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The nonlinear singular Burgers equation with small parameter and *p*-regularity theory

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Abstract

In this paper, we study a solutions existence problem of the following nonlinear singular Burgers equation

$$F(u, \varepsilon) = u'_t - u''_{xx} + uu'_x + \varepsilon u^2 = f(x, t),$$

where $F : \Omega \to \mathcal{C}([0, \pi] \times [0, \infty))$, $\Omega = \mathcal{C}^2([0, \pi] \times [0, \infty)) \times \mathbb{R}$, $u(0, t) = u(\pi, t) = 0$, $u(x, 0) = g(x)$, and F, $f(x, t)$, $g(x)$ will be describe in the text. The first derivative of operator F at the solution point is degenerate. By virtue of p-regularity theory and Michael selection theorem, we prove the existence of continuous solution for this nonlinear problem.

Keywords p-Regularity · p-Factor operator · Singularity · Burgers equation · Nonlinear boundary value problems

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1 Introduction

This paper is a continuation of the work by Medak and Tret'yakov [4, 9] devoted to solutions existence problem of singular differential equations. Now we study the structure of solutions of nonlinear Burgers equation of the form:

$$F(u,\varepsilon) = u'_t - u''_{xx} + uu'_x + \varepsilon u^2 = f(x,t), \tag{1}$$

where $F : \Omega \to \mathcal{C}([0,\pi] \times [0,\infty))$, $\Omega = \mathcal{C}^2([0,\pi] \times [0,\infty)) \times \mathbb{R}$ and $u(0,t) = u(\pi,t) = 0$, $u(x,0) = g(x)$ for $F \in \mathcal{C}^{p+1}$, $f(x,t) \in \mathcal{C}([0,\pi] \times [0,\infty))$, $g(x) \in \mathcal{C}^2[0,\pi]$. In our paper, we consider the most interesting for applications periodical case $g(x) = k\sin x$, so called the oscillating initial condition with small parameter ε. The aim of our study will be to find such k that depends on ε, so that the above problem has a solution in the neighborhood of trivial solution $(u^*, \varepsilon^*) = (0,0)$ to which corresponds $k = k^* = 0$ and give analytic approximation of this solution with the initial condition $g(x) = k\sin x$ for small parameter ε.

Now we are looking for the solution $u(x,t)$ to (1) in a traditional way by the method of separation of variable in the form $u(x,t) = \bar{v}(t)\bar{u}(x)$, where $\bar{v}(t) = ce^{-t}$, $\bar{u}(x) \in \mathcal{C}^2([0,\pi])$. In this case, this is a degenerate problem since there exists $(u^*,\varepsilon^*) = (0,0)$ such that $\operatorname{Im} F'_u(u^*,\varepsilon^*) \neq Z = \mathcal{C}([0,\pi] \times [0,\infty))$ (more careful explanation see in Sect. 4). We apply to it the p-regularity theory [6, 7, 14, 15] and examine 3-regularity. For our purposes, 3-regularity on selected elements is enough.

The problems of the form $F(u,\varepsilon) = 0$, where $F : U \times M \to Z$ is a sufficiently smooth nonlinear mapping from a Banach space $U \times M$ to a Banach space Z, we separate into two classes, called regular and irregular. Roughly speaking, regular problems are those to which implicit function theorem arguments can be applied and the irregular ones are those to which it cannot, at least not directly.

The basis for our practical applications will be the following analogue of Lyusternik theorem on tangent cone (see [10]).

Theorem 1 *Let $F(v,y) \in C^{p+1}(U \times M)$, $F : U \times M \to Z$, where M is finite-dimensional space, and U and Z are Banach spaces. Let the mappings $F_i(v,y)$, $i = 1,\ldots,p$ be defined by (8). Assume that $F(v^*,y^*) = 0$ and $\forall \bar{y} \in M$, $\|\bar{y}\| = 1$, $(0,\bar{y}) \in \bigcap_{k=1}^{p} \operatorname{Ker}^k F_k^{(k)}(v^*,y^*)$ and F is strongly p-regular with respect to M along every element $(0,\bar{y})$, $\bar{y} \in M$, that is*

$$\left\| \left\{ F'_1(v^*,y^*) + F''_2(v^*,y^*)[0,\bar{y}] + \cdots + F_p^{(p)}(v^*,y^*)[0,\bar{y}]^{p-1} \right\}^{-1} \right\| \leq C. \tag{2}$$

(Here $\{\cdot\}^{-1}$ denotes right inverse operator).
Then, there exists the continuous mapping $v = v(y)$, $y \in V_\delta(y^*)$, where $V_\delta(y^*)$ is the neighborhood of y^*, $v(y) \in C(V_\delta(y^*))$, for sufficiently small $\delta > 0$, such that $F(v(y),y) = 0$ and

$$v(y) = v^* + \omega(y), \quad \|\omega(y)\| = o(\|y - y^*\|), \tag{3}$$

$$\|v(y) - v^*\| \leq C \sum_{k=1}^{p} \|F_k(v^*, y)\|_{Z_k}^{\frac{1}{k}}, \quad \forall y \in V_\delta(y^*), \tag{4}$$

where $C > 0$—independent constant.

Above theorem is proved in [10]. This is an analogue of the Lyusternik theorem on the tangent cone, which concerns the existence of continuous solutions of the equation $F(v, y) = 0$, where $F : U \times M \to Z$, M is finite-dimensional space, U, Z are Banach spaces and $F(v, y) \in C^{p+1}(U \times M)$.

In its proof, we applied the Michael selection theorem (see [11]), which we provide in the modified form:

Theorem 2 *Let U, Z—B-spaces, $A \in L(U, Z)$, A is surjective. Then, there exists a continuous mapping $N : Z \to U$, such that $AN(z) = z$ and $\|N(z)\| \leq c\|z\|$, $z \in Z$, where $c > 0$ is a constant independent of z.*

Let us note that from the Banach theorem about surjective operator, we have $\|A^{-1}\| \leq K$ (see Definition 6).

