Letters in Mathematical Physics (2022) 112:37
https://doi.org/10.1007/s11005-022-01521-6

On the spacetime structure of infrared divergencies in QED

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Received: 16 October 2021 / Revised: 9 February 2022 / Accepted: 10 February 2022 /
Published online: 5 April 2022

Abstract
We investigate analytic properties of string-integrated massless correlation functions and propagators with emphasis on their infrared behaviour. These are relevant in various models of quantum field theory with massless fields, including QED.

Keyword Quantum field theory, QED, infrared structure, string-localized fields

Mathematics Subject Classification 81V10, 81T15

1 Motivation

A conceptually new approach to QED is presented in [12] (see also [11, 17]). It is designed to better understand the long-distance behaviour of QED, including the uncountable superselection structure of charged states due to their "asymptotic photon clouds", and the infraparticle nature of the electron. The latter is manifest in a sharp lower end of the mass spectrum with a singular set-on of the continuum due to the attached soft photons. The new approach properly addresses and solves the problem with the quantum Gauss Law on the physical Hilbert space: if the charged field were a local quantum field, the integrated electric flux at spacelike infinity would commute with it and cannot act as the generator of the $U(1)$ symmetry. The tight relations among these physical features have been known since long [3, 4, 7, 8], while a way to incorporate them into a model was so far lacking—apart from a simple model in

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1+1 spacetime dimensions [16]. The new approach proposed in [12] provides such a model and also sheds some new light on the Infrared Triangle [18].

The most prominent role in the new approach is played by an auxiliary quantum field formally defined as

$$\phi(x,e) = \int_{\Gamma_{x,e}} A^K_\mu(y)\mathrm{d}y^\mu = \int_0^\infty \mathrm{d}s\, A^K_\mu(x+se)e^\mu \tag{1.1}$$

(introduced in more detail below; $\Gamma_{x,e}$ is a straight curve from x along a direction e to infinity, and the superscript K stands for "Krein space" emphasizing the indefinite metric of the usual Feynman gauge Fock space). It is the main purpose of these notes to investigate details of its infrared behaviour in position space.

The infrared superselection sectors of QED arise by exponentiating the field $\phi(x,e)$, smeared with suitable functions $c(e)$. Because Eq. (1.1) is infrared divergent as it stands, a suitable infrared cut-off function $v(k)$ is needed that allows to extend the Fourier transform of the two-point function to $k = 0$ (as a distribution). With an appropriate regularization by a mass $m \to 0$, the correlation functions of exponentiated fields ("vertex operators")

$$N_v(c) \cdot :e^{iq\phi(x,c)}:_v \tag{1.2}$$

contain an overall factor $e^{-d_{m,v}(C,C)}$, where $d_{m,v}(C,C)$ is an integral diverging to $+\infty$ as $m \to 0$, unless $C(e) \equiv \sum_i q_i c_i(e) = 0$. In the limit, the factor converges to zero unless $C(e) = 0$:

$$e^{-d_m(C,C)} \to \delta_{C,0}. \tag{1.3}$$

This factor entails that states with different "charge functions" C are mutually orthogonal, and produces an uncountable number of superselection rules. The physical meaning of the charge functions is that of "photon clouds" attached to charged particles [12]. States created by the exponential field acting on the vacuum can formally be regarded as coherent photon states lying outside the vacuum Fock space, and these coherent states belong to inequivalent representations of the Maxwell field whenever their photon clouds (i.e. their smearing functions $c(e)$) differ.

In [12, Sect. 3.1], a "dressed Dirac field"

$$\psi_{q,c}(x) =: e^{iq\phi(x,c)} :_v \cdot \psi_0(x) \tag{1.4}$$

is introduced, where q is the unit of electric charge. This field arises by subjecting the free Dirac field to the "trivial" interaction density

$$q \cdot \partial_\mu \phi(x,c) j^\mu(x). \tag{1.5}$$

Being a total derivative, Eq. (1.5) does not contribute to the total action, and gives rise to a trivial scattering matrix. The non-perturbative construction of the dressed Dirac field is meant as a first step towards the full perturbative QED, by splitting into two parts a QED interaction density that can be defined on a positive-definite subspace of the indefinite Fock space (i.e. a Krein space). Although the interaction density Eq. (1.5) is "trivial", it drastically changes the algebraic structure of the charged field. The dressed Dirac field is string localized (see Sect. 2) and falls outside the regime of, say, Wightman quantum field theory. Unlike the free Dirac field, it creates states enjoying infrared features of QED that cannot be attained in the usual local approach to QED (including the quantum Gauss Law, the photon cloud superselection structure, and the associated breakdown of Lorentz invariance). But the dressed Dirac field does not interpolate between different scattering states. A non-trivial S-matrix is only produced when also the "true" interaction $q \cdot A^K_\mu j^\mu(x)$ is turned on.

