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Improved energy estimates for a class of time-dependent perturbed Hamiltonians

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Abstract

We consider time-dependent perturbations which are relatively bounded with respect to the square root of an unperturbed Hamiltonian operator, and whose commutator with the latter is controlled by the full perturbed Hamiltonian. The perturbation is modulated by two auxiliary parameters, one regulates its intensity as a prefactor and the other one controls its time-scale via a regular function, whose derivative is compactly supported in a finite interval. We introduce a natural generalization of energy conservation in the case of time-dependent Hamiltonians: the boundedness of the two-parameter unitary propagator for the physical evolution with respect to the $n/2$-th *power energy norm* for all $n \in \mathbb{Z}$. We provide bounds of the $n/2$-th power energy norms, uniformly in time and in the time-scale parameter, for the unitary propagators, generated by the time-dependent perturbed Hamiltonian and by the unperturbed Hamiltonian in the interaction picture. The physically interesting model of Landau-type Hamiltonians with an additional weak and time-slowly-varying electric potential of unit drop is included in this framework.

Contents

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1 Introduction

We consider the physical evolution of a quantum system in a separable Hilbert space $\mathcal{H}$ generated by the time-dependent Hamiltonian operator

$$H(\varepsilon, \eta, t) := H_0 + \varepsilon g(\eta t) H_1 \quad \text{for all } t \in \mathbb{R}, \tag{1.1}$$

where H_0 is the unperturbed Hamiltonian, H_1 is the perturbation switched on by a function g with supp $g' \subset (0, 1)$ and $g(s) = 0$ for $s < 0$, and $\varepsilon \in (0, \varepsilon_*]$, $\eta > 0$ are parameters[1] regulating respectively the *intensity* and the *time-scale* of the perturbation. The variable t here stands for time and the positive parameter η is a convenient tool to control the rate at which the system changes. The function g regulates the switch-on time of the *external* Hamiltonian εH_1 (notice that the perturbation is completely off for $t \leq 0$).

When the Hamiltonian $H(\varepsilon, \eta, t)$ is t-independent,[2] namely $H(\varepsilon, \eta, t) = H(\varepsilon)$, it is well known that, by an elementary consequence of Stone's theorem, one has that $[U_\varepsilon(t), H(\varepsilon)] = 0$, where $U_\varepsilon(t)$ denotes the unitary propagator for the self-adjoint operator $H(\varepsilon)$. In other words there is conservation of the energy and consequently one obtains that $H^{-n/2}(\varepsilon) U_\varepsilon(t) H^{n/2}(\varepsilon)$ has a bounded extension for every $n \in \mathbb{Z}$. On the other hand, if there is a non-trivial t-dependence and the perturbation commutes with the unperturbed Hamiltonian, *i.e.* $[H_1, H_0] = 0$, to establish that for all $n \in \mathbb{Z}$ the product $H^{-n/2}(\varepsilon, \eta, t) U_{\varepsilon, \eta}(t, r) H^{n/2}(\varepsilon, \eta, r)$ extends to a bounded operator, one can use the representation formula for the unitary propagator $U_{\varepsilon, \eta}(t, r) = \mathrm{e}^{-\mathrm{i} \int_r^t \mathrm{d}s \, H(\varepsilon, \eta, s)}$ (see [17, Proposition 2.5]) and rely on similar techniques developed in Proposition 2.9. In this paper, we deal with the more general case in which the commutator $[H_1, H_0] \neq 0$ and "is controlled" by the full perturbed Hamiltonian $H(\varepsilon, \eta, t)$, uniformly in (ε, η, t) (see Assumption (B(k))), beyond Assumption (A$_2$) on the perturbation H_1 to be self-adjoint and relatively bounded with respect to $H_0^{1/2}$ (see the hypotheses in the statement of Theorem 2.5).

Unlike for time-independent Hamiltonians there is no immediate notion of energy conservation, but the boundedness of the unitary propagator for the physical evolution with respect to $n/2$-*th power energy norm* arises as a natural generalization for time-dependent Hamiltonians. Specifically, fix $n \in \mathbb{N}$, defining the $n/2$-th power energy norm $\| \cdot \|_{H^{n/2}(\varepsilon, \eta, t)}$ of $H(\varepsilon, \eta, t)$ as the graph norm of $H^{n/2}(\varepsilon, \eta, t)$, namely

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[1] The value ε_* will be fixed by inequality (2.3) in order to guarantee a uniform positive lower bound, precisely 1, for $H(\varepsilon, \eta, t)$ (see condition (2.2)).

[2] In this case the η-dependence plays no role, thus we cancel it.

$$\|\psi\|_{H^{n/2}(\varepsilon,\eta,t)} := \|\psi\| + \left\| H^{n/2}(\varepsilon,\eta,t)\psi \right\| \quad \text{for any } \psi \in \mathcal{D}(H^{n/2}(\varepsilon,\eta,t))$$

and equipping $\mathcal{D}(H^{n/2}(\varepsilon,\eta,t))$ with $\|\cdot\|_{H^{n/2}(\varepsilon,\eta,t)}$, we introduce the space

$$\mathcal{L}_{\varepsilon,\eta}^{(n)}(r,t) := \{A : \mathcal{D}(H^{n/2}(\varepsilon,\eta,r)) \to \mathcal{D}(H^{n/2}(\varepsilon,\eta,t)) \text{ linear and bounded}\}.$$

Denoting by $U_{\varepsilon,\eta}(t,r)$ the unitary propagator generated by $H(\varepsilon,\eta,t)$, we will prove that for every $n \in \mathbb{N}$ one has that $U_{\varepsilon,\eta}(t,r)$ is in $\mathcal{L}_{\varepsilon,\eta}^{(n)}(r,t)$ with the corresponding operator norm $\left\| U_{\varepsilon,\eta}(t,r) \right\|_{\mathcal{L}_{\varepsilon,\eta}^{(n)}(r,t)}$ uniformly bounded in the parameters $(\eta,(t,r)) \in (0,\infty) \times \mathbb{R}^2$, which is equivalent to establish the following estimate[3]: For every $n \in \mathbb{Z}$, for all $\varepsilon \in (0,\varepsilon_*]$ and $\eta > 0$ we have that

$$\sup_{t,r\in\mathbb{R}} \sup_{\psi\in\mathcal{D}(H^{n/2}(\varepsilon,\eta,r)):\|\psi\|=1} \left\| H^{-n/2}(\varepsilon,\eta,t)U_{\varepsilon,\eta}(t,r)H^{n/2}(\varepsilon,\eta,r)\psi \right\| \le C_n(\varepsilon), \tag{1.2}$$

where the finite constant $C_n(\varepsilon)$ is η-independent. The precise assumptions and result are stated in Theorem 2.5. To the best knowledge of the author, in the standard results of well-posedness of non-autonomous linear evolution equations not even the statement $U(t,r) \in \mathcal{L}_{\varepsilon,\eta}^{(2)}(r,t)$ is shown, the only exception being [8, Theorem 5.1].

Moreover, we are interested in working in the so-called *interaction* or *intermediate picture*[4] : First one computes the unitary propagator $G(t,0) = e^{-i\frac{\varepsilon}{\eta}\phi(\eta t)H_1}$, with $\phi(s) := \int_0^s du\, g(u)$, generated by $\varepsilon g(\eta t)H_1$ (*e.g.* using again [17, Proposition 2.5]) and then one considers the time-dependent unitarily transformed[5] Hamiltonian $G(t,0)^* H_0\, G(t,0) = e^{i\frac{\varepsilon}{\eta}\phi(\eta t)H_1} H_0 e^{-i\frac{\varepsilon}{\eta}\phi(\eta t)H_1}$. Setting the *scaled time* or *macroscopic time* $s := \eta t$, we introduce

$$\hat{H}(\varepsilon,\eta,s) := e^{i\frac{\varepsilon}{\eta}\phi(s)H_1} H_0 e^{-i\frac{\varepsilon}{\eta}\phi(s)H_1}. \tag{1.3}$$

Similarly to the previous case, we will prove the following inequality: For every $n \in \mathbb{Z}$, for all $\varepsilon \in (0,\varepsilon_*]$ and $\eta > 0$ we have that

$$\sup_{s,u\in\mathbb{R}} \sup_{\psi\in\mathcal{D}(\hat{H}^{n/2}(\varepsilon,\eta,r)):\|\psi\|=1} \left\| \hat{H}^{-n/2}(\varepsilon,\eta,s)\hat{U}_{\varepsilon,\eta}(s,u)\hat{H}^{n/2}(\varepsilon,\eta,u)\psi \right\|$$

$$\le C_n(\varepsilon)(1+\varepsilon D_n), \tag{1.4}$$

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[3] We will prove this equivalent statement.

[4] Usually, the interaction picture is performed using the unitary propagator induced by the time-independent part of the time-dependent perturbed Hamiltonian (*e.g.* see [18, §X.12]). More generally, one can introduce the interaction picture via the two-parameter family of unitary operators generated by time-dependent part (see [15, §VIII.14]), fixing an initial time. In our framework, we choose the second kind of interaction picture with initial time $t_0 = 0$.

[5] In Sect. 5, where we deal with the physically interesting model of Landau-type Hamiltonians, this unitary transformation is the gauge transformation $G(t,0) = e^{-i\frac{\varepsilon}{\eta}\phi(\eta t)\Lambda_1}$, where $H_1 := \Lambda_1$ models an electric potential of negative unit drop for an electric field pointing in the negative 1-st direction (see Definition 5.1).

where $\hat{U}_{\varepsilon,\eta}(s, u)$ is the unitary propagator generated by $\hat{H}(\varepsilon, \eta, s)$ and D_n is a finite constant independent of (ε, η). This result, formulated in Corollary 2.7, is obtained as a consequence of estimate (1.2), thanks to the following identity

$$\hat{U}_{\varepsilon,\eta}(s, u) \equiv e^{i\frac{\varepsilon}{\eta}\phi(s)H_1} U_{\varepsilon,\eta}(s/\eta, u/\eta) e^{-i\frac{\varepsilon}{\eta}\phi(u)H_1}, \tag{1.5}$$

and Proposition 2.9, which guarantees that for every integer number n, $H_0^{n/2} H^{-n/2}$ (ε, η, t) and $H^{n/2}(\varepsilon, \eta, t) H_0^{-n/2}$ are bounded in the operator norm by $\mathcal{O}(\varepsilon) + 1$, uniformly in $(\eta, t) \in (0, \infty) \times \mathbb{R}$.

