Letters in Mathematical Physics (2022) 112:126
https://doi.org/10.1007/s11005-022-01602-6

On the Bott index of unitary matrices on a finite torus

Daniele Toniolo[1,2]

Received: 9 March 2022 / Revised: 27 August 2022 / Accepted: 8 October 2022 /
Published online: 8 December 2022

Abstract
This article reviews the foundations of the theory of the Bott index of a pair of unitary matrices in the context of condensed matter theory, as developed by Hastings and Loring (J. Math. Phys. **51**, 015214 (2010), Ann. Phys. **326**, 1699 (2011)), providing a novel proof of the equality with the Chern number. The Bott index is defined for a pair of unitary matrices, then extended to a pair of invertible matrices and homotopic invariance of the index is proven. An insulator defined on a lattice on a two-torus, that is a rectangular lattice with periodic boundary conditions, is considered and a pair of quasi-unitary matrices associated to this physical system are introduced. It is shown that their Bott index is well defined and the connection with the transverse conductance, the Chern number, is established proving the equality of the two quantities, in certain units.

Keywords Bott index · Chern insulator · Homotopy invariance

Mathematics Subject Classification 47B93 · 15A60 · 46N55 · 82C10

1 Introduction

The integer quantization of the transverse (Hall) conductance (IQHE) of a two-dimensional electron gas under an external perpendicular magnetic field has been experimentally discovered in 1980 [1], the fractional quantization (FQHE) a couple of years later [2]. The theoretical analysis of these phenomena has never stopped since. Initial landmarks have been established by Laughlin [3], Halperin [4] and Thouless et al. [5]. Different schools of thought originated to explain these phenomena: who focused on the two-dimensional bulk aspects of the sample [6]; who stressed the rele-

✉ Daniele Toniolo
d.toniolo@ucl.ac.uk ; danielet@alumni.ntnu.no

1 Department of Computer Science, University College London, London, UK

2 Department of Physics and Astronomy, University College London, London, UK

vance of the one-dimensional edge [7] and who analyzed the interplay between bulk and edge-physics [8]. The attention to a realistic geometrical setting is particularly relevant in the approach of Buttiker [9], while a rigorous treatment of the strong disorder needed for the quantization of the conductance is central in the work of Bellissard, summarized in [10]. The initial sections of [10] can be used as an introduction to the IQHE. Another line of research on the mathematical physics aspects of IQHE is due to Avron, Seiler and Simon [11, 12]. Haldane in 1988 formulated a lattice model with localized magnetic fluxes over the corners of a honeycomb lattice but with total magnetic fluxes per plaquette equal to zero. This model manifests a quantized transverse conductance [13] and nowadays is called Chern insulator. The Haldane model has been relevant for the theoretical formulation [14, 15] and experimental discovery [16, 17] of the topological insulators. Two relatively recent rigorous works, among others, on the nature of the invariants describing topological insulators in two and higher dimensions are [18, 19]. A rigorous discussion of the topology of one-dimensional systems with open boundary conditions has been very recently provided by the authors of [20].

The quantization of the Hall conductance on a torus geometry, meaning that periodic boundary conditions are imposed on a two- dimensional rectangular sample, is determined by a topological invariant called Chern number. This has been showed for the first time in the work of Thouless et al. [5], the emergence of the Chern number has then been made explicit by Kohmoto in [21]. Also early contributions have been made in [22] and [23].

Hastings and Loring in a set of articles [24–26] used mathematical tools including non-commutative topology, C*-algebras and K-theory to rigorously search for the topological invariants of the ten Altland and Zirnbauer symmetry classes [27] in a way that is also relevant for numerical computations. The program of classification of topological invariants of Fermi systems according to their symmetries and dimensionality started with the works of Qi et al. [28], Kitaev [29] and Ryu *et al.* [30].

One of the motivations for this article is to review the foundations of the theory of the Bott index of a pair of unitary matrices in the context of condensed matter theory, as developed by Hastings and Loring [25, 26], and in particular showing the equivalence with the Chern number, with a novel proof of the equality of the two indices, providing throughout new proofs that make this work self-contained.

In the physics literature the Bott index, following [25, 26], has been employed to characterize several topological phenomena that goes from time-reversal invariant systems [31], time-dependent systems [32–34], quasi periodic systems [35–38] and ferromagnetic systems [39]. The former list is not exhaustive.

The structure of this article is as follows: In Sect. 2 the physical setting is presented: a lattice on a two-torus, that is a finite rectangular lattice with periodic boundary conditions, is considered and an insulator is defined on it. This is modeled by the Fermi projection that fills the eigenstates of a short-ranged, bounded, gapped, single-particle Hamiltonian below a spectral gap. The Hilbert space where the Hamiltonian is defined is finite dimensional. The most important results of this section are the bounds (3) and (4). Section 3 defines and discusses the Bott index of a pair of unitary matrices according to [40–43], with the important generalization in Sect. 3.1 to a pair of invertible matrices and the proof of homotopy invariance in Sect. 3.2. Section 4 gives a sufficient condition for the vanishing of the Bott index. Section 5 inspired by the approach of [26] in 5.1 proves that the Bott index approximates the Chern number. A novel proof of the equality of the two indices is given in Sect. 5.2 through a mapping to a differential equation following an analogous proof for the infinite two-dimensional case recently presented in [44]. Three perspectives for future developments are given in Sect. 6.

I anticipate here the main new result of this work that is an equality among the Bott index of a pair of unitary matrices related to a Hamiltonian describing an insulator, on a two-torus, as specified in definition 1, and the Chern number of its Fermi projection. The proof of the theorem is in Sect. 5.2. Previously this relation has been established only approximately making use of a pair of quasi-unitary matrices, this approach is described in Sect. 5.1.

Theorem 11. Given a Hamiltonian H as in definition 1 and its Fermi projection P, the unitary matrices

$$e^{2\pi i P \frac{X}{L} P} \quad , \quad e^{2\pi i P \frac{Y}{L} P}$$

have a well-defined Bott index that satisfies:

$$\begin{aligned} \mathrm{Bott}\left(e^{2\pi i P \frac{X}{L} P}, e^{2\pi i P \frac{Y}{L} P}\right) &= 2\pi i \mathrm{Tr}\left[P\frac{X}{L}P, P\frac{Y}{L}P\right] \\ &= \frac{4\pi}{L^2}\mathrm{ImTr}\left(P[iX, P][iY, P]\right) = \mathrm{Ch}(P) \end{aligned}$$

2 Physical setting

The physical system under investigation is an insulator comprised of free fermions on a lattice on a two-torus (that is a rectangular lattice with periodic boundary conditions) filling up the energy levels of a single-particle Hamiltonian that is short-ranged, bounded and gapped. The system's fermions have in general N internal degrees of freedom. The system admits weak disorder meaning that the Hamiltonian maintains a spectral gap, the disorder is also supposed to be compatible with the periodic boundary conditions. The presence of disorder makes the concept of Brillouin zone ill defined, and therefore, I do not refer to it. The Hamiltonian has no extra symmetries. The first application of the Bott index in condensed matter theory has been provided in [24].

This section shows that from the properties of the Hamiltonian, short-range, bounded and gapped, two important estimates on norms of commutators follow that in turn allow the introduction of two suitable unitary matrices whose Bott index will be evaluated in Sect. 5.

Definition 1 The single-particle Hamiltonian $H : l^2(\Lambda) \otimes \mathbb{C}^N \to l^2(\Lambda) \otimes \mathbb{C}^N$, with Λ denoting a lattice on a two-torus, that is a lattice on a rectangle of sides L_x and L_y with periodic boundary conditions,

$$H = \sum_{l,k=1}^{N} \sum_{n,m \in \Lambda} H_{n,m,l,k} |n, l\rangle\langle m, k| \tag{1}$$

is:

- short-ranged with range R, meaning that: $H_{n,m,l,k} = 0$ when $\mathrm{dist}(n, m) > R$, with $R \ll L_x$ and $R \ll L_y$.
- bounded: $\|H\|$ is upper bounded by a finite constant independent from the system's size.
- gapped: there exists an energy gap ΔE in the spectrum of H with lower bound unaltered increasing the size of Λ.
- L_x, L_y, R, $\|H\|$ and ΔE are such that: $\frac{R\|H\|}{L_x \Delta E} \ll 1$ and $\frac{R\|H\|}{L_y \Delta E} \ll 1$.

The distance on the lattice is compatible with periodic boundary conditions. To exemplify let us consider a square lattice, with lattice distance 1 over the rectangle of sides L_x and L_y with periodic boundary conditions, then given $n = (n_x, n_y)$ and $m = (m_x, m_y)$ with $\{n_x, m_x\} \in \{0, ..., L_x - 1\}$ and $\{n_y, m_y\} \in \{0, ..., L_y - 1\}$, it is $\mathrm{dist}(n, m) = \min_{k \in \mathbb{Z}} |n_x - m_x + kL_x| + \min_{l \in \mathbb{Z}} |n_y - m_y + lL_y|$.

For the sake of simplicity from now on it is assumed $N = 1$.

We "build up" an insulator out of the single-particle Hamiltonian (1) filling from the bottom the single-particle energy levels till an energy gap of size ΔE is reached.

Definition 2 Given the Hamiltonian H as defined in 1 and the chemical potential μ fixed within the spectral gap ΔE, the orthogonal projection $P := \chi(H \leq \mu)$ is called the Fermi projection.

We now introduce well-defined position operators on the torus. To construct the torus we glue together the opposite sides of a rectangle of linear sizes L_x and L_y. We assign an ordering to the points of the rectangular lattice, such that the ith point has coordinates (x_i, y_i). We then construct the diagonal matrix X, with elements $X_{i,j} = x_i \delta_{i,j}$, and the corresponding matrix Y, $Y_{i,j} = y_i \delta_{i,j}$. The matrices X and Y have $L_x L_y$ diagonal elements. Note that points that are physically close on the lattice may have corresponding entries in the matrix X distant from each other, but at most L_x elements far away. We then define the diagonal unitary matrices that are well defined with respect to periodic boundary conditions, namely $X \to X + L_x$ and $Y \to Y + L_y$:

$$\exp\left(i \frac{2\pi}{L_x} X_{i,i}\right), \ \exp\left(i \frac{2\pi}{L_y} Y_{i,i}\right) \tag{2}$$

From now on it is set $L := L_x = L_y$.