The exponentiated escort field is reminiscent of the "dressing factor" in the Faddeev-Kulish prescription [10], that was introduced to prevent the formal vanishing of the LSZ limit, and hence of the scattering matrix of QED [5, 20]. But, as explained in more detail [12, Sect. 4.3 and 4.4], there are major differences: the dressing factor in [10] is not part of the charged field (as ours) but rather of the states in which the S-matrix has to be evaluated. The FK factor corresponds to a timelike string in the direction of the electron momentum that would arise modulo the photon momentum k (i.e. the null longitudinal photon) by averaging over spacelike strings perpendicular to p. More interestingly, our dressing factor does not *cancel* the IR divergence of QED but *interferes* with it in a way to dynamically deform the superselection rule.

Scattering theory in each of its formulations exploits the asymptotic large-time behaviour of correlation functions. Thus, a future modification of scattering theory adapted to theories with infraparticles will need detailed information about correlation functions of the infrafield. In the case of the dressed Dirac field, these involve correlation functions of the vertex operators. This is one of our motivations to study the latter.

The new approach to QED itself is not the topic of this paper, except for the short remarks in Sect. 2. For more, we refer to [12, Sect. 2]. Our topic are correlation functions of vertex operators and their analytic properties. In particular, the coefficients q in Eq. (1.2) may be regarded as free parameters, unrelated to the electric charge. The results also bear on technical aspects of the very setup of string-localized quantum field theory [13], e.g. how much smearing of the string directions is necessary.

2 Preliminaries

The basic idea of the new approach to QED [11, 13, 16] is to use "string-localized potentials" $A_\mu(x, e)$ for the free Maxwell field:

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x, e) - \partial_\nu A_\mu(x, e). \tag{2.1}$$

They depend on a spacelike four-direction e, and enjoy the axiality property

$$e^\mu A_\mu(x, e) = 0. \tag{2.2}$$

It is suggestive to think of them as “axial gauge” potentials; but the choice of e is not a gauge-fixing condition; rather the potentials $A_\mu(x,e)$ for all e coexist on the same Hilbert space where $F_{\mu\nu}$ is defined. More precisely, $A_\mu(x,e)$ is a distribution also in the variable e and requires a smearing with suitable functions $c(e)$. The latter are required to have total weight $\int d\sigma(e)\,c(e) = 1$, so that $A_\mu(x,c)$ are still potentials for $F_{\mu\nu}$.

The terminology “string-localized” refers to the fact that they can be defined as integrals (well defined in the distributional sense) over the field along a “string” of direction e:

$$A_\mu(x,e) := \int_0^\infty ds\, F_{\mu\nu}(x+se)e^\nu. \tag{2.3}$$

String-localization is an algebraic property: $A(x,e)$ commutes with $F(y)$ provided y is spacelike separated from the string $x+\mathbb{R}_+e$. The definition Eq. (2.3) ensures both Eqs. (2.1) and (2.2). Thus, “axiality” is a consequence of localization along a string.

The Hilbert space for the free Maxwell field can be directly obtained from the massless unitary Wigner representations of helicities ± 1 [20] without the detour through a potential, and can thus be seen as a primary entity. But in perturbative QED one usually starts from a local potential $A^K_\mu(x)$, say in the Feynman gauge, that is defined on an indefinite Krein space, and then defines

$$F^K_{\mu\nu}(x) = \partial_\mu A^K_\nu(x) - \partial_\nu A^K_\mu(x). \tag{2.4}$$

When the Krein space is reduced to the physical Hilbert space by the Gupta–Bleuler (or BRST) prescription, the Maxwell field becomes equivalent to the one on the Fock space over the Wigner representation; while the potential A^K_μ ceases to exist. In contrast, the string-localized potentials $A_\mu(x,e)$ (being functionals of the Maxwell field) exist on both the Krein space and on the physical Hilbert space.

Because on the Krein space both Eqs. (2.1) and (2.4) hold, the two potentials differ by an operator-valued gauge transformation

$$A^K_\mu(x,e) = A^K_\mu(x) + \partial_\mu\phi(x,e). \tag{2.5}$$

The quantity $\phi(x,e)$, baptized “escort field” in [17], turns out to be given by Eq. (1.1). Thus, it is string-localized and—as we shall see—infrared divergent. But its derivative is well defined as a string integral over ∂A^K that decays fast enough as $s\to\infty$.