Theorem 1 allows the important conclusion about existence solution to Burgers equation with respect to the boundary conditions. The existence of continuous solutions is interesting, because there are no many results connected with singular problems (see, for example, [1, 3, 5]).

2 Main constructions in *p*-regularity theory

In this section, we present some important definitions and theorems of p-regularity theory to be used in what follows [6, 7, 14, 15].

We are interested in the following nonlinear problem:

$$F(v, y) = 0, \tag{5}$$

where the mapping $F : W \times Y \to Z$ and W, Y and Z are Banach spaces.

Assume that for some point $(v^*, y^*) \in W \times Y$, $\mathrm{Im} F'(v^*, y^*) \neq Z$. Let

$$Z = Z_1 \oplus \cdots \oplus Z_p, \tag{6}$$

where $Z_1 = \mathrm{cl}(\mathrm{Im} F'(v^*, y^*))$ and $V_1 = Z$. For V_2, we use one of the closed complement of Z_1 in Z (if such one there exists). Let $P_{V_2} : Z \to V_2$ be the projector onto V_2 along Z_1. By Z_2, we denote the closure of the linear span of the image of the quadratic mapping $P_{V_2} F''(v^*, y^*)[\cdot]^2$. Then, inductively,

$$Z_i = \mathrm{cl}(\mathrm{span}\mathrm{Im} P_{V_i} F^{(i)}(v^*, y^*)[\cdot]^i) \subseteq V_i, i = 2, \ldots, p-1, \tag{7}$$

where V_i is a choice of closed complement of $Z_1 \oplus \cdots \oplus Z_{i-1}$, $i = 2, \ldots, p$ with respect to Z, and $P_{V_i} : Z \to V_i$ is a projector onto V_i along $Z_1 \oplus \cdots \oplus Z_{i-1}$,

$i = 2, \dots, p$ with respect to Z. Finally, $Z_p = V_p$. The order p is the minimal number (if it exists) for which the decomposition (6) holds.
In what follows, we will denote $\varphi^{(0)} = \varphi$ for any mapping φ.
Define the following mappings:

$$F_i : W \times Y \to Z_i, \quad F_i(v, y) = P_{Z_i} F(v, y), \quad i = 1, \dots, p, \tag{8}$$

where $P_{Z_i} : Z \to Z_i$ is the projection operator onto Z_i along $Z_1 \oplus \cdots \oplus Z_{i-1} \oplus Z_{i+1} \oplus \cdots \oplus Z_p$. Then, the mapping F can be represented as:

$$F(v, y) = F_1(v, y) + \cdots + F_p(v, y) \tag{9}$$

or

$$F(v, y) = (F_1(v, y), \dots, F_p(v, y)). \tag{10}$$

Denote $h = [h_v, h_y]$, $h_v \in W$, $h_y \in Y$.

Definition 1 The linear operator $\Psi_p = \Psi_p(h) : W \times Y \to Z$, defined by

$$\Psi_p(h) = F_1'(v^*, y^*) + F_2''(v^*, y^*)[h] + \cdots + F_p^{(p)}(v^*, y^*)[h]^{p-1} \tag{11}$$

such that for any $z = (v, y)$

$$\Psi_p(h)[z] = F_1'(v^*, y^*)[z] + F_2''(v^*, y^*)[h][z] + \cdots + F_p^{(p)}(v^*, y^*)[h]^{p-1}[z], \tag{12}$$

is called a *p-factor operator depending of h* or shortly a *p-factor operator* if it is clear from the context.

Definition 2 We say that F is *completely degenerate at (v^*, y^*) up to the order p* if $F^{(i)}(v^*, y^*) = 0$, $i = 1, \dots, p-1$.

Remark 1 In the completely degenerate case, the p-factor operator reduces to $F^{(p)}(v^*, y^*)[h]^{p-1}$.

Remark 2 For each mapping F_i, we have ([6, p. 145])

$$F_i^{(k)}(v^*, y^*) = 0, \quad k = 0, 1, \dots, i-1, \quad \forall\, i = 1, \dots, p. \tag{13}$$

Remark 3 For each mapping F_i, we have in the completely degenerate case

$$F_i^{(i)}(v^*, y^*)[h]^{i-1} = P_{Z_i} F^{(i)}(v^*, y^*)[h]^{i-1}, \quad i = 1, \dots, p. \tag{14}$$

This means that $F_i^{(i)}(v^*, y^*)[h]^{i-1}$ are i-factor operators corresponding to completely degenerate mappings F_i up to order i. Therefore, the general degeneration of F can be reduced to the study of completely degenerated mappings F_i, $i = 1, \dots, p$ and their compositions.

Let's introduce the nonlinear operator $\Psi_p[\cdot]^p$ such that

$$\Psi_p[z]^p = F_1'(v^*, y^*)[z] + F_2''(v^*, y^*)[z]^2 + \cdots + F_p^{(p)}(v^*, y^*)[z]^p$$

We can see that $\Psi_p[h]^p = \Psi_p(h)[h]$.

Definition 3 The *p-kernel* of the operator Ψ_p is a set

$$\begin{aligned} H_p(v^*, y^*) &= \operatorname{Ker}^p \Psi_p \\ &= \{h \in W \times Y : F_1'(v^*, y^*)[h] + F_2''(v^*, y^*)[h]^2 + \cdots + F_p^{(p)}(v^*, y^*)[h]^p = 0\}. \end{aligned}$$

Note that the following relation holds:

$$\operatorname{Ker}^p \Psi_p = \left\{ \bigcap_{i=1}^{p} \operatorname{Ker}^i F_i^{(i)}(v^*, y^*) \right\}.$$

The p-kernel of the operator $F^{(p)}(v^*, y^*)$ in the completely degenerate case is a set

$$\operatorname{Ker}^p F^{(p)}(v^*, y^*) = \{h \in W \times Y : F^{(p)}(v^*, y^*)[h]^p = 0\}.$$

Definition 4 A mapping F is called *p-regular at* (v^*, y^*) *along* h $(p > 1)$ if $\operatorname{Im}\Psi_p(h) = Z$ (*i.e.*, the operator $\Psi_p(h)$ is surjective).