Energy estimates in the form of (1.4) (or equivalently (1.2)) are relevant when one needs to keep track of localization in energy under the physical evolution, uniformly in the time-scale of the perturbation. More precisely, suppose that a family of operators $O(s)$ with $s \in \mathbb{R}$ *decays in energy with power $m/2$ with $m \in \mathbb{N}$*, in the sense that there exists a finite constant C_O such that

$$\left\| O(s) \hat{H}^{m/2}(\varepsilon, \eta, s) \psi \right\| \leq C_O \|\psi\| \tag{1.6}$$

for every $\varepsilon \in (0, \varepsilon_*]$, $\eta > 0$, $s \in \mathbb{R}$ and for all $\psi \in \mathcal{D}\left(\hat{H}^{m/2}(\varepsilon, \eta, s)\right)$. Then, by applying inequality (1.4) this energy localization is conserved by the evolved family of operators $\hat{U}_{\varepsilon,\eta}(u, s) O(s) \hat{U}_{\varepsilon,\eta}(s, u)$:

$$\begin{aligned}
&\left\| \hat{U}_{\varepsilon,\eta}(u, s) O(s) \hat{U}_{\varepsilon,\eta}(s, u) \hat{H}^{m/2}(u) \psi \right\| \\
&= \left\| O(s) \hat{H}^{m/2}(s) \hat{H}^{-m/2}(s) \hat{U}_{\varepsilon,\eta}(s, u) \hat{H}^{m/2}(u) \psi \right\| \\
&\leq C_O C_m(\varepsilon)(1 + \varepsilon D_m) \|\psi\|,
\end{aligned} \tag{1.7}$$

for any $s, u \in \mathbb{R}$ and for every $\psi \in \mathcal{D}\left(\hat{H}^{m/2}(\varepsilon, \eta, u)\right)$.

This work has been motivated in the first instance by the need to fill a gap in the proof of [2, Lemma 5.1], where Landau-type Hamiltonian operators with an additional weak and time-slowly-varying electric potential of unit drop are considered (see Sect. 5 for this application). While Theorem 2.5 implies [2, Lemma 5.1], Corollary 2.7 is relevant since it is explicitly used in the proof of [2, Theorem 2.2] (see [2, Remark (3), p. 599] for the case $n = 0$). The strategy proof of Theorem 2.5 is based on the one given in the aforementioned paper, with two essential differences: firstly we use $H(\varepsilon, \eta, t)$ whose time derivative is compactly supported (while $\frac{\partial}{\partial s}\hat{H}(\varepsilon, \eta, s)$ is not compactly supported) and secondly in the proof of Theorem 2.5 we establish the induction step by computing the time derivative of the bounded operator $H^{-1/2}(\varepsilon, \eta, t)$ (compare (3.8)) instead of the unbounded one $\hat{H}^{1/2}(\varepsilon, \eta, s)$. As it is briefly explained in Sect. 5, these kinds of energy estimates are used to prove the validity of the Kubo formula for the transverse conductance in the quantum Hall effect in a two-dimensional sample (*e. g.* see [1, 4–6, 11, 13, 14, 21]). But we are convinced that our results are of general conceptual interest, since we provide bounds on the growth of the $n/2$-th power energy norms for time-dependent Hamiltonian in a model-independent setting, and could

be relevant for proving the linear response in quantum Hall systems for unbounded Hamiltonians (*cf.* Sect. 5). More specifically, we require mild properties: Beyond the technical hypotheses, *i.e.* Assumptions 2.1 and ($C_2(k)$) for $k = 2$, which guarantee the self-adjointness of $H(\varepsilon, \eta, t)$ and $\hat{H}(\varepsilon, \eta, s)$ on the same t-independent domain $\mathcal{D}(H_0)$ and spectrum condition (2.2), the operator H_1 associated with the perturbation must not be bounded but only $H_0^{1/2}$-bounded (compare Assumption (A_2)), and the two parameters ε, η, related to the perturbation, are independent. Furthermore, both estimates (1.2) and (1.4) are uniform in the time-scale parameter $\eta > 0$, while for fixed $\eta > 0$ these bounds are clearly expected, due to the hypothesis supp $g' \subset (0, 1)$, with η-dependent constants. Finally, the use of the symbols ε and η is not related to a smallness assumption, as far as this paper is concerned (however our results apply to the particular case considered in [2], where the limit $\varepsilon = \eta = \frac{1}{\tau} \to 0^+$ is considered).

2 Mathematical setting and main results

In this section we set up the mathematical framework and state our main results, under different assumptions. Let $\mathcal{H}$ denote a separable Hilbert space.

Firstly, we write hypotheses on each summand of the perturbed Hamiltonian $H(\varepsilon, \eta, t)$.

Assumption 2.1 Let $H(\varepsilon, \eta, t)$ be as in (1.1) and $g \in C^k(\mathbb{R})$ with[6] $k \geq 1$, supp $g' \subset (0, 1)$ and $g(s) = 0$ for $s < 0$. We define

$$M := \max_{s \in [0,1]} |g(s)| \text{ and } M' := \max_{s \in [0,1]} |g'(s)|. \tag{2.1}$$

Here $\varepsilon \in (0, \varepsilon_*]$, where ε_* is chosen so that condition (2.3) is fulfilled, and $\eta > 0$. Furthermore, the Hamiltonian operator $H(\varepsilon, \eta, t)$ satisfies the following properties:

(A_1) $H_0: \mathcal{D}(H_0) \to \mathcal{H}$ is self-adjoint, where $\mathcal{D}(H_0) \subset \mathcal{H}$ denotes its dense domain, and[7] $H_0 \geq 1 + \gamma_0$, with $\gamma_0 > 0$.

(A_2) $H_1: \mathcal{D}(H_1) \to \mathcal{H}$ is self-adjoint, where $\mathcal{D}(H_1) \subset \mathcal{H}$ denotes its dense domain, and is $H_0^{1/2}$-bounded, namely there exists a finite constant $a > 0$ such that $\left\| H_1 H_0^{-1/2} \right\| \leq a$.

As it is explained respectively in Remark 2.4.(i) and Remark 2.4.(ii), the above assumptions ensure that $H(\varepsilon, \eta, t)$ is self-adjoint on $\mathcal{D}(H_0)$ and that $H(\varepsilon, \eta, t) \geq 1$.

Secondly, we write hypotheses on "how the perturbed Hamiltonian $H(\varepsilon, \eta, t)$ behaves with respect to the unperturbed one H_0".

Assumption 2.2 Let $H(\varepsilon, \eta, t)$ be as in Assumption 2.1.

For every $k \in \mathbb{Z}$, there exists a finite constant E_k such that for all $\varepsilon \in (0, \varepsilon_*], \eta \in (0, \infty), t \in \mathbb{R}$ we have that[8]:

if $k \geq 0$ taking any $\psi \in \mathcal{D}(H^{(k+1)/2}(\varepsilon, \eta, t))$ otherwise $\psi \in \mathcal{H}$

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[6] Notice that we do not require that supp$g$ is compact.

[7] The following hypothesis is equivalent, up to a shift of a constant, to require that H_0 is bounded from below.

[8] Notice that we are allowed to write any negative power of $H(\varepsilon, \eta, t)$ due to condition (2.2).

(B(k))

$$\left\| H^{-k/2}(\varepsilon, \eta, t)[H(\varepsilon, \eta, t), H_1]H^{(k-2)/2}(\varepsilon, \eta, t)\psi \right\| \leq E_k \|\psi\|,$$

where $[H(\varepsilon, \eta, t), H_1]$ is densely defined with $\mathcal{D}([H(\varepsilon, \eta, t), H_1]) \supset \mathcal{D}(H^{3/2}(\varepsilon, \eta, t))$, and in addition if $k \leq -1$ we require that $[H(\varepsilon, \eta, t), H_1]\colon \mathcal{D}(H^{(|k|+2)/2}(\varepsilon, \eta, t)) \to \mathcal{D}(H^{|k|/2}(\varepsilon, \eta, t))$.

Assumption 2.3 Let $H(\varepsilon, \eta, t)$ be as in Assumption 2.1.
For every $k \in \mathbb{N}$ with[9] $k \geq 2$,

(C$_1(k)$) for all $\varepsilon \in (0, \varepsilon_*], \eta \in (0, \infty), t \in \mathbb{R}$ we have that $\mathcal{D}(H^k(\varepsilon, \eta, t)) \equiv \mathcal{D}(H_0^k)$.

For every $k \in \mathbb{N}$

(C$_2(k)$) we have that the domain $\mathcal{D}(H_0^{k/2})$ is invariant under the unitary transformation $\{e^{i\lambda H_1}\}_{\lambda \in \mathbb{R}}$, namely for all $\lambda \in \mathbb{R}$ one has that

$$e^{i\lambda H_1}\colon \mathcal{D}(H_0^{k/2}) \to \mathcal{D}(H_0^{k/2}).$$

Remark 2.4 Here we explain some useful consequences of the hypotheses above.

(i) Under Assumptions (A$_1$) and (A$_2$), we have that H_1 is H_0-bounded, with relative bound $\tilde{a} < 1$. Indeed, notice that for every $C > 0$

$$\left\| H_1(H_0 + C)^{-1} \right\| = \left\| H_1 H_0^{-1/2} \cdot H_0^{1/2}(H_0 + C)^{-1/2} \cdot (H_0 + C)^{-1/2} \right\|$$
$$\leq \frac{a}{\sqrt{1+C}},$$

where a is defined in Assumption (A$_2$). Hence, for every $\psi \in \mathcal{D}(H_0)$ we obtain that

$$\|H_1\psi\| = \left\| H_1(H_0 + C)^{-1}(H_0 + C)\psi \right\| \leq \frac{a}{\sqrt{1+C}} \left(\|H_0\psi\| + C\|\psi\| \right).$$

Therefore, by the Kato–Rellich theorem $H(\varepsilon, \eta, t)$ is self-adjoint on $\mathcal{D}(H_0)$.

(ii) Observe that Assumptions (A$_1$) and (A$_2$) imply that there exists $\varepsilon_* > 0$ such that

$$\inf_{t \in \mathbb{R}, \eta > 0} \sigma(H(\varepsilon, \eta, t)) \geq 1 \quad \text{for all } \varepsilon \in (0, \varepsilon_*]. \tag{2.2}$$

In fact, for any $z < 1$, $H(\varepsilon, \eta, t) - z = \left(\mathbb{1} + \varepsilon g(\eta t)H_1(H_0 - z)^{-1}\right)(H_0 - z)$ is invertible for a suitable choice of ε_*. In view of hypothesis (A$_1$) and the previous remark, we get that

$$\left\| H_1(H_0 - z)^{-1} \right\| \leq \left\| H_1 H_0^{-1} \right\| \left(1 + \frac{|z|}{1 + \gamma_0 - z} \right) \leq \frac{3\gamma_0 + 1}{\gamma_0} \left\| H_1 H_0^{-1} \right\|$$

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[9] For $k = 1$ the following identity is implied by Assumptions (A$_1$) and (A$_2$) (see Remark 2.4.(i)).

and thus there exists $\varepsilon_* > 0$ such that

$$\frac{3\gamma_0 + 1}{\gamma_0} \varepsilon_* M \left\| H_1 H_0^{-1} \right\| < 1 \tag{2.3}$$

with M defined in (2.1).