There are now several new options [12, Sect. 2] to construct QED without the need to work in Krein spaces. The first is to replace the usual interaction density $q\,A^K j$, defined on a Krein space, by $q\,A(e)j$, defined on the Wigner Hilbert space (where the string-dependence is a total derivative and should not affect the resulting theory [17]). This option requires string-localized propagators of the potentials $A(e)$ that are the topic of Sect. 4.

The “hybrid” option indicated in the introduction is to split $q\,A(e)j$ into $q\,\partial\phi(e)j$ (which is a total derivative) and $q\,A^K j$, and study the theory with the “trivial interaction density” $q\,\partial\phi(e)j$ first. This model can actually be constructed non-perturbatively, leading to a rigorous and IR-finite definition of the vertex operators Eq. (1.2) and the dressed Dirac field Eq. (1.4). The analytic properties of vertex operators, in particular, the space–time structure of their correlation functions, is the topic of Sect. 3.

The second step, the perturbation of the dressed Dirac field with the QED interaction density $q\,A^K j$ (and the reasons why this does not reintroduce Hilbert space non-positivity) requires a rather big effort and is addressed in [12, Sect. 4].

Let us begin with an inventory of the basic quantities that are needed in the various approaches.

The two-point function W_0 and the Feynman propagator G_0^F of the massless scalar Klein–Gordon field coincide (up to a factor of i) as a function of $x = x_1 - x_2$ outside the singular support, which is the null-cone $(x_1 - x_2)^2 = 0$. As distributions, they are given in position space as boundary values of analytic functions

$$W_0(x) = \lim_{\varepsilon\downarrow 0} \frac{1}{(2\pi)^2} \frac{-1}{(x^0 - i\varepsilon)^2 - \vec{x}^2} \equiv \lim_{\varepsilon\downarrow 0} \frac{1}{(2\pi)^2} \frac{-1}{x^2 - i\varepsilon x^0} \equiv \frac{-1}{(2\pi)^2} \cdot \frac{1}{(x^2)_-}, \tag{2.6}$$

$$G_0^F(x) = \lim_{\varepsilon\downarrow 0} \frac{1}{(2\pi)^2} \frac{-i}{x^2 - i\varepsilon}, \tag{2.7}$$

respectively. The commutator function (Pauli–Jordan function) can be written as

$$C_0(x) := i(W_0(x) - W_0(-x)) = \frac{1}{2\pi}\,\mathrm{sign}(x^0)\delta(x^2). \tag{2.8}$$

For the indefinite Feynman gauge vector potential, one has

$$\langle A_\mu^K(x) A_\nu^K(x')\rangle = -\eta_{\mu\nu} W_0(x - x'), \tag{2.9}$$

$$i\langle T A_\mu^K(x) A_\nu^K(x')\rangle = -\eta_{\mu\nu} G_0^F(x - x'). \tag{2.10}$$

By Eq. (1.1), the ensuing two-point function and Feynman propagator of the escort field are given as double integrals

$$\begin{aligned} &-(ee')\int_0^\infty \mathrm{d}s' \int_0^\infty \mathrm{d}s\, F(x + se - s'e') \\ &\equiv -(ee')(I_{-e'} I_e F)(x), \qquad (F = W_0 \text{ resp. } G_0^F). \end{aligned} \tag{2.11}$$

The notation I_e stands for the string integration as in Eqs. (1.1) or (2.3). The operations I_e commute among each other and with derivatives as long as all integrals exist as distributions, and it holds

$$e^\mu \partial_\mu (I_e f)(x) = -f(x). \tag{2.12}$$

If f is a function or a distribution, $I_e f$ can only be defined if f has sufficiently rapid decay. Because W_0 and G_0^F fall off in configuration space like $1/x^2$, their first string integrations are finite, whereas the second string integration diverges logarithmically and has to be regularized.

The well-definedness as a distribution is a more subtle issue than the convergence of an integral. In Fourier space, the string integration is a multiplication with another distribution:

$$I_e e^{-ikx} \equiv \int_0^\infty \mathrm{d}s\, e^{-ik(x+se)} = \lim_{\varepsilon \downarrow 0} \frac{-i}{(ke) - i\varepsilon} \cdot e^{-ikx}. \tag{2.13}$$

The existence of a product of distributions has to be analysed by microlocal methods, such as Hörmander's criterion for the wave front sets. But the latter is only a sufficient condition, and by cancellations of singularities the product may be better behaved than the wave front sets may tell. Since in this work we are interested mainly in the behaviour in position space, we refer to [9] where the existence of certain relevant distributions has been established in Fourier space.

We just notice here that the Fourier transforms of W_0 and G_0^F scale like k^{-2}. The IR divergence arises because with two additional denominators as in Eq. (2.13), the Fourier integrals would diverge logarithmically at $k = 0$.

