that can be reached by analytic continuation along a finite length curve is algebraic. In particular, the following result holds.

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Theorem 1 [3] *Let γ be a finite length curve with z_0 as one of its endpoints and let*

$$F(z, y) = \sum_{j=0}^{n} f_j(z) y^j, \quad G(z, y) = \sum_{k=0}^{n+1} g_k(z) y^k,$$

where n is a positive integer and f_j, g_k are analytic on $\gamma \cup \{z_0\}$ and f_n is nowhere zero there. Suppose that y is a solution of Eq. (1) that is analytic on γ but cannot be analytically continued to $\gamma \cup \{z_0\}$. If, in a neihborhood of z_0, either

$$f'_{n-1} f_n - f_{n-1} f'_n + (n+1) f_{n-1} g_{n+1} - n f_n g_n = 0, \;\; n > 1, \tag{2}$$

or

$$f_0 f_1 (2g_2 - f'_1) + (2g_2 - f'_1)^2 - f_1^2 g_1 + f'_0 f_1^2 + f_1 (2g'_2 - f''_1) - f'_1 (2g_2 - f'_1) = 0, \;\; n = 1, \tag{3}$$

then y has a series expansion of the form

$$y(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^{(j-1)/n}, \tag{4}$$

where $c_0^n = -(n+1)/(n f_n(z_0))$, that converges in a neighbourhood of $z = z_0$.

Proving the existence of a recurrence relation for the coefficients of the series solution (4) is straightforward. If conditons (2) or (3) are not satisfied, then the series must include logarithmic terms.

Convergence is a more difficult problem. The idea of the proof of the theorem above was similar to the proofs in [4, 11] for equations of the form (1) with F identically equal to zero, and relied on finding an appropriate auxiliary (bounded) function and constructing with its help a corresponding regular system of first order differential equations and an initial value problem for which, after the change of dependent and independent variables, the Cauchy theorem can be applied. In this paper we revisit this result using the geometric approach (which we can apply by fixing n). We show that by taking $n = 1, 2$ and 3 and assuming that the degrees of polynomials F and G are strictly n and $n + 1$, we can easily reproduce conditions (2) and (3) and find the regularising transformation to show the convergence of (4). Although this method is computational with n fixed, we do not require any auxiliary function to find the regular system. Moreover, there are two ways to regularise the system, and one method was inspired by the integration of particular cases of the Duffing–van der Pol equation.

The Duffing–van der Pol oscillator is given by [2, p. 21]

$$y'' + (ay^2 + b)y' - cy + dy^2 + \beta y^3 = 0. \tag{5}$$

This is a particular case of the Liénard equation (1) with $n = 2$ and $g_0 = f_1 = 0$, $g_1 = c$, $g_2 = -d$, $g_3 = -\beta$, $f_0 = -b$, $f_2 = -a$. It admits two families $y \sim y_0(z - z_0)^{-1/2}$, $y_0^2 = 3/(2a)$ without logarithmic terms provided $d = 0$ and $a \neq 0$

(which also follows from (2)). In the following we assume that $d = 0$. The Duffing–van der Pol oscillator has the first integral in one particular case [2, Sect. 3.2.6], namely, $3ab\beta+a^2c-9\beta^2 = 0$ with $K = (3ay'+(3ab-9\beta)y+a^2u^3)e^{3\beta y/a}$. Equation $K = 0$ can be mapped to an ODE with the Painlevé property via the so-called hodograph transformation

$$(y,\ z) \to (Y,\ Z): \quad dz = y^{-1}dZ, \quad y = Y. \tag{6}$$

Using this transformation the Abel equation $K = 0$ is mapped to the Riccati equation $dY/dZ + aY^2/3 + b - 3\beta/a = 0$. The case $K =$ const is similar (see [2] for details).