Definition 5 A mapping F is called *p-regular at* (v^*, y^*) $(p > 1)$ if it is p-regular along every $h \in H_p(v^*, y^*)\backslash\{0\}$ or $H_p(v^*, y^*) = \{0\}$.

Let $A \in L(U, Z)$ and $AU = Z$.

Let $\{\cdot\}^{-1}$ denote right inverse operator, i.e.,

$$A^{-1}z = \{u \in U : \quad Au = z\}$$

and

$$\|A^{-1}z\| = \inf_{u \in U}\{\|u\| : Au = z\}.$$

Obviously the operator $\{\cdot\}^{-1}$ is multivalued.

Definition 6 Define

$$\|A^{-1}\| = \sup_{z \in Z, \|z\|=1} \inf_{u \in U}\{\|u\| : Au = z\}. \tag{15}$$

Definition 7 Let $F : W \times Y \to Z = Z_1 \oplus \cdots \oplus Z_p$. The mapping $F(v, y)$ is called *strongly p-regular at the point* (v^*, y^*) if there exist $\gamma > 0$ and $c > 0$ such that

$$\sup_{h \in H_\gamma} \|\{\Psi_p(h)\}^{-1}\| \leq c < \infty,$$

where

$$H_\gamma = \big\{h = (h_v, h_y) \in W \times Y : \big\| F_k^{(k)}(v^*, y^*)[h]^k \big\|_{Z_k} \le \gamma,$$
$$\forall k = 1, \ldots, p, \quad \|h\|_{W\times Y} = 1\big\}.$$

Define *the solution set* for the mapping F as the set

$$S = \{(v, y) \in W \times Y : F(v, y) = F(v^*, y^*) = 0\} \tag{16}$$

and let $T_{(v^*,y^*)}S$ denote the tangent cone to the set S at the point (v^*, y^*), *i.e.*,

$$T_{(v^*,y^*)}S = \big\{h \in W\times Y : (v^*, y^*)+\varepsilon h+r(\varepsilon) \in S, \|r(\varepsilon)\| = o(\varepsilon), \varepsilon \in [0, \delta], \delta > 0\big\} \tag{17}$$

The following theorems describe the tangent cone to the solution set of equation (5) in the p-regular case.

Theorem 3 *Let W, Y and Z be the Banach spaces, and let the mapping $F \in C^p(W \times Y, Z)$ be p-regular at $(v^*, y^*) \in W \times Y$ along h. Then, $h \in T_{(v^*,y^*)}S$.*

Theorem 4 (Generalized Lyusternik Theorem, [6]) *Let W, Y and Z be the Banach spaces, and let the mapping $F \in C^p(W \times Y, Z)$ be p-regular at $(v^*, y^*) \in W \times Y$. Then,*

$$T_{(v^*,y^*)}S = H_p(v^*, y^*). \tag{18}$$

Let us explain that here (for Banach spaces U and Z) $F \in C^p(U, Z)$ means that $F : U \to Z$ is p times continuously Frechét differentiable.

The following Lemma will be important in the study of the surjectivity of p-factor operators.

Lemma 1 *Suppose that $Z = Z_1 \oplus Z_2$, where Z_1 and Z_2 are closed subspaces in Z, $A, B \in \mathcal{L}(U, Z)$, and $\mathrm{Im}A = Z_1$. Let P_2 also be the projection onto Z_2 along Z_1. Then, $(A + P_2B)U = Z \Leftrightarrow (P_2B)\mathrm{Ker}A = Z_2$.*

This lemma is a consequence of the following.

Lemma 2 *Suppose that $Z = Z_1 \oplus Z_2$, where Z_1 and Z_2 are closed subspaces in Z, $A_1, A_2 \in \mathcal{L}(U, Z)$, $A_1U \subset Z_1$, and $A_2U \subset Z_2$. Then, $(A_1 + A_2)U = Z$ iff $A_1\mathrm{Ker}A_2 = Z_1$ and $A_2\mathrm{Ker}A_1 = Z_2$.*

The proof is obvious. Lemma 1 follows from Lemma 2 if we put $A_1 = A$ and $A_2 = P_2B$.

Lemma 3 is the generalization of lemma 1.

Lemma 3 [8] *Let $A_1, A_2, \ldots, A_p \in \mathcal{L}(U, Z)$, $Z = Z_1\oplus\cdots\oplus Z_p$. Let $\mathrm{Im}\Pi_k A_k = Z_k$, where $\Pi_k : Z \to Z_k$ is a projection operator from the space Z onto Z_k along $Z_1 \oplus \cdots \oplus Z_{k-1} \oplus Z_{k+1} \oplus \cdots \oplus Z_p$, $k = 1, \ldots, p$ and $\Pi_1 A_1 = A_1$. Then*

$$\big(\Pi_1 A_1 + \Pi_2 A_2 + \cdots + \Pi_p A_p\big)U = Z \Leftrightarrow \big(\Pi_p A_p\big)\left(\bigcap_{i=1}^{p-1} \mathrm{Ker}\Pi_i A_i\right) = Z_p.$$

3 Auxiliary *p*-factor implicit function theorems

Now, we consider two theorems, which are the modification of analogical theorems in [6].