Lemma 3

$$\|[e^{2\pi i \frac{X}{L}}, H]\| \leq \mathcal{O}\left(\frac{R}{L}\|H\|\right) \tag{3}$$

and

$$\|[e^{2\pi i \frac{X}{L}}, P]\| \leq \mathcal{O}\left(\frac{R}{L}\frac{\|H\|}{\Delta E}\right) \tag{4}$$

Proof We employ the Holmgren bound for the norm of a bounded operator A, that is:

$$\|A\| \leq \max\left(\sup_{m\in\Lambda}\sum_{n\in\Lambda}|\langle m|A|n\rangle|, m \leftrightarrow n\right) \tag{5}$$

$m \leftrightarrow n$ denotes the exchange of the indices m and n in the sup and the $\sum$. A proof of this bound can be found in chapter 16 of [45], for convenience a proof is also presented in "Appendix A."

$$\|[e^{2\pi i\frac{X}{L}}, H]\| \leq \max\left(\sup_{m\in\Lambda}\sum_{n\in\Lambda}|\langle m|[e^{2\pi i\frac{X}{L}}, H]n\rangle|, m \leftrightarrow n\right) \tag{6}$$

We notice that

$$\langle m|[e^{2\pi i\frac{X}{L}}, H]|n\rangle = \langle m|(e^{2\pi i\frac{X}{L}}H - He^{2\pi i\frac{X}{L}})|n\rangle = (e^{2\pi i\frac{m_x}{L}} - e^{2\pi i\frac{n_x}{L}})\langle m|H|n\rangle \tag{7}$$

Therefore

$$\|[e^{2\pi i\frac{X}{L}}, H]\| \leq \max\left(\sup_{m\in\Lambda}\sum_{\mathrm{dist}(n,m)\leq R}|e^{2\pi i\frac{m_x}{L}} - e^{2\pi i\frac{n_x}{L}}||\langle m|H|n\rangle|, m \leftrightarrow n\right) \tag{8}$$

In Eq. (8) we took into account that given a fixed point $m \in \Lambda$ only the points of the lattice within the range R contribute to $\langle m|H|n\rangle$. We see that

$$\begin{aligned}&|e^{2\pi i\frac{m_x}{L}} - e^{2\pi i\frac{n_x}{L}}|\\ &\quad = |e^{2\pi i\frac{m_x}{L}}\left(1 - e^{2\pi i\frac{n_x - m_x}{L}}\right)| \leq \frac{2\pi}{L}\min\{|n_x - m_x|, L - |n_x - m_x|\}\end{aligned} \tag{9}$$

Let us illustrate the bound (9) with $m_x = 1$, $n_x = L - 1$. The points $(1, y)$ and $(L - 1, y)$ are on the opposite sides of the square lattice but they are close by on the torus because of the periodic boundary conditions and therefore within the range R of the Hamiltonian.

$$\begin{aligned}|e^{2\pi i\frac{m_x}{L}} - e^{2\pi i\frac{n_x}{L}}| &= |e^{2\pi i\frac{1}{L}} - e^{2\pi i\frac{L-1}{L}}| = |e^{2\pi i\frac{1}{L}} - e^{-2\pi i\frac{1}{L}}|\\ &= |1 - e^{-2\pi i\frac{2}{L}}| \leq \frac{4\pi}{L}\end{aligned} \tag{10}$$

It follows that

$$\|[e^{2\pi i\frac{X}{L}}, H]\| \leq \mathcal{O}\left(\frac{R}{L}\|H\|\right)$$

To obtain the bound (4) we start considering, with $z \in \rho(H)$, and A any matrix, the equality:

$$0 = [A, \mathbb{1}] = [A, (H - z)(H - z)^{-1}] = [A, (H - z)](H - z)^{-1} \\ + (H - z)[A, (H - z)^{-1}] \tag{11}$$

$$[A, (H - z)^{-1}] = (H - z)^{-1}[(H - z), A](H - z)^{-1} \\ = (H - z)^{-1}[H, A](H - z)^{-1} \tag{12}$$

The projection P on the occupied energy levels, with the loop Γ in the complex plane enclosing them, can be written as

$$P = \frac{1}{2\pi i} \oint_{\Gamma} \mathrm{d}z (z - H)^{-1} \tag{13}$$

then:

$$[e^{2\pi i \frac{X}{L}}, P] = \frac{1}{2\pi i} \oint_{\Gamma} \mathrm{d}z \left[e^{2\pi i \frac{X}{L}}, (z - H)^{-1}\right] \tag{14}$$

$$= \frac{1}{2\pi i} \oint_{\Gamma} \mathrm{d}z (H - z)^{-1}[H, e^{2\pi i \frac{X}{L}}](H - z)^{-1} \tag{15}$$

$$\|[e^{2\pi i \frac{X}{L}}, P]\| \leq \frac{1}{2\pi} \|[H, e^{2\pi i \frac{X}{L}}]\| \oint_{\Gamma} \|(H - z)^{-1}\|^2 |\mathrm{d}z| \tag{16}$$

$\|(H - z)^{-1}\| = \mathrm{dist}(z, \sigma(H))^{-1}$

Let us consider the positively oriented loop Γ in Fig. 1. Along the edge of the loop aligned with the imaginary axis of the complex-plane, assuming for simplicity that the energy gap is located around zero, as in Fig. 1, we have that $\|(H - z)^{-1}\|^2 = \mathrm{dist}(z, \sigma(H))^{-2} = 1/\left[(\frac{\Delta E}{2})^2 + (\mathrm{Im} z)^2\right]$. Sending $R \to \infty$ the only contribution to

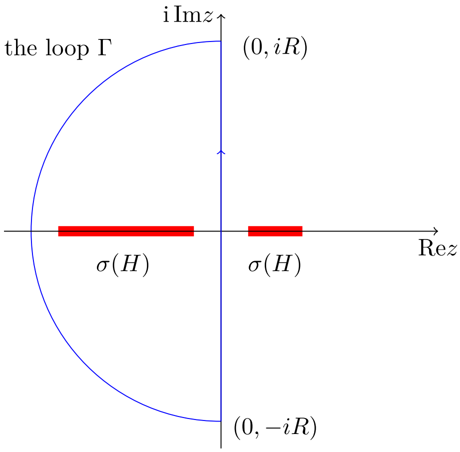

Fig. 1 The red stripes enclose the spectrum of H

the loop-integral comes from the edge along the imaginary axis, then:

$$\oint_\Gamma \|(H-z)^{-1}\|^2|\mathrm{d}z| = \int_{-\infty}^{\infty} \frac{1}{\left(\frac{\Delta E}{2}\right)^2 + (\mathrm{Im}z)^2}\mathrm{d}(\mathrm{Im}z) = \frac{2\pi}{\Delta E} \tag{17}$$

Combining (3) and (16), this implies:

$$\|[e^{2\pi i \frac{X}{L}}, P]\| \leq \mathcal{O}\left(\frac{R\|H\|}{L\Delta E}\right)$$

□

3 Bott index

The Bott index arose as an index to distinguish pairs of unitary matrices that can be approximated by commuting unitary matrices from those that cannot. It was established both as a winding number and a K-theoretic invariant in the early works of Exel and Loring [40, 41, 46]. For a discussion of these aspects of the Bott index see [42] and references therein.

The Bott index has been employed in the context of condensed matter physics by Hastings and Loring in a set of papers [24–26].

In the following the logarithm of a matrix is defined according to the holomorphic (Dunford) functional calculus, for a discussion see, for example, [47]. Denoting ρ an invertible matrix and Γ a contour enclosing its spectrum but not the origin of the complex plane it is:

$$\log\rho := \frac{1}{2\pi i}\oint_\Gamma \log z(z\mathbb{1}-\rho)^{-1}\mathrm{d}z \tag{18}$$

If the spectrum of ρ does not contain real negative values then the contour Γ is chosen to not intersect the real negative axis of the complex plane and $\log z$ is the principal logarithm of z.

Definition 4 Given two unitary matrices U and V, such that $\{-1\} \notin \sigma\left(UVU^{-1}V^{-1}\right)$, or equivalently such that $\|[U, V]\| < 2$, their Bott index is defined as:

$$\mathrm{Bott}(U, V) := \frac{1}{2\pi i}\mathrm{Tr}\log\left(UVU^{-1}V^{-1}\right) \tag{19}$$

Remark the equivalence of $\{-1\} \notin \sigma\left(UVU^{-1}V^{-1}\right)$ and $\|[U, V]\| < 2$ follows from:

$$\|UVU^{-1}V^{-1} - \mathbb{1}\| = \|(UV - VU)U^{-1}V^{-1}\| = \|[U, V]\| \tag{20}$$

Lemma 5 *The Bott index of two unitary matrices, as in* Definition 4*, is an integer.*

Proof We denote $\{e^{i\theta_j}\}$, with $\theta_j \in (-\pi, \pi)$, the elements of the spectrum of the unitary matrix $UVU^{-1}V^{-1}$. From $\det\left(UVU^{-1}V^{-1}\right) = 1$, it follows that: $1 = \prod_j e^{i\theta_j} = e^{i\sum_j \theta_j}$. This implies $\mathrm{Bott}(U, V) = \frac{1}{2\pi}\sum_j \theta_j \in \mathbb{Z}$. □

It is immediate to see that when U and V are commuting their index is vanishing.