The geometric approach for the Painlevé equations was developed in the papers [9, 10] (see also [7]). The method of blowing up points of indeterminacy of certain systems of two ordinary differential equations was also applied to obtain information about the singularity structure of solutions of the corresponding non-linear differential equations in [5, 8]. Originally, the notion of a blow-up comes from algebraic geometry. Algebraically, the blow-up is a certain bi-rational transformation, namely, the blow-up at a point $(p, q) = (a, b)$ is defined by the following construction. One introduces new coordinate charts, $p = a+u = a+UV$ and $q = b+uv = b+V$. The exceptional line corresponds to $u = 0$ or $V = 0$. For a system of differential equations for $p = p(z)$, $q = q(z)$ the points of indeterminacy of the vector field a and b may depend on z. To take into account the infinite values of p and q one should study the system for instance on $\mathbb{P}^1 \times \mathbb{P}^1$. Performing the blow-up at the point (a, b) one re-writes the system in new coordinates (u, v) and (U, V) and examines whether new indeterminacy points arise on the exceptional lines. A sequence of blow-ups (or the cascade) may be very long, even possibly infinite. When the cascade is finite, we examine the resulting systems. If they are not regular on the last exceptional line, one may find a way to regularise them (for instance, by interchanging the dependent and independent variables as in [8] or examining the n-th order regularisability in case of algebroid solutions [6]) and extract information about the nature of singularities of the original system (e.g., solutions have movable algebraic branch points either locally or globally). Keeping track of the changes of variables introduced by the blow-ups in a given finite cascade one then finds a bi-rational regularising transformation for the system under consideration. For the Painlevé equations the process of repeatedly blowing up base points is finite and leads to the construction of regularising transformations [9]. For certain classes of equations with the quasi-Painlevé property this process is also finite but the regularisation involves the interchanging of dependent and independent variables [8]. A natural question is how to extend such results to other classes of equations.

In this paper we examine Eq. (1). We re-write this equation in the form of the system suitable for the geometric method to apply and find regularising transformations for $n = 1, 2, 3$. We present the formulas only for $n = 1$ and $n = 2$ since $n = 3$ case is analogous but very cubersome. We show that for the system equivalent to (1) there is an infinite sequence of blow-ups and another one that terminates, which further gives a Laurent expansion of the solution around a movable pole. We also show how the resonance conditions (2), (3) arise. We discuss two ways to regularise the system, one by interchanging the dependent and independent variables and the other one using the hodograph transformation. We conjecture that the last transformation can also be used in a wider classes of equations, for instance in equatons with solutions possessing movable algebraic singularities as in [8].

2 Main results

First, let us consider the case $n = 1$. Take $y' = p$, $y = q$ in the Liénard Eq. (1). This yields the system

$$q' = p, \quad p' = (f_0 + f_1 q)p + g_0 + g_1 q + g_2 q^2 \tag{7}$$

with coefficients being functions of z. We further assume that f_1, g_2 are not identically zero as otherwise the geometry of the system may change. Considering the system over $\mathbb{P}^1 \times \mathbb{P}^1$ (that is, introducing charts (q, P), (Q, p) and (Q, P) with $Q = 1/q$, $P = 1/p$ in addition to the original chart (q, p)), we find the first indeterminacy point $p_1 = (Q = 0,\ P = 0)$. Next we find $p_2 = (u_1 = 0,\ v_1 = 0)$ and $p_5 = (u_1 = 0,\ v_1 = -f_1/g_2)$. The cascade from p_5 leads to more and more complicated expressions for points of indeterminacy of the corresponding systems and it does not stop after a reasonable number of steps. In general, as discussed in [5], the infinite cascades possibly indicate the presence of more complicated singularities or some special solutions. Blowing up p_2 we find the point $p_3 = (u_2 = 0,\ v_2 = 2/f_1)$ on the exceptional line. Finally, we find $p_4 = (u_3 = 0,\ v_3 = -4(f_0 f_1 + 2g_2 - f_1')/f_1^3)$. Blowing up this point we see that the first equation regularises at $u_4 = 0$ and the second equation also regularises provided condition (3) is satisfied. We have

$$u_4' = R_1(z, u_4, v_4), \quad v_4' = R_2(z, u_4, v_4)$$

with R_1 and R_2 rational in u_4 and v_4 and $R_1(z, 0, v_4)$ and $R_2(z, 0, v_4)$ being nonzero. Further points of indeterminacy do not arise. Thus, the first result is the following statement.