Because $\frac{1}{(y-i\varepsilon)^2}$ (where $y = (ke)$) is well defined, $(I_e I_e f)(x)$ is well defined as a distribution in x, provided the decay of f is fast enough. But because $\frac{1}{(y-i\varepsilon)(y+i\varepsilon)}$ is ill defined, $(I_{-e} I_e f)(x)$ is always ill defined. In position space, this can be easily understood because the integrand in Eq. (2.11) depends only on $s - s'$. In particular, $\langle A(x,e)A(x',e')\rangle$ is ill defined at $e = e'$. Not least for this reason, one should consider the string integrals also as distributions in e and e'. We shall see in Sect. 3.2 that the singularity at $e = e'$ is integrable in e and e' w.r.t. the invariant measure of S^2, so that smeared expressions like $\langle A(x,c)A(x',c')\rangle$ are well defined even when the supports of $c(e)$ and $c'(e')$ overlap.

Because W_0 and G_0^F are homogeneous distributions in x of degree -2, $I_e W_0$ and $I_e G_0^F$ are homogeneous both in e and in x of degree -1. We shall (most of the time) restrict e to the open set of spacelike vectors, because these are the directions needed in the intended applications [12]. Because of homogeneity in e, we may as well restrict e to the unit spacelike hyperboloid $H_1 = \{e \in \mathbb{R}^4 : e^2 = -1\}$ [9]; but it will be advantageous to display factors "e^2" explicitly, so as to maintain homogeneity in e. Yet, smearing in e is always understood with normalized e.

The quantities of interest in various applications [12] are:

(i) Two-point functions of $A_\mu(x,e) = (I_e F_{\mu\nu})(x)e^\nu$ or $A_\mu^K(x,e) = A_\mu^K(x) + \partial_\mu \phi(x,e)$. (These distributions are identical in the Wigner Hilbert space version and the Krein space version.)
(ii) Mixed two-point functions between $A_\mu^K(x)$ and $\phi(x,e)$.
(iii) IR-regularized two-point functions of $\phi(x,e)$ and their exponentials.
(iv) Propagators (= time-ordered two-point functions) of $A_\mu(x,e)$.

Propagators of ϕ with itself would require time-ordering *and* IR regularization, which would be very delicate to implement simultaneously. Fortunately, such objects do not occur [12, Sect. 4.2].

(i) define a string-localized free quantum field theory. Apart from a local contribution, they involve contributions with one or two string integrations over derivatives of

Eq. (2.6). Thanks to the derivatives, these string integrations are IR finite and do not need a regularization. They are far simpler than (iii) without derivatives.

(ii) and (iii) arise in the non-perturbative "dressing model" [12, Sect. 3.1] and the "hybrid" approach to QED outlined in [12, Sect. 2, Eq. (1.15)], where the escort field without derivative appears in the exponent of a regularized normal-ordered Weyl operator (vertex operator). Their computation and analysis will be our first main topic in Sect. 3.

(iv) are needed in perturbation theory when a current is coupled to a string-localized potential on the Wigner Hilbert space [12, Sect. 2, Eq. (1.5)]. They involve one or two string integrations over derivatives of Eq. (2.7). Again, thanks to the derivatives, an IR regularization is not needed. Their computation and analysis will be the second main topic of this work in Sect. 4.

Thus, we shall study the IR-finite expressions

$$(I_e W_0)(x), \quad (I_{e_2} I_{e_1} \partial W_0)(x), \quad (I_{e_2} I_{e_1} \partial G_0^F)(x) \tag{2.14}$$

from which the other IR-finite quantities of interest arise via

$$(I_e \partial W_0)(x) = \partial (I_e W_0)(x), \qquad (I_{e_2} I_{e_1} \partial\partial W_0)(x) = \partial (I_{e_2} I_{e_1} \partial W_0)(x); \tag{2.15}$$

and the IR-regularized expression

$$(I_{e_2} I_{e_1} W_0)_v(x). \tag{2.16}$$

3 Two-point functions and vertex operator correlations

3.1 One string integration

The distribution $W_0(x)$ is defined as the boundary value of the analytic function $-\frac{1}{(2\pi)^2}\frac{1}{z^2}$ in the complex forward tube $\operatorname{Im} z^0 < 0$ (see [14, Thm. IX.16], which we shall repeatedly refer to). We may thus write $z = x - i\varepsilon u$ where x is real and u a forward timelike (unit) vector. By Lorentz invariance, the distributional limit $\varepsilon \downarrow 0$ is independent of u. So, because e is spacelike, one may choose u perpendicular to e. Let in this section $F(x) = \frac{1}{(x-i\varepsilon u)^2}$, where the distributional limit $\varepsilon \downarrow 0$ is always understood.