3.1 The Bott index of two invertible matrices

The Bott index of two invertible matrices, S and T, can be defined similarly as done for unitary matrices. To ensure that $\log\left(STS^{-1}T^{-1}\right)$ is well defined, having chosen the branch cut of the logarithm on the real negative axis, we need that $\sigma(STS^{-1}T^{-1})$ does not contain any real negative value: $\sigma(STS^{-1}T^{-1}) \bigcap \mathbb{R}^- = \emptyset$. Denoting with $\lambda_j = |\lambda_j|e^{i\theta_j}$, $\theta_j \in (-\pi, \pi)$, the set of eigenvalues of $STS^{-1}T^{-1}$ we get:

$$1 = \det\left(STS^{-1}T^{-1}\right) = \prod_j \lambda_j = \prod_j |\lambda_j|e^{i\theta_j} = \prod_j e^{i\theta_j} \Rightarrow \frac{1}{2\pi}\sum_j \theta_j \in \mathbb{Z} \tag{21}$$

In Eq. (21) it has been used: $1 = \prod_j \lambda_j = |\prod_j \lambda_j| = \prod_j |\lambda_j|$, implying $0 = \sum_j \log|\lambda_j|$, then

$$\begin{aligned}\mathrm{Bott}(S, T) &:= \frac{1}{2\pi i}\mathrm{Tr}\log\left(STS^{-1}T^{-1}\right) = \frac{1}{2\pi i}\sum_j \log\lambda_j \\ &= \frac{1}{2\pi i}\sum_j\left(\log|\lambda_j| + i\theta_j\right) = \frac{1}{2\pi}\sum_j \theta_j \in \mathbb{Z}\end{aligned} \tag{22}$$

Remark The unitary matrices are invertible matrices; the reason why two separate definitions are given for their Bott index is that the condition $\sigma(STS^{-1}T^{-1}) \bigcap \mathbb{R}^- = \emptyset$ reduces in the unitary case to $\{-1\} \notin \sigma\left(UVU^{-1}V^{-1}\right)$ and in turn to $\|[U, V]\| < 2$ that might be easier to check both analytically and numerically.

3.2 Homotopy invariance of the Bott index of two invertible matrices

Homotopy invariance of the Bott index of two unitary matrices has been previously shown by Exel and Loring [40, 41, 46], in their approach this follows from casting the Bott index as a winding number or a K-theoretic invariant. Here I take a direct approach looking at the derivative of the index.

Lemma 6 *Given two maps* $U(s) : [0, 1] \to GL(N, \mathbb{C})$ *and* $V(s) : [0, 1] \to GL(N, \mathbb{C})$*, continuous with respect to the operator norm, such that* $\sigma(U(s)V(s)U(s)^{-1}V(s)^{-1}) \bigcap \mathbb{R}^- = \emptyset$, $\forall s \in [0, 1]$ *it is:*

$$Bott(U(s), V(s)) = Bott(U(0), V(0)) \tag{23}$$

Proof A continuous path of invertible matrices can be approximated in norm by a differentiable path, [48] proposition 1.7.2. Let us consider the partial derivatives $\partial_s U(s)$ and $\partial_s V(s)$ of such a differentiable path.

$$\begin{aligned}
&\partial_s \mathrm{Tr}\log(U(s)V(s)U^{-1}(s)V^{-1}(s)) \\
&\quad = \mathrm{Tr}\left[\partial_s(U(s)V(s)U^{-1}(s)V^{-1}(s))(V(s)U(s)V^{-1}(s)U^{-1}(s))\right] \\
&\quad = \mathrm{Tr}[(\partial_s U(s))U^{-1}(s) + U(s)(\partial_s V(s))V^{-1}(s)U^{-1}(s) \\
&\qquad + U(s)V(s)(\partial_s U^{-1}(s))U(s)V^{-1}(s)U^{-1}(s) \\
&\qquad + U(s)V(s)U^{-1}(s)(\partial_s V^{-1}(s))V(s)U(s)V^{-1}(s)U^{-1}(s)] \qquad (24) \\
&\quad = \mathrm{Tr}[(\partial_s U(s))U^{-1}(s)] + \mathrm{Tr}[(\partial_s V(s))V^{-1}(s)] \\
&\qquad + \mathrm{Tr}[(\partial_s U^{-1}(s))U(s)] + \mathrm{Tr}[(\partial_s V^{-1}(s))V(s)] \qquad (25)
\end{aligned}$$

Since $(\partial_s U^{-1}(s))U(s) = -U^{-1}(s)\partial_s U(s)$, we obtain: $\partial_s \mathrm{Bott}(U(s), V(s)) = 0$. □

4 A sufficient condition for the vanishing of the Bott index of a pair of unitary matrices

Lemma 7 *Given U and V a pair of unitary matrices such that $\|[U,V]\| < 2$, if $\|[U,V]\|_1 < 4$ then Bott$(U,V) = 0$. $\|\cdot\|_1$ denotes the trace norm, namely the sum of the singular values.*

Proof The statement follows from the inequality

$$|\mathrm{Bott}(U,V)| \le \frac{1}{2\pi}\|\log(UVU^{-1}V^{-1})\|_1 \le \frac{1}{4}\|[U,V]\|_1 \qquad (26)$$

Since Bott(U,V) is an integer, if its modulus has an upper bound strictly smaller than 1 then it vanishes.

Let us prove (26). Given the set of eigenvalues of $UVU^{-1}V^{-1}$, $\{e^{i\theta_j}\}$ with $\theta_j \in (-\pi,\pi)$, it holds:

$$\begin{aligned}
\frac{1}{2\pi}\|\log(UVU^{-1}V^{-1})\|_1 &= \frac{1}{2\pi}\sum_j |\theta_j| \qquad (27) \\
&\le \frac{1}{4}\sum_j |e^{i\theta_j} - 1| = \frac{1}{4}\|UVU^{-1}V^{-1} - \mathbb{1}\|_1 \\
&= \frac{1}{4}\|(UV - VU)U^{-1}V^{-1}\|_1 \le \frac{1}{4}\|[U,V]\|_1 \qquad (28)
\end{aligned}$$

The singular values of a normal matrix are the modulus of the eigenvalues; this is used in (27). The inequality $|\theta| \le \frac{\pi}{2}|e^{i\theta} - 1|$, with $\theta \in [-\pi,\pi]$, has been used in (28). □

5 Equivalence of the Bott index and the Chern number on a finite torus

The aim of this section is to introduce a pair of suitable quasi-unitary matrices, given by (29), and a pair of unitary matrices, given by (72) that arise from the physical system considered in Sect. 2 and prove that their Bott index equals the transverse conductance. This implies that the transverse conductance is an integer.

In Sect. 5.1 I follow the ideas of Hastings and Loring that realized how to remove the log in the expression of the Bott index up to corrections of order $\frac{R\|H\|}{L\Delta E}$. In Sect. 5.2 I adopt a novel approach showing the exact equality of the Bott index of two unitary matrices, given by (72) with the Chern number. The invertible matrices (29) and the unitary matrices (72) are shown to be related by a homotopy within the invertible matrices, implying the equality of their Bott indices. In Sect. 5.2 I also discuss subtleties concerning homotopies and periodic boundary conditions on the lattice.

According to sections 2F and 4 of [10], the wordings "Chern number" and "transverse conductance" will be used in here as synonymous.

In the following the notation is that of Sect. 2.

5.1 The Hastings–Loring approach, and more

Hastings and Loring considered in [26] the pair of almost unitary matrices $Pe^{\left(i\frac{2\pi X}{L}\right)}P$ and $Pe^{\left(i\frac{2\pi Y}{L}\right)}P$ as acting on the subspace $\mathrm{Ran}(P)$. I will consider instead the pair of almost unitary matrices over $l^2(\Lambda)$, already introduced by Loring in section 9 of [43] given by (29).

Lemma 8 *Given a Hamiltonian H as in 1, with P the Fermi projection, $P = \chi(H \leq \mu)$, and defining $\theta_x := \frac{2\pi X}{L}$, $\theta_y := \frac{2\pi Y}{L}$ and $P^\perp := \mathbb{1} - P$, the matrices*

$$P^\perp + Pe^{i\theta_x}P \quad , P^\perp + Pe^{i\theta_y}P \tag{29}$$

are almost unitary, namely it holds:

$$\|(P^\perp + Pe^{i\theta_x}P)^*(P^\perp + Pe^{i\theta_x}P) - \mathbb{1}\| \ll 1 \tag{30}$$

$$\|(P^\perp + Pe^{i\theta_x}P)(P^\perp + Pe^{i\theta_x}P)^* - \mathbb{1}\| \ll 1 \tag{31}$$

The same is true replacing θ_x with θ_y. Moreover, the pair (29) almost commute:

$$\|[P^\perp + Pe^{i\theta_x}P, P^\perp + Pe^{i\theta_y}P]\| \ll 1 \tag{32}$$

Proof

$$\|(P^\perp + Pe^{i\theta_x}P)^*(P^\perp + Pe^{i\theta_x}P) - \mathbb{1}\| = \|P^\perp + Pe^{-i\theta_x}Pe^{i\theta_x}P - \mathbb{1}\| \tag{33}$$

$$= \|Pe^{-i\theta_x}Pe^{i\theta_x}P - P\| = \|P(e^{-i\theta_x}(P - \mathbb{1})e^{i\theta_x})P\| \tag{34}$$

$$= \| Pe^{-i\theta_x} P^{\perp} e^{i\theta_x} P \| = \| [P, e^{-i\theta_x}] P^{\perp} [e^{i\theta_x}, P] \| \tag{35}$$

$$\leq \| [P, e^{-i\theta_x}] \|^2 \leq \mathcal{O}\left(\frac{R}{L} \frac{\|H\|}{\Delta E} \right)^2 \ll 1 \tag{36}$$

Analogously $\| (P^{\perp} + Pe^{i\theta_x} P)(P^{\perp} + Pe^{i\theta_x} P)^* - \mathbb{1} \| \ll 1$. Moreover,

$$\| [P^{\perp} + Pe^{i\theta_x} P, P^{\perp} + Pe^{i\theta_y} P] \| = \| [Pe^{i\theta_x} P, Pe^{i\theta_y} P] \| \tag{37}$$

$$= \| P(e^{i\theta_x} Pe^{i\theta_y} - e^{i\theta_y} Pe^{i\theta_x}) P \| = \| P(e^{i\theta_y} P^{\perp} e^{i\theta_x} - e^{i\theta_x} P^{\perp} e^{i\theta_y}) P \| \tag{38}$$

$$= \| [P, e^{i\theta_y}] P^{\perp} [e^{i\theta_x}, P] - [P, e^{i\theta_x}] P^{\perp} [e^{i\theta_y}, P] \| \tag{39}$$

$$\leq 2 \| [P, e^{i\theta_x}] \| \| [P, e^{i\theta_y}] \| \leq \mathcal{O}\left(\frac{R}{L} \frac{\|H\|}{\Delta E} \right)^2 \ll 1 \tag{40}$$

□

We now want to evaluate the Bott index of the pair (29). With the aid of Eq. (44), already stated in section 5.3 of [26], we get rid of the log in Eq. (45) introducing a $\mathcal{O}(\lambda^2)$ correction, with $\lambda := \frac{R\|H\|}{L\,\Delta E}$.