Theorem 2 *The bi-rational transformation*

$$u_4 = \frac{1}{q}, \quad v_4 = \frac{q^4}{p} + \frac{q}{f_1^3}(4f_0 f_2 - 2f_1^2 q + 8g_2 - 4f_1')$$

regularises (7) *provided* (3) *is satisfied. The inverse transformation is given by*

$$q = \frac{1}{u_4}, \quad p = \frac{f_1^3}{u_4^2(2f_1^2 - 4f_0 f_1 u_4 + f_1^3 u_4^2 v_4 + 4(f_1' - 2g_2)u_4)}.$$

The polar expansions (4) with $n = 1$ correspond to the usual Taylor series for u_4 and v_4 with $u_4(z_0) = 0$. Since the coordinates of the blow-up in (u, v) chart are related to the coordinates in the (U, V) chart, we have a similar statement for the system in the other chart (U_4, V_4). This agrees with the results given in [3].

We repeat the procedure for $n = 2$ assuming $f_2 g_3 \neq 0$. We start from the system

$$q' = p, \quad p' = \sum_{j=0}^{2} f_j q^j p + \sum_{j=0}^{3} g_j q^j. \tag{8}$$

In this case the finite cascade is longer. We find $p_1 = (Q = 0,\ P = 0)$, $p_2 = (u_1 = 0,\ v_1 = 0)$, $p_3 = (u_2 = 0,\ v_2 = 0)$, $p_4 = (u_3 = 0,\ v_3 = 3/f_2)$, $p_5 = (u_4 = 0,\ v_4 = -9f_1/(2f_2^2))$, $p_6 = (u_5 = 0,\ v_5 = 9(3f_1^2 + 4(f_2' - 3g_3 - f_0 f_2))/(4f_2^3))$. After blowing up p_6 the system after imposing condition (2) becomes

$$u_6' = \frac{1}{u_6} R_1(z, u_6, v_6), \ \ v_6' = \frac{1}{u_6} R_2(z, u_6, v_6) \tag{9}$$

with R_1 and R_2 rational in u_6 and v_6 and $R_1(z, 0, v_6)$ and $R_2(z, 0, v_6)$ being nonzero. Note that these functions R_1 and R_2 are clearly different from the case $n = 1$ above. The second cascade from the point $p_7 = (u_1 = 0,\ v_1 = -f_2/g_3)$ is infinite.

System (9) can then be regularised in two ways. The first one, as in [3] (see also [4, 11]), is by interchanging the dependent and independent variables leading to

$$\frac{dz}{du_6} = u_6 \tilde{R}_1(z, u_6, v_6), \ \ \frac{dv_6}{du_6} = \tilde{R}_2(z, u_6, v_6), \tag{10}$$

which is regular with $z(0) = z_0$. Expansion $y = \sum_{j=-1}^{\infty} a_j (z - z_0)^{j/2}$ with $a_{-1}^2 = -3/(2f_2(z_0))$, $a_0 = -f_1(z_0)/(2f_2(z_0))$ and a_2 arbitrary, which we write as $y \sim a_{-1}(z - z_0)^{-1/2}$ giving only the leading term in the Puiseux series, corresponds to the Puiseux expansions $u_6 \sim (z - z_0)^{1/2}/a_{-1}$ and $v_6(z_0) = v_6^0$. Thus, inverting the series we have a solution to the initial value problem (10) with $u_6 = 0$, $z(0) = z_0$ and $v_6(0) = v_6^0$.

The second way to regularise system (9), which is inspired by the integrable case of the Duffing–van der Pol equation, is the following. Take a new independent variable Z defined by $dZ = u_6^{-1} dz$ and take $u_6(z) = U(Z),\ v_6(z) = V(Z)$. Then system (9) becomes

$$\frac{dU}{dZ} = \hat{R}_1(Z, U, V), \ \ \frac{dV}{dZ} = \hat{R}_2(Z, U, V), \tag{11}$$

which is regular and to which the Cauchy theorem can be applied. One can calculate that $Z - Z_0 \sim 2a_{-1}(z - z_0)^{1/2}$, hence, we get the Taylor series $z - z_0 \sim (Z - Z_0)^2/(4a_{-1}^2)$ so we can replace the coefficients in the system depending on z and obtain analytic coefficients in Z. Recalculating $U(Z)$ and $V(Z)$ we have the usual Taylor series with $U(Z_0) = 0$, $U(Z) \sim (Z - Z_0)/(2a_{-1}^2)$ and $V(z_0) = V_0$. This shows how the algebraic expansions (4) arise from the regular systems.

Therefore, we have the following result.