The string-integral over $F(x)$ can be written as

$$\begin{aligned} f(x,e) &:= -(I_e F)(x) \equiv -\int_0^\infty \frac{\mathrm{d}s}{(x+se-i\varepsilon u)^2} \\ &= -\int_0^\infty \frac{\mathrm{d}s}{(x-i\varepsilon u)^2 + 2s(xe) + s^2 e^2}. \end{aligned} \tag{3.1}$$

The point is that the complex denominator cannot vanish for real s, and $i\varepsilon$ appears only in the parameter $(x - i\varepsilon u)^2$. The elementary integration gives

Lemma 3.1

$$(2\pi)^2(I_eW_0)(x)=f(x,e)=\frac{\frac{1}{2}\log\frac{-(xe)+i\sqrt{\det_{x,e}}}{-(xe)-i\sqrt{\det_{x,e}}}}{i\sqrt{\det_{x,e}}}=\frac{\frac{1}{2}\log\frac{-(xe)+\sqrt{-\det_{x,e}}}{-(xe)-\sqrt{-\det_{x,e}}}}{\sqrt{-\det_{x,e}}}, \tag{3.2}$$

where

$$\det{}_{x,e} := (x-i\varepsilon u)^2e^2-(xe)^2 \tag{3.3}$$

is the Gram determinant, with the imaginary shift of x and the distributional limit $\varepsilon \downarrow 0$ in the sense of [14, Thm. IX.16] *being implicitly understood.*

The logarithm of the quotient is understood as the difference of two logarithms with their branch cuts along $\mathbb{R}_-$. The same applies for logarithm's of products or quotients throughout.[1] In particular, Eq. (3.2) does *not* depend on the choice of the branch of the square root (because numerator and denominator would simultaneously switch sign). As a function on $\mathbb{R}^4 \times H_1 \times H_1$ (i.e. putting $\varepsilon = 0$), Eq. (3.2) is ill defined only when $x^2 = 0$ or $\det_{x,e} = 0$. The $i\varepsilon$-prescription in $\det_{x,e}$ defines Eq. (3.2) as a distribution.

When x and e lie in a common spacelike plane, one may without loss of generality assume $x^0 = 0$ and $e^0 = 0$. Then,

$$\sqrt{x^2e^2}\cdot f(x,e) = \frac{\alpha}{\sin\alpha}, \tag{3.4}$$

where $\alpha = \angle(\vec{x},\vec{e}) \in [0,\pi)$. It has a singularity at $\alpha = \pi$ reflecting the fact that the string $x + \mathbb{R}_+e$ passes through the origin. Yet, upon smearing in $\vec{e}$, this singularity is integrable w.r.t. the invariant measure of S^2 and does not need an $i\varepsilon$ prescription to define it as a distribution in $\vec{e}$.

More generally, the defining integral Eq. (3.1) may be singular whenever $(x+se)^2$ can become zero for $s \geq 0$, i.e. geometrically, when the string $x + \mathbb{R}_+e$ hits the null-cone. This happens necessarily if x is timelike or lightlike (the string starts inside or on the null-cone). If x is spacelike, the string may touch the null-cone or pierce it twice. However, inspection of Eq. (3.2) shows that with the $i\varepsilon$-prescription, the singular support of $f(x,e)$ is at $x^2 = 0$ (the string starts on the null-cone) and at $\det_{x,e} = 0$, $(xe) \geq 0$ (the string touches the null-cone).

Somewhat unexpected from its unsymmetric definition $f(x,e) = -(I_eF)(x)$, this function (where it is regular) is symmetric in $x \leftrightarrow e$. The symmetry can be understood by a change of integration variables $s \to \frac{1}{s}$ in Eq. (3.1).

3.2 Two string integrations

The two-point function of the escort field is the twofold string-integral over $W_0(x)$, multiplied by the factor $-(ee')$. The presence of this factor jeopardizes the positivity of

[1] In particular, negative factors must not be cancelled under the logarithm: this would produce errors of $2\pi i$!

the inner product defined by the two-point function. The latter is essential to produce the superselection structure, that arises by the divergence to $+\infty$ of the exponent $d_{m,v}(C, C)$ in Eq. (3.29) below. To secure positivity, one has to impose that e and e' are smeared within a spacelike surface [12, Sect. 3.1] perpendicular to any timelike unit vector u, so that $-(ee')$ becomes positive definite. Without loss of generality, we may pick

$$u = u_0 := \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \quad \Rightarrow \quad e = \begin{pmatrix} 0 \\ \vec{e} \end{pmatrix}, \quad e' = \begin{pmatrix} 0 \\ \vec{e}\,' \end{pmatrix}. \tag{3.5}$$

Thus, smearing functions $c(e) = c(\vec{e})$ are elements of $C^\infty(S^2)$. Vectors y with $y^0 = 0$ will be called "purely spatial".