Lemma 9 *Given U and V two invertible matrices over $\mathbb{C}^N$, $N \gg 1$, and given a parameter g with $g^2 \propto \frac{1}{N}$, with U and V satisfying:*

$$\| U^* U - \mathbb{1} \| = \mathcal{O}(g^2), \qquad \| UU^* - \mathbb{1} \| = \mathcal{O}(g^2) \tag{41}$$

$$\| V^* V - \mathbb{1} \| = \mathcal{O}(g^2), \qquad \| VV^* - \mathbb{1} \| = \mathcal{O}(g^2) \tag{42}$$

$$\| [U, V] \| = \mathcal{O}(g^2), \tag{43}$$

namely they are almost unitary and they almost commute. This implies that:

$$\| \log(UVU^{-1}V^{-1}) - (UVU^{-1}V^{-1} - \mathbb{1}) \| \leq \mathcal{O}(g^4) \tag{44}$$

$$|\mathrm{Bott}(U, V) - \frac{1}{2\pi} \mathrm{ImTr}(UVU^{-1}V^{-1})| \leq \mathcal{O}(g^2) \tag{45}$$

Proof Let us consider the log series, given A with $\|A - \mathbb{1}\| < 1$, that implies A invertible, it holds:

$$\log A = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(A - \mathbb{1})^n}{n} \tag{46}$$

Then:

$$\| \log A - (A - \mathbb{1}) \| = \| \sum_{n=2}^{\infty} (-1)^{n+1} \frac{(A - \mathbb{1})^n}{n} \| \leq \| A - \mathbb{1} \|^2 \sum_{n=2}^{\infty} \frac{\| A - \mathbb{1} \|^{n-2}}{n} \tag{47}$$

Equation (44) follows from (47) with $A = UVU^{-1}V^{-1}$. The trace of a matrix is less equal than the norm of the matrix itself times the dimension of the space the matrix is acting upon, then:

$$|\mathrm{Tr}[\log(UVU^{-1}V^{-1}) - (UVU^{-1}V^{-1} - \mathbb{1})]| \leq \mathcal{O}(g^4)\mathcal{O}(N) = \mathcal{O}(g^2) \quad (48)$$

Noticing that

$$\mathrm{Bott}(U, V) = \frac{1}{2\pi}\mathrm{ImTr}\log(UVU^{-1}V^{-1}) \quad (49)$$

equation (45) follows. □

To show the equality of the Bott index of the pair (29) with the Chern number of P we make use of an expression of the latter that is suitable for this proof. The name Chern number arises within the theory of Chern class, see [49, 50], as an invariant of manifolds. For infinite systems in dimension two when the Hilbert space of the system is $l^2(\mathbb{Z}^2)$ the authors of [10] have shown that given a Hamiltonian with a local Fermi projection P, $P\frac{X+iY}{|X+iY|}P + P^{\perp}$, is a Fredholm operator, with index equal, after average over the disorder distribution, to the transverse conductance. The same authors in section 2F and section 4 of their work [10] develop the linear response theory that provides the form of the transverse conductance, this coincides with the Chern character (number) of the projection P defined as follows:

$$\mathrm{Ch}(P) = -2\pi i \mathrm{Tr}_{u.a.} P\left[\partial_x P, \partial_y P\right] \quad (50)$$

∂_x and ∂_y denote non commutative derivatives, namely $\partial_x A := [-iX, A]$, $\partial_y A := [-iY, A]$, with X and Y the position operators. The trace $\mathrm{Tr}_{\mathrm{u.a.}}$ stays for the trace per unit area, that is: $\mathrm{Tr}_{\mathrm{u.a.}} := \lim_{A\to\infty} \frac{\mathrm{Tr}_A}{A}$. From (50) it follows that

$$\mathrm{Ch}(P) = -4\pi \mathrm{ImTr}_{u.a.} P\,[X, P]\,[Y, P] \quad (51)$$

In fact:

$$\begin{aligned} 2\pi i \mathrm{Tr}_{u.a.} P\left[\partial_x P, \partial_y P\right] =& 2\pi i \mathrm{Tr}_{u.a.} P\,[[-iX, P], [-iY, P]] & (52)\\ =& 2\pi i \mathrm{Tr}_{u.a.} P\,([-iX, P][-iY, P] - [-iY, P][-iX, P]) & \\ =& -4\pi \mathrm{ImTr}_{u.a.} P\,[-iX, P]\,[-iY, P] & (53) \end{aligned}$$

I remark that in here I follow a sign convention for the Chern number different from [10], but in agreement with definitions 6.3 and 6.6 of [12], and in agreement also with appendix C of [29]. The definition of Chern number also generalizes to higher dimensions as can be seen in chapter 6 of [48]. Prodan is discussed in theorem 5.11 and corollary 5.12 of [51] the stability properties of the Chern number with respect to deformations of the Hamiltonian that gives rise to the Fermi projection P.

As far as regards the physical system on the torus that we are considering, as described in Sect. 2, the definition of Chern number is as in Eq. (51) with $\mathrm{Tr}_{\mathrm{u.a.}}(\cdot) = \frac{1}{L^2}\mathrm{Tr}_{l^2(\Lambda)}(\cdot)$.

Theorem 10 *Given a Hamiltonian H as in* Definition 1 *and its Fermi projection P, the approximated expression of the Bott index, as in* Eq. (45)*, of the almost unitary matrices $P^\perp + Pe^{\left(i\frac{2\pi X}{L}\right)}P$ and $P^\perp + Pe^{\left(i\frac{2\pi Y}{L}\right)}P$ equals the Chern number of the projection P, as given in Eq. (51), within a correction of order $O\left(\frac{R\|H\|}{L\,\Delta E}\right)$.*

$$\mathrm{Bott}\left(P^\perp + Pe^{\left(i\frac{2\pi X}{L}\right)}P,\, P^\perp + Pe^{\left(i\frac{2\pi Y}{L}\right)}P\right) = \mathrm{Ch}(P) + \mathcal{O}(\lambda) \tag{54}$$

with $\lambda := \frac{R\|H\|}{L\,\Delta E}$.

Proof With an eye toward operators in infinite-dimensional Hilbert spaces, it is a good idea to have a control on the order of magnitude of a trace, therefore starting from Eq. (56) all the terms appearing under a trace are of order L^{-2}, being in our model the Hilbert space of dimension of order L^2, it means that we are handling traces of $\mathcal{O}(1)$.

$$\mathrm{Bott}\left(P^\perp + Pe^{\left(i\frac{2\pi X}{L}\right)}P,\, P^\perp + Pe^{\left(i\frac{2\pi Y}{L}\right)}P\right)$$
$$= \frac{1}{2\pi}\mathrm{ImTr}\left(\mathbb{1} + [U,V]U^{-1}V^{-1}\right) + \mathcal{O}(\lambda^2) = \frac{1}{2\pi}\mathrm{ImTr}\left(\mathbb{1} + [U,V]U^*V^*\right) + \mathcal{O}(\lambda^2) \tag{55}$$
$$= \frac{1}{2\pi}\mathrm{ImTr}\left([Pe^{i\theta_x}P, Pe^{i\theta_y}P]Pe^{-i\theta_x}Pe^{-i\theta_y}P\right) + \mathcal{O}(\lambda^2) \tag{56}$$
$$= \frac{1}{2\pi}\mathrm{ImTr}\left[\left(Pe^{i\theta_y}P^\perp e^{i\theta_x}P - e^{i\theta_x}P^\perp e^{i\theta_y}P\right)e^{-i\theta_x}Pe^{-i\theta_y}P\right] + \mathcal{O}(\lambda^2) \tag{57}$$
$$= \frac{1}{2\pi}\mathrm{ImTr}\left(Pe^{-i\theta_y}Pe^{i\theta_y}P^\perp e^{i\theta_x}Pe^{-i\theta_x}P - Pe^{-i\theta_y}Pe^{i\theta_x}P^\perp e^{i\theta_y}Pe^{-i\theta_x}P\right) + \mathcal{O}(\lambda^2) \tag{58}$$

The terms inside the trace in Eq. (58) have quite a similar structure:

$$Pe^{-i\theta_y}Pe^{i\theta_y}P^\perp e^{i\theta_x}Pe^{-i\theta_x}P = P[-i\theta_y, P]P^\perp[i\theta_x, P]P + \mathcal{O}(\lambda)^3 \tag{59}$$

Inserting a couple of identities $\mathbb{1} = e^{i\theta_y}e^{-i\theta_y} = e^{-i\theta_x}e^{i\theta_x}$ in the second term of (58) we get:

$$Pe^{-i\theta_y}Pe^{i\theta_x}P^\perp e^{i\theta_y}Pe^{-i\theta_x}P = Pe^{-i\theta_y}Pe^{i\theta_y}e^{-i\theta_y}e^{i\theta_x}P^\perp e^{i\theta_y}e^{-i\theta_x}e^{i\theta_x}Pe^{-i\theta_x}P \tag{60}$$
$$= P\left(P + [-i\theta_y, P]\right)(P^\perp + [i(\theta_x - \theta_y), P^\perp])(P + [i\theta_x, P])P + \mathcal{O}(\lambda)^3 \tag{61}$$
$$= -P[i\theta_x, P]P^\perp[i\theta_x, P]P + P[i\theta_y, P]P^\perp[i\theta_x, P]P - P[-i\theta_y, P]P^\perp[i\theta_x, P]P$$
$$+ P[-i\theta_y, P]P^\perp[i\theta_y, P]P + P[-i\theta_y, P]P^\perp[i\theta_x, P]P + \mathcal{O}(\lambda)^3 \tag{62}$$

