A ribbon graph derivation of the algebra of functional renormalization for random multi-matrices with multi-trace interactions

Carlos I. Pérez-Sánchez[1,2]

Abstract

We focus on functional renormalization for ensembles of several (say $n \geq 1$) random matrices, whose potentials include multi-traces, to wit, the probability measure contains factors of the form $\exp[-\mathrm{Tr}(V_1) \times \cdots \times \mathrm{Tr}(V_k)]$ for certain noncommutative polynomials $V_1, \ldots, V_k \in \mathbb{C}_{\langle n \rangle}$ in the n matrices. This article shows how the "algebra of functional renormalization"—that is, the structure that makes the renormalization flow equation computable—is derived from ribbon graphs, only by requiring the one-loop structure that such equation (due to Wetterich) is expected to have. Whenever it is possible to compute the renormalization flow in terms of $\mathrm{U}(N)$-invariants, the structure gained is the matrix algebra $M_n(\mathcal{A}_{n,N}, \star)$ with entries in $\mathcal{A}_{n,N} = (\mathbb{C}_{\langle n \rangle} \otimes \mathbb{C}_{\langle n \rangle}) \oplus (\mathbb{C}_{\langle n \rangle} \boxtimes \mathbb{C}_{\langle n \rangle})$, being $\mathbb{C}_{\langle n \rangle}$ the free algebra generated by the n Hermitian matrices of size N (the flowing random variables) with multiplication of homogeneous elements in $\mathcal{A}_{n,N}$ given, for each $P, Q, U, W \in \mathbb{C}_{\langle n \rangle}$, by

$$(U \otimes W) \star (P \otimes Q) = PU \otimes WQ,$$
$$(U \boxtimes W) \star (P \otimes Q) = U \boxtimes PWQ,$$
$$(U \otimes W) \star (P \boxtimes Q) = WPU \boxtimes Q,$$
$$(U \boxtimes W) \star (P \boxtimes Q) = \mathrm{Tr}(WP)U \boxtimes Q,$$

which, together with the condition $(\lambda U) \boxtimes W = U \boxtimes (\lambda W)$ for each complex λ, fully define the symbol $\boxtimes$.

✉ Carlos I. Pérez-Sánchez
  perez@thphys.uni-heidelberg.de

[1] Institute for Theoretical Physics, University of Heidelberg, Philosophenweg 19, 69120 Heidelberg, Germany

[2] Faculty of Physics, University of Warsaw, ul. Pasteura 5, 02-093 Warsaw, Poland

1 Introduction and motivation

By the *Functional Renormalization Group* (FRG), physicists refer to a certain flow in the renormalization time t, usually the logarithm $t = \log k$ of the energy scale k, which in the "nonperturbative" [1] setting is governed by Wetterich equation [2]

$$\text{``}\partial_t \Gamma_k[\phi] = \frac{1}{2}\text{STr}\left(\frac{\partial_t R_k}{\text{Hess}\,\Gamma_k[\phi] + R_k}\right)\text{''}.$$

This is satisfied by the "effective action $\Gamma_k[\phi]$, infrared-regulated by R_k up to the energy scale k, on some space of fields ϕ" (quotation marks, since mathematical details follow for the system of our interest). This article addresses functional renormalization for ensembles of n-tuples of Hermitian matrices; the particular type of ensembles we analyze have clear physical motivations (Sects. 1.1 and 1.2).

While there is no better way to compute it, the denominator in the right-hand side of Wetterich equation is a Neumann expansion (geometric series) in "Hess $\Gamma_k[\phi]/R_k$", essentially, the Hessian of the fields. For matrix ensembles, this Hessian is an object of four indices, two from each of the two derivatives. The question is which is the meaning of the product $\star$ implied in powers $(\text{Hess}\,\Gamma_k[\phi])^{\star m}$ of the Hessian; we call the algebra defined by such product the *algebra of functional renormalization*.[1]

Of course, this question can be answered directly by looking at the proof of Wetterich equation; for the field theory in question, see [3]. For multi-matrix ensembles with probability measures defined, as is usual, solely in terms of single traces of matrix polynomials, part of the answer relies on observing that the Hessian is spanned, as $\sum_\alpha F_\alpha \otimes G_\alpha$, by couples of noncommutative polynomials F_α, G_α. The (so far, unsurprising) answer is that powers of the Hessian are obtained by the product rule

$$(U \otimes W)\star(P \otimes Q) = PU \otimes WQ.$$

Notice the "inversion" in one of the first tensor-factors, which starts to reflect the inner boundary and the outer one of the one-loop, relevant in this note (see Figs. 1 and 2). Interestingly, the incorporation of double traces yields a less trivial answer, for a "second product" appears (if one wants, a twisted tensor product) that also satisfies bilinearity $(zP) \boxtimes Q = P \boxtimes (zQ)$, $z \in \mathbb{C}$, but which differs from the usual tensor product only in the way one multiplies it with another element, $U \otimes W$ or $T \boxtimes V$. From interactions of two (or more) traces, then the Hessian of the effective action

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[1] We should probably write the *functional renormalization group algebra*, but "group algebra" can be confused with $\mathbb{C}[G]$, for a group G; or *functional renormalization algebra*, which would suggest, that the renormalization group (which is none) is upgraded to an algebra.
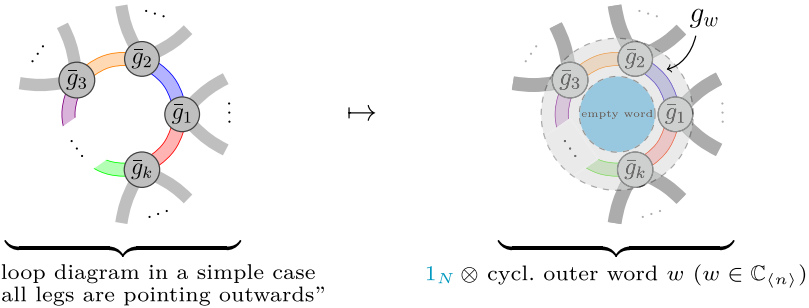
Fig. 1 The colored legs correspond to M_n-block entries $\mathrm{Hess}_{a,b}\,\Gamma$ of the effective action Γ. Left: Unrenormalized interactions $\bar{g}_i$ appearing in a kth power of the Hessian. Right: The contribution to the β_w-function, w formed by reading off clockwise the legs
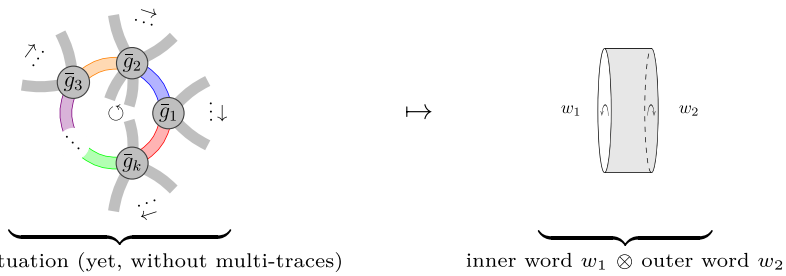

Fig. 2 How the one-loop structure of the FRG is encoded in $M_n(\mathcal{A}_n, \star)$. *Left:* Unrenormalized interactions $\bar{g}_i$ appearing in a kth power of the Hessian. *Right:* Unlike Fig. 1, this situation leads to a cylindric topology. Each word w_1 and w_2 distributed at the boundary is oriented in a consistent way with an orientation of the cylindric surface (determined by the cyclic clockwise order in the interaction vertices)

turns out to be spanned by noncommutative polynomials in a more general position[2] $\sum_\alpha F_\alpha \otimes G_\alpha + \sum_\rho H_\rho \boxtimes I_\rho$. The product reads

$$(U \otimes W + Y \boxtimes Z)\star(P \otimes Q) = PU \otimes WQ + Y \boxtimes PZQ\,, \tag{1a}$$

$$(U \otimes W + Y \boxtimes Z)\star(T \boxtimes V) = WTU \boxtimes V + \mathrm{Tr}_N(WT)U \boxtimes V\,. \tag{1b}$$

The aim of this article is to prove, using graphs, that the sole assumption that the contributions to the rhs of Wetterich equation have *all* a "one-loop structure" implies that the rhs of Wetterich equation satisfies Eq. (1). Next, we justify the appearance of noncommutative (nc) polynomials and of double traces, relating both with other theories (in Sects. 1.1 and 1.2, respectively). In Sect. 2, before presenting the precise statement, we give a short, but self-contained account of the ribbon graph theory needed to prove, in Sect. 3, the main statement (Theorem 2). "Appendix A" explains the construction of the infrared-regulated effective action.

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[2] These, moreover, might have traces as factors, but being these scalar functions of the matrices that will not be derived again, this is irrelevant.

1.1 The origin of the noncommutative polynomials and potential applications

Ensembles of several matrices with probability laws given by ordinary (commutative) potentials are extensively studied in high-energy physics. An important family of models solved by Eynard–Orantin [4], using their topological recursion, is the *two-matrix model*, which refers to measures $d\mu$ on $\mathcal{H}_N^2$ of the form:

$$d\mu(A, B) = \exp[-\operatorname{Tr}_N(AB)]$$
$$\times \exp\big\{-\operatorname{Tr}_N[V_1(A)]\big\}(dA)_{\text{LEB}} \times \exp\big\{-\operatorname{Tr}_N[V_2(B)]\big\}(dB)_{\text{LEB}}. \quad (2)$$

Modulo the first factor, this is still a product of measures, each of which on the space $\mathcal{H}_N$ of $N \times N$ Hermitian matrices. Here, $V_1(x)$ and $V_2(x)$ are polynomials in a real variable x and $\operatorname{Tr}_N(X) = \sum_{i=1}^N X_{i,i}$ is the unnormalized trace. We will keep this notation in the sequel.

The simplest addressed and (using the character expansion method [5]) solved model with a genuinely noncommutative law, generalizing Eq. (2), is the $ABAB$-model with measure

$$d\mu(A, B) = \exp\big\{-N\operatorname{Tr}_N(g_{A^4}A^4 + g_{B^4}B^4 + g_{ABAB}ABAB)\big\}d\gamma(A, B)$$
$$=: \exp(-S^{\text{INT}}[A, B])d\gamma(A, B), \quad (3)$$

where

$$d\gamma(A, B) = \exp\big\{-\frac{N}{2}\operatorname{Tr}_N(A^2 + B^2)\big\}(dA)_{\text{LEB}}(dB)_{\text{LEB}} \quad (4)$$

is the product Gaussian measure on $\mathcal{H}_N^2$. The action S that defines the probability measure $d\mu = \exp(-S[A, B])(dA)_{\text{LEB}}(dB)_{\text{LEB}}$ is the *bare action*. Hermitian ensembles with wildly non-factorizable measures, as those relevant in this paper, generalizing (3), are studied in free probability [6].

A more recent application of nc polynomial interactions concerns ensembles of Dirac operators

$$\mathcal{Z} = \int_{\text{Dirac}} \exp[-S(D)]dD \quad (5)$$

which aim at the quantization of the spectral action $S(D) = \operatorname{Tr} f(D)$ in noncommutative geometry. This problem was posed since [7,Ch. 19] and finite approximations to smooth geometries that allow to make precise sense of the partition function (5) recently reawakened interest in the problem [8–11]. In the spectral action [12, 13], traces of $f(D)$ yield (nontrivially noncommutative) polynomial interactions as well as double traces, cf. Eq. (6).

Applications of nc polynomial interactions were relevant for a better understanding of the Temperley–Lieb algebra.[3] From a Temperley–Lieb vertex $\mathcal{B}$, i.e., a rooted, planar

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[3] I thank Bertrand Eynard for pointing out the Temperley–Lieb algebra in the context of nc polynomial matrix interactions.

chord diagram one obtains a nc polynomial as in the next example:

$$\mathcal{B} = \raisebox{-1em}{\includegraphics[width=2em]{}}_* \;\mapsto\; \mathcal{B}(X_1, \ldots, X_n) = \sum_{a,b,c,d,e=1}^{n} X_a X_b X_c^2 X_d^2 X_b X_a X_e^2.$$

Nc polynomial matrix interactions are also auxiliary in the description of more general planar algebras [14] and $O(\mathfrak{n})$-loop models.

1.2 On multi-trace interactions

We will see later that not including multi-trace interactions in renormalization is unnatural (since generic radiative corrections include more traces than the bare action did). This short section mentions theories that contain multi-traces even before addressing renormalization (whenever possible).