Because the two-point function $W_0(x)$ is homogeneous of degree -2, $(I_e W_0)(x)$ is homogeneous of degree -1, and the second string integration would diverge logarithmically. In momentum space, with the Lorentz invariant measure $d\mu_0(k) = (2\pi)^{-3} d^4k\, \delta(k^2)\theta(k^0) = (2\pi)^{-3} \frac{d^3k}{2k^0}$,

$$(I_{-e'} I_e W_0)(x) \overset{?}{=} \lim_{\varepsilon \downarrow 0} \lim_{\varepsilon' \downarrow 0} \int d\mu_0(k) \frac{e^{-ikx}}{((ke) - i\varepsilon)((ke') + i\varepsilon')} \tag{3.6}$$

diverges at $k = 0$. We therefore have to regularize it in the infrared. The regularization extends the momentum space distribution to $k = 0$. This is done by replacing e^{-ikx} by $e^{-ikx} - v(k)$ where $v(k)$ is any smooth test function with $v(0) = 1$. Thus, we define

$$(I_{-e'} I_e W_0)_v(x) := \lim_{\varepsilon \downarrow 0} \lim_{\varepsilon' \downarrow 0} \int d\mu_0(k) \frac{e^{-ikx} - v(k)}{((ke) - i\varepsilon)((ke') + i\varepsilon')}. \tag{3.7}$$

Because of the symmetry $e \leftrightarrow -e'$, we shall in the sequel write $e = e_1$ and $-e' = e_2$, so that $(I_{e_2} I_{e_1} W_0)_v(x)$ is symmetric in $e_1 \leftrightarrow e_2$.

We want to gain insight into the distribution $(I_{e_2} I_{e_1} W_0)_v(x)$ in position space, leaving the regulator function v unspecified. It is clearly not possible to compute the integral Eq. (3.7) when $v(k)$ is not specified. The strategy is therefore to compute instead the cut-off integral

$$(I^a_{e_2} I_{e_1} W_0)(x) \equiv \int_0^a ds_2\, (I_{e_1} W_0)(x + s_2 e_2) = \frac{1}{(2\pi)^2} \int_0^a ds\, f(x + se_2, e_1) \tag{3.8}$$

—which can be done analytically—and use that

$$\begin{aligned} \partial_\mu (I_{e_2} I_{e_1} W_0)_v(x) &= (I_{e_2} I_{e_1} \partial_\mu W_0)(x) = \lim_{a\to\infty} (I^a_{e_2} I_{e_1} \partial_\mu W_0)(x) \\ &= \lim_{a\to\infty} \partial_\mu (I^a_{e_2} I_{e_1} W_0)(x), \end{aligned} \tag{3.9}$$

where the first equality follows from the definition Eq. (3.7), the second holds because $\partial_\mu W_0$ decays sufficiently fast to make the integral converge in a, and the last is obvious from the definition of the integral operations. Thus, the difference is independent of x, and the result for $(I_{e_2} I_{e_1} W_0)_v(x)$ is obtained by replacing the cut-off dependent but x-independent term by another (unknown) x-independent term. Specifically, we will show in the remainder of this section

$$(I^a_{e_2} I_{e_1} W_0)(x) = \frac{1}{(2\pi)^2}\left[\frac{1}{2} f(e_1, e_2) \cdot \log\left(\frac{4(ae_2)^2}{(x - i\varepsilon u)^2}\right) + \frac{H(x; e_1, e_2)}{(e_1 e_2)}\right] + O\left(\frac{1}{a}\right), \tag{3.10}$$

where $f(e_1, e_2) = f(e_2, e_1)$ is the same distribution as in Eq. (3.1), and the distribution H is symmetric in $e_1 \leftrightarrow e_2$ and homogeneous of degree 0 separately in x, e_1, and e_2. We conclude

$$(I_{e_2} I_{e_1} W_0)_v(x) = \frac{1}{(2\pi)^2}\left[-\frac{1}{2} f(e_1, e_2) \cdot \log(-\mu_v^2 \cdot (x - i\varepsilon u)^2) + \frac{H(x; e_1, e_2)}{(e_1 e_2)}\right], \tag{3.11}$$

where μ_v carries the dependence on the regulator function v and may depend on e_1 and e_2.