In Eq. (62) there are two Hermitian terms, therefore they have imaginary part of the trace vanishing, and two other terms cancel. Going back to (58), we get:

$$\mathrm{Bott}\left(P^\perp + Pe^{\left(i\frac{2\pi X}{L}\right)}P,\, P^\perp + Pe^{\left(i\frac{2\pi Y}{L}\right)}P\right)$$

$$= \frac{1}{\pi}\mathrm{ImTr}\left(P[\theta_y, P]P^{\perp}[\theta_x, P]\right) + \mathcal{O}(\lambda) \tag{63}$$

$$= -\frac{1}{\pi}\mathrm{ImTr}\left(P[\theta_x, P][\theta_y, P]\right) + \mathcal{O}(\lambda) = -\frac{4\pi}{L^2}\mathrm{ImTr}\left(P[X, P][Y, P]\right) + \mathcal{O}(\lambda) \tag{64}$$

We see that Eq. (64) coincides with Eq. (51) up to $\mathcal{O}(\lambda)$. Some remarks from the equations above: Eqs. (59) and (61) follow from the identity, with A Hermitian:

$$e^{iA}Be^{-iA} - B = [iA, B] + \frac{1}{2}[iA, [iA, B]] + \ldots + \frac{1}{n!}[iA \ldots [iA, B] \ldots] + \ldots \tag{65}$$

$\|[\theta_x, P]\| \leq \mathcal{O}(\lambda)$ is equation 5.5 of [25], this has been used in (59) and (61). This also follows from the application of the Holmgren bound, in a similar fashion to what done in Lemma 3, to $\|[X, H]\|$, with $|\langle m|[X, H]|n\rangle| = \mathrm{dist}(m_x, n_x)|\langle m|H|n\rangle|$. With m_x and n_x in the set $[0, L-1] \cap \mathbb{Z}$ the distance dist reflects the periodic boundary conditions, namely

$$\mathrm{dist}(m_x, n_x) = \min\{|n_x - m_x|, L - |n_x - m_x|\} \tag{66}$$

Application of the same ideas leads to $\frac{1}{L^2}\|[X, [X, H]]\| = \mathcal{O}(\lambda^2)$.

The first equation in (64) follows from:

$$\begin{aligned}\mathrm{ImTr}\left(P[\theta_y, P]P^{\perp}[\theta_x, P]\right) &= \mathrm{ImTr}\left([\theta_x, P][\theta_y, P]P^{\perp}\right) \\ &= -\mathrm{ImTr}\left(P[\theta_x, P][\theta_y, P]\right)\end{aligned} \tag{67}$$

In (67) it has been used: $[A, P] = P[A, P]P^{\perp} + P^{\perp}[A, P]P$ with A bounded and P a projection, this implies that $P[A, P]P = 0$. It has also been used: given B and C skew adjoint matrices then $\mathrm{ImTr}(BC) = -\mathrm{ImTr}(BC)^* = -\mathrm{ImTr}(C^*B^*) = -\mathrm{ImTr}(CB) = -\mathrm{ImTr}(BC)$, implying $\mathrm{ImTr}(BC) = 0$.

We stress that $[X, P]$ is well defined with periodic boundary conditions but neither XP nor PX is well defined, if singularly taken. This implies that (64) is well defined on a finite torus and coincides up to corrections of order $\mathcal{O}(\lambda)$ with the transverse conductance. □

Hastings and Loring have developed the theory of the linear response on the torus in section 5.3 of [26] making use of the current operator, with $\theta_x = 2\pi\frac{X}{L}$:

$$J_x = \frac{1}{2}\left(e^{i\theta_x}He^{-i\theta_x} - e^{-i\theta_x}He^{i\theta_x}\right) = \frac{2\pi}{L}[iX, H] + \mathcal{O}(L^{-3}) \tag{68}$$

They show that the Bott index equals the transverse conductance on the torus up to correction of order $\mathcal{O}(L^{-1})$.

It is worth mentioning that the Chern number of a finite-dimensional projection P, $\mathrm{rank}(P) < \infty$, defined on an infinite-dimensional Hilbert space or on a finite-dimensional Hilbert space with open boundary conditions is vanishing, in fact:

$$\mathrm{ImTr}\,(P[X,P][Y,P]) = \mathrm{ImTr}\left(P[X,P]P^{\perp}[Y,P]\right) \tag{69}$$

$$= \mathrm{ImTr}\left(PXP^{\perp}YP\right) = \mathrm{ImTr}\,(PXYP - PXPYP) \tag{70}$$

$$= -\mathrm{ImTr}\,(PXPYP) = -\mathrm{ImTr}\,(PYPXP) = 0 \tag{71}$$

In the first equality of (71) I have used the cyclic property of the trace with respect the two blocks PXP, PYP; in the second equality the fact that if the trace of a (trace class) operator coincides with the trace of its adjoint then it is real. This has also been used in (70).

On the torus, namely with periodic boundary conditions, we cannot "open" the commutator and take the trace of each operator, like it has been done in (70), in fact in that case $PXYP$ and $PXPYP$ are not well defined with respect to periodic boundary conditions.

A proof of the Bott index - Chern number equivalence based on a "momentum space" approach has been discussed in [33].

5.2 An exact approach: homotopies on the torus: the right and the wrong way

As stated at the beginning of Sect. 5, I will provide here a novel proof of the Bott index - Chern number correspondence, showing the equality among the Bott index of the pair of unitary matrices given in Eq. (72) and the Chern number $\mathrm{Ch}(P)$ taking advantage of an approach developed in [44] for bounded operators. The operators (72) have been already suggested in the context of the integer quantum Hall effect by Kitaev in appendix C of [52], and more recently, with a suitable modification, in the context of infinite-dimensional Hilbert spaces by the authors of [53, 54]. Being the Bott index an integer this shows that the correction $\mathcal{O}(\lambda)$ in Eq. (54) is actually vanishing. As an application of the homotopy invariance of the Bott index I will also show in lemma 12 that the two almost unitary matrices 29 and the two unitary matrices (72) are connected by an homotopy within the invertible matrices, providing another proof of the equality of their Bott indices.

Theorem 11 *Given a Hamiltonian H as in* Definition 1 *and its Fermi projection P, the unitary matrices*

$$e^{2\pi i P\frac{X}{L}P} \quad , \quad e^{2\pi i P\frac{Y}{L}P} \tag{72}$$

have a well-defined Bott index that satisfies:

$$
\begin{aligned}
Bott\left(e^{2\pi i P\frac{X}{L}P}, e^{2\pi i P\frac{Y}{L}P}\right) &= 2\pi i \mathrm{Tr}\left[P\frac{X}{L}P, P\frac{Y}{L}P\right] \\
&= \frac{4\pi}{L^2}\mathrm{ImTr}\left(P[iX,P][iY,P]\right) = \mathrm{Ch}(P)
\end{aligned}
\tag{73}
$$

Proof First of all we start noticing that $e^{2\pi i P\frac{X}{L}P} = Pe^{2\pi i P\frac{X}{L}P}P + P^{\perp}$, implying that $[P, e^{2\pi i P\frac{X}{L}P}] = 0$. Let us show that the unitary matrices in (72) almost commute implying that their Bott index is well defined. This is implied by $\| P^{\perp} + Pe^{i2\pi\frac{X}{L}}P - e^{2\pi i P\frac{X}{L}P}\| \leq \mathcal{O}(\lambda^2)$ that can be shown as follows:

$$
P^{\perp} + Pe^{i2\pi\frac{X}{L}}P - e^{2\pi i P\frac{X}{L}P} = P\left(e^{2\pi i\frac{X}{L}} - e^{2\pi i P\frac{X}{L}P}\right)P \tag{74}
$$

$$
= P\left(\sum_{n=2}^{\infty}\frac{(2\pi i)^n}{n!}\left[\left(\frac{X}{L}\right)^n - \left(P\frac{X}{L}\right)^n\right]\right)P \tag{75}
$$

Using the following equality of bounded operators A and B

$$
A^n - B^n = \sum_{j=0}^{n-1} B^j(A-B)A^{n-j-1} \tag{76}
$$

We obtain with $n \geq 2$

$$
P\left[\left(\frac{X}{L}\right)^n - \left(P\frac{X}{L}\right)^n\right]P
$$

$$
= P\left[\sum_{j=0}^{n-1}\left(P\frac{X}{L}\right)^j\left(\frac{X}{L} - P\frac{X}{L}\right)\left(P\frac{X}{L}\right)^{n-j-1}\right]P \tag{77}
$$

$$
= P\left[\sum_{j=0}^{n-1}\left(P\frac{X}{L}\right)^j P^{\perp}\frac{X}{L}\left(P\frac{X}{L}\right)^{n-j-1}\right]P \tag{78}
$$

$$
= P\left[\sum_{j=1}^{n-1}\left(P\frac{X}{L}\right)^j P^{\perp}\frac{X}{L}\left(P\frac{X}{L}\right)^{n-j-1}\right]P \tag{79}
$$

$$
= P\left[\sum_{j=1}^{n-1}\left(P\frac{X}{L}\right)^{(j-1)} P\frac{X}{L}P^{\perp}\frac{X}{L}P\left(P\frac{X}{L}\right)^{n-j-1}\right]P \tag{80}
$$

$$
= P\left[\sum_{j=1}^{n-1}\left(P\frac{X}{L}\right)^{(j-1)}\left[P,\frac{X}{L}\right]P^{\perp}\left[\frac{X}{L},P\right]\left(P\frac{X}{L}\right)^{n-j-1}\right]P \tag{81}
$$