(iii) For $k \in \mathbb{N}$ with $k \geq 2$, Assumption $(C_1(k))$ and [9, Supplementary notes, V.7] imply that for all $\varepsilon \in (0, \varepsilon_*]$, $\eta \in (0, \infty)$, $t \in \mathbb{R}$ one has that $\mathcal{D}(H^{k/2}(\varepsilon, \eta, t)) \equiv \mathcal{D}(H_0^{k/2})$. The same result holds true automatically for $k = 1$ due to $\mathcal{D}(H(\varepsilon, \eta, t)) \equiv \mathcal{D}(H_0)$ by Remark 2.4.(i).

Before stating the main results, namely Theorem 2.5 and Corollary 2.7, it is convenient to recall the problem of well-posedness of non-autonomous linear evolution equations. As it is emphasized in [18, Notes of Section X.12], the Cauchy problem for linear evolution equations

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(t) = A(t)\psi(t), 0 \leq t \leq T, \text{ in a Banach space}$$

where $A(\cdot)$ is an unbounded-operator valued function and the domain $\mathcal{D}(A(t)) \equiv \mathcal{D}$ of $A(t)$ is independent of t, under general suitable conditions, was solved first by Kato [7] and then by Yosida [23] (for the comparison of these works see [19]). For more general results, considering that $A(t)$ has domain which does depend on time, see *e. g.* [8, 20, 22] and references therein. In the present setting, under Assumption 2.1 one has that the domain of self-adjointness $\mathcal{D}(H(\varepsilon, \eta, t))$ of $H(\varepsilon, \eta, t)$ is independent of t by Remark 2.4.(i). Hence, under additional hypotheses (*e. g.* assumptions in [7, Theorem 3]) one can prove that there exists the unitary propagator $U_{\varepsilon,\eta}(t, r)$ generated by $H(\varepsilon, \eta, t)$. This means that $U_{\varepsilon,\eta}(t, r)$ is the two-parameter family of unitary operators, jointly strongly continuous in $t \in \mathbb{R}$ and $r \in \mathbb{R}$, such that for every $t, r, u \in \mathbb{R}$

$$U_{\varepsilon,\eta}(t, r)U_{\varepsilon,\eta}(r, u) = U_{\varepsilon,\eta}(t, u), \quad U_{\varepsilon,\eta}(t, t) = \mathbb{1}, \quad U_{\varepsilon,\eta}(t, u)\mathcal{D}(H_0) = \mathcal{D}(H_0),$$

$$\mathrm{i}\frac{\partial U_{\varepsilon,\eta}}{\partial t}(t, u)\psi = H(\varepsilon, \eta, t)U_{\varepsilon,\eta}(t, u)\psi \quad \text{for all } \psi \in \mathcal{D}(H_0),$$

$$-\mathrm{i}\frac{\partial U_{\varepsilon,\eta}}{\partial u}(t, u)\psi = U_{\varepsilon,\eta}(t, u)H(\varepsilon, \eta, u)\psi \quad \text{for all } \psi \in \mathcal{D}(H_0).$$

In order to keep the reader's attention on the main results, *i. e.* Theorem 2.5 and Corollary 2.7, we postpone their proofs to Sect. 3.

Theorem 2.5 *Consider the Hamiltonian* $H(\varepsilon, \eta, t) = H_0 + \varepsilon g(\eta t)H_1$ *satisfying Assumption 2.1 and let* $U_{\varepsilon,\eta}(t, r)$ *be the unitary propagator generated by* $H(\varepsilon, \eta, t)$. *Let* $n \in \mathbb{Z}$. *If* $|n| \geq 2$ *we assume in addition Assumption $(B(k))$ for all* $0 \leq k \leq |n| - 2$. *Then for every* $\varepsilon \in (0, \varepsilon_*]$ *we have that*

$$\sup_{t,r\in\mathbb{R}} \sup_{\psi\in\mathcal{D}(H^{n/2}(\varepsilon,\eta,r)):\|\psi\|=1} \left\| H^{-n/2}(\varepsilon, \eta, t)U_{\varepsilon,\eta}(t, r)H^{n/2}(\varepsilon, \eta, r)\psi \right\| \leq C_n(\varepsilon) \quad \forall \eta > 0,$$

$$\tag{2.4}$$

where $C_n(\varepsilon)$ is defined iteratively as

$$\begin{cases} C_0(\varepsilon) := C_0 = 1 \\ C_n(\varepsilon) := C_{n-1}(\varepsilon)e^{C_{n-1}(\varepsilon)(\alpha+\beta\varepsilon+\gamma_n)\varepsilon} \text{ for all } n \geq 1 \end{cases} \quad (2.5)$$

with α, β and γ_n finite constants defined as

$$\alpha + \varepsilon\beta := M'(a + \varepsilon Ma^2), \quad \gamma_1 := 0 \text{ and } \gamma_n := M' \sum_{k=0}^{n-2} E_k \text{ for } n \geq 2, \quad (2.6)$$

and $C_{-n}(\varepsilon) := C_n(\varepsilon)$ for all $n \in \mathbb{N}$.

Remark 2.6 Both in the Gell-Mann and Low [3] and the Kubo [10] formula the standard choice for the switch-on procedure in time is to make use of the exponential function for the non-positive time-axis $\mathbb{R}_- := (-\infty, 0]$. More specifically, in our setting of reference, we replace the function g with the exponential and restrict the whole real time-axis to the non-positive one, *i. e.* one considers the time-dependent Hamiltonian operator

$$H_{\exp}(\varepsilon, \eta, t) := H_0 + \varepsilon e^{\eta t} H_1 \quad \text{for all } t \in \mathbb{R}_-.$$

Clearly, the main difference between $H_{\exp}(\varepsilon, \eta, t)$ and $H(\varepsilon, \eta, t) \equiv H_g(\varepsilon, \eta, t)$, defined in (1.1), is that the the switch-on process acts respectively on the infinite time-interval $\mathbb{R}_-$ and on a finite time-interval (precisely, under Assumption 2.1: supp $g' \subset (0, 1)$). Under the assumptions of Theorem 2.5 except for the substitution of g with the exponential and the restriction to $\mathbb{R}_-$ (applying the these two replacements everywhere in the nested hypotheses), a type of inequality similar to (2.4) still holds true. Precisely, denoting by $U_{\exp,\varepsilon,\eta}(t, r)$ the unitary propagator generated by $H_{\exp}(\varepsilon, \eta, t)$, we have

$$\sup_{t,r\in\mathbb{R}_-} \sup_{\psi\in\mathcal{D}(H_{\exp}^{n/2}(\varepsilon,\eta,r)):\|\psi\|=1} \left\| H_{\exp}^{-n/2}(\varepsilon, \eta, t) U_{\exp,\varepsilon,\eta}(t, r) H_{\exp}^{n/2}(\varepsilon, \eta, r)\psi \right\| \leq \widetilde{C}_n(\varepsilon),$$

$$(2.7)$$

for all $\eta > 0$, where $\widetilde{C}_n(\varepsilon)$ is defined iteratively as

$$\begin{cases} \widetilde{C}_0(\varepsilon) := \widetilde{C}_0 = 1 \\ \widetilde{C}_n(\varepsilon) := \widetilde{C}_{n-1}(\varepsilon)e^{\widetilde{C}_{n-1}(\varepsilon)(\widetilde{\alpha}+\widetilde{\beta}\varepsilon+\widetilde{\gamma}_n)\varepsilon} \text{ for all } n \geq 1 \end{cases} \quad (2.8)$$

with $\widetilde{\alpha}$, $\widetilde{\beta}$ and $\widetilde{\gamma}_n$ finite constants, and $\widetilde{C}_{-n}(\varepsilon) := \widetilde{C}_n(\varepsilon)$ for all $n \in \mathbb{N}$.
Here, for completeness we sketch a proof of the above statement. We follow the argument of the proof of Theorem 2.5 in Sect. 3.1, excepting the restriction of the

times t, r to a finite time-interval depending on η (see (3.1)). Similarly, defining for every $t, r \in \mathbb{R}_-$

$$\widetilde{C}_{\varepsilon,\eta,n}(t,r) := \sup_{\psi \in \mathcal{D}(H_{\exp}^{n/2}(\varepsilon,\eta,r)):\|\psi\|=1} \left\| H_{\exp}^{-n/2}(\varepsilon,\eta,t) U_{\exp,\varepsilon,\eta}(t,r) H_{\exp}^{n/2}(\varepsilon,\eta,r)\psi \right\|,$$

one arrives at the inequality

$$\widetilde{C}_{\varepsilon,\eta,N}(t,r) \leq \widetilde{C}_{N-1}(\varepsilon) \left(1 + (\widetilde{\alpha} + \widetilde{\beta}\varepsilon + \widetilde{\gamma}_N)\varepsilon \int_t^r d\tau\, \eta e^{\eta\tau} \widetilde{C}_{\varepsilon,\eta,N}(\tau,r) \right),$$

for $-\infty < t \leq r \leq 0$. By using Grönwall's inequality and that $\int_{-\infty}^0 d\tau\, \eta e^{\eta\tau} = 1$, we conclude that

$$\begin{aligned}
\widetilde{C}_{\varepsilon,\eta,N}(t,r) &\leq \widetilde{C}_{N-1}(\varepsilon) e^{\widetilde{C}_{N-1}(\varepsilon)(\widetilde{\alpha}+\widetilde{\beta}\varepsilon+\widetilde{\gamma}_N)\varepsilon \int_t^r d\tau\, \eta e^{\eta\tau}} \\
&\leq \widetilde{C}_{N-1}(\varepsilon) e^{\widetilde{C}_{N-1}(\varepsilon)(\widetilde{\alpha}+\widetilde{\beta}\varepsilon+\widetilde{\gamma}_N)\varepsilon} =: \widetilde{C}_N(\varepsilon).
\end{aligned}$$

Therefore, it emerges that the crucial properties of the switch-on procedure modeled by a generic function $f: I \to \mathbb{R}$, $f \in C^k(I)$ with $k \geq 1$ on a subset $I \subseteq \mathbb{R}$ to deduce a type of inequality in the form of (2.4), which is uniform in the time-scale parameter η, is to have that both $\|f\|_{L^\infty(I)}$ and $\|f'\|_{L^1(I)}$ are finite.