Theorem 3 *The bi-rational transformation*

$$u_6 = \frac{1}{q}, \quad v_6 = \frac{q^6}{p} - \frac{3q}{4f_2^3}(9f_1^2 - 6f_1 f_2 q + 4(3f_2' - 9g_3 + f_2^2 q^2 - 3f_0 f_2))$$

leads to the regularisation of (8) *in the sense of* (9) *provided* (2) *with* $n = 2$ *is satisfied. The inverse transformation is given by*

$$q = \frac{1}{u_6},$$
$$p = \frac{4f_2^3}{u_6^3(12f_2^2 - 18f_1f_2u_6 + 27f_1^2u_6^2 - 36f_0f_2u_6^2 - 108g_3u_6^2 + 4f_2^3u_6^3v_6 + 36f_2'u_6^2)}.$$

In particular, for the Duffing–van der Pol oscillator with $d = 0$ *we have the bi-rational transformation*

$$u_6 = \frac{1}{q}, \quad v_6 = \frac{q^6}{p} + \frac{3q}{a^3}(a^2q^2 - 3ab + 9\beta).$$

which leads to the regularisation in the sense of (9). *The inverse transformation is*

$$q = \frac{1}{u_6}, \quad p = \frac{a^3}{u_6^3(9abu_6^2 - 27\beta u_6^2 - 3a^2 + a^3u_6^3v_6)}.$$

For $n = 3$ case the calculations are similar, but expressions are more cumbersome, so we omit them. The only differences with the case $n = 2$ are that the finite cascade is longer and instead of the first power of u_6 in the denominators of (9) we have the second power of u_8 and so we need to modify all other arguments accordingly. This leads us to conjecture that similar behaviour can be observed for general n in the $(n, n+1)$-Liénard equations and the transformation leading to the system

$$u' = \frac{1}{u^{n-1}}R_1(z, u, v), \quad v' = \frac{1}{u^{n-1}}R_2(z, u, v)$$

that can be regularised in two ways, by interchanging of dependent and independent variables or by an appropriate hodograph transformation, will be obtained by the geometric method.

3 Discussion and open problems

As already mentioned in [5], Liénard type equations may possess cascades which do not finish after a reasonable number of steps. In some cases expressions of consequitive points become very cumbersome and the complexity of symbolic computations rise singificantly. In other cases the form of the equations does not change much and similar points appear over and over again. One such example is the Smith equation $y'' = 4y^3y' + y$ discussed in [3, 5]. Another example is the maximum balance equation $y'' = \mu y^n y' + \nu y^{2n+1}$ from [3] with $n = 2$: $y'' = \mu y^2 y' + \nu y^5$. It has one splitting cascade after the third point and both branches are long. In addition, the form of equations in the systems does not change much after consequitive blowups with the same points with zero coordinates appearing after each step which leads us to believe that the cascades are infinite. Therefore, in these and all cases discussed in this paper we can speak about partial regularisability (in comparison with [8] where all cascades of points of indeterminacy are finite). It is an interesting problem to understand this phenomenon better. Perhaps some combination of changes of variables including the hodograph transformation is needed in such cases.

Another open problem is how to recover auxiliary functions bounded as $1/y \to 0$ needed in the proof of the quasi-Painlevé property (see [4, 11]), which are almost first integrals plus certain correction terms, using the geometric approach. In addition, it remains open whether there exists a regularising transformation leading to a system with polynomial right-hand sides, not rational, as in the case of the Painlevé equations after some modifications in the blow-up procedure. This is currently work in progress.

In this paper we have studied the case with generic coefficients of the Liénard equation. However, there might be some additional relations between the coefficients. For instance, in the case of the Duffing–van der Pol equation with $a = b = d = 0$, integrable using elliptic functions, all cascased become finite and the final systems become regular. The full regularization also happens if $a = d = 0$ and $c = -2b^2/9$. Note that if one differentiates the Abel equation $y' = \sum_{j=0}^{3} f_i(z)y^i$, then one finds a particular Liénard equation (1) with $n = 2$ and the resonance condition (2) will be automatically satisfied. Special relations between the coefficients might lead to the changes in the points of indeterminacy (for instance, new points might appear in comparison with the generic case). Therefore, one more interesting question is whether the geometric approach can help to distinguish some integrable cases of such Liénard equations (see, for instance, [1] and the references therein).

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Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest There is no conflict of interest.

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