The norm of the matrix in Eq. (81) is bounded by $(n-2)\mathcal{O}(\lambda^2)$. Using the series expansion of the exponential it is possible to show that the norm of (75) is bounded by $\mathcal{O}(\lambda^2)$. In a more concise way we can obtain an upper bound of $\mathcal{O}(\lambda)$ for (74), as follows:

$$\begin{aligned}
&\|P^{\perp} + Pe^{i2\pi\frac{X}{L}}P - e^{2\pi i P\frac{X}{L}P}\| \\
&\quad = \|P\left(e^{i\frac{2\pi X}{L}} - e^{2\pi i P\frac{X}{L}P}\right)P\| && (82)\\
&\quad = \|2\pi i P\int_0^1 e^{i2\pi\frac{X}{L}t}\left(\frac{X}{L} - P\frac{X}{L}P\right)e^{2\pi i P\frac{X}{L}P(1-t)}\mathrm{d}t\,P\| && (83)\\
&\quad = \|2\pi i P\int_0^1 e^{i2\pi\frac{X}{L}t}\left(\frac{X}{L} - P\frac{X}{L}P\right)Pe^{2\pi i P\frac{X}{L}P(1-t)}\mathrm{d}t\| && (84)\\
&\quad \le 2\pi\|\frac{X}{L}P - P\frac{X}{L}P\| = 2\pi\|P^{\perp}\frac{X}{L}P\| = 2\pi\|P^{\perp}\left[\frac{X}{L}, P\right]\| && (85)\\
&\quad \le \mathcal{O}(\lambda) && (86)
\end{aligned}$$

In Eq. (83) a DuHamel formula has been used. Then:

$$\begin{aligned}
&\|\left[e^{2\pi i P\frac{X}{L}P}, e^{2\pi i P\frac{Y}{L}P}\right]\| && (87)\\
&= \|P\left[e^{2\pi i P\frac{X}{L}P} - Pe^{i2\pi\frac{X}{L}}P + Pe^{i2\pi\frac{X}{L}}P, e^{2\pi i P\frac{Y}{L}P} - Pe^{i2\pi\frac{Y}{L}}P + Pe^{i2\pi\frac{Y}{L}}P\right]P\| && (88)\\
&\le \|\left[Pe^{i2\pi\frac{X}{L}}P, Pe^{i2\pi\frac{Y}{L}}P\right]\| + \mathcal{O}(\lambda) \le \mathcal{O}(\lambda) && (89)
\end{aligned}$$

To prove Eq. (73) we consider the maps of unitary matrices $U(t) : [0, 1] \to e^{2\pi i t P\frac{X}{L}P}$ and $V(t) : [0, 1] \to e^{2\pi i t P\frac{Y}{L}P}$. With $t \in (0, 1)$, $U(t)$ and $V(t)$ do not satisfy periodic boundary conditions therefore they are not admissible homotopies, according to lemma 6, meaning that along the paths $U(t)$ and $V(t)$, Bott $(U(t), V(t))$ is allowed to change. This can be seen for example looking at Eq. (25), we see that $U^{-1}(t)\partial_t U(t)$ must be a well-defined matrix over the given Hilbert space, in our case $l^2(\Lambda)$ periodic boundary conditions. On the contrary we see that

$$U^{-1}(t)\partial_t U(t) = 2\pi i P\frac{X}{L}P. \tag{90}$$

We now map the problem of determining the LHS of Eq. (73) into the solution of a first-order different equation.

To simplify the notation we denote $\phi_x := 2\pi P\frac{X}{L}P$ and $\phi_y := 2\pi P\frac{Y}{L}P$. Let us define $g(t) := \operatorname{Tr}\log\left(e^{it\phi_x}e^{it\phi_y}e^{-it\phi_x}e^{-it\phi_y}\right)$; the argument of the log satisfies periodic boundary conditions, in fact with $X \to X + nL\mathbb{1}$ and $Y \to Y + mL\mathbb{1}$, with n and $m \in \mathbb{Z}$, we have that

$$e^{2\pi i t P\left(\frac{X}{L}+n\mathbb{1}\right)P} = e^{2\pi i t n P}e^{2\pi i t P\frac{X}{L}P} \tag{91}$$

$$e^{2\pi itP\left(\frac{Y}{L}+m\mathbb{1}\right)P} = e^{2\pi itmP}e^{2\pi itP\frac{Y}{L}P} \tag{92}$$

This implies that the trace defining $g(t)$ is well posed.

$$\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}t} &= \mathrm{Tr}[(i\phi_x e^{it\phi_x}e^{it\phi_y}e^{-it\phi_x}e^{-it\phi_y} + e^{it\phi_x}i\phi_y e^{it\phi_y}e^{-it\phi_x}e^{-it\phi_y} \\
&\quad - e^{it\phi_x}e^{it\phi_y}i\phi_x e^{-it\phi_x}e^{-it\phi_y} &(93)\\
&\quad - e^{it\phi_x}e^{it\phi_y}e^{-it\phi_x}i\phi_y e^{-it\phi_y})e^{it\phi_y}e^{it\phi_x}e^{-it\phi_y}e^{-it\phi_x}] &(94)\\
&= \mathrm{Tr}\Big(i\phi_x + e^{it\phi_x}i\phi_y e^{-it\phi_x} - e^{it\phi_x}e^{it\phi_y}i\phi_x e^{-it\phi_y}e^{-it\phi_x} \\
&\qquad -e^{it\phi_x}e^{it\phi_y}e^{-it\phi_x}i\phi_y e^{it\phi_x}e^{-it\phi_y}e^{-it\phi_x}\Big) \\
&= \mathrm{Tr}\left(i\phi_x + i\phi_y - e^{it\phi_y}i\phi_x e^{-it\phi_y} - e^{it\phi_y}e^{-it\phi_x}i\phi_y e^{it\phi_x}e^{-it\phi_y}\right) &(95)\\
&= \mathrm{Tr}\left(e^{-it\phi_y}i\phi_x e^{it\phi_y} + i\phi_y - i\phi_x - e^{-it\phi_x}i\phi_y e^{it\phi_x}\right) &(96)\\
&= \mathrm{Tr}\left(e^{-it\phi_y}i\phi_x e^{it\phi_y} - i\phi_x\right) + \mathrm{Tr}\left(i\phi_y - e^{-it\phi_x}i\phi_y e^{it\phi_x}\right) &(97)\\
&= \mathrm{Tr}\left(\int_0^1 \mathrm{d}s e^{-ist\phi_y}[-it\phi_y, i\phi_x]e^{ist\phi_y}\right) + \mathrm{Tr}\left(\int_0^1 \mathrm{d}s e^{-ist\phi_x}[i\phi_y, -ti\phi_x]e^{ist\phi_x}\right) &(98)\\
&= \mathrm{Tr}\left([-it\phi_y, i\phi_x]\right) + \mathrm{Tr}\left([i\phi_y, -it\phi_x]\right) = 2t\mathrm{Tr}\left([i\phi_x, i\phi_y]\right) &(99)
\end{aligned}$$

Observing that $g(0) = 0$

$$g(t) = \int_0^t g'(s)\mathrm{d}s = \int_0^t 2s\mathrm{Tr}[i\phi_x, i\phi_y]\mathrm{d}s = t^2\mathrm{Tr}[i\phi_x, i\phi_y] \tag{100}$$

This implies that

$$\mathrm{Bott}(e^{i\phi_x}, e^{i\phi_y}) = \frac{1}{2\pi i}g(1) = \frac{1}{2\pi i}\mathrm{Tr}[i\phi_x, i\phi_y] \tag{101}$$

Then

$$\mathrm{Bott}\left(e^{2\pi iP\frac{X}{L}P}, e^{2\pi iP\frac{Y}{L}P}\right) = 2\pi i\mathrm{Tr}\left[P\frac{X}{L}P, P\frac{Y}{L}P\right] \tag{102}$$

Moreover,

$$\begin{aligned}
2\pi i\mathrm{Tr}\left[P\frac{X}{L}P, P\frac{Y}{L}P\right] &= \frac{2\pi i}{L^2}\mathrm{Tr}\left(PXPYP - PYPXP\right) &(103)\\
&= \frac{2\pi i}{L^2}\mathrm{Tr}\left(PYP^{\perp}XP - PXP^{\perp}YP\right) = \frac{4\pi}{L^2}\mathrm{ImTr}PXP^{\perp}YP &(104)\\
&= \frac{4\pi}{L^2}\mathrm{ImTr}[P, X]P^{\perp}[Y, P] = -\frac{4\pi}{L^2}\mathrm{ImTr}P[X, P][Y, P] &(105)
\end{aligned}$$

□

Lemma 12 *Given H as in* Definition 1 *and P its Fermi projection, it holds:*

$$Bott\left(Pe^{2\pi i\frac{X}{L}}P+P^{\perp}, Pe^{2\pi i\frac{Y}{L}}P+P^{\perp}\right)=Bott\left(e^{2\pi iP\frac{X}{L}P}, e^{2\pi iP\frac{Y}{L}P}\right) \quad (106)$$

Proof Equation (106) follows from the relation with the Chern number established in Theorems 10 and 11. To exemplify the construct of a homotopy, Eq. (106) can also be proven considering the paths:

$$\rho(s) := (1-s)e^{2\pi iP\frac{X}{L}P}+s\left(Pe^{2\pi i\frac{X}{L}}P+P^{\perp}\right) \quad (107)$$

$$\eta(s) := (1-s)e^{2\pi iP\frac{Y}{L}P}+s\left(Pe^{2\pi i\frac{Y}{L}}P+P^{\perp}\right) \quad (108)$$

with $s\in[0,1]$. It is important to stress that $\rho(s)$ and $\eta(s)$ are well defined for all $s\in[0,1]$ with respect to periodic boundary conditions, namely $\rho(s)$ is invariant when $X\to X+nL, \forall n\in\mathbb{Z}$. $\rho(s)$ is almost unitary: $\rho(s)\rho^*(s)=\mathbb{1}+\mathcal{O}(\lambda)$. This can be seen as follows:

$$\begin{aligned}\rho(s) &= (1-s)e^{2\pi iP\frac{X}{L}P}+s\left(Pe^{2\pi i\frac{X}{L}}P+P^{\perp}\right)\\ &= e^{2\pi iP\frac{X}{L}P}+sP\left(e^{2\pi i\frac{X}{L}}-e^{2\pi iP\frac{X}{L}P}\right)P\end{aligned} \quad (109)$$

According to Eq. (74), the term proportional to s in (109) is upper bounded by $\mathcal{O}(\lambda)$. The same holds for $\eta(s)$.