Theorem 5 (The p-order implicit function theorem for nontrivial p-kernel) *Let W, Y and Z be Banach spaces, $F \in C^{p+1}(W \times Y) \to Z$, $F(v^*, y^*) = 0$, $F_i(v, y)$, $i = 1, \dots, p$ be defined by (8) and the p-factor operator $\Psi_p(h)$ be given by (11). Assume that there exists an element $\bar{h} \in \bigcap_{r=1}^{p} \operatorname{Ker}^r F_{r_v}^{(r)}(v^*, y^*)$, $\|\bar{h}\| = 1$ such that $\operatorname{Im}\Psi_p(\bar{h}) = Z$, that is the mapping F is p-regular along the element $\bar{h}$. Then for a sufficiently small $\alpha > 0$, $\nu > 0$ and $\delta = \alpha\nu^p$ there exists the continuous mapping $\varphi(y) : U_\delta(y^*) \to U_\nu(v^*)$ and constant $K > 0$ such that the following hold:*

1. $\varphi(y^*) = v^*$;
2. $F(\varphi(y), y) = 0$ *for all* $y \in U_\delta(y^*)$;
3. $\varphi(y) = v^* + h(y) + v(y)$, *where* $h(y) = \gamma(y)\bar{h}$, $\gamma(\cdot) : U_\delta(y^*) \to \mathbb{R}$ *and* $\gamma(\cdot)$ *is arbitrary, continuous function such that*

$$\frac{\|y - y^*\|^{\frac{1}{p}}}{\alpha^{\frac{1}{p}}} \le \gamma(y) \le \nu.$$

Moreover, $v(y)$ satisfies

$$\|v(y)\|_W \le K \sum_{r=1}^{p} \frac{\|F_r(v^* + h(y), y)\|_{Z_r}}{\gamma(y)^{r-1}} \tag{19}$$

for all $y \in U_\delta(y^*)$, $\gamma(y) \ne 0$, or

$$\|v(y)\|_W = O(\gamma^2(y)). \tag{20}$$

The proof of the above theorem is similar to the proof of analogous theorem in [6].

Remark 4 Estimate (19) can be replaced by the following

$$\|v(y)\| \le K \sum_{r=1}^{p} \|F_r(v^* + h(y), y)\|_{Z_r}^{\frac{1}{r}} \quad y \in U_\delta(y^*), y \ne y^*. \tag{21}$$

The following theorem is some generalization of Theorem 5.

Theorem 6 *Let W, Y and Z be Banach spaces, $F \in C^{p+1}(W \times Y) \to Z$, $F(v^*, y^*) = 0$, $F_i(v, y)$, $i = 1, \dots, p$ be defined by (8) and the p-factor operator $\Psi_p(h)$ be given by (11). Assume that there exists an element $\bar{h} \in \bigcap_{r=1}^{p} \operatorname{Ker}^r F_{r_{(v,y)}}^{(r)}(v^*, y^*)$, $\|\bar{h}\| = 1$, $\bar{h} \in W \times Y$, $\bar{h} = (\bar{h}_v, \bar{h}_y)$, $\bar{h}_y = 0$ such that $\operatorname{Im}\Psi_p(\bar{h}) \cdot (0 \times Y) = Z$, that is the mapping F is p-regular along the element $\bar{h}$. Then for a sufficiently small $\alpha > 0$, $\nu > 0$ and $\delta = \alpha\nu^p$ there exists the continuous mapping $\varphi(y) : U_\delta(y^*) \to U_\nu(v^*)$ and constant $K > 0$ such that the following hold:*

1. $\varphi(y^*) = v^*$;
2. $F(\varphi(y), y) = 0$ *for all* $y \in U_\delta(y^*)$;
3. $\varphi(y) = v^* + h(y) + v(y)$, *where* $h(y) = \gamma(y)\bar{h}_y$, $\gamma(\cdot) : U_\delta(y^*) \to \mathbb{R}$ *and* $\gamma(\cdot)$ *is arbitrary, continuous function such that*

$$\frac{\|y - y^*\|^{\frac{1}{p}}}{\alpha^{\frac{1}{p}}} \leq \gamma(y) \leq \nu.$$

Moreover, $v(y)$ *satisfies*

$$\|v(y)\|_W \leq K \sum_{r=1}^{p} \frac{\|F_r(v^* + h(y), y)\|_{Z_r}}{\gamma(y)^{r-1}} \tag{22}$$

for all $y \in U_\delta(y^*)$, $\gamma(y) \neq 0$, *or*

$$\|v(y)\|_W = O(\gamma^2(v)). \tag{23}$$

The difference between Theorems 5 and 6 is that in Theorem 5 we take derivation with respect to v, but in Theorem 6 with respect to (v, y).

4 Solutions to Burgers equations

In this section, we will present the main result of this work.

Consider the Burgers equation

$$F(u, \varepsilon) = u'_t - u''_{xx} + uu'_x + \varepsilon u^2 = 0, \tag{24}$$

$F : \Omega \to \mathcal{C}([0, \pi] \times [0, \infty))$ where F is sufficiently smooth (at least up to order $p + 1$) and $u(0, t) = u(\pi, t) = 0$.

We will apply Theorem 6 denoting

$$F(v, y) = F(u, \varepsilon), \quad v := u, \quad y := \varepsilon, \tag{25}$$

The following result will hold

Theorem 7 *The mapping* $F(u, \varepsilon)$ *is* 3*-regular along* $\bar{h} = (\bar{h}_u, 0_\varepsilon)$, *where* $\bar{h}_u = e^{-t}\sin x$. *Moreover, for sufficiently small* $\alpha > 0$, $\nu > 0$, $\varepsilon \in (-\alpha\nu^3, \alpha\nu^3)$ *and* $k(\varepsilon) \in \left[\frac{\varepsilon^{\frac{1}{3}}}{\alpha^{\frac{1}{3}}}, \nu\right]$ *there exists continuous solution of* (24) *in the following form*

$$u(x, t, \varepsilon) = \gamma(\varepsilon)\bar{h}_u + y(x, t, \varepsilon), \tag{26}$$

$$u(x, 0) = k(\varepsilon)\sin x, \tag{27}$$

where $\gamma(\varepsilon)$ *is some continuous function such that* $\gamma(\varepsilon) = O(k(\varepsilon))$ *and* $\|y(x, t, \varepsilon)\| = o(\varepsilon^{\frac{1}{3}})$.

Proof We introduce the mapping $\Phi(u, \varepsilon, k)$ and equation

$$\Phi(u, \varepsilon, k) = (F(u, \varepsilon), u(x, 0) - k \sin x) = 0 \tag{28}$$

with boundary conditions $u(0, t) = u(\pi, t) = 0$.