- *Dirac ensembles* always yield double trace interactions (then renormalization creates even more traces). The (bare) Dirac ensemble measure is of the type

$$\mathrm{d}\mu(X_1, \ldots, X_n) = \exp\big\{ -N\,\mathrm{Tr}_N(P) - \mathrm{Tr}_N^{\otimes 2}(Q_{(1)} \otimes Q_{(2)}) \big\} \mathrm{d}(X_1, \ldots, X_n)_{\mathrm{LEB}} \tag{6}$$

  for P, $Q_{(1)}$ and $Q_{(2)}$ also[4] nc polynomials in the matrices $X_1, \ldots, X_n$ and $\mathrm{d}(X_1, \ldots, X_n)_{\mathrm{LEB}}$ is the product Lebesgue measure, now on $\mathcal{H}_N^n$. Even though P has an extra factor of N with respect to the double trace, observe that the latter cannot be neglected, since $\mathrm{Tr}_N(Q_{(1)}) \times \mathrm{Tr}_N(Q_{(2)})$ contains a double sum too.
- *Face-worded, stuffed maps.* Combinatorial maps ("gluing of polygons" dual to ribbon graphs) are counted with the aid of matrix partition functions [15]. In the presence of two random matrices with the probability law (2), the faces of these maps can be uniformly colored (and interpreted as Ising model) [16,§8]. If the potentials are noncommutative polynomials, this is no longer possible, and the partition function generates maps whose faces are labelled by "cyclic words" in the matrices (thus the maps could be called *face-worded*, as presented in Fig. 3 for the alphabet $\{A, B\}$). If the interaction vertices have several traces, the generated maps are said to be *stuffed* [11, 17] (independent of whether the potentials are ordinary or noncommutative). The terminology reflects that one now allows maps to have elementary cells of a topology that need not be that of a disk, i.e., one has "maps stuffed with bordered Riemann surfaces". The renormalization flow we study yields equations for the β-functions for matrix ensembles whose partition function generate "face-worded, stuffed maps". The fixed-point solution of the β-function system Eq. (14) could be useful to compute critical exponents (see Remark 1).
- *"Touching interactions"*. In several quantum gravity approaches, multi-trace operators appear, to name only few:

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[4] This notation has been inspired by Sweedler notation in quantum groups, and avoids to write sums like $Q_{(1)} \otimes Q_{(2)} = \sum_\alpha Q_{1,\alpha} \otimes Q_{2,\alpha}$ where each $Q_{1,\alpha}, Q_{2,\alpha} \in \mathbb{C}_{\langle n \rangle}$.
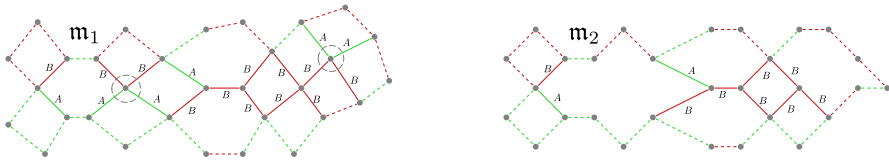
Fig. 3 Example of *(face-)worded maps* $\mathfrak{m}_1$ and $\mathfrak{m}_2$ dual to ribbon graphs generated by multi-matrix models with noncommutative polynomial interactions. Each *r*-agon of sides marked with letters $X_{i_1} \ldots X_{i_r}$ is generated by the interaction vertex $\mathrm{Tr}_N(X_{i_1} \ldots X_{i_r})$. The relation between $\mathfrak{m}_1$ and $\mathfrak{m}_2$ is the renormalization flow. In the cross graining process, the one-loop configurations at the nodes marked with dashed circles in $\mathfrak{m}_1$ yield the effective (in this case, higher-degree) interactions in $\mathfrak{m}_2$. This is the dual version of the cross graining depicted in Fig. 1 (Dashed edges mean that the maps can extend in that direction and might get some non-planar topology)

- in *Liouville gravity*, multi-trace one-matrix models are interpreted as generating functions of surfaces that might touch at isolated points. (The planar sector, for instance, is grasped, according to [18], as trees of spheres that can touch other spheres at most once.)
- multi-trace interactions appear in *curvature matrix models* [19]. Double traces appear in the effective description of a matrix model with a kinetic term $\mathrm{Tr}(\phi E \phi E)$ (with broken symmetry by a constant matrix E).
- another interpretation in terms of wormholes appears in (a certain two-matrix model description of) 3-dimensional *Causal Dynamical Triangulations* [20]
- under the *AdS/CFT*-correspondence, the AdS-object matching multi-trace operators in CFT are multi-particle states. In this context, for those states [21] defines the natural boundary conditions at ∞.

2 Terminology and main statement

Since our aim is to connect combinatorics and algebra in matrix models, on the one hand, with renormalization on the other, this article is somewhat interdisciplinary. Therefore, it is convenient to precisely define our framework and notation.

2.1 Ribbon graphs and the noncommutative Hessian on single trace interactions

The next points introduce our notation and present some definitions:

- The space of Hermitian $N \times N$ matrices is denoted by $\mathcal{H}_N$. The size of matrices $X_1^{(N)}, X_2^{(N)}, \ldots, X_n^{(N)} \in \mathcal{H}_N$ (which will become the random variables) will be relevant, but the lighter notation $X_1, X_2, \ldots, X_n$ is convenient. The number n of matrices remains fixed, and we will denote the n-tuple $(X_1, X_2, \ldots, X_n)$ by $\mathbb{X}$.
- $\mathbb{C}_{\langle n \rangle} = \mathbb{C}\langle X_1, X_2, \ldots, X_n \rangle$ is the *free algebra*. Any element of $\mathbb{C}_{\langle n \rangle}$ is spanned by words in the alphabet $\mathbb{X}$, and $\mathbb{C}_{\langle n \rangle}$ is endowed with the concatenation product. We actually should write $\mathbb{C}_{\langle n \rangle, N}$ instead of $\mathbb{C}_{\langle n \rangle}$ emphasizing that the generators X_a are matrices of size N, but the only manifestation of it is the empty word being the unit matrix 1_N, and we opt again for a light notation.

- The *noncommutative derivative* with respect to A, $\partial_A : \mathbb{C}_{\langle n \rangle} \to \mathbb{C}_{\langle n \rangle}^{\otimes 2}$ on a word w containing A is the sum over "replacements of A in w by the $\otimes$ tensor product symbol" in the middle of the word; if A occurs at the left (resp. right) end, then one additionally attaches the empty word (or in $\mathbb{C}_{\langle n \rangle, N}$ a unit 1_N) to the left (resp. right) of $\otimes$. For example, in a free algebra with enough letters

$$\partial_A(PAAR) = P \otimes AR + PA \otimes R \,,$$
$$\text{but} \quad \partial_A(ALGEBRA) = 1 \otimes LGEBRA + ALGEBR \otimes 1 \,.$$

- The noncommutative derivative defined on "cyclic words" $\mathrm{Tr}\, P$, $P \in \mathbb{C}_{\langle n \rangle}$, is given by the sum of all possible excisions $P \setminus A$ of A from P, rooting (i.e., starting) the remaining word at the letter after the removed A

$$\partial_A : \mathrm{im}\, \mathrm{Tr} \to \mathbb{C}_{\langle n \rangle}, \quad \mathrm{Tr}\, P \mapsto \sum_{\substack{\text{rootings at} \\ A\text{'s next letter}}} P \setminus A \,.$$
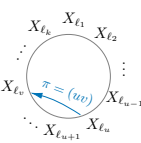
The result $\partial_A \mathrm{Tr}\, P =: \mathscr{D}_A P$ defines the *cyclic derivative* $\mathscr{D}_A$ of P and is due to Rota-Sagan-Stein [22] and Voiculescu [23]. For instance, $\partial_A \mathrm{Tr}(PAAR) = ARP + RPA = \mathscr{D}_A(PAAR)$. The adjective "cyclic" for $\mathscr{D}$ comes from the property $\mathscr{D}_{X_a} P = \mathscr{D}_{X_a}[\sigma(P)]$, which holds for any cyclic permutation $\sigma(P)$ of the letters of P ($P \in \mathbb{C}_{\langle n \rangle}$ and any $a = 1 \ldots, n$).

- Grasping Tr as the trace in $\mathbb{C}_{\langle n \rangle}$ induced by that of $M_N(\mathbb{C})$, define the *noncommutative Hessian* [3] of a cyclic word

$$\mathrm{Hess} : \mathrm{im}\, \mathrm{Tr} \to M_n(\mathbb{C}_{\langle n \rangle} \otimes \mathbb{C}_{\langle n \rangle})$$
$$\mathrm{Tr}\, P \mapsto (\mathrm{Hess}_{a,b}\, \mathrm{Tr}\, P)_{a,b=1,\ldots,n} := (\partial_{X_a} \circ \partial_{X_b}\, \mathrm{Tr}\, P)_{a,b=1,\ldots,n} \,. \quad (7)$$

Referring to the block M_n-matrix structure, i.e., to indices $a, b = 1 \ldots, n$, notice that in general the nc Hessian is not a symmetric matrix, $\mathrm{Hess}_{a,b}\, \mathrm{Tr}\, P \neq \mathrm{Hess}_{b,a}\, \mathrm{Tr}\, P$.

The (b, a)-entry in the M_n-matrix block structure of the Hessian of a cyclic word $\mathrm{Tr}\, W$ can be represented graphically by summing over all the *ordered* double markings of X_a and X_b inside a word W. On $W = X_{\ell_1} X_{\ell_2} \cdots X_{\ell_k} \in \mathbb{C}_{\langle n \rangle, N}$ (with $k \geq 2$), according to Eq. (7), this is given for $a, b = 1, \ldots, n$ by (for a proof see [3,Prop. 2.3])

$$(\partial_{X_b} \circ \partial_{X_a}) \mathrm{Tr}_N\, W = \sum_{\pi=(uv)} \delta_{\ell_u}^a \delta_{\ell_v}^b \pi_1(W) \otimes \pi_2(W) = \sum_{\pi=(uv)} \delta_{\ell_u}^a \delta_{\ell_v}^b \quad , \tag{8}$$

We sum over all oriented pairings $\pi = (uv)$ between the letters of the cyclic word $\mathrm{Tr}\, W$ (which explains the circle in the second equality). In Eq. (8), $\pi_1(W)$ is the ordinary word between X_{ℓ_u} and X_{ℓ_v} and $\pi_2(W)$ that between X_{ℓ_v} and X_{ℓ_u}, and because of the

deltas $X_{\ell_v} = X_b$ and $X_{\ell_u} = X_a$ must hold, and the empty word in either case leads to writing 1_N.

Example 1 To simplify the drawings, we expose the case $n = 2$. We compute the nc Hessian entry corresponding to $X_a = A$ and $X_b = B$ on $\mathrm{Tr}_N W = \mathrm{Tr}_N(ABAABABB)$. The entry reads:

$$\mathrm{Hess}_{b,a}(\mathrm{Tr}\, W) = \partial_B \partial_A \left(\begin{array}{c} \text{\tiny graph} \end{array} \right)$$

$$= 1_N \otimes \left(\begin{array}{c} \text{\tiny graph} \end{array} + \begin{array}{c} \text{\tiny graph} \end{array} + \begin{array}{c} \text{\tiny graph} \end{array} \right)$$

$$+ \left(\begin{array}{c} \text{\tiny graph} \end{array} + \begin{array}{c} \text{\tiny graph} \end{array} + \begin{array}{c} \text{\tiny graph} \end{array} \right) \otimes 1_N$$

$$+ \text{ polynomials of the form } P \otimes Q \text{ with } Q \neq 1_N \neq P$$

The cyclicity is lost due to each cut (dashed line). The word represented by each excision is read starting from the letter right after[5] the cut: the first one is $AABABB, \ldots$, and the sixth $BBABAA$. These terms that are listed arise from contiguous appearances of AB and BA in W and in each case the empty word between the letters originates the 1_N tensor factor. According to Eq. (8), the rest of the polynomials (last line) are computed by cutting the circle into two non-trivial words. For instance, the next cut yields

$$\begin{array}{c} \text{\tiny graph} \end{array} \rightarrow BAA \otimes ABB \qquad (9)$$

The order of the derivatives $\partial_B \partial_A$ (to the left of the cut "from A to B", $\Rightarrow$ first factor, to the left of "from B to A" $\Rightarrow$ second factor) determines which word is placed in which tensor factor.