It is $\forall s\in[0,1]$, $\|[\rho(s),\eta(s)]\|<2$. Let us verify this. Setting as before $\theta_x=2\pi\frac{X}{L}$ and $\theta_y=2\pi\frac{X}{L}$, we have:

$$\begin{aligned}&\|[\rho(s),\eta(s)]\|\\ &=\|[(1-s)e^{iP\theta_xP}+s\left(Pe^{i\theta_x}P+P^{\perp}\right),(1-s)e^{iP\theta_yP}+s\left(Pe^{i\theta_y}P+P^{\perp}\right)]\|\\ &\le(1-s)^2\|[e^{iP\theta_xP},e^{iP\theta_yP}]\|+(1-s)s\|[e^{iP\theta_xP},Pe^{i\theta_y}P]\|\\ &\quad+(1-s)s\|[e^{iP\theta_yP},Pe^{i\theta_x}P]\|+s^2\|[Pe^{i\theta_x}P,Pe^{i\theta_y}P]\| &(110)\\ &\le(1-s)^2\|[P\theta_xP,P\theta_yP]\||+(1-s)s\|[e^{iP\theta_xP},e^{i\theta_y}]\|\\ &\quad+(1-s)s\|[e^{iP\theta_yP},e^{i\theta_x}]\|+s^2\mathcal{O}\left(\lambda^2\right) &(111)\\ &\le(1-s)^2\mathcal{O}\left(\lambda^2\right)+(1-s)s\|[P\theta_xP,\theta_y]\|\\ &\quad+(1-s)s\|[P\theta_yP,\theta_x]\|+s^2\mathcal{O}\left(\lambda\right)^2 &(112)\\ &\le(1-s)^2\mathcal{O}(\lambda^2)+2(1-s)s\mathcal{O}(\lambda)+s^2\mathcal{O}(\lambda^2)\ll 1 &(113)\end{aligned}$$

□

As a final remark about the subtleties of homotopies let us consider $W(s) := Pe^{i2\pi(1-sP^{\perp})\frac{X}{L}(1-sP^{\perp})}P + P^{\perp}$ that is a unitary map with the same initial and final point of $\rho(s)$. Nevertheless, $W(s)$ does not satisfy periodic boundary conditions with $s \in (0, 1)$ therefore it is not admissible as an homotopy for the torus geometry.

6 Discussion and perspectives

The formulation of the theory that leads to the construction of the Bott index is compatible with weak disorder, meaning that disorder is admitted in the description of the system by the model Hamiltonian as far as a spectral gap is present. The role of the disorder in system with topological features is essential in fact it is the presence of strong disorder that makes possible the presence of plateaus in the shape of the Hall conductance as the external magnetic field is varied. See the introduction of [10] for a discussion. The Chern number admits a formulation developed in [10], based on the use of tools from non commutative geometry that shows its quantization even in the presence of a disorder as strong as to close the band gap, for a visual illustration of this see figure 1 of reference [55]. The presence of a mobility gap, meaning that at the chemical potential there are only localized states, is still required, otherwise the system would lose its insulating nature. A different rigorous approach to strong-disordered systems is that of the so-called deterministic disorder, see for example [56] and section 7 of [57] with the definition of SULE, and the more recent works [54, 58]. In the formulation of the Bott index a band gap has been assumed. It seems natural to ask if the extension to strong disorder leads to a quantization of the Bott index.

A periodic driven Hamiltonians may host a peculiar topological invariant named, after [59], W invariant that does not have a counterpart in the static case. The authors of [60–62], among others, have extended the definition of the W invariant to the weak- and strong-disorder cases for infinite systems in two dimensions. A formulation of the W invariant for finite and disordered systems, in particular, the case of periodic boundary conditions considered here, as far as I know, is missing.

The last point of this perspective is to offer a connection with the spectral localizer, see [63] and references therein. Is it possible to replace the Fermi projection with the Hamiltonian, or a computationally straightforward function of the Hamiltonian, in the evaluation of the Bott index, for example, via homotopy? This would make the index faster to compute affording also larger samples. The reference [64] provides a direct comparison among the two indices.

Acknowledgements I acknowledge financial support by the UK's Engineering and Physical Sciences Research Council (grant number EP/R012393/1 Masanes).

Data availability statement Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The author states that there is no conflict of interest.

Appendices

Appendix A: Holmgren bound

$A : \mathcal{H} \to \mathcal{H}$, a bounded operator over a separable Hilbert space $\mathcal{H}$. It holds:

$$\|A\| \leq \sqrt{\sup_m \sum_n |A_{n,m}|}\sqrt{\sup_n \sum_m |A_{n,m}|} \tag{A1}$$

$A_{n,m} := \langle \chi_n, A\chi_m \rangle$, with $\{\chi_n\}$ any ONB of $\mathcal{H}$.

Proof $\|A\| := \sup_{\{\|\phi\|=1, \|\psi\|=1\}} |\langle \phi, A\psi \rangle|$

$$|\langle \phi, A\psi \rangle| = |\sum_{n,m} \phi_n^* A_{n,m} \psi_m| \leq \sum_{n,m} |\phi_n||A_{n,m}||\psi_m| = \sum_{n,m} |\phi_n| \sqrt{|A_{n,m}|}\sqrt{|A_{n,m}|}|\psi_m| \tag{A2}$$

$$\leq \sum_{n,m} |\phi_n| \left(\sup_m \sqrt{|A_{n,m}|}\right)\left(\sup_n \sqrt{|A_{n,m}|}\right)|\psi_m| \tag{A3}$$

$$= \sum_{n} |\phi_n| \left(\sup_m \sqrt{|A_{n,m}|}\right)\sum_m\left(\sup_n \sqrt{|A_{n,m}|}\right)|\psi_m| \tag{A4}$$

Applying Cauchy–Schwarz to the sums on n and m considered each as a scalar product, we get:

$$|\langle \phi, A\psi \rangle| \leq \sup_m \sqrt{\sum_a |\phi_a|^2}\sqrt{\sum_b |A_{b,m}|} \sup_n \sqrt{\sum_c |A_{n,c}|}\sqrt{\sum_d |\psi_d|^2} \tag{A5}$$

It follows that:

$$\|A\| \leq \sup_m \sqrt{\sum_b |A_{b,m}|} \sup_n \sqrt{\sum_c |A_{n,c}|} \tag{A6}$$

□

References

1. Klitzing, K.V., Dorda, G., Pepper, M.: New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance. Phys. Rev. Lett. **45**, 494 (1980). https://doi.org/10.1103/PhysRevLett.45.494
2. Tsui, D.C., Stormer, H.L., Gossard, A.C.: Two-dimensional magnetotransport in the extreme quantum limit. Phys. Rev. Lett. **48**, 1559 (1982). https://doi.org/10.1103/PhysRevLett.48.1559
3. Laughlin, R.B.: Quantized hall conductivity in two dimensions. Phys. Rev. B **23**, 5632 (1981). https://doi.org/10.1103/PhysRevB.23.5632
4. Halperin, B.I.: Quantized hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential. Phys. Rev. B **25**, 2185 (1982). https://doi.org/10.1103/PhysRevB.25.2185
5. Thouless, D.J., Kohmoto, M., Nightingale, M.P., den Nijs, M.: Quantized Hall conductance in a two-dimensional periodic potential. Phys. Rev. Lett. **49**, 405 (1982). https://doi.org/10.1103/PhysRevLett.49.405
6. Thouless, D.J.: Topological interpretations of quantum hall conductance. J. Math. Phys. **35**, 5362 (1994). https://doi.org/10.1063/1.530757
7. Wen, X.-G.: Theory of the edge states in fractional quantum hall effects. Int. J. Mod. Phys. B **06**, 1711 (1992). https://doi.org/10.1142/S0217979292000840
8. Fröhlich, J., Studer, U.M.: Gauge invariance and current algebra in nonrelativistic many-body theory. Rev. Mod. Phys. **65**, 733 (1993). https://doi.org/10.1103/RevModPhys.65.733
9. Büttiker, M.: Absence of backscattering in the quantum hall effect in multiprobe conductors. Phys. Rev. B **38**, 9375 (1988). https://doi.org/10.1103/PhysRevB.38.9375
10. Bellissard, J., van Elst, A., Schulz-Baldes, H.: The noncommutative geometry of the quantum Hall effect. J. Math. Phys. **35**, 5373 (1994). https://doi.org/10.1063/1.530758
11. Avron, J.E., Seiler, R.: Quantization of the hall conductance for general, multiparticle schrödinger hamiltonians. Phys. Rev. Lett. **54**, 259 (1985). https://doi.org/10.1103/PhysRevLett.54.259
12. Avron, J., Seiler, R., Simon, B.: Charge deficiency, charge transport and comparison of dimensions. Commun. Math. Phys. **159**, 399 (1994). https://doi.org/10.1007/BF02102644
13. Haldane, F.D.M.: Model for a quantum Hall effect without Landau levels: condensed-matter realization of the "parity anomaly". Phys. Rev. Lett. **61**, 2015 (1988). https://doi.org/10.1103/PhysRevLett.61.2015
14. Kane, C.L., Mele, E.J.: $\mathbb{Z}_2$ topological order and the quantum spin hall effect. Phys. Rev. Lett. **95**, 146802 (2005). https://doi.org/10.1103/PhysRevLett.95.146802
15. Kane, C.L., Mele, E.J.: Quantum spin hall effect in graphene. Phys. Rev. Lett. **95**, 226801 (2005). https://doi.org/10.1103/PhysRevLett.95.226801
16. König, M., et al.: Quantum spin Hall insulator state in HgTe quantum wells. Science **318**, 766 (2007)
17. Knez, I., Rettner, C.T., Yang, S.-H., Parkin, S.S.P., Du, L., Du, R.-R., Sullivan, G.: Observation of edge transport in the disordered regime of topologically insulating InAs/GaSb quantum wells. Phys. Rev. Lett. **112**, 026602 (2014). https://doi.org/10.1103/PhysRevLett.112.026602
18. Fiorenza, D., Monaco, D., Panati, G.: $\mathbb{Z}_2$ invariants of topological insulators as geometric obstructions. Commun. Math. Phys. **343**, 1115 (2016). https://doi.org/10.1007/s00220-015-2552-0
19. Großmann, J., Schulz-Baldes, H.: Index pairings in presence of symmetries with applications to topological insulators. Commun. Math. Phys. **343**, 477 (2016). https://doi.org/10.1007/s00220-015-2530-6
20. Jezequel, L., Tauber, C., Delplace, P.: Estimating bulk and edge topological indices in finite open chiral chains (2022). arXiv:2203.17099 [math.ph]
21. Kohmoto, M.: Topological invariant and the quantization of the hall conductance. Ann. Phys. **160**, 343 (1985). https://doi.org/10.1016/0003-4916(85)90148-4
22. Avron, J.E., Seiler, R., Simon, B.: Homotopy and quantization in condensed matter physics. Phys. Rev. Lett. **51**, 51 (1983). https://doi.org/10.1103/PhysRevLett.51.51
23. Simon, B.: Holonomy, the quantum adiabatic theorem, and Berry's phase. Phys. Rev. Lett. **51**, 2167 (1983). https://doi.org/10.1103/PhysRevLett.51.2167
24. Loring, T.A., Hastings, M.B.: Disordered topological insulators via C* -algebras. EPL (Europhys. Lett.) **92**, 67004 (2010)
25. Hastings, M.B., Loring, T.A.: Almost commuting matrices, localized wannier functions, and the quantum hall effect. J. Math. Phys. **51**, 015214 (2010). https://doi.org/10.1063/1.3274817