Here $(u^*, \varepsilon^*, k^*) = (0, 0, 0)$ is the trivial solution of this equation and the tangent cone goes out from this point. Note that the analysis of the first derivative of the mapping Φ comes to the analysis of 3-regularity of the mapping F.

Denoting $u'_t - u''_{xx}$ by Lu (parabolic operator) and u'_x by $\frac{\partial}{\partial x}u$ (operator of differentiation), we obtain

$$F(u, \varepsilon) = Lu + u\frac{\partial}{\partial x}u + \varepsilon u^2 = 0. \tag{29}$$

Denote by F'_u, F'_ε partial derivatives of F with respect to u, ε and analogically higher-order partial derivatives by F''_{uu}, $F''_{u\varepsilon}$, $F''_{\varepsilon u}$, $F''_{\varepsilon\varepsilon}$, F''_{uuu}, $F''_{uu\varepsilon}$, ..., etc.

We have

$$F'(u, \varepsilon) = \big(F'_u(u, \varepsilon), F'_\varepsilon(u, \varepsilon)\big), \tag{30}$$

where

$$F'_u(u, \varepsilon)(\cdot)_u = L(\cdot)_u + (\cdot)_u\frac{\partial}{\partial x}u + u\frac{\partial}{\partial x}(\cdot)_u + 2\varepsilon u(\cdot)_u \tag{31}$$

and

$$F'_\varepsilon(u, \varepsilon)(\cdot)_\varepsilon = (\cdot)_\varepsilon u^2. \tag{32}$$

Note that

$$F'_u(0, 0)(\cdot)_u = L(\cdot)_u \tag{33}$$

and

$$\operatorname{Ker} F'_u(0, 0) = \big\{u \in \mathcal{C}^2([0, \pi] \times [0, \infty)) : Lu = 0, u(0, t) = u(\pi, t) = 0\big\}. \tag{34}$$

Taking into account the Fourier method of solving second-order partial differential equations and bearing in mind that we are looking for at least one solution we can determine

$$\operatorname{Ker} F'_u(0, 0) = \operatorname{span}\left\{\frac{2}{\sqrt{\pi}}e^{-t}\sin x\right\}. \tag{35}$$

The image of the operator $F'_u(0, 0)$ is defined as follows:

$$\begin{aligned}\operatorname{Im} F'_u(0, 0) &= \big\{z \in \mathcal{C}([0, \pi] \times [0, \infty)) : \exists u \in \mathcal{C}^2([0, \pi] \times [0, \infty))\ F'_u(0, 0)u = z,\\ &\quad u(0, t) = u(\pi, t) = 0\big\}\\ &= \big\{z \in \mathcal{C}([0, \pi] \times [0, \infty)) : \exists u \in \mathcal{C}^2([0, \pi] \times [0, \infty))\ Lu = z,\\ &\quad u(0, t) = u(\pi, t) = 0\big\}.\end{aligned}$$

We will look for a solution to the equation $F(u, \varepsilon) = 0$ in the form

$$u(x, t) = e^{-t}\bar{u}(x). \tag{36}$$

Then

$$Lu = -e^{-t}\bar{u} - e^{-t}\bar{u}''_{xx} = -e^{-t}\big(\bar{u} + \bar{u}''_{xx}\big) \tag{37}$$

One can show that the boundary value problem

$$Lu = -e^{-t}\big(\bar{u} + \bar{u}''_{xx}\big) = e^{-t}\sin x, \quad u(0,t) = u(\pi,t) = 0,$$

i.e.,

$$\bar{u}''_{xx} + \bar{u} = -\sin x, \quad \bar{u}(0) = \bar{u}(\pi) = 0,$$

does not have a solutions.

Therefore, the operator $F'_u(0,0)$ is not surjective and

$$\mathrm{Im} F'_u(0,0) \neq \mathcal{C}([0,\pi]\times[0,\infty)) = Z.$$

This implies that $Z = Z_1 \oplus V_2$, where $Z_1 = \mathrm{Im} F'_u(0,0)$ and $V_2 = Z_1^{\perp}$.

The projector $P_{V_2} : Z \to V_2$ can be described as

$$\begin{aligned} P_{V_2} z &= \frac{2}{\sqrt{\pi}} e^{-t}\sin x < z, \frac{2}{\sqrt{\pi}} e^{-t}\sin x > \\ &= \frac{4}{\pi} e^{-t}\sin x \int_0^{\infty} e^{-t}\mathrm{d}t \int_0^{\pi} z(\tau,t)\sin\tau\,\mathrm{d}\tau. \\ & \quad z \in Z. \end{aligned}$$

This implies that

$$\begin{aligned} Z_2 = \mathrm{span}\big(\mathrm{Im} P_{V_2} F''(0,0)[\cdot]^2\big) = \mathrm{span}\Big\{ & z(x,t) \in Z : z(x,t) \\ = \frac{4}{\pi} e^{-t}\sin x \int_0^{\infty} e^{-t}\mathrm{d}t & \int_0^{\pi} F''(0,0)[p(\tau,t)]^2 \sin\tau\,\mathrm{d}\tau, \\ p(\tau,t) \in \mathcal{C}^2([0,\pi]\times[0,\infty)) \Big\} & \subseteq V_2. \end{aligned}$$