Let the scaled time $s = \eta t$, consider the unperturbed Hamiltonian in the interaction picture $\hat{H}(\varepsilon, \eta, s)$, defined in (1.3), which is self-adjoint on $\mathcal{D}(H_0)$ under Assumptions 2.1 and $(C_2(k))$ for $k = 2$. Let us briefly recall the notion of the corresponding unitary propagation, whose existence and uniqueness are guaranteed again by [7, Theorem 3], under additional regularity hypotheses. Let $\hat{U}_{\varepsilon,\eta}(s, r)$ be the unitary propagator generated by $\hat{H}(\varepsilon, \eta, s)$, namely $\hat{U}_{\varepsilon,\eta}(s, r)$ is the two-parameter family of unitary operators, jointly strongly continuous in $s \in \mathbb{R}$ and $r \in \mathbb{R}$, such that for every $s, r, u \in \mathbb{R}$

$$\hat{U}_{\varepsilon,\eta}(s,r)\hat{U}_{\varepsilon,\eta}(r,u) = \hat{U}_{\varepsilon,\eta}(s,u), \quad \hat{U}_{\varepsilon,\eta}(s,s) = \mathbb{1}, \quad \hat{U}_{\varepsilon,\eta}(s,u)\mathcal{D}(H_0) = \mathcal{D}(H_0),$$

$$i\eta \frac{\partial \hat{U}_{\varepsilon,\eta}}{\partial s}(s,u)\psi = \hat{H}(\varepsilon,\eta,s)\hat{U}_{\varepsilon,\eta}(s,u)\psi \quad \forall \psi \in \mathcal{D}(H_0),$$

$$-i\eta \frac{\partial \hat{U}_{\varepsilon,\eta}}{\partial u}(s,u)\psi = \hat{U}_{\varepsilon,\eta}(s,u)\hat{H}(\varepsilon,\eta,u)\psi \quad \forall \psi \in \mathcal{D}(H_0).$$

$$(2.9)$$

Corollary 2.7 *Under Assumptions 2.1 and $(C_2(k))$ for $k = 2$, consider $\hat{H}(\varepsilon, \eta, s) = e^{i\frac{\varepsilon}{\eta}\phi(s)H_1} H_0 e^{-i\frac{\varepsilon}{\eta}\phi(s)H_1}$, where $s = \eta t$ is the scaled time. Let $\hat{U}_{\varepsilon,\eta}(s, u)$ be the unitary propagator generated by $\hat{H}(\varepsilon, \eta, s)$. Let $n \in \mathbb{Z}$. Let Assumption $(C_2(k))$ for $k = |n|$ hold true. If $|n| \geq 3$ we assume in addition Assumption $(C_1(k))$ for all $3 \leq k \leq |n|$ and Assumption $(B(k))$ for $k = 0$. If $|n| \geq 4$ we assume further Assumption $(B(k))$ for all $2 - |n| \leq k \leq -2$. Then there exists a finite constant D_n such that for every $\varepsilon \in (0, \varepsilon_*]$ and $\eta \in (0, \infty)$ we have that*

$$\sup_{s,u\in\mathbb{R}} \sup_{\psi\in\mathcal{D}\left(\hat{H}^{n/2}(\varepsilon,\eta,r)\right):\|\psi\|=1} \left\|\hat{H}^{-n/2}(\varepsilon,\eta,s)\hat{U}_{\varepsilon,\eta}(s,u)\hat{H}^{n/2}(\varepsilon,\eta,u)\psi\right\|$$

$$\leq C_n(\varepsilon)(1+\varepsilon D_n),$$

where $C_n(\varepsilon)$ is defined in (2.5).

Here, we state two auxiliary results whose technical proofs are deferred to Sect. 4. Specifically, the following lemma shows that H_1 is actually $H^{1/2}(\varepsilon,\eta,t)$-bounded with a relative bound independent of the parameters $(\eta,t)\in(0,\infty)\times\mathbb{R}$, not only $H_0^{1/2}=H^{1/2}(\varepsilon,\eta,r)$-bounded with $r\leq 0$ (compare Assumption (A$_2$)).

Lemma 2.8 *Let $H(\varepsilon,\eta,t)$ be as in Assumption* 2.1. *Then for every $\varepsilon\in(0,\varepsilon_*]$, $\eta\in(0,\infty)$ and $t\in\mathbb{R}$ we have that*

$$\left\|H_1 H^{-1/2}(\varepsilon,\eta,t)\right\|\leq a+\varepsilon Ma^2.$$

On the other hand, the next proposition turns out to be useful to deduce the energy estimates for the unperturbed Hamiltonian in the interaction picture $\hat{H}(\varepsilon,\eta,s)$ from the ones for the perturbed Hamiltonian $H(\varepsilon,\eta,t)$.

Proposition 2.9 *Let $H(\varepsilon,\eta,t)$ be as in Assumption* 2.1. *Let $n\in\mathbb{Z}$. If $|n|\geq 3$ we assume in addition Assumption (C$_1(k)$) for all $3\leq k\leq|n|$ and Assumption (B(k)) for $k=0$. If $|n|\geq 4$ we assume further Assumption (B(k)) for all $2-|n|\leq k\leq -2$. Then there exist finite constants A_n, B_n such that for every $\varepsilon\in(0,\varepsilon_*]$, $\eta\in(0,\infty)$ and $t\in\mathbb{R}$:*

(i) *for any $\psi\in\mathcal{D}(H^{-n/2}(\varepsilon,\eta,t))$ we have that*

$$\left\|H_0^{n/2}H^{-n/2}(\varepsilon,\eta,t)\psi\right\|\leq(1+A_n\varepsilon)\|\psi\|, \tag{2.10}$$

(ii) *for any $\psi\in\mathcal{D}(H_0^{-n/2})$ we have that*

$$\left\|H^{n/2}(\varepsilon,\eta,t)H_0^{-n/2}\psi\right\|\leq(1+B_n\varepsilon)\|\psi\|. \tag{2.11}$$

3 Proof of the main results

3.1 Proof of Theorem 2.5

First of all, notice that it suffices to check inequality (2.4) for $n\in\mathbb{N}_0$ due to the Riesz Lemma. In view of the hypothesis supp $g'\subset(0,1)$, for any $\psi\in\mathcal{D}(H_0)$ the map $t\mapsto H(\varepsilon,\eta,t)\psi$ is time-independent for $t\leq 0$ and $t\geq 1/\eta$. Therefore, it is enough to prove that for all $n\in\mathbb{N}_0$

$$\sup_{t,r\in[0,1/\eta]} \sup_{\psi\in\mathcal{D}(H^{n/2}(\varepsilon,\eta,r)):\|\psi\|=1} \left\|H^{-n/2}(\varepsilon,\eta,t)U_{\varepsilon,\eta}(t,r)H^{n/2}(\varepsilon,\eta,r)\psi\right\|\leq C_n(\varepsilon).$$

$$\tag{3.1}$$

Indeed, defining

$$C_{\varepsilon,\eta,n}(t,r) := \sup_{\psi \in \mathcal{D}(H^{n/2}(\varepsilon,\eta,r)):\|\psi\|=1} \left\| H^{-n/2}(\varepsilon,\eta,t)U_{\varepsilon,\eta}(t,r)H^{n/2}(\varepsilon,\eta,r)\psi \right\|, \tag{3.2}$$

we have

$$\sup_{t,r\in\mathbb{R}} C_{\varepsilon,\eta,n}(t,r) = \sup_{t,r\in[0,1/\eta]} C_{\varepsilon,\eta,n}(t,r). \tag{3.3}$$

To prove the last equality it suffices to notice that for all $t \in \mathbb{R}$: if $r < 0$ then $C_{\varepsilon,\eta,n}(t,r) = C_{\varepsilon,\eta,n}(t,0)$, and similarly if $r > 1/\eta$ then $C_{\varepsilon,\eta,n}(t,r) = C_{\varepsilon,\eta,n}(t,1/\eta)$, using that $H(\varepsilon,\eta,r)$ is constant for $r \in \mathbb{R} \setminus (0,1/\eta)$ and $U_{\varepsilon,\eta}(t,r) = U_{\varepsilon,\eta}(t,s)U_{\varepsilon,\eta}(s,r)$ for all $t,s,r \in \mathbb{R}$. One obtains analogous identities exchanging the roles of r and t. In order to prove inequality (3.1), we proceed by induction over $n \in \mathbb{N}_0$. For $n = 0$ it is trivial. Now we take some $N \in \mathbb{N}_0$ with $N \geq 1$. We assume that the thesis holds true for $n = N - 1$ and we prove it for $n = N$. Let us start by noticing that for every $\psi \in \mathcal{D}(H_0)$, we have that

$$U_{\varepsilon,\eta}(t,r)H^{-1/2}(\varepsilon,\eta,r)U_{\varepsilon,\eta}(r,t)\psi = H^{-1/2}(\varepsilon,\eta,t)\psi$$
$$+ \int_t^r d\tau\, U_{\varepsilon,\eta}(t,\tau)\frac{\partial}{\partial\tau}\left(H^{-1/2}(\varepsilon,\eta,\tau)\right)U_{\varepsilon,\eta}(\tau,t)\psi, \tag{3.4}$$

by using that $U_{\varepsilon,\eta}(s,u)\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$ for all $s,u \in \mathbb{R}$ and $\frac{\partial}{\partial\tau}\left(H^{-1/2}(\varepsilon,\eta,\tau)\right)$ is a bounded operator, computed as follows. By applying [9, V-§3.11 equation (3.43)] one has that

$$H^{-1/2}(\varepsilon,\eta,\tau) = \frac{2}{\pi}\int_0^\infty dx\,\left(x^2 + H(\varepsilon,\eta,\tau)\right)^{-1}, \tag{3.5}$$

and thus

$$\frac{\partial}{\partial\tau}H^{-1/2}(\varepsilon,\eta,\tau) = -\frac{2\varepsilon\eta g'(\eta\tau)}{\pi}\int_0^\infty dx\left(x^2 + H(\varepsilon,\eta,\tau)\right)^{-1}H_1\left(x^2 + H(\varepsilon,\eta,\tau)\right)^{-1}. \tag{3.6}$$