26. Hastings, M.B., Loring, T.A.: Topological insulators and C*-algebras: theory and numerical practice. Ann. Phys. **326**, 1699 (2011). https://doi.org/10.1016/j.aop.2010.12.013. (**july 2011 Special Issue**)
27. Altland, A., Zirnbauer, M.R.: Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures. Phys. Rev. B **55**, 1142 (1997). https://doi.org/10.1103/PhysRevB.55.1142
28. Qi, X.-L., Hughes, T.L., Zhang, S.-C.: Topological field theory of time-reversal invariant insulators. Phys. Rev. B **78**, 195424 (2008). https://doi.org/10.1103/PhysRevB.78.195424
29. Kitaev, A.: Periodic table for topological insulators and superconductors. AIP Conf. Proc. **1134**, 22 (2009). https://doi.org/10.1063/1.3149495
30. Ryu, S., Schnyder, A.P., Furusaki, A., Ludwig, A.W.W.: Topological insulators and superconductors: tenfold way and dimensional hierarchy. New J. Phys. **12**, 065010 (2010). https://doi.org/10.1088/1367-2630/12/6/065010
31. Huang, H., Liu, F.: Theory of spin bott index for quantum spin hall states in nonperiodic systems. Phys. Rev. B **98**, 125130 (2018). https://doi.org/10.1103/PhysRevB.98.125130
32. Titum, P., Lindner, N.H., Rechtsman, M.C., Refael, G.: Disorder-induced floquet topological insulators. Phys. Rev. Lett. **114**, 056801 (2015). https://doi.org/10.1103/PhysRevLett.114.056801
33. Ge, Y., Rigol, M.: Topological phase transitions in finite-size periodically driven translationally invariant systems. Phys. Rev. A **96**, 023610 (2017). https://doi.org/10.1103/PhysRevA.96.023610
34. Toniolo, D.: Time-dependent topological systems: a study of the Bott index. Phys. Rev. B **98**, 235425 (2018). https://doi.org/10.1103/PhysRevB.98.235425
35. Huang, H., Liu, F.: Quantum spin Hall effect and spin Bott index in a quasicrystal lattice. Phys. Rev. Lett. **121**, 126401 (2018). https://doi.org/10.1103/PhysRevLett.121.126401
36. Loring, T.A.: Bulk spectrum and k-theory for infinite-area topological quasicrystals. J. Math. Phys. **60**, 081903 (2019). https://doi.org/10.1063/1.5083051
37. Duncan, C.W., Manna, S., Nielsen, A.E.B.: Topological models in rotationally symmetric quasicrystals. Phys. Rev. B **101**, 115413 (2020). https://doi.org/10.1103/PhysRevB.101.115413
38. Yoshii, M., Kitamura, S., Morimoto, T.: Topological charge pumping in quasiperiodic systems characterized by the Bott index. Phys. Rev. B **104**, 155126 (2021). https://doi.org/10.1103/PhysRevB.104.155126
39. Wang, X.S., Brataas, A., Troncoso, R.E.: Bosonic Bott index and disorder-induced topological transitions of magnons. Phys. Rev. Lett. **125**, 217202 (2020). https://doi.org/10.1103/PhysRevLett.125.217202
40. Exel, R., Loring, T.A.: Almost commuting unitary matrices. Proc. Am. Math. Soc. **106**, 913–5 (1989). https://doi.org/10.2307/2047274
41. Exel, R., Loring, T.A.: Invariants of almost commuting unitaries. J. Funct. Anal. **95**, 364 (1991). https://doi.org/10.1016/0022-1236(91)90034-3
42. Loring, T.A.: Quantitative k-theory related to spin chern numbers. Sigma **10**, 077 (2014). https://doi.org/10.3842/SIGMA.2014.077
43. Loring, T.A.: K-theory and pseudospectra for topological insulators. Ann. Phys. **356**, 383 (2015). https://doi.org/10.1016/j.aop.2015.02.031
44. Toniolo, D.: The Bott index of two unitary operators and the integer quantum Hall effect (2021). arXiv:2112.01339 [math-ph]
45. Lax, P.D.: Functional Analysis. Wiley (2002)
46. Exel, R.: The soft torus and applications to almost commuting matrices. Pacific J. Math. **160**, 207–17 (1993)
47. Kato, T.: Perturbation Theory for Linear Operators, 2nd edn. Springer (1980)
48. Prodan, E., Schulz-Baldes, H.: Bulk and Boundary Invariants for Complex Topological Insulators. Springer (2016). arXiv:1510.08744
49. Choquet-Bruhat, Y., DeWitt-Morette, C., Dillard-Bleick, M.: Analysis. Manifold and Physics, Part I. North Holland (1982)
50. Dubrovin, B., Fomenko, A., Novikov, S.: Modern Geometry: Methods and Applications, Part II. Springer (1985)
51. Prodan, E.: Disordered topological insulators: a non-commutative geometry perspective. J. Phys. A: Math. Theor. **44**, 113001 (2011)
52. Kitaev, A.: Anyons in an exactly solved model and beyond. Ann. Phys. **321**, 2 (2006)
53. Fonseca, E., Shapiro, J., Sheta, A., Wang, A., Yamakawa, K.: Two-dimensional time-reversal-invariant topological insulators via fredholm theory. Math. Phys. Anal. Geom. (2020). https://doi.org/10.1007/s11040-020-09342-6

54. Bols, A., Schenker, J., Shapiro, J.: Fredholm homotopies for strongly-disordered 2d insulators (2021). arXiv:2110.07068 [math.ph]
55. Prodan, E., Hughes, T.L., Bernevig, B.A.: Entanglement spectrum of a disordered topological chern insulator. Phys. Rev. Lett. **105**, 115501 (2010). https://doi.org/10.1103/PhysRevLett.105.115501
56. del Rio, R., Jitomirskaya, S., Last, Y., Simon, B.: What is localization? Phys. Rev. Lett. **75**, 117 (1995). https://doi.org/10.1103/PhysRevLett.75.117
57. del Rio, R., Jitomirskaya, S., Last, Y., Simon, B.: Operators with singular continuous spectrum, iv: hausdorff dimensions, rank one perturbations, and localization. J. d'Analyse Math **69**, 153 (1996). https://doi.org/10.1007/BF02787106
58. Elgart, A., Graf, G., Schenker, J.: Equality of the bulk and edge hall conductances in a mobility gap. Commun. Math. Phys. **259**, 185 (2005). https://doi.org/10.1007/s00220-005-1369-7
59. Rudner, M.S., Lindner, N.H., Berg, E., Levin, M.: Anomalous edge states and the bulk-edge correspondence for periodically driven two-dimensional systems. Phys. Rev. X **3**, 031005 (2013). https://doi.org/10.1103/PhysRevX.3.031005
60. Graf, G.M., Tauber, C.: Bulk-edge correspondence for two-dimensional floquet topological insulators. Annales Henri Poincaré **19**, 709 (2018). https://doi.org/10.1007/s00023-018-0657-7
61. Sadel, C., Schulz-Baldes, H.: Topological boundary invariants for floquet systems and quantum walks. Math. Phys. Anal. Geometry **20**, 22 (2017). https://doi.org/10.1007/s11040-017-9253-1
62. Shapiro, J., Tauber, C.: Strongly disordered floquet topological systems. Annales Henri Poincaré **20**, 1837 (2019). https://doi.org/10.1007/s00023-019-00794-3
63. Lozano Viesca, E., Schober, J., Schulz-Baldes, H.: Chern numbers as half-signature of the spectral localizer. J. Math. Phys. **60**, 072101 (2019). https://doi.org/10.1063/1.5094300
64. Loring, T.: A guide to the Bott index and localizer index (2019). arXiv:1907.11791 [math-ph]