To prepare the computation of $H(x; e_1, e_2)$ in Eqs. (3.10) and (3.11), we need some definitions. We shall denote by

$$\begin{aligned}\det{}_{y_1,y_2,y_3} := {} & y_1^2 y_2^2 y_3^2 - y_1^2 (y_2 y_3)^2 - y_2^2 (y_1 y_3)^2 \\ & - y_3^2 (y_1 y_2)^2 + 2(y_1 y_2)(y_1 y_3)(y_2 y_3)\end{aligned} \tag{3.12}$$

the Gram determinant of vectors y_1, y_2, y_3. For $i, j, k \in \{1, 2, 3\}$ pairwise distinct, the cofactors of y_i^2 are the 2×2 Gram determinants $\det_{y_j,y_k}$, and we shall denote the cofactors of $(y_i y_j)$ by

$$\Lambda_k = (y_i y_k)(y_j y_k) - y_k^2 (y_i y_j). \tag{3.13}$$

We shall need a few trivial facts, proven by elementary computation.

Lemma 3.2 *It holds*

$$\partial_{y_i} \det{}_{y_1,y_2,y_3} = 2 \begin{pmatrix} \det_{y_2,y_3} & \Lambda_3 & \Lambda_2 \\ \Lambda_3 & \det_{y_1,y_3} & \Lambda_1 \\ \Lambda_2 & \Lambda_1 & \det_{y_1,y_2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{3.14}$$

If G is the Gram matrix and L the matrix in Eq. (3.14), then $GL = LG = \det_{y_1,y_2,y_3}$, i.e. if $\det_{y_1,y_2,y_3} \neq 0$, then $\det^{-1}_{y_1,y_2,y_3} L = G^{-1}$.

Lemma 3.3 *For $i, j, k \in \{1, 2, 3\}$ pairwise distinct, it holds*

$$y_i^2 \det{}_{y_i,y_j,y_k} = \det{}_{y_i,y_k} \det{}_{y_i,y_j} - \Lambda_i^2. \tag{3.15}$$

Notice that with Lorentzian metric the vanishing of a Gram determinant does not require the linear dependence of the vectors, see, however Lemmas 4.2 and 4.3.

The inverse of the Gram determinant $\det_{x,e_1,e_2}$ will play a major role. Similar to Eq. (3.2), it is understood as the distributional boundary value from the forward tube $x - i\varepsilon u$. Because $(e_i u) = 0$, this simply means that x^2 is understood as $(x - i\varepsilon u)^2$ while all other scalar products are real. For properties of Gram determinants in Lorentzian metric, see Sect. 4.

It is convenient to define $\gamma = \angle(\vec{e}_1, \vec{e}_2)$, so that $\sqrt{e_1^2 e_2^2}\cos\gamma = -(e_1 e_2)$ and $\det_{e_1,e_2} = e_1^2 e_2^2 \sin^2\gamma$. One trivially has

Lemma 3.4 *The distribution $f(x, e_2)$ in Lemma 3.1 with x substituted by e_1 equals*

$$\sqrt{e_1^2 e_2^2} \cdot f(e_1, e_2) = \frac{\gamma}{\sin\gamma}. \tag{3.16}$$

The singularity at $\gamma = \pi$ is integrable w.r.t. the invariant measure on $S^2 \times S^2$.

Remark 3.5 The singularity is *not* integrable along one-dimensional submanifolds of S^2. Thus, strings must not be further restricted than $e^2 = -1$ (which is trivial by homogeneity) and $e^0 = 0$.

Next, by using Lemma 3.3 with $y_1 = e_1$, $y_2 = e_2$, $y_3 = x$, and $i = 1$ and $i = 2$, respectively, one can define the homogeneous functions $\zeta_1(x, e_1, e_2)$ and $\zeta_2(x, e_1, e_2)$ by

$$\pm e^{\pm\zeta_1} = \frac{\Lambda_1 \pm \sqrt{\det_{e_1,e_2}\det_{x,e_1}}}{\sqrt{e_1^2 \det_{x,e_1,e_2}}}, \qquad \pm e^{\pm\zeta_2} = \frac{\Lambda_2 \pm \sqrt{\det_{e_1,e_2}\det_{x,e_2}}}{\sqrt{e_2^2 \det_{x,e_1,e_2}}}. \tag{3.17}$$

When e_1, e_2 and x are purely spatial, the geometry is Euclidean. Then, all diagonal cofactors (2×2 Gram determinants) are ≥ 0 and $e_1^2 \det_{x,e_1,e_2} \geq 0$. In this case, ζ_1 and ζ_2 are real functions.

We can now state the result.