Notice that in the above computation we have exchanged the derivative and the integral since by using condition (2.2) and Lemma 2.8, we obtain that

$$\left|g'(\eta\tau)\right|\left\|\left(x^2 + H(\varepsilon,\eta,\tau)\right)^{-1}H_1\left(x^2 + H(\varepsilon,\eta,\tau)\right)^{-1}\right\|$$
$$\leq M'\left\|\left(x^2 + H(\varepsilon,\eta,\tau)\right)^{-1}\right\|\left\|H_1H^{-1/2}(\varepsilon,\eta,\tau)\right\|\cdot$$
$$\cdot\left\|H^{1/2}(\varepsilon,\eta,\tau)\left(x^2 + H(\varepsilon,\eta,\tau)\right)^{-1}\right\|$$
$$\leq \frac{M'}{1+x^2}(a + \varepsilon Ma^2) \qquad \text{for all } \tau \in \mathbb{R},$$

where the right-hand term is integrable on $[0,\infty)$. Obviously, the previous bound implies that $\frac{\partial}{\partial\tau}H^{-1/2}(\varepsilon,\eta,\tau)$ is bounded uniformly in time. Moreover, notice that

$$\frac{\partial}{\partial \tau} \left(H^{-1/2}(\varepsilon, \eta, \tau) \right) \mathcal{D}(H_0) \subset \mathcal{D}(H_0). \tag{3.7}$$

Indeed for every $\phi \in \mathcal{D}(H_0) = \mathcal{D}(H(\varepsilon, \eta, \tau))$ there exists $\varphi \in \mathcal{H}$ such that $\phi = H^{-1}(\varepsilon, \eta, \tau)\varphi$ thus

$$\frac{\partial}{\partial \tau} H^{-1/2}(\varepsilon, \eta, \tau)\phi = -\frac{2\varepsilon\eta g'(\eta\tau)}{\pi} \int_0^\infty \mathrm{d}x \left(x^2 + H(\varepsilon, \eta, \tau) \right)^{-1} H_1 H^{-1/2}(\varepsilon, \eta, \tau)\cdot$$
$$\cdot \left(x^2 + H(\varepsilon, \eta, \tau) \right)^{-1} H^{-1/2}(\varepsilon, \eta, \tau)\varphi,$$

by using condition (2.2) and Lemma 2.8, inclusion (3.7) is obtained. Therefore, we are allowed to apply $H^{1/2}(\varepsilon, \eta, \tau)$ on the left-hand side of (3.4), getting that for every $\psi \in \mathcal{D}(H_0)$

$$H^{1/2}(\varepsilon, \eta, t)U_{\varepsilon,\eta}(t, r)H^{-1/2}(\varepsilon, \eta, r)\psi = U_{\varepsilon,\eta}(t, r)\psi$$
$$+ \int_t^r \mathrm{d}\tau \, H^{1/2}(\varepsilon, \eta, t)U_{\varepsilon,\eta}(t, \tau)\frac{\partial}{\partial \tau} \left(H^{-1/2}(\varepsilon, \eta, \tau) \right) U_{\varepsilon,\eta}(\tau, r)\psi.$$

By multiplying the above equality on the left-hand side by $H^{-N/2}(\varepsilon, \eta, t)$ and applying it to a particular subset of $\mathcal{D}(H_0) \ni \psi = H^{N/2}(\varepsilon, \eta, r)\phi$, where $\phi \in \mathcal{D}(H^{(N+2)/2}(\varepsilon, \eta, r))$, we obtain that for every $\phi \in \mathcal{D}(H^{(N+2)/2}(\varepsilon, \eta, r))$

$$H^{-N/2}(\varepsilon, \eta, t)U_{\varepsilon,\eta}(t, r)H^{N/2}(\varepsilon, \eta, r)\phi$$
$$= H^{-(N-1)/2}(\varepsilon, \eta, t)U_{\varepsilon,\eta}(t, r)H^{(N-1)/2}(\varepsilon, \eta, r)\phi$$
$$- \int_t^r \mathrm{d}\tau \, H^{-(N-1)/2}(\varepsilon, \eta, t)U_{\varepsilon,\eta}(t, \tau)\frac{\partial}{\partial \tau} \left(H^{-1/2}(\varepsilon, \eta, \tau) \right) \cdot$$
$$\cdot U_{\varepsilon,\eta}(\tau, r)H^{N/2}(\varepsilon, \eta, r)\phi. \tag{3.8}$$

Therefore, in view of the induction hypothesis for $n = N - 1$ we have that

$$\left\| H^{-N/2}(\varepsilon, \eta, t)U_{\varepsilon,\eta}(t, r)H^{N/2}(\varepsilon, \eta, r)\phi \right\| \le C_{N-1}(\varepsilon) \|\phi\|$$
$$+ C_{N-1}(\varepsilon) \int_t^r \mathrm{d}\tau \left\| H^{-(N-1)/2}(\varepsilon, \eta, \tau)\frac{\partial}{\partial \tau} \left(H^{-1/2}(\varepsilon, \eta, \tau) \right) H^{N/2}(\varepsilon, \eta, \tau) \cdot \right.$$
$$\left. \cdot H^{-N/2}(\varepsilon, \eta, \tau)U_{\varepsilon,\eta}(\tau, r)H^{N/2}(\varepsilon, \eta, r)\phi \right\|, \tag{3.9}$$

for $0 \leq t \leq r \leq 1/\eta$. Being $\mathcal{D}(H^{(N+2)/2}(\varepsilon, \eta, r))$ a core[10] of $H^{N/2}(\varepsilon, \eta, r)$, it suffices to prove the induction step on this set. In order to conclude the proof, it is enough to observe that: For every $m \geq 1$, being α, β and γ_m defined in (2.6), for all $\tau \in [0, 1/\eta]$, for all $\psi \in \mathcal{D}(H^{m/2}(\varepsilon, \eta, \tau))$, we have that

$$\left\| H^{-(m-1)/2}(\varepsilon, \eta, \tau) \frac{\partial}{\partial \tau} \left(H^{-1/2}(\varepsilon, \eta, \tau) \right) H^{m/2}(\varepsilon, \eta, \tau) \psi \right\| \leq (\alpha + \beta\varepsilon + \gamma_m)\varepsilon\eta \, \|\psi\|.$$
(3.10)

Indeed, notice that

$$\left\| H^{-(m-1)/2}(\varepsilon, \eta, \tau) \frac{\partial}{\partial \tau} \left(H^{-1/2}(\varepsilon, \eta, \tau) \right) H^{m/2}(\varepsilon, \eta, \tau)\psi \right\|$$

$$\leq \left\| \frac{\partial}{\partial \tau} \left(H^{-1/2}(\varepsilon, \eta, \tau) \right) H^{1/2}(\varepsilon, \eta, \tau)\psi \right\| +$$

$$+ \left\| \left[H^{-(m-1)/2}(\varepsilon, \eta, \tau), \frac{\partial}{\partial \tau} \left(H^{-1/2}(\varepsilon, \eta, \tau) \right) \right] H^{m/2}(\varepsilon, \eta, \tau)\psi \right\|, \quad (3.11)$$

where each of the summands on the right-hand side is uniformly bounded in time as follows. Being $\mathcal{D}(H(\varepsilon, \eta, \tau))$ a core of $\mathcal{D}(H^{1/2}(\varepsilon, \eta, \tau))$ [9, V-§3.11 Lemma 3.38], in view of (3.6), above the first summand is bounded since for every $\widetilde{\psi} \in \mathcal{D}(H(\varepsilon, \eta, \tau))$

$$\left\| \int_0^\infty dx \, \left(x^2 + H(\varepsilon, \eta, \tau) \right)^{-1} H_1 \left(x^2 + H(\varepsilon, \eta, \tau) \right)^{-1} H^{1/2}(\varepsilon, \eta, \tau)\widetilde{\psi} \right\|$$

$$\leq \int_0^\infty dx \, \left(x^2 + 1 \right)^{-1} \left\| H_1 H^{-1/2}(\varepsilon, \eta, \tau) \right\| \left\| \left(x^2 + H(\varepsilon, \eta, \tau) \right)^{-1} H(\varepsilon, \eta, \tau)\widetilde{\psi} \right\|$$

$$\leq \frac{\pi}{2}(a + \varepsilon M a^2) \left\| \widetilde{\psi} \right\|.$$

---

[10] First of all, notice that $\left(\mathbb{1} + \frac{1}{n} H^{(N+2)/2}(\varepsilon, \eta, r) \right)^{-1}$ converges strongly to $\mathbb{1}$. Indeed, in view of $\left\| \left(\mathbb{1} + \frac{1}{n} H^{(N+2)/2}(\varepsilon, \eta, r) \right)^{-1} \right\| \leq 1$, if $v \in \mathcal{D}(H^{(N+2)/2}(\varepsilon, \eta, r))$ then

$$\left\| \left(\mathbb{1} + \frac{1}{n} H^{(N+2)/2}(\varepsilon, \eta, r) \right)^{-1} v - v \right\| \leq \frac{1}{n} \left\| \left(\mathbb{1} + \frac{1}{n} H^{(N+2)/2}(\varepsilon, \eta, r) \right)^{-1} \right\| \left\| H^{(N+2)/2}(\varepsilon, \eta, r)v \right\|$$

$$\leq \frac{1}{n} \left\| H^{(N+2)/2}(\varepsilon, \eta, r)v \right\|.$$

By density of $\mathcal{D}(H^{(N+2)/2}(\varepsilon, \eta, r))$ in $\mathcal{H}$ the strong convergence follows. Therefore, for every $u \in \mathcal{D}(H^{N/2}(\varepsilon, \eta, r))$ defining $u_n := \left(\mathbb{1} + \frac{1}{n} H^{(N+2)/2}(\varepsilon, \eta, r) \right)^{-1} u \in \mathcal{D}(H^{(N+2)/2}(\varepsilon, \eta, r))$ one has that

$$\lim_{n \to \infty} H^{N/2}(\varepsilon, \eta, r)u_n = \lim_{n \to \infty} \left(\mathbb{1} + \frac{1}{n} H^{(N+2)/2}(\varepsilon, \eta, r) \right)^{-1} H^{N/2}(\varepsilon, \eta, r)u = H^{N/2}(\varepsilon, \eta, r)u,$$

and thus by using that $H^{-N/2}(\varepsilon, \eta, r)$ is bounded we obtain that $\lim_{n \to \infty} u_n = u$ as well.