2.2 Multi-trace interaction vertices, effective vertices

The interaction vertices in the measures $\mathrm{d}\mu(\mathbb{X}) = \exp\left\{ -N\,\mathrm{Tr}_N(P) - \mathrm{Tr}_N^{\otimes 2}(Q_{(1)} \otimes Q_{(2)}) \right\}\mathrm{d}\gamma(\mathbb{X})$ are represented by ribbon vertices framed with a dashed circle. This is unusual, but in view of the multiple products of traces in the measure, a helpful notation. The coupling constant $\bar{g}$ of multiple trace interactions is what prevents the multiple traces from being interpreted as different, disconnected polygonal building

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[5] "After" is determined by the scissors pointing in that direction. This is clearer in the graph in Eq. (9).

blocks (and are interpreted as "touching-interactions" [18–21] in other settings). Their relation to the free algebra is explained with the following examples (where green/light means the A matrix and red/dark represents B)

$$\bar{g}_1 \operatorname{Tr}_N(ABBBAB) \quad \leftrightarrow \quad \text{} \tag{10}$$

$$\bar{g}_2 \operatorname{Tr}_N^{\otimes 2}(AABABA \otimes AA) \quad \leftrightarrow \quad \text{} \tag{11}$$

$$\bar{g}_3 \operatorname{Tr}_N^{\otimes 2}(BBABB \otimes A) \quad \leftrightarrow \quad \text{} \tag{12}$$

The convention is that the label of the coupling constants applies to everything inside the dashed circle, i.e., simultaneously both traces (see also Example 5). This representation also reflects the mathematical nature of the effective action $\Gamma_N[\mathbb{X}]$ as (for now, at least) a formal series (with the coupling constants as parameters) of the form:

$$\Gamma_N[\mathbb{X}] = \sum_{\alpha} O_{\alpha} \qquad O_{\alpha} = \bar{g}_{\alpha} \prod_{r=1}^{t_{\alpha}} \operatorname{Tr}_N(w_{\alpha,r}),$$

$$\text{for certain monomials } w_{\alpha,r} \in \mathbb{C}_{\langle n \rangle} = \mathbb{C}\langle \mathbb{X} \rangle \tag{13}$$

so t_{α} is the number of traces in the *operators* O_{α}. The monomials $w_{\alpha,r}$ need not be monic; as a matter of fact, one usually normalizes $w_{\alpha,r}$ with symmetry factors. The coefficient of the kinetic operator $\operatorname{Tr}(X_c^2/2)$ (for each $c = 1, \ldots, n$) is called the wave function renormalization (of the matrix X_c) and, since it is special, it is usually denoted not by a $\bar{g}$ but by Z_c. Else, we call *interaction vertices* the remaining O_{α}'s. The bar on the coupling constant $\bar{g}_i = \bar{g}_i(N)$, which are functions of N, denotes that it will still be rescaled $\bar{g}_{\alpha} \to g_{\alpha} = Z^{\lambda_{\alpha}} N^{\kappa_{\alpha}} \bar{g}_{\alpha}$, solving for λ_{α} and κ_{α}, in order to render finite and Z-independent the next system (only in the large-N differential[6]) equations

$$\left\{ \eta_c := -\partial_t \log Z_c = -Z_c^{-1} \times \text{coeff. of } \operatorname{Tr}_N(X_c^2/2) \text{ in rhs of Eq.(28.)} \right\}_{c=1,\ldots,n} \tag{14a}$$

$$\left\{ \beta_{\alpha} := \partial_t g_{\alpha} = \text{coeff. of } \prod_{r=1}^{t_{\alpha}} \operatorname{Tr}_N(w_{\alpha,r}) \text{ in the rhs of Eq.(28)} \right\}_{\alpha}, \ t = \log N \tag{14b}$$

of (η-functions and) β-function equations for the interaction vertices α, determined by Wetterich equation. This list of operators appearing in Eq. (14) includes those of the original (bare) action S, but additionally those generated from it by "radiative

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[6] I thank Alexander Schenkel for pointing out that the parameter t is still discrete for finite N and thus Eq. (14) is not yet a system of differential equations.

corrections" to S. For instance[7] if the initial model is given by

$$S = N \operatorname{Tr}_N \left\{ \frac{1}{2} A^2 + \frac{1}{2} B^2 + g_{A^4} \frac{1}{4} A^4 + g_{B^4} \frac{1}{4} B^4 + \frac{1}{2} g_{ABAB} ABAB \right\} \tag{15}$$

then the radiative corrections

$$\cdots \tag{16}$$
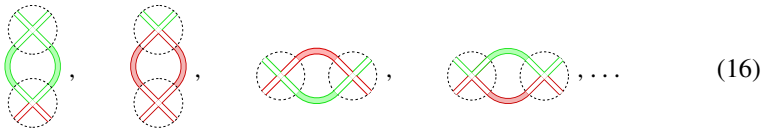
"generate" the *effective vertex* $N \operatorname{Tr}_N(ABBA)$ (see below, how). Also disconnected vertices are generated; for instance, $\operatorname{Tr}_N(A) \times \operatorname{Tr}(A)$ is generated from A^4 (by contracting non-consecutive half-edges) and from $ABAB$ (by contracting the B edges).

Therefore, the effective action should include these (and all corrections), and becomes[8]

$$\Gamma_N[A, B] = \operatorname{Tr}_N \left\{ \overbrace{\frac{Z_A}{2} A^2 + \frac{Z_B}{2} B^2 + \bar{g}_{A^4} \frac{1}{4} A^4 + \bar{g}_{B^4} \frac{1}{4} B^4 + \frac{1}{2} \bar{g}_{ABAB} ABAB}^{\text{operators from the bare action (but with "running couplings")}} \right.$$

$$\left. + \underbrace{\frac{1}{2} \bar{g}_{ABBA} ABBA + \frac{1}{2} \bar{g}_{A|A} \operatorname{Tr}_N(A) \times A + \cdots}_{\text{radiative corrections}} \right\} \tag{17}$$

The effective vertices are obtained by taking the boundary graph of the radiative corrections. In other words, they are constructed from a Feynman graph—as were those in (16) for the model (15)—as defined next, and explained with examples immediately thereafter.

Definition 1 (Effective interaction vertex) Given a Feynman graph G of a multi-trace multi-matrix model, first single out the traces $\operatorname{Tr}_N(U_1), \ldots, \operatorname{Tr}_N(U_r)$ that are not contracted by a propagator. Second, pick an arbitrary side of a ribbon-propagator and travel along the diagram with the orientation induced by the clockwise orientation of the interaction vertices, listing in that order the letters that label the half-edges of these (cf. Example 3) until one comes back to the initial, chosen propagator (on the same side); call the thus obtained word w_1. Repeat this process picking an unvisited side of a propagator, and iterate until all ribbon propagators visited once by both sides (and thus all uncontracted half-edges are listed exactly once), say, at the sth iteration. The

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[7] These graphs are based on the next comment: [24].

[8] Notice that the N can be re-absorbed in Z and the bar-coupling constants, but other conventions are possible.

effective vertex O_G^{eff} of the graph G is defined by

$$O_G^{\text{eff}} = \underbrace{\text{Tr}_N(w_1) \times \text{Tr}_N(w_2) \times \cdots \times \text{Tr}_N(w_s)}_{\text{from vertices contracted with propagators}} \times \underbrace{\text{Tr}_N(U_1) \times \text{Tr}_N(U_2) \cdots \times \text{Tr}_N(U_r)}_{\text{from vertices uncontracted with propagators}}$$

Since the words appear inside the trace, the construction is evidently independent of the propagators we started with to construct each word $w_1, \ldots, w_s$.

Example 2 (Graphs containing an empty loop.) To illustrate the effective vertex construction of a two-matrix model, consider the graph on the right, which corresponds to a correction from the operators

$$O_1 = \bar{g}_1 \, \text{Tr}_N(B^3 ABA), \tag{18}$$

$$O_2 = \bar{g}_2 \, \text{Tr}_N(A^2) \, \text{Tr}_N(A^3 BAB). \tag{19}$$

The effective vertex is $N\bar{g}_1\bar{g}_2 \, \text{Tr}_N(A^2) \, \text{Tr}_N(B^3 A^2 BA^2)$. The quadratic trace comes from the uncontracted trace in O_2; the long word comes from the "outward" loop and the factor $N = \text{Tr}_N \mathbb{1}_N$ from the inner, empty word.
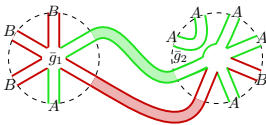

Example 3 (Orientation of loops.) With the operators

$$O_1 = \bar{g}_1 \, \text{Tr}_N(CFBDEA), \tag{20}$$

$$O_2 = \bar{g}_2 \, \text{Tr}_N(CDFABCEA), \tag{21}$$

we now illustrate the orientation of the loops. Each operator endows the interaction vertex with an orientation. The effective vertex should be read off respecting it. This means that outward loops are clockwise oriented and inward loops anti-clockwise. The effective vertex is $\bar{g}_1\bar{g}_2 \, \text{Tr}_N(CFCE) \times \text{Tr}_N(CDFADE)$.
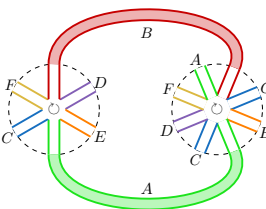
Example 4 (Propagators joining different traces in the same interaction vertex.) Consider now the graph on the right.
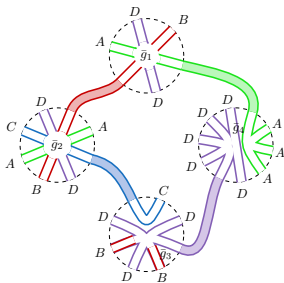
$$O_1 = \bar{g}_1 \operatorname{Tr}_N(ADBADB), \tag{22}$$

$$O_2 = \bar{g}_2 \operatorname{Tr}_N(BACDBACD), \tag{23}$$

$$O_3 = \bar{g}_3 \operatorname{Tr}_N(C^2) \times \operatorname{Tr}_N(D^3BDB), \tag{24}$$

$$O_4 = \bar{g}_4 \operatorname{Tr}_N(A^4) \times \operatorname{Tr}_N(D^6), \tag{25}$$

(and possibly other more operators making the action real). The effective vertex is $\bar{g}_1\bar{g}_2\bar{g}_3\bar{g}_4 \times \operatorname{Tr}_N(BDBD^7)\operatorname{Tr}_N(A^3DACDBACDADB)$. This graph is also a one-loop (see Definition 2 for the subtleties that appear in the presence of multi-trace interactions).

In the presence of multi-traces, the one-loop condition cannot be formulated purely in terms of the first Betti-number $b_1(G)$. Instead

Definition 2 Let G be a ribbon graph of a multi-trace multi-matrix model. We denote by G° the one-dimensional skeleton obtained after collapsing the interaction vertices[9] to points and the propagators (edges between interaction vertices) to ordinary edges. A *one-loop graph* of a multi-matrix model with multi-traces is a ribbon graph G whose skeleton G° is one-particle irreducible (1PI; or, equivalently, a 2-edge connected graph) and which additionally has a first Betti-number $b_1(G^\circ) = 1$.

Example 5 The next three diagrams are all one-loop graphs:
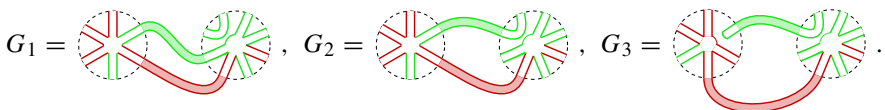
(We omit the coupling constants $\bar{g}_i$ by now, since we care about topology in this example). First, G_1 has $b_1(G_1^\circ) = 1$; next, although $b_1(G_2) \neq 1$, since thinning the edges and collapsing the stars (dashed circles) to points yields a circle, G_2° does have

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[9] In the single-trace random matrices literature these are sometimes called "stars" [6]. In the definition of the one-dimensional skeleton, it is implicit that we ignore discrete spaces obtained from the many traces that might be floating around the one-dimensional complex. This will be clear in Example 5.

first Betti-number 1. The same argument holds for G_3. Having these graphs explained the subtleties of the multiple traces, we give now ordinary examples. Regarding
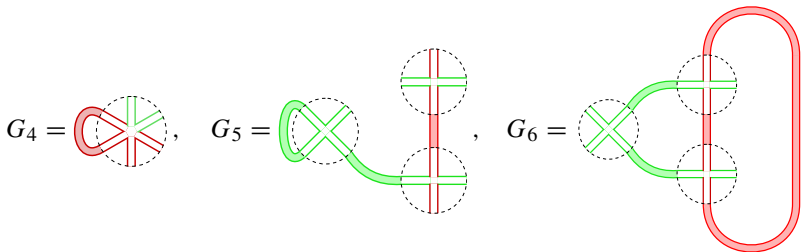
only the *tadpole* G_4 is a one-loop graph; although $b_1(G_5^\circ) = 1$, G_5 is not 1PI. And G_6 is such that G_6° has two loops, $b_1(G_6^\circ) = 2$, so neither G_5 nor G_6 satisfy Definition 2.

2.3 Including multi-traces and the main result

We denote by $N_\infty \in \mathbb{N}$ the energy scale at which the bare action describes the system.[10] The renormalization flow modifies then the probability measure used to compute observables as follows. The starting point of the flow is the bare action S (or the measure $d\mu_{N_\infty}^{\mathrm{UV}}$ defined by it)

$$S : \mathcal{H}_{N_\infty}^n \to \mathbb{R} \qquad d\mu_{N_\infty}^{\mathrm{UV}} = \exp\left\{ - S[\mathbb{X}^{(N_\infty)}] \right\} d\mathbb{X}_{\mathrm{LEB}}^{(N_\infty)} . \qquad (26)$$

The "UV" in the measure emphasizes that the action S that defines the probability measure $d\mu_N^{\mathrm{UV}} = \exp\left\{ - S[\mathbb{X}] \right\}(d\mathbb{X})_{\mathrm{LEB}}$ is the bare action. In order to flow toward a lower energy scale $N < N_\infty$, a regulator R_N takes care of integrating the *higher modes* (i.e., the matrix entries $N < i, j \leq N_\infty$ of each of the n matrices; see Appendix A). This smoothens the idea of step-by-step integration [25] of the $N + 1$th momentum shell, in order to obtain from ensembles of matrix of size $N + 1$, effective ensembles of $N \times N$ matrices. This idea was put forward in [26] for the one-matrix model in a quantum gravity context. Other renormalization theories based on Polchinski equation have been addressed in [27].

The system at that lower scale N is described by the effective action Γ_N and by the respective measure $d\mu_N^{\mathrm{eff}}$ at the scale N,

$$\Gamma_N : \begin{cases} \mathcal{H}_N^n \to \mathbb{R}, \\ \mathbb{X}^{(N)} \mapsto \Gamma_N[\mathbb{X}^{(N)}] \end{cases} \qquad d\mu_N^{\mathrm{eff}}(\mathbb{X}^{(N)}) = \exp\left\{ - \Gamma_N[\mathbb{X}^{(N)}] \right\} d\mathbb{X}_{\mathrm{LEB}}^{(N)} . \quad (27)$$

---

[10] The reason for the notation N_∞ is that, at the end, that integer can be thought of as being ∞. One computes first all with finite N_∞ and then takes the limit $N_\infty \to \infty$.