Let us evaluate the second derivative of the mapping F

$$F''(u,\varepsilon) = \Big(\big(F''_{uu}(u,\varepsilon), F''_{u\varepsilon}(u,\varepsilon)\big), \big(F''_{\varepsilon u}(u,\varepsilon), F''_{\varepsilon\varepsilon}(u,\varepsilon)\big)\Big), \tag{38}$$

where

$$F''_{uu}(u,\varepsilon)(\cdot)_u(\cdot)_u = (\cdot)_u \frac{\partial}{\partial x}(\cdot)_u + (\cdot)_u \frac{\partial}{\partial x}(\cdot)_u + 2\varepsilon(\cdot)_u(\cdot)_u, \tag{39}$$

$$F''_{u\varepsilon}(u,\varepsilon)(\cdot)_u(\cdot)_\varepsilon = 2u(\cdot)_u(\cdot)_\varepsilon, \tag{40}$$

$$F''_{\varepsilon u}(u,\varepsilon)(\cdot)_\varepsilon(\cdot)_u = 2u(\cdot)_\varepsilon(\cdot)_u, \tag{41}$$

$$F''_{\varepsilon\varepsilon}(u,\varepsilon)(\cdot)_\varepsilon(\cdot)_\varepsilon = 0. \tag{42}$$

From this, we obtain

$$F''(u,\varepsilon)=\left(\left(2\frac{\partial}{\partial x}+2\varepsilon,2u\right),(2u,0)\right),$$
$$F''(0,0)=\left(\left(2\frac{\partial}{\partial x},0\right),(0,0)\right)$$

and

$$Z_2=\operatorname{span}\big(\operatorname{Im}P_{V_2}F''(0,0)[\cdot]^2\big)=\operatorname{span}\Bigg\{z(x,t)\in Z: z(x,t)$$
$$=\frac{4}{\pi}e^{-t}\sin x\int_0^\infty e^{-t}\mathrm{d}t\int_0^\pi 2\frac{\partial}{\partial\tau}h_u^2\sin\tau\,\mathrm{d}\tau\Bigg\}.$$

Substituting $h_u=se^{-t}\sin\tau$, where $s\in\mathbb{R}$, we get

$$Z_2=\{0\},\quad V_3=(Z_1\oplus\{0\})^{\perp},$$

since

$$\int_0^\pi\sin^2\tau\cos\tau\,\mathrm{d}\tau=0.$$

We continue

$$P_{V_3}z=\frac{2}{\sqrt{\pi}}e^{-t}\sin x<z,\frac{2}{\sqrt{\pi}}e^{-t}\sin x>$$
$$=\frac{2}{\sqrt{\pi}}e^{-t}\sin x\int_0^\pi\int_0^\infty z(\tau,t)\frac{2}{\sqrt{\pi}}e^{-t}\sin\tau\,\mathrm{d}\tau\,\mathrm{d}t,\quad z\in Z.$$
$$Z_3=\operatorname{span}(\operatorname{Im}P_{V_3}F'''(0,0)[\cdot]^3)=\operatorname{span}\Bigg\{z(x,t)\in Z: z(x,t)$$
$$=\frac{2}{\sqrt{\pi}}e^{-t}\sin x\int_0^\pi\int_0^\infty F'''(0,0)[p(\tau,t)]^3\frac{2}{\sqrt{\pi}}e^{-t}\sin\tau\,\mathrm{d}\tau\,\mathrm{d}t,$$
$$p(\tau,t)\in\mathcal{C}^2([0,\pi]\times[0,\infty))\Bigg\}\subseteq V_3.$$

Let us evaluate the third derivative of the mapping F:

$$F'''(u,\varepsilon)=(((F'''_{uuu},F'''_{uu\varepsilon}),(F'''_{u\varepsilon u},F'''_{u\varepsilon\varepsilon})),((F'''_{\varepsilon uu},F'''_{\varepsilon u\varepsilon}),(F'''_{\varepsilon\varepsilon u},F'''_{\varepsilon\varepsilon\varepsilon})))$$
$$F'''_{uuu}(\cdot)_u(\cdot)_u(\cdot)_u=0,\qquad F'''_{uu\varepsilon}(\cdot)_u(\cdot)_u(\cdot)_\varepsilon=2(\cdot)_\varepsilon(\cdot)_u(\cdot)_u,$$
$$F'''_{u\varepsilon u}(\cdot)_u(\cdot)_\varepsilon(\cdot)_u=2(\cdot)_u(\cdot)_\varepsilon(\cdot)_u,\qquad F'''_{u\varepsilon\varepsilon}(\cdot)_u(\cdot)_\varepsilon(\cdot)_\varepsilon=0,$$
$$F'''_{\varepsilon uu}(\cdot)_\varepsilon(\cdot)_u(\cdot)_u=2(\cdot)_u(\cdot)_\varepsilon(\cdot)_u,\qquad F'''_{\varepsilon u\varepsilon}(\cdot)_\varepsilon(\cdot)_u(\cdot)_\varepsilon=0,$$
$$F'''_{\varepsilon\varepsilon u}(\cdot)_\varepsilon(\cdot)_\varepsilon(\cdot)_u=0,\qquad F'''_{\varepsilon\varepsilon\varepsilon}(\cdot)_\varepsilon(\cdot)_\varepsilon(\cdot)_\varepsilon=0 \tag{43}$$

and

$$F'''(u,\varepsilon) = F'''(0,0).$$

Note that

$$F^{(4)}(u,\varepsilon) = 0,$$

and $V_3 = Z_3$, $P_{V_3}z = \Pi_{Z_3}z$.

Therefore, we will show that the mapping F is 3-regular on some elements that belong to the 3-kernel of the 3-factor operator and next we will describe the solutions of Burgers equation.