On the other hand for the second summand in (3.11) for $m \geq 2$, we have that

$$
\left[H^{-(m-1)/2}(\varepsilon, \eta, \tau), \frac{\partial}{\partial \tau} \left(H^{-1/2}(\varepsilon, \eta, \tau) \right) \right] H^{m/2}(\varepsilon, \eta, \tau) \psi
$$

$$
= \sum_{k=0}^{m-2} H^{-k/2}(\varepsilon, \eta, \tau) \left[H^{-1/2}(\varepsilon, \eta, \tau), \frac{\partial}{\partial \tau} \left(H^{-1/2}(\varepsilon, \eta, \tau) \right) \right] H^{(k+2)/2}(\varepsilon, \eta, \tau) \psi
$$

$$
= \frac{4\varepsilon \eta\, g'(\eta \tau)}{\pi^2} \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, (x^2 + H(\varepsilon, \eta, \tau))^{-1} (y^2 + H(\varepsilon, \eta, \tau))^{-1}.
$$

$$
\cdot \sum_{k=0}^{m-2} H^{-k/2}(\varepsilon, \eta, \tau) [H(\varepsilon, \eta, \tau), H_1] H^{(k-2)/2}(\varepsilon, \eta, \tau)
$$

$$
\cdot H(\varepsilon, \eta, \tau)(x^2 + H(\varepsilon, \eta, \tau))^{-1} H(\varepsilon, \eta, \tau)(y^2 + H(\varepsilon, \eta, \tau))^{-1} \psi.
$$

Clearly, the operator at right-hand side is uniformly bounded in τ, since $(x^2 + H(\varepsilon, \eta, \tau))^{-1}$ and $(y^2 + H(\varepsilon, \eta, \tau))^{-1}$ ensure the uniform convergence of the integrals, hypothesis (B(k)) for $0 \leq k \leq m - 2$ guarantees the boundedness of the middle factor and $\left\| H(\varepsilon, \eta, \tau)(z^2 + H(\varepsilon, \eta, \tau))^{-1} \right\| \leq 1$ for all $z \in [0, \infty)$. Therefore, we obtain that

$$
\left\| \left[H^{-(m-1)/2}(\varepsilon, \eta, \tau), \frac{\partial}{\partial \tau} \left(H^{-1/2}(\varepsilon, \eta, \tau) \right) \right] H^{m/2}(\varepsilon, \eta, \tau) \psi \right\| \leq \varepsilon \eta M' \sum_{k=0}^{m-2} E_k \left\| \psi \right\|.
$$

Finally, plugging estimate (3.10) into inequality (3.9), we have

$$
C_{\varepsilon, \eta, N}(t, r) \leq C_{N-1}(\varepsilon) \left(1 + (\alpha + \beta \varepsilon + \gamma_N) \varepsilon \eta \int_t^r \mathrm{d}\tau\, C_{\varepsilon, \eta, N}(\tau, r) \right),
$$

for $0 \leq t \leq r \leq 1/\eta$. Applying Grönwall's inequality, we conclude that

$$
C_{\varepsilon, \eta, N}(t, r) \leq C_{N-1}(\varepsilon) \mathrm{e}^{C_{N-1}(\varepsilon)(\alpha + \beta \varepsilon + \gamma_N) \varepsilon \eta |t - r|}
$$
$$
\leq C_{N-1}(\varepsilon) \mathrm{e}^{C_{N-1}(\varepsilon)(\alpha + \beta \varepsilon + \gamma_N) \varepsilon} =: C_N(\varepsilon)
$$

for all $t, r \in [0, 1/\eta]$. $\square$

3.2 Proof of Corollary 2.7

Notice that identity (1.5) holds true since for every $\varphi \in \mathcal{D}(H_0)$ one has that

$$
\mathrm{i} \frac{\partial}{\partial s} \left(\mathrm{e}^{\mathrm{i} \frac{\varepsilon}{\eta} \phi(s) H_1} U_{\varepsilon, \eta}(s/\eta, u/\eta) \mathrm{e}^{-\mathrm{i} \frac{\varepsilon}{\eta} \phi(u) H_1} \varphi \right)
$$

$$
= \mathrm{e}^{\mathrm{i} \frac{\varepsilon}{\eta} \phi(s) H_1} \left(\frac{1}{\eta} H(\varepsilon, \eta, s/\eta) - \frac{\varepsilon}{\eta} g(s) H_1 \right) U_{\varepsilon, \eta}(s/\eta, u/\eta) \mathrm{e}^{-\mathrm{i} \frac{\varepsilon}{\eta} \phi(u) H_1} \varphi
$$

$$= \frac{1}{\eta} e^{i\frac{\varepsilon}{\eta}\phi(s)H_1} H_0 e^{-i\frac{\varepsilon}{\eta}\phi(s)H_1} e^{i\frac{\varepsilon}{\eta}\phi(s)H_1} U_{\varepsilon,\eta}(s/\eta, u/\eta) e^{-i\frac{\varepsilon}{\eta}\phi(u)H_1} \varphi$$

$$= \frac{1}{\eta} \hat{H}(\varepsilon, \eta, s) \hat{U}_{\varepsilon,\eta}(s, u)\varphi,$$

due to strong differentiability of $U_{\varepsilon,\eta}(t, r)$ on $\mathcal{D}(H_0)$, Assumption (C$_2$(k)) for $k = 2$ and $\mathcal{D}(H_0) \subset \mathcal{D}(H_1)$ by Assumption (A$_2$), and similarly one verifies the other properties in (2.9). Therefore, fixed any $n \in \mathbb{N}$, in view of Assumption (C$_2$(k)) for $k = n$, for every $\psi \in \mathcal{D}(H_0^{n/2})$ we have that

$$\hat{H}^{-n/2}(\varepsilon, \eta, s) \hat{U}_{\varepsilon,\eta}(s, u) \hat{H}^{n/2}(\varepsilon, \eta, u)\psi$$

$$= e^{i\frac{\varepsilon}{\eta}\phi(s)H_1} H_0^{-n/2} e^{-i\frac{\varepsilon}{\eta}\phi(s)H_1} e^{i\frac{\varepsilon}{\eta}\phi(s)H_1} U_{\varepsilon,\eta}(s/\eta, u/\eta) \cdot$$

$$\cdot e^{-i\frac{\varepsilon}{\eta}\phi(u)H_1} e^{i\frac{\varepsilon}{\eta}\phi(u)H_1} H_0^{n/2} e^{-i\frac{\varepsilon}{\eta}\phi(u)H_1}\psi$$

$$= e^{i\frac{\varepsilon}{\eta}\phi(s)H_1} H_0^{-n/2} U_{\varepsilon,\eta}(s/\eta, u/\eta) H_0^{n/2} e^{-i\frac{\varepsilon}{\eta}\phi(u)H_1}\psi.$$

Thus, we deduce that

$$\left\| \hat{H}^{-n/2}(\varepsilon, \eta, s) \hat{U}_{\varepsilon,\eta}(s, u) \hat{H}^{n/2}(\varepsilon, \eta, u)\psi \right\|$$

$$= \left\| H_0^{-n/2} H^{n/2}(\varepsilon, \eta, s/\eta) \cdot H^{-n/2}(\varepsilon, \eta, s/\eta) U_{\varepsilon,\eta}(s/\eta, u/\eta) H^{n/2}(\varepsilon, \eta, u/\eta) \cdot \right.$$

$$\left. \cdot H^{-n/2}(\varepsilon, \eta, u/\eta) H_0^{n/2} e^{-i\frac{\varepsilon}{\eta}\phi(u)H_1}\psi \right\|$$

$$\leq C_n(\varepsilon)(1 + \varepsilon D_n),$$

by using Theorem 2.5 and Proposition 2.9. Finally, the Riesz Lemma implies the thesis for all $n = -|n| \in \mathbb{Z}$. $\qquad\square$

4 Proof of the auxiliary results

4.1 Proof of Lemma 2.8

In view of $\mathcal{D}(H^{1/2}(\varepsilon, \eta, t)) = \mathcal{D}(H_0^{1/2})$ by Remark 2.4.(iii), equality (3.5) and the second resolvent identity, we have that

$$H_1 H^{-1/2}(\varepsilon, \eta, t) = \frac{2}{\pi} \int_0^\infty dx\, H_1 \left(x^2 + H(\varepsilon, \eta, t)\right)^{-1}$$

$$= H_1 H_0^{-1/2} - \frac{2\varepsilon g(\eta t)}{\pi} \int_0^\infty dx\, H_1 \left(x^2 + H_0\right)^{-1} H_1 \left(x^2 + H(\varepsilon, \eta, t)\right)^{-1}.$$

$$(4.1)$$

In the last expression, for the second summand we observe that

$$\left\| \int_0^\infty \mathrm{d}x \, H_1 H_0^{-1/2} \cdot H_0^{1/2} \left(x^2 + H_0\right)^{-1} H_0^{1/2} \cdot H_0^{-1/2} H_1 \cdot \left(x^2 + H(\varepsilon, \eta, t)\right)^{-1} \right\| \leq \frac{a^2 \pi}{2},$$

where we have used the hypothesis $\left\| H_1 H_0^{-1/2} \right\| = a < \infty$, condition (2.2) and $\left\| H_0^{-1/2} H_1 \varphi \right\| = \left\| \left(H_1 H_0^{-1/2}\right)^* \varphi \right\| \leq a \, \|\varphi\|$ for all $\varphi \in \mathcal{D}(H_1) \supseteq \mathcal{D}(H_0)$. Using the last inequality in (4.1) the thesis is obtained. $\square$