It can be proven—rigorously, at least in the sense of formal series—that the effective action satisfies Wetterich equation [3],

$$\partial_t \Gamma_N[\mathbb{X}] = \frac{1}{2} \mathrm{STr}\Big(\frac{\partial_t R_N}{\mathrm{Hess}\, \Gamma_N[\mathbb{X}] + R_N}\Big), \tag{28}$$

but as pointed out in the introduction, this is not the approach we follow in this article. We rather assume that the renormalization flow is governed by an equation of the form (28) and let ribbon graph theory dictate us the several objects that appear, specially the algebra obeyed by the Hessian. If an expansion in $\mathrm{U}(N)$-invariant operators exist, one is able to split the supertrace as follows:

$$\frac{1}{2}\mathrm{STr}\Big\{\frac{\partial_t R_N}{\mathrm{Hess}\,\Gamma_N[\mathbb{X}] + R_N}\Big\} = \sum_{k=0}^{\infty} \overbrace{\bar{h}_k(N, \eta_1, \ldots, \eta_n)}^{R_N\text{-dependent part}}$$
$$\times \underbrace{\frac{1}{2}(-1)^k \mathrm{STr}\big\{(\mathrm{Hess}\,\Gamma_N^{\mathrm{INT}}[\mathbb{X}])^{\star k}\big\}}_{\text{regulator-independent part}}, \tag{29}$$

where $\bar{h}_k(N, \{\eta_1, \ldots, \eta_n\})$ is a function of N and the anomalous dimensions $\eta_c = -\partial_t \log Z_c$; finally, $\Gamma_N^{\mathrm{INT}}[\mathbb{X}]$ is the interaction part of Γ_N, which will be constructed below. Since we are looking for a "universal" algebras (not in the usual sense, but in the sense that they will appear independent on the regulator R_N), details on R_N are placed in "Appendix A".

In order to find the algebra $\mathscr{A}$ where the Hessian of the effective action lies, let us search for the identity element of $\mathscr{A}$. Because this algebra should contain $M_n(\mathbb{C}_{\langle n \rangle} \otimes \mathbb{C}_{\langle n \rangle})$ (still seen as a vector space), we assume that $\mathscr{A}$ is also a matrix algebra of the form $\mathscr{A} = M_n(\mathcal{A}_n)$ for certain $\mathcal{A}_n$, and define the *supertrace*[11] STr on a matrix $\mathcal{P} = (P_{a,b})_{a,b=1,\ldots,n} \in M_n(\mathcal{A}_n)$, $P_{a,b} \in \mathcal{A}_n$ by

$$\mathrm{STr}(\mathcal{P}) = \sum_{a=1}^{n} \mathrm{Tr}_{\mathcal{A}_n}(P_{a,a}) \tag{30}$$

in terms of $\mathrm{Tr}_{\mathcal{A}_n}$, where $\mathcal{A}_n$, its product $\star$ and its trace $\mathrm{Tr}_{\mathcal{A}_n}$ are to be determined.

For this purpose, we observe that the effect of the kinetic terms, at a graph level, is just elongating the ribbons, and since all R_N-dependence has been absorbed in the coefficients $\bar{h}_k$ in Eq. (29), we conclude that the Hessian of the kinetic terms cannot

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[11] This is a historical terminology which should not evoke supersymmetry.

modify the effective vertex at all: since, for $a, b, c, d \in \{1, \ldots, n\}$,

$$\cdots \operatorname{Hess} O_{d,b} \star \operatorname{Hess} \left\{ \frac{1}{2} \operatorname{Tr}(X_c^2) \right\}_{b,a} \cdots = \cdots$$

$$= \cdots$$ $$\times \delta_a^c \delta_b^c \cdots$$

(31a)

$$\cdots \operatorname{Hess} \left\{ \frac{1}{2} \operatorname{Tr}(X_c^2) \right\}_{a,b} \star \operatorname{Hess} O_{b,d} \cdots = \cdots$$

$$= \cdots$$ $$\times \delta_a^c \delta_b^c \cdots$$

(31b)

for any interaction vertex O. On the other hand, the double trace terms $[\operatorname{Tr} X_c]^2$ "cut" the interaction vertex:

$$\cdots \operatorname{Hess} \left\{ \frac{1}{2} [\operatorname{Tr}(X_c)]^2 \right\}_{a,b} \star \operatorname{Hess} O_{b,d} \cdots = \cdots \delta_a^c \delta_b^c$$ $$\cdots$$

(32a)

$$\cdots \operatorname{Hess} O_{d,a} \star \operatorname{Hess} \left\{ \frac{1}{2} [\operatorname{Tr}(X_c)]^2 \right\}_{a,b} \cdots = \cdots \delta_a^c \delta_b^c$$ $$\cdots$$

(32b)

By Eq. (31), $\operatorname{Hess}_{c,c} \frac{1}{2} \operatorname{Tr}(X_c^2) = 1_N \otimes 1_N$ (no sum) is the left and right identity of $\mathcal{A}_{n,N}$, and by Eq. (32) there is another constant generator in $\mathcal{A}_{n,N}$ that, by the previous graph argument, is not proportional to $1_N \otimes 1_N$ (and therefore cannot be the identity) and which we denote by $1_N \boxtimes 1_N$.

Definition 3 We define $\mathcal{A}_n := \mathbb{C}_{\langle n \rangle}^{\otimes 2} \oplus \mathbb{C}_{\langle n \rangle}^{\boxtimes 2} = [\mathbb{C}_{\langle n \rangle} \otimes \mathbb{C}_{\langle n \rangle}] \oplus [\mathbb{C}_{\langle n \rangle} \boxtimes \mathbb{C}_{\langle n \rangle}]$. Again, this is simplified notation for $\mathcal{A}_{n,N}$ defined as $\mathcal{A}_n$, but with $\mathbb{C}_{\langle n \rangle, N}$ instead of $\mathbb{C}_{\langle n \rangle}$.

So far, $\mathcal{A}_n$ is only a vector space and $\boxtimes$ is just a symbol which will be different from $\otimes$ when we leave the category of vector spaces and grasp $\mathcal{A}_n$ already as an algebra. The bilinearity of $\boxtimes$ is due to the coupling constants $\bar{g}$ of interaction vertices $O = \operatorname{Tr}_N[\bar{g} Q_1] \operatorname{Tr}_N Q_2 = \operatorname{Tr}_N Q_1 \operatorname{Tr}_N[\bar{g} Q_2]$, which can "enter into any trace". Thus, $\boxtimes$ must satisfy $(\lambda U) \boxtimes W = U \boxtimes (\lambda W)$ for complex λ and $U, W \in \mathbb{C}_{\langle n \rangle}$. The noncommutative Hessian can be extended to products of traces as follows:

Definition 4 On double traces $\operatorname{Hess} : \operatorname{im} \operatorname{Tr}^{\otimes 2} \to M_n(\mathcal{A}_n)$ is given by

$$\operatorname{Hess} \left\{ \operatorname{Tr}^{\otimes 2}(P \otimes Q) \right\} = \operatorname{Hess} P \times \operatorname{Tr} Q + \operatorname{Hess} Q \times \operatorname{Tr} P + \Delta(P, Q), \quad (33)$$

where $\Delta(P, Q) = (\Delta_{a,b}(P, Q))_{a,b=1,\dots,n}$ has the following M_n-matrix entries:

$$\Delta_{a,b}(P, Q) = \partial_{X_a} \operatorname{Tr}_N P \boxtimes \partial_{X_b} \operatorname{Tr}_N Q + \partial_{X_a} \operatorname{Tr}_N Q \boxtimes \partial_{X_b} \operatorname{Tr}_N P$$
$$= \mathscr{D}_{X_a} P \boxtimes \mathscr{D}_{X_b} Q + \mathscr{D}_{X_a} Q \boxtimes \mathscr{D}_{X_b} P . \tag{34}$$

Lemma 1 *The trace* $\operatorname{Tr}_{\mathcal{A}_n}$ *on* $\mathcal{A}_n$ *is defined*[12] *in terms of* Tr_N *by linear extension of*

$$\operatorname{Tr}_{\mathcal{A}_n}(P \otimes Q) = \operatorname{Tr}_N^{\otimes 2}(P \otimes Q) = \operatorname{Tr}_N(P) \times \operatorname{Tr}_N(Q) \tag{35a}$$
$$\operatorname{Tr}_{\mathcal{A}_n}(P \boxtimes Q) = \operatorname{Tr}_N(PQ) \tag{35b}$$

Proof The tadpoles yield the desired relations. To obtain the first, for any fixed $c \in \{1, \dots, n\}$, consider an interaction vertex $O = \bar{g} \operatorname{Tr}(X_c P X_c Q)$ with $P, Q \in \mathbb{C}_{\langle n \rangle}$ satisfying $\partial_{X_c} P = \partial_{X_c} Q = 0$ (e.g., take $P, Q \in \mathbb{C}_{\langle n-1 \rangle} = \mathbb{C}\langle X_1, \dots, X_{c-1}, X_{c+1}, \dots, X_n \rangle$). The contribution to the rhs of the flow equation is

$$\frac{1}{2}\operatorname{STr} \operatorname{Hess} O = \frac{\bar{g}}{2} \operatorname{Tr}_{\mathcal{A}_n}(P \otimes Q + Q \otimes P) + \text{terms without } c\text{-propagators} . \tag{36}$$

The value of the first two summands is determined by the effective vertex of the graphs that the Hessian computes according to Eq. (8). These are such that the two ribbons are attached at the only two X_c matrices in O,

$$\operatorname{Hess}_{c,c} O = \quad \quad + \quad $$
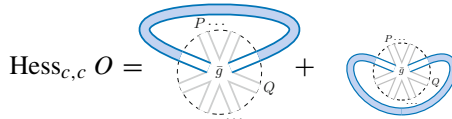
The ellipsis means that in the graphs, P is the word after the contracted X_c running clockwise until the next X_c, after which Q begins. The seemingly different propagator contraction is just an attempt to reflect that in the first graph P is inside the loop and Q outside, with these words in the other way around for the second graph. However, these two graphs are indistinguishable, thus, for each graph the effective vertex reads $\bar{g} \operatorname{Tr}_N(P) \times \operatorname{Tr}_N(Q)$, so by Eqs. (36), (35a) follows. To obtain the other product, we consider tadpoles with the ends of the propagator on different traces of the same operator. Let

$$O' = \bar{g}' \operatorname{Tr}_N(PX_c) \operatorname{Tr}_N(QX_c) = \quad $$

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[12] Just as the operators $\mathscr{D}_A$ and ∂_A, this abstract trace is the result of matrix-trace calculations with entries.

whose Hessian (cc)-entry reads

$$\text{Hess}_{c,c}\, O' = \quad \text{} \quad + \quad \text{} \quad .$$

By Eqs. (33) and (34),

$$\frac{1}{2}\text{STr Hess}\, O = \frac{\bar{g}'}{2}\,\text{Tr}_{\mathcal{A}_n}(P \boxtimes Q + Q \boxtimes P) + \text{terms without } c\text{-propagators}\,. \quad (37)$$

Ignoring the two coupling, the effective vertex of each graph is $\text{Tr}_N(PQ)$, which must be the value of $\text{Tr}_{\mathcal{A}_n}(P \boxtimes Q)$, but since the graphs are indistinguishable, also of $\text{Tr}_{\mathcal{A}_n}(Q \boxtimes P)$, Therefore, Eq. (37) implies Eq. (35b).