Now let us take $p(\tau,t) = (h_u, h_\varepsilon)$. For such defined vector $p(\tau,t)$ the following relations hold:

$$F'''(0,0)[p(\tau,t)]^2 = \left(4h_\varepsilon h_u, 2h_u^2\right), \tag{44}$$

$$F'''(0,0)[p(\tau,t)]^3 = 6h_u^2 h_\varepsilon. \tag{45}$$

Substituting $h_u = se^{-t}\sin\tau$, where $s \in \mathbb{R}$ and bearing in mind that $\int_0^\pi \sin^3\tau d\tau = \frac{4}{3}$ we get

$$\begin{aligned} Z_3 &= \operatorname{span}\left\{z(x,t) \in Z : z(x,t) = \frac{2}{\sqrt{\pi}}e^{-t}\sin x \int_0^\pi \int_0^\infty 6h_u^2 h_\varepsilon \frac{2}{\sqrt{\pi}} \sin\tau \mathrm{d}\tau \mathrm{d}t\right\} \\ &= \operatorname{span}\left\{\frac{2}{\sqrt{\pi}}e^{-t}\sin x\right\} = \operatorname{Ker} F_u'(0,0), \end{aligned}$$

3-factor operator

$$\begin{aligned} &\forall [h_{\bar{u}}, h_\lambda] \in C^2([0,\pi]\times[0,\infty)) \times \mathbb{R} \\ &\Psi_3(h)[h_{\bar{u}}, h_\lambda] = \Psi_3((0,0),[h_u,h_\varepsilon])[h_{\bar{u}}, h_\lambda] \\ &\quad = Lh_{\bar{u}} + \frac{2}{\sqrt{\pi}}e^{-t}\sin x \int_0^\pi \int_0^\infty (4h_\varepsilon h_u h_{\bar{u}} + 2h_u^2 h_\lambda) \frac{2}{\sqrt{\pi}}e^{-t}\sin\tau \mathrm{d}\tau \mathrm{d}t \end{aligned}$$

and 3-kernel of 3-factor operator $\Psi_3(h)$

$$\begin{aligned} \operatorname{Ker}^3\Psi_3(h) = \Big\{ & h = [h_u, h_\varepsilon] \in C^2([0,\pi]\times[0,\infty)) \times \mathbb{R} : \\ & Lh_u + \frac{2}{\sqrt{\pi}}e^{-t}\sin x \int_0^\pi \int_0^\infty (6h_u^2 h_\varepsilon) \frac{2}{\sqrt{\pi}}e^{-t}\sin\tau \mathrm{d}\tau \mathrm{d}t = 0\Big\}. \end{aligned}$$

Taking into account the equation

$$\int_0^\pi \sin^3\tau \mathrm{d}\tau = \frac{4}{3},$$

and fact, that $h_u = ce^{-t}\sin\tau$, since $h_u \in \operatorname{Ker} F'_u(0,0)$, $c \in \mathbb{R}$, we solve the following equation of unknowns c, h_ε:

$$\frac{4}{\pi}e^{-t}\sin x h_\varepsilon \int_0^\pi \int_0^\infty 6c^2 e^{-3t}\sin^3\tau \mathrm{d}\tau \mathrm{d}t = 0. \tag{46}$$

From here we get $c = 0$ or $h_\varepsilon = 0$ and the locus $\operatorname{Ker}^3\Psi_3(h)$ we describe clearly as follows:

$$\operatorname{Ker}^3\Psi_3(h) = \left\{(ce^{-t}\sin x, 0)\right\} \cup \{(0,\varepsilon)\},\ \|(h_u,\varepsilon)\| = 1,$$

where h_u is equal $ce^{-t}\sin x$ or 0.

Now we verify whether the 3-factor is surjective onto $C([0,\pi]\times[0,\infty))$ for element H belongs to the 3-kernel of the 3-factor operator.

Let $H = (h_u, h_\varepsilon) = (ce^{-t}\sin x, 0)$. We examine that

$$\forall z \in C([0,\pi]\times[0,\infty)) \quad \exists\left[\bar{h}_{\bar{u}}, \bar{h}_\lambda\right] \in C^2([0,\pi]\times[0,\infty))\times\mathbb{R} \quad \Psi_3(H)\left[\bar{h}_{\bar{u}}, \bar{h}_\lambda\right] = z. \tag{47}$$

Using Lemma 3, i.e., putting $z = z_3 = a\frac{2}{\sqrt{\pi}}e^{-t}\sin x \in Z_3$, we find the element $\bar{h}_{\bar{u}} = be^{-t}\sin x \in \operatorname{Ker} F'_u(0,0)$. We obtain

$$\frac{2}{\sqrt{\pi}}e^{-t}\sin x \int_0^\pi \int_0^\infty \left(4\cdot 0\cdot ce^{-t}\sin\tau\cdot\bar{h}_{\bar{u}} + 2\left(ce^{-t}\sin\tau\right)^2\bar{h}_\lambda\right)\frac{2}{\sqrt{\pi}}e^{-t}\sin\tau \mathrm{d}\tau\mathrm{d}t$$
$$= a\frac{2}{\sqrt{\pi}}e^{-t}\sin x, \tag{48}$$

$$\int_0^\pi \int_0^\infty 2(ce^{-t}\sin\tau)^2\bar{h}_\lambda \frac{2}{\sqrt{\pi}}e^{-t}\sin\tau \mathrm{d}\tau\mathrm{d}t = a, \tag{49}$$

$$\bar{h}_\lambda c^2 \frac{4}{3\sqrt{\pi}}\int_0^\pi (\sin^3\tau)\mathrm{d}\tau = a, \tag{50}$$

$$\bar{h}_\lambda = \frac{9a\sqrt{\pi}}{16c^2}, \quad c \neq 0. \tag{51}$$

Therefore, $(\bar{h}_{\bar{u}}, \bar{h}_\lambda) = (be^{-t}\sin x, \frac{9a\sqrt{\pi}}{16c^2})$ for any b and $c \neq 0$. Then, 3-factor operator $\Psi_3(H)$ is surjective for $H \neq (0,0)$. This implies that the mapping F is 3-regular at the point $(0,0)$ with respect to the element $H = (ce^{-t}\sin x, 0)$.

It turns out that the mapping F is not 3-regular at the point $(0,0)$ with respect to the element $H = (0,\varepsilon)$. We show it in Appendix. But for the existence of solutions, regularity on selected elements is enough.

Now we consider the equation $\Phi(u,\varepsilon,k) = 0$, i.e., to find the mapping $u(x,t,\varepsilon)$ and $k(\varepsilon)$ such that

$$\Phi(u(x,t,\varepsilon),\varepsilon,k(\varepsilon)) = 0 \Leftrightarrow \begin{bmatrix} F(u(x,t,\varepsilon)) = 0, \\ u(x,0,\varepsilon) = 0. \end{bmatrix} \tag{52}$$

Element $\tilde{h}_u = (\bar{h}_u, 0, 1) \in \operatorname{Ker}\Phi'(0,0,0) \cap \operatorname{Ker}^3\Phi_3^{(3)}(0,0,0)$ and the mapping Φ is regular along $\tilde{h}_u$.