4.2 Proof of Proposition 2.9

First of all, notice that for any $k \in \mathbb{N}$ if one supposes Assumption $(C_1(k))$ then Remark 2.4.(iii) ensures that the products of operators $H_0^{k/2} H^{-k/2}(\varepsilon, \eta, t)$ and $H(\varepsilon, \eta, t)^{k/2} H_0^{-k/2}$ are well defined on $\mathcal{H}$. We are going to prove inequality (2.10) for every $n \in \mathbb{N}_0$, proceeding by induction. The induction step will be proved by using the base cases for $0 \leq n \leq 3$ and estimate (2.11) for $n = 1$. For $n = 0$ it is trivial. For $n = 1$, in view of equality (3.5) and the second resolvent identity we obtain that

$$\left\| H_0^{1/2} H^{-1/2}(\varepsilon, \eta, t) \right\| = \frac{2}{\pi} \left\| H_0^{1/2} \int_0^\infty \mathrm{d}x \left(x^2 + H(\varepsilon, \eta, t)\right)^{-1} \right\|$$

$$\leq 1 + \frac{2\varepsilon M}{\pi} \int_0^\infty \mathrm{d}x \left\| H_0^{1/2} \left(x^2 + H_0\right)^{-1} H_0^{1/2} \right\| \left\| H_0^{-1/2} H_1 \left(x^2 + H(\varepsilon, \eta, t)\right)^{-1} \right\|$$

$$\leq 1 + \varepsilon M a,$$

where we have used the hypothesis $\left\| H_1 H_0^{-1/2} \right\| = a < \infty$ and condition (2.2). Analogously, by virtue of Lemma 2.8 and condition (2.2), one obtains (2.11) for $n = 1$. For $n = 2$ rewriting

$$H_0 H^{-1}(\varepsilon, \eta, t) = (H_0 + \varepsilon g(\eta t) H_1 - \varepsilon g(\eta t) H_1) H^{-1}(\varepsilon, \eta, t)$$
$$= \mathbb{1} - \varepsilon g(\eta t) H_1 H^{-1}(\varepsilon, \eta, t),$$

thus by applying Lemma 2.8 and condition (2.2), inequality (2.10) is obtained. For $n = 3$ notice that

$$H_0^{3/2} H^{-3/2}(\varepsilon, \eta, t) = H_0^{1/2} H_0 H^{-1/2}(\varepsilon, \eta, t) H^{-1}(\varepsilon, \eta, t)$$
$$= H_0^{1/2} H^{-1/2}(\varepsilon, \eta, t) H_0 H^{-1}(\varepsilon, \eta, t) + H_0^{1/2}[H_0, H^{-1/2}(\varepsilon, \eta, t)] H^{-1}(\varepsilon, \eta, t),$$
$$\tag{4.2}$$

where on the right-hand side the first summand is bounded[11] by $1 + \mathcal{O}(\varepsilon)$ by applying the base cases for $1 \leq n \leq 2$. For the second summand in (4.2), Leibniz's rule and equality (3.5) imply that

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[11] In this proof when we write that an operator is bounded by a constant we mean it in the sense of the operator norm, and $\mathcal{O}(\varepsilon)$ is understood in the sense of the operator norm as well.

$$H_0^{1/2}[H_0, H^{-1/2}(\varepsilon, \eta, t)]H^{-1}(\varepsilon, \eta, t)$$
$$= \frac{2\varepsilon g(\eta t)}{\pi}\int_0^\infty dx\, H_0^{1/2}\left(x^2 + H(\varepsilon, \eta, t)\right)^{-1} \cdot [H_1, H(\varepsilon, \eta, t)]H^{-1}(\varepsilon, \eta, t)$$
$$\cdot \left(x^2 + H(\varepsilon, \eta, t)\right)^{-1}$$

where in the last equality the first factor is uniformly bounded in x since

$$\left\|H_0^{1/2}\left(x^2 + H(\varepsilon, \eta, t)\right)^{-1}\right\| \le \left\|H_0^{1/2}H^{-1/2}(\varepsilon, \eta, t)H^{1/2}(\varepsilon, \eta, t)\left(x^2 + H(\varepsilon, \eta, t)\right)^{-1}\right\|$$
$$\le 1 + A_1\varepsilon,$$

the second factor is bounded by virtue of hypothesis (B(k)) for $k = 0$ and the last one ensures the convergence of the integral. Now we take some $N \in \mathbb{N}_0$. We assume that inequality (2.10) holds true for $n \in \{1, \ldots, N-1\}$ and we prove it for $n = N$. We split the cases for even and odd N. Let $N = 2m$ for $m \ge 2$, we get that

$$H_0^{N/2}H^{-N/2}(\varepsilon, \eta, t) = H_0^m H^{-m}(\varepsilon, \eta, t)$$
$$= H_0^{m-1}[H_0, H^{1-m}(\varepsilon, \eta, t)]H^{-1}(\varepsilon, \eta, t) + H_0^{m-1}H^{1-m}(\varepsilon, \eta, t)H_0 H^{-1}(\varepsilon, \eta, t).$$
$$(4.3)$$

In (4.3) the second summand is bounded by $1 + \mathcal{O}(\varepsilon)$ by applying the induction hypothesis for $n = N - 2$ and the base case for $n = 2$. On the other hand, for the first summand in (4.3) Leibniz's rule implies that

$$H_0^{m-1}[H_0, H^{1-m}(\varepsilon, \eta, t)]H^{-1}(\varepsilon, \eta, t)$$
$$= \varepsilon g(\eta t)H_0^{m-1}H^{1-m}(\varepsilon, \eta, t)H^{-1}(\varepsilon, \eta, t)\cdot$$
$$\cdot \sum_{h=0}^{m-2} H^{m-h-1}(\varepsilon, \eta, t)[H_1, H(\varepsilon, \eta, t)]H^{h-m}(\varepsilon, \eta, t),$$

which is $\mathcal{O}(\varepsilon)$ thanks to the induction hypothesis for $n = N - 2$, condition (2.2) and hypothesis (B(k)) for all $2 - N \le k := 2(h - m) + 2 \le -2$. Let $N = 2m + 1$ for $m \ge 2$, similarly we have that

$$H_0^{N/2}H^{-N/2}(\varepsilon, \eta, t) = H_0^{m-1/2}H_0 H^{-m+1}(\varepsilon, \eta, t)H^{-3/2}(\varepsilon, \eta, t)$$
$$= H_0^{m-1/2}[H_0, H^{-m+1}(\varepsilon, \eta, t)]H^{-3/2}(\varepsilon, \eta, t)$$
$$+ H_0^{m-1/2}H^{1/2-m}(\varepsilon, \eta, t)H^{1/2}(\varepsilon, \eta, t)H_0^{-1/2}H_0^{3/2}H^{-3/2}(\varepsilon, \eta, t),$$

where in the last equality the second summand can be bounded by $1 + \mathcal{O}(\varepsilon)$ due to the induction hypothesis for $n = N - 2$, inequality (2.11) for $n = 1$ and the base case for $n = 3$. While, the first summand can be rewritten as

$$H_0^{m-1/2}[H_0, H^{-m+1}(\varepsilon, \eta, t)]H^{-3/2}(\varepsilon, \eta, t)$$
$$= \varepsilon g(\eta t)H_0^{m-1/2}H^{-m+1/2}(\varepsilon, \eta, t)\cdot$$
$$\cdot H^{-1}(\varepsilon, \eta, t)\sum_{h=0}^{m-2}H^{m-h-1/2}(\varepsilon, \eta, t)[H_1, H(\varepsilon, \eta, t)]H^{-m+h-1/2}(\varepsilon, \eta, t),$$

where last term is $\mathcal{O}(\varepsilon)$ in view of the induction hypothesis for $n = N - 2$ and assumption (B(k)) for every $2 - N \leq k := 2(h-m)+1 \leq -3$. Thus, inequality (2.10) is proved for every $n \in \mathbb{N}_0$. Similarly, one proves estimate (2.11) for all $n \in \mathbb{N}_0$. Finally, to show inequality (2.10) for negative integer numbers, we notice that for any $n \in \mathbb{N}$, for every $\psi \in \mathcal{D}(H^{n/2}(\varepsilon, \eta, t))$

$$\left\| H_0^{-n/2}H^{n/2}(\varepsilon, \eta, t)\psi \right\| = \left\| \left(H^{n/2}(\varepsilon, \eta, t)H_0^{-n/2}\right)^*\psi \right\|$$
$$\leq \left\| H^{n/2}(\varepsilon, \eta, t)H_0^{-n/2} \right\| \|\psi\|,$$

where the right-hand side is bounded by $(1 + B_n\varepsilon) \|\psi\|$ in view of estimate (2.11) for positive integers. Analogously, one proves estimate (2.11) for negative integers. □

5 Application of the general strategy to Landau-type Hamiltonians

Among magnetic Schrödinger operators associated with non-interacting electrons in the plane, with (constant) magnetic field perpendicular to the plane, the Landau model is emblematic for the understanding of the quantum Hall effect (QHE) [5]. For the model of Landau-type Hamiltonians an explanation for the QHE is provided [2, Theorem 2.2] by relying on adiabatic perturbation theory [16], which allows to compute rigorously the response of the intensity current being linear in the perturbation determined by the voltage difference (for recent topical reviews see [6, 12]). Here, first we briefly explain why the energy estimates established in the general mathematical framework of Sect. 2 are useful in this respect. Then, we verify that this model satisfies the assumptions previously stated.

This class of perturbed Hamiltonians is specified by [2, Equation (1.1)]. For the sake of clarity, we recall some definitions.

Definition 5.1 Let be $j \in \{1, 2\}$ and $l_j > 0$, a l_j-**switch function in the** j-th **direction** is a smooth function $\Lambda_j : \mathbb{R}^2 \to [0, 1]$ that depends only on the variable x_j and satisfies

$$\Lambda_j(x_j) = \begin{cases} 0 & \text{if } x_j < -l_j \\ 1 & \text{if } x_j > l_j. \end{cases}$$

We consider the unperturbed Hamiltonian H_0, defined as[12]

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[12] We use Hartree atomic units, and moreover we reabsorb the factor $\frac{e}{c}$, where e is the charge of the electron and c is the speed of light, in the definition of the magnetic potential **A**.