Now let us consider the general case, where P might depend on X_c (the dependence of Q on X_c can be likewise implemented, additionally, but the argument is the same in essence). Suppose that $P = P_L X_c P_R$, where $P_L, P_R \in \mathbb{C}_{\langle n\rangle}$ are monomials independent of X_c. In this simple case, the rhs of Eq. (36) receives the correction $\bar{g}\,\text{Tr}_{\mathcal{A}_n}[P_L \otimes P_R X_c Q + P_R X_c Q \otimes P_L]$, by the formula (8) for the Hessian. However, since P, Q are arbitrary, these terms cannot contribute to the coefficient of $\text{Tr}_N P\, \text{Tr}_N Q$ in $\frac{1}{2}\text{STr}(\text{Hess}\, O)$, since none of the graphs in such correction comply with having effective vertex (proportional to) $\text{Tr}_N P\, \text{Tr}_N Q$, and the word P has been split. Therefore, such contributions can be ignored (they do contribute, but to other effective vertices). A similar treatment for a generic word P and Q that might contain X_c concludes also the proof of (37) without restrictions on P and Q imposed above. □

In order to justify Eq. (29), we now define both $\mathcal{C}$ and $\Gamma_N^{\text{INT}}[\mathbb{X}]$ by

$$R_N + \text{Hess}\, \Gamma_N[\mathbb{X}] =: \mathcal{C}^{-1} + \text{Hess}\, \Gamma_N^{\text{INT}}[\mathbb{X}]\,, \quad (38)$$

where $\Gamma_N^{\text{INT}}[\mathbb{X}]$ contains only interaction vertices (and $[\text{Tr}_N X_c]^2$ counts as such; $\mathcal{C}$ is the correlation or inverse propagator). That is, Γ_N^{INT} is defined in such a way that the Gaussian part $\text{d}\gamma_N^{\text{eff}}$ in the effective measure is factorized out:

$$\text{d}\mu_N^{\text{eff}}(\mathbb{X}) = e^{-\Gamma_N[\mathbb{X}]}\text{d}\mathbb{X}_{\text{LEB}} = e^{-\Gamma_N^{\text{INT}}[\mathbb{X}]}\text{d}\gamma_N^{\text{eff}}(\mathbb{X})\,, \quad (39a)$$

$$\text{d}\gamma_N^{\text{eff}}(\mathbb{X}) = \prod_{c=1}^{n} e^{-Z_c\,\text{Tr}_N(X_c^2/2)}(\text{d}X_c)_{\text{LEB}}\,. \quad (39b)$$

Notice that one could have been tempted, inspired by [28], to separate the Hessian in its field-independent part (defined by its vanishing when $\mathbb{X} = 0$) and the field dependent part as performed in the functional renormalization treatment to one-matrix models by [26]. The "field part" of the algebra $\mathcal{A}_{n,N}$ consists of non-trivial words (i.e., except multiples of $1_N \otimes 1_N$ and $1_N \boxtimes 1_N$). But the presence of double-trace quadratic operators

$\frac{1}{2}[\mathrm{Tr}(X_c)]^2$, whose Hessian is $\frac{1}{2}\mathrm{Hess}\{(\mathrm{Tr}\,X_c)^2\} = \mathrm{diag}_n[0,\dots,1_N\boxtimes 1_N,0,\dots,0]$, with the non-zero in the (cc)th entry of the M_n-block diagonal diag_n, lies in the field-independent part, and this impairs (as we see now) the Neumann expansion. On the other hand, the definition (38) guarantees that the propagator $\mathcal{C}^{-1}$ is $1_n \otimes 1_N \otimes 1_N$ times a function (on $[1,\dots,N]^2$), due to

$$\sum_{c=1}^{n}\mathrm{Hess}\left\{\frac{1}{2}\mathrm{Tr}(X_c^2)\right\} = \sum_{c=1}^{n}\mathrm{diag}_n[0,\dots,\underbrace{1_N\otimes 1_N}_{c\text{th place}},0,\dots,0] = 1_n\otimes 1_N\otimes 1_N\,.$$

When the wave function renormalization constant Z_c is supposed to be equal for all matrices, $Z_c = Z$, then $\bar{h}_k(N,\eta)$, $\eta = -\partial_t \log Z$, and the sums in $\bar{h}_k$ can be approximated by integrals of the form $\frac{1}{N^2}\int (\partial_t r_N)_{\sigma,\tau}\mathcal{C}^{k+1}_{\tau,\sigma}\mathrm{d}\sigma\,\mathrm{d}\tau$ that remain finite as $N\to\infty$. We do not study the space of possible regulators (in itself, interesting), but we stress that the expansion (29) in unitary invariants is an assumption. Ideally, as commented in [26], since R_N breaks the symmetry, Eq. (29) should include operators $\mathrm{STr}(\partial_t R_N\mathcal{C}[\mathrm{Hess}\,\Gamma_N{}^{\mathrm{INT}}[\mathbb{X}]\mathcal{C}]^{\star k})$. However, identifying these operators with broken unitary symmetry is out of our present scope and for now the best one can do is to split, as in Eq. (29), the rhs of Wetterich equation in R_N-dependent and R_N-independent part. The main result of this article is the unique description of the latter.

Theorem 2 *For multiple-trace self-adjoint n-matrix ensembles, assume the rhs of Wetterich equation to be computable in terms of $\mathrm{U}(N)$-invariants as the geometric series (29) in the Hessian. Moreover, require that in Eq. (29) only one-loop graphs are generated. Then, the powers $(\mathrm{Hess}\,\Gamma_N^{\mathrm{INT}}[\mathbb{X}])^{\star k}$ are taken in the algebra $M_n(\mathcal{A}_{n,N},\star)$ of $n\times n$ matrices with entries in $\mathcal{A}_{n,N}$, explicitly*

$$M_n(\mathcal{A}_{n,N}) = M_n(\mathbb{C})\otimes\mathcal{A}_{n,N}\,,\quad \mathcal{A}_{n,N} = \mathbb{C}^{\otimes 2}_{\langle n\rangle,N}\oplus\mathbb{C}^{\boxtimes 2}_{\langle n\rangle,N}\,,\tag{40}$$

whose product is given entry-wise by $(\mathcal{P}\star\mathcal{Q})_{a,c} = \sum_{b=1}^{n}P_{a,b}\star Q_{b,c}$ for $\mathcal{P} = (P_{a,b})_{a,b=1,\dots,n}$, and $\mathcal{Q} = (Q_{a,b})_{a,b=1,\dots,n}\in M_n(\mathcal{A}_{n,N})$, and each entry $P_{a,b}$ and $Q_{b,c}$ obeys the following multiplication rule, given here on homogeneous elements of $\mathcal{A}_{n,N}$: for any $P,Q,U,W\in\mathbb{C}_{\langle n\rangle}$,

$$(U\otimes W)\star(P\otimes Q) = PU\otimes WQ\,,\tag{41a}$$

$$(U\boxtimes W)\star(P\otimes Q) = U\boxtimes PWQ\,,\tag{41b}$$

$$(U\otimes W)\star(P\boxtimes Q) = WPU\boxtimes Q\,,\tag{41c}$$

$$(U\boxtimes W)\star(P\boxtimes Q) = \mathrm{Tr}_N(WP)U\boxtimes Q\,.\tag{41d}$$

Proof Sect. 3 is the proof. □

In other words, if one computes functional renormalization of matrix models with a product different from Eq. (41), *either* contributions that do not have the one-loop structure appear in the β-functions (14), *or* it is impossible to compute the renormalization flow by splitting, in regulator-dependent and regulator-independent parts as in (29)—regardless of what $\bar{h}_k$ might be.

Remark 1 There are two interesting limiting cases,[13] large-N (together with the initial scale of the bare action $N_\infty \to \infty$) and small-N. From Fig. 1, it is evident that N-factors appear only when one-loop graphs have the "empty word" 1_N at any side. This suggests that the algebra of Theorem 2 could be reduced to Eq. (41a), but actually double-traces appear again in Eq. (41d), and $\mathrm{Tr}_N(Q_1) \times \mathrm{Tr}_N(Q_2)$ compete with terms of the form $N \, \mathrm{Tr}_N(P)$. Further, this argument should be thoroughly investigated, since the ensemble in the large-N depends also on the power-counting, that is, on the solution for the κ_α and λ_α; see the discussion just above Eq. (14). For β-functions computed with the algebra Eq. (41), see [3,Thm. 7.2] and [24]. The critical behavior could be explored in the sense of [26] as eigenvalues of the stability matrix, namely $-\mathrm{Eig}\big\{(\partial\beta_\alpha(\eta^\cdot, \{g^\cdot\})/\partial g_{\alpha'})\big\}_{\alpha,\alpha'}$, where the bullet means the fixed-point solutions of the system (14), $\beta_\alpha(\eta^\cdot, \{g^\cdot\}) = 0$ and $\eta_c(\eta^\cdot, \{g^\cdot\}) = 0$ for all interactions α and all matrices $c = 1, \ldots, n$. In the large-N, for the two-matrix model with 48 operators (that is the number of operators in a sextic truncation) compatible with the symmetries of the $ABAB$-model, the unique fixed point solution with a single positive eigenvalue of the stability matrix happens when two coupling constants have the value 0.07972 ($1/4\pi = 0.07957\ldots$ is the critical value for the coupling constants in [5], when one takes their sign and normalization conventions) and some double-trace operators like $\mathrm{Tr}_N^2(A)$, $\mathrm{Tr}_N^2(A^2)$, $\mathrm{Tr}_N(A) \times \mathrm{Tr}_N(B^3)$, do contribute to the flow (at least so with the regulator of App. A). The limit $N \to 1$ ($t \to 0$) should yield the full effective action (see limits in App. A), but this is unexplored here and needs an independent study. In the worst of the cases, the full algebra (41) is needed to next-to-leading-order or NLO corrections, but bounds on those NLO-terms are precisely the beginning of an analytic approach.

Remark 2 (The product $\star$ in terms of matrix entries) Consider the permutation $\tau = (13) \in \mathrm{Sym}(4)$ and denote by id the identity of the symmetric group $\mathrm{Sym}(4)$. Let $\rho, \pi \in \{\mathrm{id}, \tau\}$. Then, if $a, b, c, d = 1, \ldots, N$, and $Y_1, Y_2, Y_3, Y_4 \in \mathbb{C}_{\langle n \rangle} \subset M_N(\mathbb{C})$ are monomials, the four products of Theorem 2 are summarized in the following equation, where the sum over $x, y = 1, \ldots, N$ is implicit:

$$(Y_1 \otimes_\rho Y_2) \star (Y_3 \otimes_\pi Y_4)_{ab;cd} = (Y_1)_{\rho(a)\rho(b)} (Y_2)_{\rho(x)\rho(y)} (Y_3)_{\pi(y)\pi(x)} (Y_4)_{\pi(c)\pi(d)} \tag{42}$$
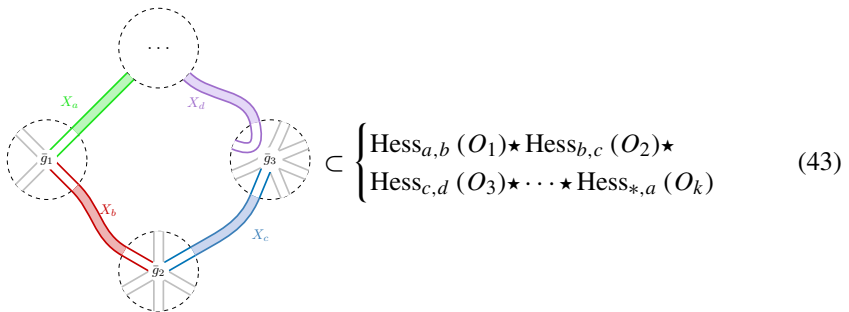
where for $\varpi \in \{\mathrm{id}, \tau\}$, $\otimes_\varpi = \otimes$ if $\varpi = \tau = (13)$ and $\otimes_\varpi = \otimes_{\mathrm{id}} = \boxtimes$ if ϖ is the trivial permutation.. Also ρ acts as element of $\mathrm{Sym}(a, b, x, y)$ and π on $\mathrm{Sym}(y, x, c, d)$. For instance, in the nontrivial case $\rho = \tau$, $\tau(a, b, x, y) = (x, b, a, y)$. We remark that in order to keep the Hessian simple in this paper, the convention is the opposite of [3], i.e., $\otimes_\tau$ there is $\otimes$ here; and the $\otimes$ of [3] corresponds to the $\boxtimes$ here. The particular permutation $\tau = (13)$ might seem at first arbitrary, but it is actually natural and can be found in *op.cit.* or in [6,Eq. 5].

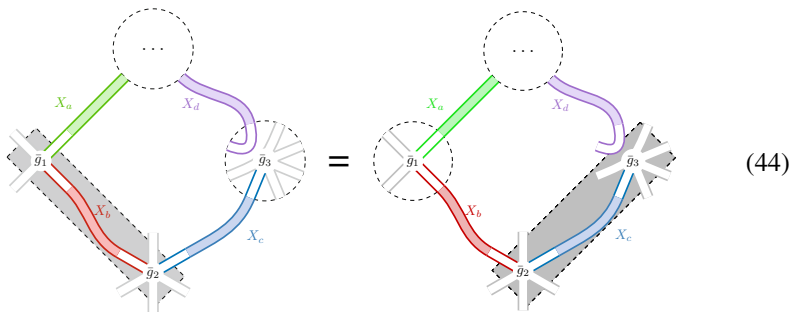[13] I thank Răzvan Gurău for questions that motivated this remark (which gives partial answers).

3 The proof of the main statement

Figure 4 gives the logic structure in the proof. By "$s \subset \mathrm{Hess}_{a,b}(O)$" we abbreviate that s is a summand in $\mathrm{Hess}_{a,b}(O)$. Further, $M, L, P, Q, R, S, T, U, V, W \in \mathbb{C}_{\langle n \rangle}$ are arbitrary monomials.

Proof of Theorem 2 Start with the kth power of a Hessian. First, we argue that we can simplify this situation and deduce the behavior regarding the kth power for any k from the square of a Hessian. Supertraces of products of Hessians will be sums over terms of the following form:

$$\subset \begin{cases} \mathrm{Hess}_{a,b}(O_1) \star \mathrm{Hess}_{b,c}(O_2) \star \\ \mathrm{Hess}_{c,d}(O_3) \star \cdots \star \mathrm{Hess}_{*,a}(O_k) \end{cases} \tag{43}$$

The associativity of the product $\star$ follows from the definition of effective vertices (but should be verified purely algebraically, after the product is constructed):

$$\tag{44}$$

where the gray boxes with uncontracted, protruding ribbon edges mean the new interactions formed from the two grouped interaction vertices. The new cyclic order is determined by the propagator, together with the half-edges it is attached to, being shrunk. The left corresponds to the $[\mathrm{Hess}_{a,b}(O_1) \star \mathrm{Hess}_{b,c}(O_2)] \star \mathrm{Hess}_{c,d}(O_3)$ bracketing, while the right one to $\mathrm{Hess}_{a,b}(O_1) \star [\mathrm{Hess}_{b,c}(O_2) \star \mathrm{Hess}_{c,d}(O_3)]$.