By Theorem 6 for all $\varepsilon \in (-\alpha\nu^3, \alpha\nu^3)$ and for all $\gamma(\varepsilon)$ such that $\frac{\varepsilon^{\frac{1}{3}}}{\alpha^{\frac{1}{3}}} \leq \gamma(\varepsilon) \leq \nu$ and by Michael selection theorem, we obtain the continuous solution in the following form

$$\begin{bmatrix} u(x,t,\varepsilon) \\ k(\varepsilon) \end{bmatrix} = \gamma(\varepsilon)\tilde{h}_u + y(x,t,\varepsilon) = \begin{bmatrix} \gamma(\varepsilon)\bar{h}_u + y_1(x,t,\varepsilon) \\ \gamma(\varepsilon)\cdot 1 + y_2(x,t,\varepsilon) \end{bmatrix} \tag{53}$$

such that $F(u(x,t,\varepsilon),\varepsilon) = 0$, $u(x,0,t) - k(\varepsilon)\sin x = 0$.

From here if $\gamma(\varepsilon) = k(\varepsilon) + o(k(\varepsilon))$, then for all $k(\varepsilon) \in \left(\frac{\varepsilon^{\frac{1}{3}}}{\alpha^{\frac{1}{3}}}\right)$, there exists $\gamma(\varepsilon) = k(\varepsilon) + o(k(\varepsilon))$ such that $u(x,t,\varepsilon) = \bar{h}_u + \tilde{y}(x,t,\varepsilon)$.

This completes the proof. □

For the Burgers equation of the form

$$\begin{aligned} &F(u,\varepsilon,f) = u'_t - u''_{xx} + uu'_x + \varepsilon u^2 - f(x,t) = 0, \\ &u^* = 0,\ \varepsilon^* = 0,\ f^* = 0, \end{aligned} \tag{54}$$

where $F : \Omega \times \mathcal{C}([0,\pi]\times[0,\infty)) \to \mathcal{C}([0,\pi]\times[0,\infty))$ and $u(0,t) = u(\pi,t) = 0$, without loss of generality, consider the homogeneous initial condition $u(x,0) = 0$. We will apply Theorem 6 denoting $F(v,y) = F(u,\varepsilon,f)$, $v := (u,f)$, $y := \varepsilon$.

Note that the kernel $\mathrm{Ker}F'(0,0,0)$ is determined by solution to

$$u'_t - u''_{xx} - f(x,t) = 0,$$

which we denote by $\bar{u}(x,t,f)$. Then

$$\bar{h} = (\bar{u}(x,t,f), 0_\varepsilon, f(x,t)) \in \mathrm{Ker}F'(0,0,0) \cap \mathrm{Ker}^3 PF'''(0,0,0) = \mathrm{Ker}^3\Psi_3(h)$$

and

$$\bar{u}(x,t,f) = \int_0^t \int_0^\pi G(x,\xi,t,\tau) f(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau,$$

where $G(x,\xi,t,\tau)$ is Green's function (see, for example, [12, 13]).

Then the following result will be hold.

Theorem 8 (Classical case) *The mapping $F(u,\varepsilon,f)$ is 3-regular along $\bar{h}_u$ and for α, ν sufficiently small, $\varepsilon \in (-\alpha\nu^3, \alpha\nu^3)$ and $\|f(x,t)\| \in \left(\frac{\varepsilon^{\frac{1}{3}}}{\alpha}, \nu\right)$ there exists continuous solution of (54) in the following form*

$$u(x,t,\varepsilon,f) = \bar{u}(x,t,f) + y(x,t,\varepsilon,f), \tag{55}$$

where

$$\|y(x,t,\varepsilon,f\| = o(\varepsilon^{\frac{1}{3}}). \tag{56}$$

The proof for Theorem 8 is similar to Theorem 7.

5 Conclusion

Our research was inspired by works [9, 10]. We obtain analytical formulas for solving the nonlinear Burgers equation of the form (1) based on the p-regularity theory. Additionally by Theorem 1 and Michael selection theorem 2, we conclude that there exists a continuous solution.

Analogously may be investigated the following equations

$$u'_t - \varepsilon u''_{xx} + uu'_x + \phi(u) - f(x,t) = 0, \tag{57}$$
$$u'_t - u''_{xx} + \varepsilon uu'_x + \phi(u) - f(x,t) = 0 \tag{58}$$

and other, where $\phi^{(k)}(0) = 0$, $k = 1, \dots, p$ (see [2]).

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

For the clarity of the calculations, we show that the mapping F from Theorem 7 is not 3-regular at the point $(0,0)$ with respect to the element $H = (0, \varepsilon)$. Let $H = (h_u, h_\varepsilon) = (0, \varepsilon)$. We examine that

$$\forall z \in C([0,\pi]\times[0,\infty)) \quad \exists\big[\bar{h}_{\bar{u}}, \bar{h}_\lambda\big] \in C^2([0,\pi]\times[0,\infty))\times\mathbb{R} \quad \Psi_3(H)\big[\bar{h}_{\bar{u}}, \bar{h}_\lambda\big] = z. \tag{59}$$

Using Lemma 3, i.e., putting $z = z_3 = ae^{-t}\sin x \in Z_3$, we must find the element $\bar{h}_{\bar{u}} = be^{-t}\sin x \in \operatorname{Ker} F'_u(0,0)$. We obtain

$$0 = ae^{-t}\sin x \tag{60}$$

But the above equation has no solutions. Then, 3-factor operator $\Psi_3(H)$ is not surjective. This implies that the mapping F is not 3-regular at the point $(0, 0)$ with respect to the element $H = (0, \varepsilon)$.

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