$$H_0 := \frac{1}{2}\mathbf{p}_\mathbf{A}^2 + \lambda V \quad \text{acting in } L^2(\mathbb{R}^2, \mathrm{dx}), \tag{5.1}$$

where $\mathbf{p}_\mathbf{A} := (\mathbf{p} - \mathbf{A}(\mathbf{x}))$ with $\mathbf{p} := -\mathrm{i}\nabla = -\mathrm{i}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ and $\mathbf{A}(x_1, x_2) := B/2(-x_2, x_1)$ with $B > 0, \lambda \in \mathbb{R}$ and the potential V is such that $\|V\|_\infty$ is finite.[13] The perturbed Hamiltonian is defined as[14] $H(\varepsilon, t) := H(\varepsilon, \eta = \varepsilon, t) = H_0 + \varepsilon g(\varepsilon t)\Lambda_1$, where $0 < \varepsilon \ll 1$, Λ_1 is a l_1-switch function in the 1-st direction and g fulfills the hypotheses in Assumption 2.1. The multiplication operator Λ_1 models an electric potential of negative unit drop for an electric field pointing in the negative 1-st direction. One is interested in computing the Hall conductance G_{Hall}, defined as a ratio between the (excess of) induced current intensity when the perturbation is fully switched on and the voltage difference. More precisely, one introduces the operator $\mathrm{i}[H_0, \Lambda_2]$ standing for the current intensity in the 2-nd direction and $\rho_\varepsilon(t)$ the density operator, representing the state of the system at time t evolving from the Fermi projection P_0 of the unperturbed Hamiltonian H_0 with associated Fermi energy in a spectral gap of H_0. Thus, one is in shape to define the Hall conductance as

$$\text{Tr}\left(\mathrm{i}[H_0, \Lambda_2](\rho_\varepsilon(t) - P_0)\right) =: -G_{\text{Hall}}\,\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0, \tag{5.2}$$

for any $t \geq 1/\varepsilon$ (when the perturbation is fully switched on). In [2] first, by exploiting the invariance of the trace under unitary conjugation, one rewrites[15]

$$\text{Tr}\left(\mathrm{i}[H_0, \Lambda_2](\rho_\varepsilon(s/\varepsilon) - P_0)\right) = \text{Tr}\left(\mathrm{e}^{\mathrm{i}\phi(s)\Lambda_1}\mathrm{i}[H_0, \Lambda_2](\rho_\varepsilon(s/\varepsilon) - P_0)\mathrm{e}^{-\mathrm{i}\phi(s)\Lambda_1}\right)$$
$$= \text{Tr}\left(\mathrm{i}[\hat{H}(s), \Lambda_2](\hat{\rho}_\varepsilon(s) - \hat{P}_0(s))\right) \tag{5.3}$$

where $s := \varepsilon t$ is the scaled time, $\hat{H}(s) := \mathrm{e}^{\mathrm{i}\phi(s)\Lambda_1}H_0\mathrm{e}^{-\mathrm{i}\phi(s)\Lambda_1}$, $\hat{\rho}_\varepsilon(s) := \mathrm{e}^{\mathrm{i}\phi(s)\Lambda_1}\rho_\varepsilon(s/\varepsilon)\mathrm{e}^{-\mathrm{i}\phi(s)\Lambda_1}$ and $\hat{P}_0(s) = \mathrm{e}^{\mathrm{i}\phi(s)\Lambda_1}P_0\mathrm{e}^{-\mathrm{i}\phi(s)\Lambda_1}$. Then, in order to derive an explicit formula for the Hall conductance G_{Hall}, they use an asymptotic expansion in ε powers of $\hat{\rho}_\varepsilon(s)$

$$\hat{\rho}_\varepsilon(s) = \sum_{j=0}^{k} \varepsilon^j B_j - \varepsilon^k \int_0^s \mathrm{d}r\, \hat{U}_\varepsilon(s, r)\dot{B}_k(r)\hat{U}_\varepsilon(r, s), \tag{5.4}$$

where $\hat{U}_\varepsilon(r, s) := \hat{U}_{\varepsilon, \eta=\varepsilon}(r, s)$. Clearly, by plugging (5.4) into (5.3) and (5.2), beyond controlling the terms involving the B_j's, one needs to estimate for $k > 1$

$$\varepsilon^{k-1}\text{Tr}\left(\int_0^s \mathrm{d}r\, \mathrm{i}[H_0, \Lambda_2]\hat{U}_\varepsilon(s, r)\dot{B}_k(r)\hat{U}_\varepsilon(r, s)\right).$$

---

[13] In [2, Theorem 2.2] a stronger hypothesis is assumed, namely $|\lambda|\,\|V\|_\infty < B$ to ensure that the spectrum of H_0 consists of a infinite sequence of bands, separated from each other by finite gaps.

[14] Notice that in this case we are imposing that the intensity of the perturbation and time-scale parameter, respectively ε and η, are equal.

[15] The advantage of working with $\hat{H}(s)$ instead of $H(\varepsilon, t)$ is the isospectrality of the former Hamiltonians.

In the continuum to prove that the trace of an operator O is finite it suffices to show that O has suitable localization both in energy and space [2, Proposition 3.2]. Since $\dot{B}_k(r)$ decays fast enough in energy in the sense of (1.6), this energy localization is retained by the corresponding evolved operator $\hat{U}_\varepsilon(s, r)\dot{B}_k(r)\hat{U}_\varepsilon(r, s)$ as in (1.7) by exploiting the energy estimate in the form of (1.4) (compare the inequality after [2, Equation (3.12)] and [2, Remark (3), p. 599] for the case $n = 0$).

Now we are going to verify that the general assumptions of Sect. 2 are fulfilled by this specific model. Clearly, $H(\varepsilon, t)$ satisfies Assumptions (A_1) and (A_2). Assumptions $(B(k))$, $(C_1(k))$ and $(C_2(k))$ hold true under certain regularity conditions on V. Fix any $k \in \mathbb{Z}$, assume that the Sobolev norm[16] $\|V\|_{|k|+1,\infty}$ is finite then hypothesis $(B(k))$ holds true. Indeed, since $[\Lambda_1, H(\varepsilon, t)] = \frac{i}{2}\left(p_{\mathbf{A},1}\Lambda_1' + \Lambda_1' p_{\mathbf{A},1}\right)$, applying [2, Proposition 3.1.(i)] we deduce that there exists a finite constant e_k:

$$\left\| H^{-k/2}(\varepsilon, t)[\Lambda_1, H(\varepsilon, t)]H^{(k-2)/2}(\varepsilon, t) \right\| \le e_k \|\Lambda_1\|_{|k|+2,\infty},$$

for all $\varepsilon \in (0, 1)$ and $t \in \mathbb{R}$.

Now let $k \in \mathbb{N}$ with $k \ge 2$, assume that $\|V\|_{2(k-1),\infty}$ is finite then it follows that for all $\varepsilon \in (0, 1)$ and $t \in \mathbb{R}$

$$\mathcal{D}(H^k(\varepsilon, t)) \equiv \mathcal{D}(H_0^k),$$

namely the hypothesis $(C_1(k))$ is fulfilled. Indeed, observe that

$$H^k(\varepsilon, t) = H_0^k + (\varepsilon g(\varepsilon t))^k \Lambda_1^k + \sum_{j=1}^{2^k-2} M_j, \tag{5.5}$$

where each operator M_j is such that there exist $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_k) \in \{0, 1\}^k$ with $\boldsymbol{\alpha} \ne 0 \ne \boldsymbol{\beta}$ and $\sum_{j=1}^k \alpha_j + \beta_j = k$:

$$M_j = (\varepsilon g(\varepsilon t))^{\sum_{j=1}^k \beta_j} H_0^{\alpha_1} \Lambda_1^{\beta_1} \cdots H_0^{\alpha_k} \Lambda_1^{\beta_k}.$$

We are going to show that $\mathcal{D}(H_0^k) \subseteq \mathcal{D}(H^k(\varepsilon, t))$. It suffices to observe that every M_j is densely defined on $\mathcal{D}(H_0^{k-1}) \supseteq \mathcal{D}(H_0^k)$. In fact, rewriting[17]

$$H_0^{\alpha_1} \Lambda_1^{\beta_1} \cdots H_0^{\alpha_k} \Lambda_1^{\beta_k} H_0^{-k+1} = H_0^{\sum_{j=1}^k \alpha_j - k + 1}$$

$$\cdot \prod_{m=1}^{k-1} H_0^{k-1-\sum_{j=0}^{m-1} \alpha_{k-j}} \Lambda_1^{\beta_{k-m}} H_0^{\sum_{j=0}^{m-1} \alpha_{k-j} - k + 1}$$

$$\cdot H_0^{k-1} \Lambda_1^{\beta_k} H_0^{-k+1}$$

---

[16] Let us recall that for $k \in \mathbb{N}$ the Sobolev norm $\|f\|_{k,\infty}$ of a scalar function f on $\mathbb{R}^2$ is defined as $\|f\|_{k,\infty} := \sum_{\substack{\alpha_1,\alpha_2 \in \mathbb{N} \\ \alpha_1+\alpha_2 \le k}} \left\|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} f\right\|_\infty$, where $\|f\|_\infty := \sup_{\mathbf{x} \in \mathbb{R}^2} |f(\mathbf{x})|$.

[17] As in the previous sections, up to a shift of a constant, we can assume that $H_0 \ge 1$.

here the product $\prod_{m=1}^{k-1}$ is ordered in the sense that a factor with larger index m stands to the left of ones with smaller m and, hence [2, Proposition 3.1.(i).(b)] implies that

$$\left\| H_0^{\alpha_1} \Lambda_1^{\beta_1} \cdots H_0^{\alpha_k} \Lambda_1^{\beta_k} H_0^{-k+1} \right\| \leq C_{k-1} \left\| H_0^{\sum_{j=1}^{k} \alpha_j - k + 1} \right\| \left\| \Lambda_1^{\beta_k} \right\|_{2k-2,\infty} \cdot$$
$$\cdot \prod_{m=1}^{k-1} C_{k-1-\sum_{j=0}^{m-1} \alpha_{k-j}} \left\| \Lambda_1^{\beta_{k-m}} \right\|_{2k-2-\sum_{j=0}^{m-1} 2\alpha_{k-j},\infty},$$

which is finite, because $\sum_{j=1}^{k} \alpha_j - k + 1 \leq 0$ and any Sobolev norm of $\Lambda_1^{\beta_j}$ for all $\beta_j \in \{0, 1\}$ is bounded. On the other hand, rewriting $H_0^k = (H(\varepsilon, t) - \varepsilon g(\varepsilon t) \Lambda_1)^k$ and applying again [2, Proposition 3.1.(i).(b)], we deduce that $\mathcal{D}(H_0^k) \supseteq \mathcal{D}(H^k(\varepsilon, t))$. Now let $k \in \mathbb{N}$, suppose that $\|V\|_{k,\infty}$ is finite then Assumption $(C_2(k))$ is satisfied. In fact, consider the gauge transformation $\mathrm{e}^{\mathrm{i}\lambda\Lambda_1}$ with $\lambda \in \mathbb{R}$, thus by virtue of [2, Proposition 3.1.(i).(b)] we obtain that $\left\| H_0^{k/2} \mathrm{e}^{\mathrm{i}\lambda\Lambda_1} H_0^{-k/2} \right\| \leq C_{k/2} \left\| \mathrm{e}^{\mathrm{i}\lambda\Lambda_1} \right\|_{k,\infty} < \infty$. Thus, for every $n \in \mathbb{Z}$, if $|n| \geq 2$ assuming that $\|V\|_{|n|-1,\infty}$ is finite, then Theorem 2.5 implies that the inequality in [2, Lemma 5.1] holds true. Furthermore, assuming that $\|V\|_{2,\infty}$ is finite, then fixing any $n \in \mathbb{Z}$, if $|n| \geq 2$ supposing in addition that $\|V\|_{2|n|-2,\infty}$ is finite one can apply Corollary 2.7 as well.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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