We have four cases, depending on the way the four propagators in the loop connect the interaction vertices of $k = 2$ interaction vertices. The fact that $\mathscr{A} = M_n(\mathcal{A}_n)$ is an associative algebra (or recursive application of (44)) allows us not to consider more cases. However, to determine the product, $k = 3, 4$ will yield also useful information too.
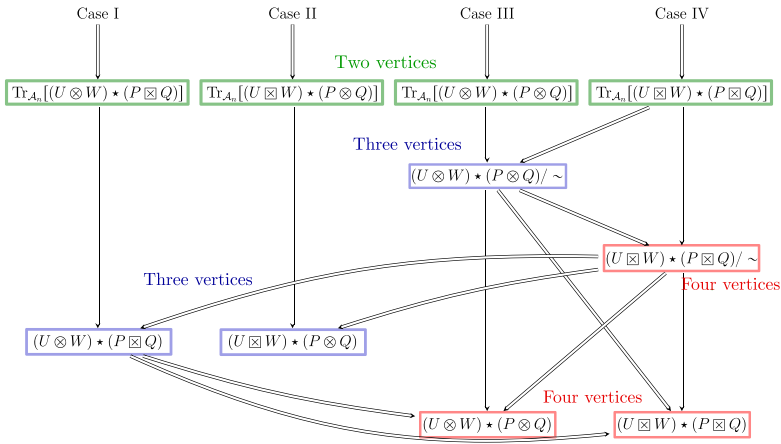
(a) In the first row (top to bottom), we have the four different cases in one-loop diagrams in multi-trace matrix models. From each, we can recognize a product type in the lhs of Eq. (41). Starting from each case one can, autonomously, determine the value for the trace $\mathrm{Tr}_{\mathcal{A}_n}$ of each of these products using two interaction vertices. As one wants to determine the product itself, more information (which turns out to be delivered by looking at other cases) is needed: For instance, knowing both traces for cases III and IV, one can determine the product in case III up to the transformation $X \otimes Y + V \boxtimes Z \mapsto \widetilde{X \otimes Y} + \widetilde{V \boxtimes Z} = Y \otimes X + Z \boxtimes V$ (which is marked with "modulo $\sim$") by using a third interaction vertex, et cetera, as determined by this diagram. Observe that, although we combine the cases, and present each column at once, there is no (oriented) loop.
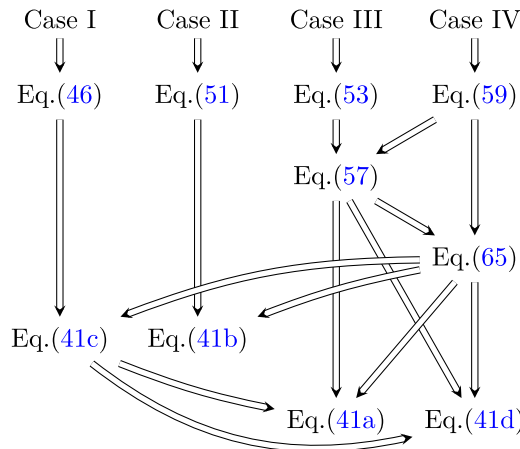
(b) The equation numbers corresponding with Figure 4a.

Fig. 4 The "topology" of the proof of Theorem 2 showing the absence of logic loops, notwithstanding the mix of cases in the proof. The arrows are implications. These diagrams show how we "bootstrap" the algebra

- CASE I: *When two ribbons in the loop lie in the same trace in the first interaction vertex, but in different traces in the second:*

$$\subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,a}\, O_2 \qquad (45)$$
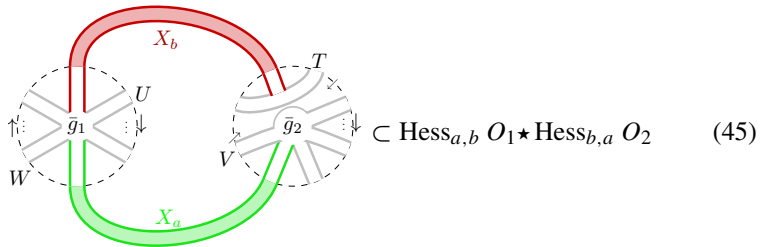
Suppose that the interaction vertices have one and two traces, respectively. In fact they might have more traces, but these not being implied in the loop for the present case, they remain intact; thus, we do not loose generality by this simplification. There exist then words T, U, V, W (which might be empty) such that

$$O_1 = \bar{g}_1 \,\mathrm{Tr}(U X_a W X_b) \quad \text{and} \quad O_2 = \bar{g}_2 \,\mathrm{Tr}(X_b T)\,\mathrm{Tr}(X_a V)\,.$$

The words $T, U, V, W \in \mathbb{C}_{\langle n \rangle}$ might contain the letters X_a, X_b, but we are analyzing only the summand in the lhs of (45). To compute the contribution of the two Hessians to this precise summand, we get by Eq. (8)

$$\mathrm{Hess}_{a,b}\, O_1 \supset \bar{g}_1 \partial_{X_a} \circ \partial_{X_b}\, \mathrm{Tr}(U X_a W X_b) = \bar{g}_1 U \otimes W\,,$$

and by Eq. (34),

$$\mathrm{Hess}_{b,a}\, O_2 \supset \bar{g}_2\, \mathscr{D}_{X_b}\, \mathrm{Tr}(X_b T) \boxtimes \mathscr{D}_{X_a}\, \mathrm{Tr}(X_a V) = \bar{g}_2 T \boxtimes V\,.$$

Now, since the effective vertex of (45) is formed by shrinking the green and red propagators and merging the rest of the ribbon half-edges while preserving the order, the graph (45) implies that the effective vertex is $\bar{g}_1 \bar{g}_2\, \mathrm{Tr}_N(WTUV)$. By Wetterich equation,

$$\mathrm{Tr}_{\mathcal{A}_n}[(U \otimes W)\star(V \boxtimes T)] = \mathrm{Tr}_N(WTUV)\,. \qquad (46)$$

Since the lhs is a single trace, this is enough to conclude that the result of $(U \otimes W)\star(V \boxtimes T)$ must be "a $\boxtimes$ inserted somewhere in the cyclic word $WTUV$", otherwise it would be a product of the form $w_1 \otimes w_2$ which, when traced, would yield a N-factor, in case that any of the words w_1 or w_2 is trivial, and a double trace if both are not trivial. We also know that the result of $(U \otimes W)\star(V \boxtimes T)$ must be an ordinary and not a cyclic word; thus, so far, we need to know how to

root it, i.e., the expression for $(U \otimes W) \star (V \boxtimes T)$ should be listed in

$$
\begin{array}{lllll}
1 \boxtimes WTUV, & W \boxtimes TUV, & WT \boxtimes UV, & WTU \boxtimes V, & WTUV \boxtimes 1, \\
1 \boxtimes TUVW, & T \boxtimes UVW, & TU \boxtimes VW, & TUV \boxtimes W, & TUVW \boxtimes 1, \\
1 \boxtimes UVWT, & U \boxtimes VWT, & UV \boxtimes WT, & UVW \boxtimes T, & UVWT \boxtimes 1, \\
1 \boxtimes VWTU, & V \boxtimes WTU, & VW \boxtimes TU, & VWT \boxtimes U, & VWTU \boxtimes 1.
\end{array}
$$
$$(47)$$

To discard the wrong ones, we first consider the following interaction vertices:

$$
\begin{aligned}
O_1 &= \bar{g}_1 \operatorname{Tr}_N (X_b W X_a U), \\
O_2 &= \bar{g}_2 \operatorname{Tr}_N (X_b T) \operatorname{Tr}_N (X_c V), \\
O_3 &= \bar{g}_3 \operatorname{Tr}_N (X_c R) \operatorname{Tr}_N (S X_a).
\end{aligned}
$$

and the corresponding product of Hessians of each of these (in that order), which contains in particular, the next graph:

 $\subset \operatorname{Hess}_{a,b} O_1 \star \operatorname{Hess}_{b,c} O_2 \star \operatorname{Hess}_{c,a} O_3$
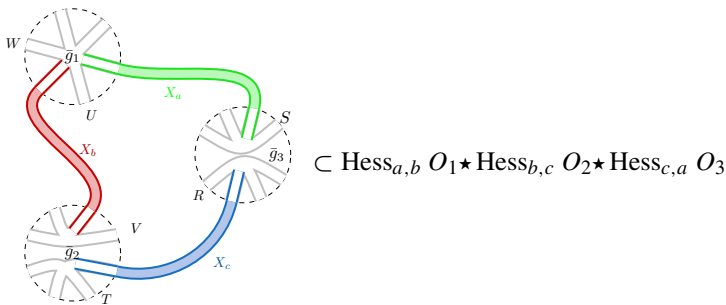
The effective vertex must be $\operatorname{Tr}_N(VWSU) \operatorname{Tr}_N(RT)$, so

$$
\operatorname{Tr}_{\mathcal{A}_n} \left\{ [(W \otimes U) \star (V \boxtimes T)] \star (R \boxtimes S) \right\} = \operatorname{Tr}_N (VWSU) \times \operatorname{Tr}_N (RT). \quad (48)
$$

One can use the previous graph to discard elements in the list (47). For instance, we suppose that $(W \otimes U) \star (V \boxtimes T) = W \boxtimes TUV$. For the product inside curly brackets $\{\ldots\}$, using Eq. (60c) or Eq. (60d) (equivalently, Eq. (65); Fig. 4b), one gets the following possibilities:

$$
= \begin{cases} S \boxtimes W \operatorname{Tr}_N (RTUV) & \text{if Eq. (60c) holds}, \\ W \boxtimes S \operatorname{Tr}_N (RTUV) & \text{if Eq. (60d.) holds}. \end{cases} \quad (49)
$$

But the trace of it yields in either case $\operatorname{Tr}_N(SW) \operatorname{Tr}_N(RTUV)$ which differs from Eq. (48). Thus, $(W \otimes U) \star (V \boxtimes T) = W \boxtimes TUV$ is impossible. By the same token, with the same counterexample above, one discards the possibilities that do not contain a factor of the empty word 1, except $(W \otimes U) \star (V \boxtimes T) = UVW \boxtimes T$.

Regarding those possibilities containing the factor of 1, following any of the prescription of the leftmost column in (47) for the square brackets product, and the Case IV, which is to say either Eq. (60c) or Eq. (60d), for the resulting multiplication of the form $w_1 \boxtimes w_2 \star w_3 \boxtimes w_4$, one easily sees that these generate a factor $\mathrm{Tr}_N(S)$; likewise, those possible products on the rightmost columns (47) generate a factor $\mathrm{Tr}_N(R)$. Both lead then to contradiction with the previous graph. Therefore, indeed $(W \otimes U) \star (V \boxtimes T) = UVW \boxtimes T$, i.e., Eq. (41c) holds.

- CASE II: *When two ribbons in the loop lie in the same trace in the first interaction vertex, but in different traces in the second:* This case is proven by swapping the roles of the first and second interaction vertices in Case I. Since the proof is analogous, we rather sketch it. The next graph can be used

$$
\text{} \quad \subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,a}\, O_2 \tag{50}
$$

for suitable operators O_1 and O_1 (constructed in a similar way to Case I) to obtain

$$
\mathrm{Tr}_{\mathcal{A}_n}\left(U \boxtimes W \star P \otimes Q\right) = \mathrm{Tr}_N(QUPW). \tag{51}
$$

From Eqs. (60c) and (60d) (see Fig. 4) and the graph

$$
\text{} \quad \subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,c}\, O_2 \star \mathrm{Hess}_{c,a}\, O_3 ,
$$

one can create suitable operators O_1, O_2, O_3 to deduce Eq. (41b).

- CASE III: *When two ribbons in the loop lie on the same trace in both the first and second interaction vertices:*

$$
\text{} \quad \subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,a}\, O_2 . \tag{52}
$$

We consider operators $O_1 = \bar{g}_1 \operatorname{Tr}_N(X_a W X_b U)$ and $O_2 = \bar{g}_2 \operatorname{Tr}_N(X_a P X_b Q)$. These might have more traces, but as depicted above, these being outside the loop, do not suffer any transformation (in that summand) and can be ignored. Then, $\operatorname{Hess}_{a,b} O_1 \star \operatorname{Hess}_{b,a} O_2 = (U \otimes W) \star (P \otimes Q)$. According to Wetterich equation, the effective vertex must be

$$\operatorname{Tr}_{\mathcal{A}_n}[(U \otimes W) \star (P \otimes Q)] = \operatorname{Tr}_N(PU) \operatorname{Tr}_N(WQ). \tag{53}$$

which implies either of the following possibilities:

$$(U \otimes W) \star (P \otimes Q) = \operatorname{Tr}_N(PU) W \boxtimes Q \tag{54a}$$
$$(U \otimes W) \star (P \otimes Q) = \operatorname{Tr}_N(PU) Q \boxtimes W \tag{54b}$$
$$(U \otimes W) \star (P \otimes Q) = PU \otimes QW \tag{54c}$$
$$(U \otimes W) \star (P \otimes Q) = UP \otimes QW \tag{54d}$$
$$(U \otimes W) \star (P \otimes Q) = UP \otimes WQ \tag{54e}$$
$$(U \otimes W) \star (P \otimes Q) = PU \otimes WQ \tag{54f}$$

To obtain the right one(s), we consider now the third power of the Hessian, but in the contraction with the additional vertex Case IV shall be here useful. By contradiction to each of the cases, we suppose that Eq. (54a) holds. Then, consider the following interaction vertices:

$$O_1 = \bar{g}_1 \operatorname{Tr}_N(X_a W X_b U),$$
$$O_2 = \bar{g}_2 \operatorname{Tr}_N(X_b Q X_c P),$$
$$O_3 = \bar{g}_3 \operatorname{Tr}_N(X_c R) \operatorname{Tr}_N(S X_a).$$

By a similar ribbon graph argument, we obtain, using the hypothesis, that $(\operatorname{Hess}_{a,b} O_1 \star \operatorname{Hess}_{b,c} O_2) \star \operatorname{Hess}_{c,a} O_3$ which is, modulo the coupling constants $[(U \otimes W) \star (P \otimes Q)] \star (R \boxtimes S) = [\operatorname{Tr}_N(PU) W \boxtimes Q] \star (R \boxtimes S)$. Applying $\operatorname{Tr}_{\mathcal{A}_n}$ to this quantity we deduce, according to the partial conclusion in Case IV (recalling that the bracketing is irrelevant due to Eq. (44)), that

$$\operatorname{Tr}_{\mathcal{A}_n} \left\{ (U \otimes W) \star (P \otimes Q) \star (R \boxtimes S) \right\} = \operatorname{Tr}_N(PU) \operatorname{Tr}_{\mathcal{A}_n} \left\{ (W \boxtimes Q) \star (R \boxtimes S) \right\}$$
$$= \operatorname{Tr}_N(PU) \operatorname{Tr}_N(QR) \operatorname{Tr}_N(WS). \tag{55}$$

However, by looking at the graph that the product in curly brackets represents, the previous equation cannot be true, for the graph leads to a single trace, namely $\operatorname{Tr}_N(WQRPUS)$. This is a contradiction with the supposition that $(U \otimes W) \star (P \otimes Q) = \operatorname{Tr}_N(PU) W \boxtimes Q$. Hence, we discard Eq. (54a). By the same argument in

number of traces, we discard also Eq. (54b). Further, with the next counterexample

$$\subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,c}\, O_2 \star \mathrm{Hess}_{c,a}\, O_3\,, \qquad (56)$$

obtained from the Hessians of the operators

$$O_1 = \bar{g}_1 \operatorname{Tr}_N(X_a W X_b U)\,,\ \ O_2 = \bar{g}_2 \operatorname{Tr}_N(X_b Q X_c P)\,,\ \ O_3 = \bar{g}_3 \operatorname{Tr}_N(X_a R X_c S)\,.$$

Then, two further possibilities are ruled out, for, on the one hand, Eq. (54c) implies that the effective vertex is $\operatorname{Tr}_N(RPU)\operatorname{Tr}_N(SQW)$; and Eq. (54e), on the other hand, implies that it is $\operatorname{Tr}_N(UPR)\operatorname{Tr}_N(WQS)$, modulo coupling constants. Either is different from the effective vertex for the graph (56), namely $\operatorname{Tr}_N(WQS)\operatorname{Tr}_N(RPU)$. So only the next two are possible:

$$(U \otimes W)\star(P \otimes Q) = UP \otimes QW \qquad (U \otimes W)\star(P \otimes Q) = PU \otimes WQ\,. \tag{57}$$

We solve now Case IV and then determine which of the two is the right expression.

- CASE IV: *When two ribbons in the loop lie on different traces in both interaction vertices:*

$$\subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,a}\, O_2 \qquad (58)$$

Notice that, notwithstanding the disconnectedness of the ribbons in this graph G_{IV}, as a graph in the field theory context, what matters is the connectivity of its skeleton G_{IV}° ($\bar{g}_i$ are the coupling constants for both traces inside the dashed circle, cf. Def 2). We construct now operators that yield the desired product. Let $O_1 = \bar{g}_1 \operatorname{Tr}_N(UX_a)\operatorname{Tr}_N(X_b W)$ and $O_2 = \bar{g}_2 \operatorname{Tr}_N(QX_a)\operatorname{Tr}_N(X_b P)$. Then, the product of Hessians in Eq. (58) contains $(U \boxtimes W)\star(P \boxtimes Q)$ as a summand. The effective vertex must be what we obtain by shrinking the propagators. In turn, in the rhs of Wetterich Eq. (28) this effective vertex is obtained by tracing over[14] $\mathcal{A}_n$,

---

[14] Actually one has to trace over $M_n(\mathcal{A}_n)$, i.e., take the supertrace. But the trace corresponding to the M_n-block matrix was already taken in Eq. (58).

so

$$\text{Tr}_{\mathcal{A}_n}[(U \boxtimes W)\star(P \boxtimes Q)] = \text{Tr}_N(WP) \times \text{Tr}_N(UQ). \tag{59}$$

This means that the quantity in square brackets must be either of the following product formulas for $\star$:

$$(U \boxtimes W) \star (P \boxtimes Q) = P \boxtimes W \,\text{Tr}_N(QU) \tag{60a}$$
$$(U \boxtimes W) \star (P \boxtimes Q) = W \boxtimes P \,\text{Tr}_N(QU) \tag{60b}$$
$$(U \boxtimes W) \star (P \boxtimes Q) = Q \boxtimes U \,\text{Tr}_N(PW) \tag{60c}$$
$$(U \boxtimes W) \star (P \boxtimes Q) = U \boxtimes Q \,\text{Tr}_N(PW) \tag{60d}$$
$$(U \boxtimes W) \star (P \boxtimes Q) = PW \otimes QU \tag{60e}$$
$$(U \boxtimes W) \star (P \boxtimes Q) = WP \otimes QU \tag{60f}$$
$$(U \boxtimes W) \star (P \boxtimes Q) = WP \otimes UQ \tag{60g}$$
$$(U \boxtimes W) \star (P \boxtimes Q) = PW \otimes UQ \tag{60h}$$

To determine the correct product, we consider higher powers of the Hessian, and one of the partial conclusion of the Case III, Eq. (53).

By contradiction, suppose that Eq. (60a) holds. Then applying twice this equation,

$$[(U \boxtimes W)\star(P \boxtimes Q)]\star(R \boxtimes S) = \text{Tr}_N(QU)P \boxtimes W\star R \boxtimes S$$
$$= \text{Tr}_N(QU)\,\text{Tr}(SP)R \boxtimes W \tag{61}$$

which when is traced in $\mathcal{A}_n$ yields

$$\text{Tr}_{\mathcal{A}_n}\left\{[(U \boxtimes W)\star(P \boxtimes Q)]\star(R \boxtimes S)\right\} = \text{Tr}_N(QU)\,\text{Tr}_N(RW)\,\text{Tr}_N(SP). \tag{62}$$

However, if we pick the next observables,

$$O_1 = \bar{g}_1 \,\text{Tr}_N(UX_a)\,\text{Tr}_N(WX_b)$$
$$O_2 = \bar{g}_2 \,\text{Tr}_N(PX_b)\,\text{Tr}_N(QX_c)$$
$$O_3 = \bar{g}_3 \,\text{Tr}_N(RX_c)\,\text{Tr}_N(SX_a)$$

the effective vertex for the summand

 $\subset \text{Hess}_{a,b}\, O_1 \star \text{Hess}_{b,c}\, O_2 \star \text{Hess}_{c,a}\, O_3\,,$
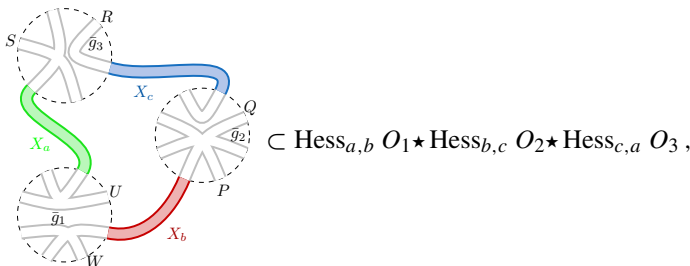
must be $\mathrm{Tr}_N(SU)\,\mathrm{Tr}_N(PW)\,\mathrm{Tr}_N(QR)$, which is a contradiction with Eq. (62). Thus, Eq. (60a) is impossible. By the same token, one sees that Eq. (60b) leads to an effective vertex $\mathrm{Tr}_N(PR)\,\mathrm{Tr}_N(QU)\,\mathrm{Tr}_N(SW)$, which differs from $\mathrm{Tr}_N(SU)\,\mathrm{Tr}_N(PW)\,\mathrm{Tr}_N(QR)$. Thus, Eq. (60b) is not the right product either.

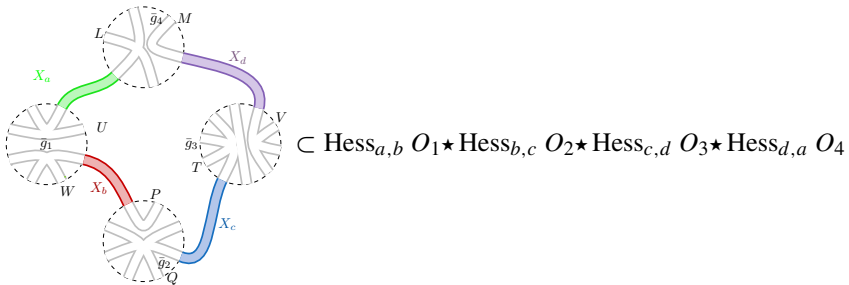To rule out further products, we go to fourth degree in the Hessian. Suppose that Eq. (60e) holds. Then,

$$
[U \boxtimes W \star P \boxtimes Q] \star [T \boxtimes V \star M \boxtimes L] = PW \otimes QU \star MV \otimes LT
$$
$$
= \begin{cases} PWMV \otimes LTQU \\ MVPW \otimes QULT \end{cases} \tag{63}
$$

where the last equality lists the possibilities Eq. (54d) or Eq. (54f). In either case, Eq. (60e) holds, then the trace of Eq. (63) reads $\mathrm{Tr}_N(PWMV) \times \mathrm{Tr}_N(LTQU)$. Again, if we consider the operators

$$
O_1 = \bar{g}_1 \,\mathrm{Tr}_N(UX_a)\,\mathrm{Tr}_N(WX_b) \qquad O_2 = \bar{g}_2 \,\mathrm{Tr}_N(PX_b)\,\mathrm{Tr}_N(QX_c) \tag{64a}
$$
$$
O_3 = \bar{g}_3 \,\mathrm{Tr}_N(TX_c)\,\mathrm{Tr}_N(VX_d) \qquad O_4 = \bar{g}_4 \,\mathrm{Tr}_N(MX_c)\,\mathrm{Tr}_N(LX_a). \tag{64b}
$$

we get from the summand

$$
\subset \mathrm{Hess}_{a,b}\,O_1 \star \mathrm{Hess}_{b,c}\,O_2 \star \mathrm{Hess}_{c,d}\,O_3 \star \mathrm{Hess}_{d,a}\,O_4
$$

in the fourth power of the Hessian the effective vertex $\mathrm{Tr}_N(LU)$ $\mathrm{Tr}_N(WP)\,\mathrm{Tr}_N(QT)\,\mathrm{Tr}_N(VM)$. Since not even the number of traces coincides, Eq. (60e) is impossible. By the same trace-counting argument, the same operators (64) serve as a counterexample for the products Eqs. (60f), (60g) and (60h). This leaves us only with possibilities Eqs. (60c) and (60d):

$$
(U \boxtimes W) \star (P \boxtimes Q) = \begin{cases} Q \boxtimes U \,\mathrm{Tr}_N(PW) \\ U \boxtimes Q \,\mathrm{Tr}_N(PW) \end{cases} \tag{65}
$$

To finish the proof, we have to determine which of are the correct products, consider the operators

$$O_1 = \bar{g}_1 \operatorname{Tr}_N(X_a V X_b T),$$
$$O_2 = \bar{g}_2 \operatorname{Tr}_N(X_b W X_c U),$$
$$O_3 = \bar{g}_3 \operatorname{Tr}_N(X_c P) \operatorname{Tr}_N(X_d Q),$$
$$O_4 = \bar{g}_4 \operatorname{Tr}_N(X_d R) \operatorname{Tr}_N(X_a S),$$

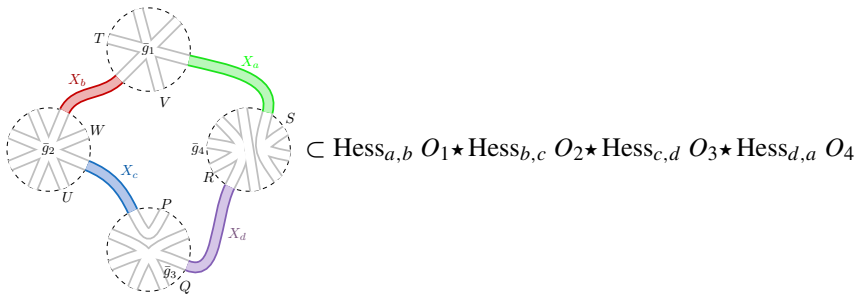and the product of their Hessian (entries)

$$(T \otimes V \star U \otimes W) \star (P \boxtimes Q \star R \boxtimes S)$$

This expression is given by

$$
= \begin{cases} TU \otimes WV & \text{if Eq. (60c) holds} \\ UT \otimes VW & \text{if Eq. (60d) holds} \end{cases} \star \begin{cases} S \otimes P \operatorname{Tr}_N(QR) & \text{if Eq. (54d) holds} \\ P \otimes S \operatorname{Tr}_N(QR) & \text{if Eq. (54f) holds} \end{cases}
$$

$$
= \begin{cases} \operatorname{Tr}_N(QR) WV STU \boxtimes P & \text{if Eq. (60c)\&Eq. (54d) hold} \\ \operatorname{Tr}_N(QR) WV PTU \boxtimes S & \text{if Eq. (60c)\&Eq. (54f) hold} \\ \operatorname{Tr}_N(QR) VW SUT \boxtimes P & \text{if Eq. (60d)\&Eq. (54d) hold} \\ \operatorname{Tr}_N(QR) VW PUT \boxtimes S & \text{if Eq. (60d)\&Eq. (54f) hold} \end{cases} \tag{66}
$$

However, since the graph

 $\subset \operatorname{Hess}_{a,b} O_1 \star \operatorname{Hess}_{b,c} O_2 \star \operatorname{Hess}_{c,d} O_3 \star \operatorname{Hess}_{d,a} O_4$

has an $\mathcal{A}_n$-trace equal to $\operatorname{Tr}_N(QR) \times \operatorname{Tr}_N(WPUTSV)$, only the last choice is possible. This proves at once Eqs. (60d) and (54f), that is Eqs. (41d) and (41a), respectively.

Remark 3 (On well-definedness of the graphical representation.) Example 1 shows a phenomenon that is more general: the asymmetry of the M_n-block structure of the nc Hessian matrix, $\operatorname{Hess}_{a,b} \neq \operatorname{Hess}_{b,a}$. Nevertheless, a weaker symmetry persists. Since the swap of X_a and X_b in Eq. (8) leads to the exchange π_1 with π_2, we conclude that for any interaction vertex O,

$$\operatorname{Hess}_{a,b} O = \widetilde{\operatorname{Hess}_{b,a} O} \quad \text{where} \quad \widetilde{(P \otimes Q)} = Q \otimes P \text{ and } \widetilde{(P \boxtimes Q)} = Q \boxtimes P, \tag{67}$$

for each $P, Q \in \mathbb{C}_{\langle n \rangle}$; the exchange $P \boxtimes Q \to Q \boxtimes P$ follows by Definition 4. This makes the present construction independent of the choice of "inner" and "outer" loop, as well as the orientation of the interaction vertices in the one-loop (whether the Hessians of $O_1, O_2, \ldots, O_k$ being multiplied means we draw $\bar{g}_1, \ldots, \bar{g}_k$ clockwise or anti-clockwise as in Fig. 2) for the following reason. First, observe that using the algebra obtained in Theorem 2 one can easily derive

$$\widetilde{\mathfrak{a} \star \mathfrak{b}} = \tilde{\mathfrak{b}} \star \tilde{\mathfrak{a}} \qquad \text{for each } \mathfrak{a}, \mathfrak{b} \in \mathcal{A}_n, \tag{68}$$

Let $\mathfrak{h} := \mathrm{Hess}_{a_1,a_2} O_1 \star \mathrm{Hess}_{a_2,a_3} O_2 \star \cdots \star \mathrm{Hess}_{a_k,a_1} O_k$, for fixed $a_i = 1, \ldots, n$ and for some fixed interaction vertices O_i $(i = 1, \ldots, k)$. Then, by Eq. (67)

$$\mathrm{Hess}_{a_1,a_k} O_k \star \cdots \star \mathrm{Hess}_{a_3,a_2} O_2 \star \mathrm{Hess}_{a_2,a_1} O_1$$
$$= \widetilde{\mathrm{Hess}_{a_k,a_1} O_k} \star \widetilde{\mathrm{Hess}_{a_2,a_3} O_2} \star \cdots \star \widetilde{\mathrm{Hess}_{a_1,a_2} O_1},$$

and because of Eq. (68), this last expression equals $\tilde{\mathfrak{h}}$. But according to Lemma 35, $\mathrm{Tr}_{\mathcal{A}_n} \tilde{\mathfrak{h}} = \mathrm{Tr}_{\mathcal{A}_n} \mathfrak{h}$. Thus, if $\mathfrak{h}$ contributes to the flow, so does $\tilde{\mathfrak{h}}$, and in an equal way, yielding our description independent on the cyclic orientation we choose for drawing the interaction vertices. What we just proved can also be pictorially justified:

$$\mathfrak{h} = \qquad \Rightarrow \qquad = \tilde{\mathfrak{h}} \tag{69}$$
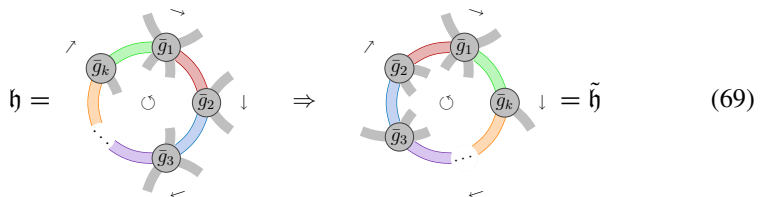
Notice that these are the only two representations that the cyclic orientation of each vertex O_i allows (meaning, if one inverts the order of the interaction vertices).

The final piece of well-definedness is that the product found here is indeed associative, without using graphs. The purely algebraic proof is routine (and can be found in [3]).

Example 6 Once proven the main statement, we can use the algebra to exemplify a typical contribution to the renormalization flow in a Hermitian 3-matrix model. Consider two operators $O_1 = \frac{\bar{g}_1}{2}[\mathrm{Tr}_N(\frac{A^2}{2})]^2$ and $O_2 = \bar{g}_2 \mathrm{Tr}_N(ABC)$. Suppose we wish to determine the $\bar{g}_1 \bar{g}_2^2$-coefficient of the rhs of Wetterich equation. We need (essentially) the Hessian of O_1 and $[\mathrm{Hess}\, O_2]^{\star 2}$. The former has only one nonzero entry,

$$\mathrm{Hess}_{I,J}\, O_1 = \delta_I^J \delta_I^A \bar{g}_1 \{ \underbrace{\mathrm{Tr}_N(A^2/2)[1_N \otimes 1_N]}_{} + \underbrace{A \boxtimes A}_{} \}, \tag{70}$$

where a "filled ribbon" means that that half-edge is contracted in the one-loop graph, and an "empty ribbon" that it is not (and therefore contributes to the final effective

vertex). We also have

$$\text{Hess } O_2 = \bar{g}_2 \begin{bmatrix} 0 & C \otimes 1_N & B \otimes 1_N \\ 1_N \otimes C & 0 & A \otimes 1_N \\ 1_N \otimes B & 1_N \otimes A & 0 \end{bmatrix} \tag{71}$$

getting

$$[\text{Hess } O_2]^{\star 2} = \bar{g}_2^2 \begin{bmatrix} C \otimes C + B \otimes B & B \otimes A & C \otimes A \\ A \otimes B & A \otimes A + C \otimes C & C \otimes B \\ A \otimes C & B \otimes C & B \otimes B + A \otimes A \end{bmatrix}. \tag{72}$$

Only the black-colored entry will contribute, since Hess O_1's (11)-entry is the only non-vanishing. In the (11)-entry, the term

$C \otimes C$ corresponds to the graph and $B \otimes B$ to

(the horizontal green edges still to be contracted in the loop with those in Hessian (70) that are also filled). Finally, we extract the coefficients

$$[\bar{g}_1 \bar{g}_2]\text{STr}\{\text{Hess } O_1[\text{Hess } O_2]^{\star 2}\}$$
$$= \text{Tr}_{\mathcal{A}_n}\left\{[\text{Tr}_N(A^2/2)[1_N \otimes 1_N] + A \boxtimes A]\star[C \otimes C + B \otimes B]\right\}$$
$$= \text{Tr}_{\mathcal{A}_n}\left\{\text{Tr}_N(A^2/2)(C \otimes C + B \otimes B) + A \boxtimes CAC + A \boxtimes BAB\right\}$$
$$= \text{Tr}_N(A^2/2) \times [\text{Tr}_N^2 C + \text{Tr}_N^2 B] + \text{Tr}_N(ACAC + ABAB),$$

which are effective vertices of the four one-loop graphs that can be formed with the contractions of (the filled ribbon half-edges of)

any of $\left\{ \text{}, \text{} \right\}$ with any of $\left\{ \text{}, \text{} \right\}$.

Of course, this is a toy-example: the algebraic structure pays off with higher-power interactions and/or higher number of matrices (whose flow becomes inaccessible by traditional methods and can hardly be cross-checked using graphs, due to the large amount of these; cf. the supplementary material of [3]).

4 Conclusion

The algebraic structure of functional renormalization of Hermitian n-matrix models with interactions containing several traces has been addressed. Under the assumption that it is possible to compute the flow in terms of U(N)-invariant operators, the present

result completely describes the regulator-independent part of the flow. This paper complements[15] [3]. There, for multi-matrix models with multi-traces, Wetterich equation was proven, and in the middle of the proof one were able to read off the algebraic structure, (41). Computations of β-functions using (41) revealed a one-loop structure in [3]. Here we showed the converse: the one-loop structure requires the algebra of functional renormalization (i.e., the structure that makes the rhs of Wetterich equation computable for such matrix models) to be Eq. (41), showing its uniqueness.

As a final perspective, the present results can be useful to connect different renormalization theories, e.g., [30, 31]. Also, Fig. 1 is strikingly reminiscent of the Connes–Kreimer residue defining the coproduct (of their renormalization Hopf-algebra [32]). Between those, the algebraic language could build a shorter bridge than graph theory—all the more, algebra can be coded more directly than graphs.

Declarations

Conflict of interest statement The author states that there is no conflict of interest.

Appendix A On Γ_N and R_N

Although our aim in this paper is rather algebraic, we sketch how the FRG works, for sake of completeness. Starting from the bare action $S[\Phi]$ on $S : \mathcal{H}_N^n \to \mathbb{R}$, in order to construct the effective action $\Gamma_N[\mathbb{X}]$ one first regulates the connected partition function $\log \mathcal{Z}[\mathbb{J}] = \log \int_{\mathcal{H}_N^n} \exp[-S[\Phi] + \sum_{c=1}^{n} \mathrm{Tr}(J_c \Phi_c)] \mathrm{d}\Phi_{\mathrm{LEB}}$ (where $\mathbb{J} = (J_c) \in \mathcal{H}_N^n$)

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[15] To fully implement the flow for Dirac ensembles, which is the aim of [3] either operators with broken unitary symmetry could be introduced. Another perspective is to implement a Ward-constrained flow [29].

by adding a mass-like term, $\frac{1}{2} r_{c,N} \Phi_c^2$, to each matrix:

$$\mathcal{W}_N[\mathbb{J}] = \log \int_{\mathcal{H}_N^n} \exp\left\{ -S[\Phi] - \sum_{c=1}^n \operatorname{Tr}_N \left[\frac{1}{2} r_{c,N} \Phi_c^2 + J_c \Phi_a \right] \right\} d\Phi_{\mathrm{LEB}}$$

where $r_{c,N}$ is the next *infrared regulator*, given in terms of the Heaviside or indicator function $\Theta_{\mathbb{D}_N}$ on the disk $\mathbb{D}_N = \{(a, b) \in \mathbb{N}^2 \mid a^2 + b^2 \leq N^2\}$ by

$$r_{c,N}(a, b) = Z_c \cdot \left[\frac{N^2}{a^2 + b^2} - 1 \right] \cdot \Theta_{\mathbb{D}_N}(a, b), \text{ or plotted:}$$
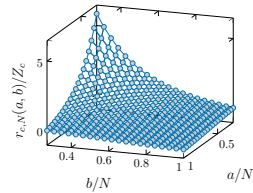
$$(\mathrm{A1})$$

with Z_c the "wave function renormalization" of the matrix X_c (see [3] for details on the dependence on the "classical fields" $\mathbb{X} = (X_1, \ldots, X_n) := (\partial_{J_1} \mathcal{W}, \ldots, \partial_{J_n} \mathcal{W}) \in \mathcal{H}_N^n$). Other regulators are not discussed here, but another regulator should comply with the following limits: in order for $r_{c,N}$ to integrate out only the "higher modes", i.e., matrix entries $\left\{ (X_c)_{i,j} \right\}_{i,j \geq N; c=1,\ldots,n}$ above the *energy scale* N, one imposes that $r_{c,N}(a, b) = 0$ if $a > N$ or $b > N$, which explains the presence of $\Theta_{\mathbb{D}_N}$ in this particular choice. The additional condition $r_{c,N} > 0$ creates a mass-like term for the low-energy modes $\left\{ (X_c)_{i,j} \right\}_{0 < i,j < N; c=1,\ldots,n}$ that protects these from being integrated out. One wants, moreover, to recover the bare action via saddle point approximation as $N \to \infty$ and thus $r_{c,N} \to \infty$ is necessary in that limit. The interpolating *effective action* is then constructed by taking the Legendre transform, namely $\Gamma_N[\mathbb{X}] = \sup_{J_1,\ldots,J_n} \sum_{c=1}^n \left\{ \operatorname{Tr}_N(X_c J_c) - \mathcal{W}_N[J_1, \ldots, J_n] - \frac{1}{2} \operatorname{Tr}_N(r_{c,N} X_c^2) \right\}$. The regulator R_N that appears in the main text is

$$R_N = \begin{bmatrix} r_{1,N} 1_N \otimes 1_N & 0 & \ldots & 0 \\ 0 & r_{2,N} 1_N \otimes 1_N & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & r_{n,N} 1_N \otimes 1_N \end{bmatrix}.$$

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