Proposition 3.6 *Let e_i $(i = 1, 2)$ be purely spatial and linearly independent* ($\det_{e_1,e_2} \neq 0$)*. Denote by $D = \frac{\det_{x,e_1,e_2}}{(x - i\varepsilon u)^2 e_1^2 e_2^2}$ the normalized Gram determinant. Then, the distribution H in Eqs.* (3.10) *and* (3.11) *is*

$$\begin{aligned} H(x; e_1, e_2) = \; & -\frac{\cos\gamma}{2\sin\gamma}\Bigg[\gamma \log\left(\frac{\sin^4\gamma}{D}\right) + \pi(\zeta_1 + \zeta_2) \\ & -\frac{i}{2}\left\{ \mathrm{Li}_2\left(e^{i\gamma} e^{\zeta_1} e^{\zeta_2}\right) \begin{matrix} +(e^{\zeta_1} \leftrightarrow -e^{-\zeta_1}) \\ +(e^{\zeta_2} \leftrightarrow -e^{-\zeta_2}) \\ -(e^{i\gamma} \leftrightarrow e^{-i\gamma}) \end{matrix} \right\}\Bigg]. \end{aligned} \tag{3.18}$$

The limit $\gamma \to 0$ (e_1 and e_2 parallel) is regular, while the limit $\gamma \to \pi$ (e_1 and e_2 antiparallel) is singular, but integrable w.r.t. the invariant measure on $S^2 \times S^2$.

Via Eq. (3.11), this formula determines the regularized double string-integrated distribution $(I_{e_2} I_{e_1} W_0)_v(x)$ on $\mathbb{R}^4 \times S^2 \times S^2$, up to the unknown additive dependence on the regulator function v via $\mu_v(e_1, e_2)$. Its relevance for the intended applications to QED will be discussed at the end of Sect. 3.3.

Sketch of proof By the argument preceding Eq. (3.10), we need to compute the cut-off integral over $f(x + se_2, e_1)$ in Eq. (3.8) in order to compute the regularized integral Eq. (3.7). We begin with $x^\perp$ (the component of x perpendicular to e_1 and e_2) spacelike. Then, there is a boost preserving e_1 and e_2, such that $(u\Lambda x) = 0$. Thus, we may without loss of generality assume that also x is purely spatial, and ζ_1 and ζ_2 are real.

The clue to compute Eq. (3.8) analytically is the change of integration variable

$$C(s) = e^{\operatorname{arcsinh}(s\Gamma_1 + \Gamma_2)} = s\Gamma_1 + \Gamma_2 + \sqrt{(s\Gamma_1 + \Gamma_2)^2 + 1}, \qquad \frac{1}{C}\frac{\mathrm{d}C}{\mathrm{d}s} = \frac{2}{C + C^{-1}}, \tag{3.19}$$

where

$$\Gamma_1 := \frac{\det_{e_1,e_2}}{\sqrt{e_1^2 \det_{x,e_1,e_2}}}, \qquad \Gamma_2 := -\frac{\Lambda_1}{\sqrt{e_1^2 \det_{x,e_1,e_2}}} \tag{3.20}$$

with Λ_i defined as in Eq. (3.13) for $y_1 = e_1$, $y_2 = e_2$, $y_3 = x$. Then, $C(0) = e^{-\zeta_1}$ and $C(a) = \log(2a\Gamma_1) + O(\frac{1}{a})$ for large a, and one finds

$$f(x + se_2, e_1) = \frac{\Gamma_1}{\sqrt{\det_{e_1,e_2}}} \frac{\log \dfrac{\frac{e^{-\zeta_2}}{2\Gamma_1}(1 + e^{i\gamma} e^{\zeta_2} C(s))(1 - e^{-i\gamma} e^{\zeta_2} C(s)^{-1})}{\frac{e^{-\zeta_2}}{2\Gamma_1}(1 + e^{-i\gamma} e^{\zeta_2} C(s))(1 - e^{i\gamma} e^{\zeta_2} C(s)^{-1})}}{i(C(s) + C(s)^{-1})}. \tag{3.21}$$

Because ζ_i are real and Γ_1 is positive, one can cancel the factor $\frac{e^{-\zeta_2}}{2\Gamma_1}$ in the difference of logarithms without risk of changing the branches, see footnote 1. The integral is solved by

$$\begin{aligned} &2\Gamma_1 \frac{\log\left(1 + e^{i\gamma} e^{\zeta_2} C(s))(1 - e^{-i\gamma} e^{\zeta_2} C(s)^{-1}\right)}{C(s) + C(s)^{-1}} \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}s}\Big[\mathrm{Li}_2\left(e^{-i\gamma} e^{\zeta_2} C(s)^{-1}\right) - \mathrm{Li}_2\left(-e^{i\gamma} e^{\zeta_2} C(s)\right)\Big], \end{aligned} \tag{3.22}$$

and likewise for $\gamma \to -\gamma$. (Eq. (3.22) holds in fact with the sum of two logarithms on the left-hand side. Because the complex phases of the two factors under the logarithm always have the same sign and do not add up beyond $\pm\pi$, the branch is the same when the sum is written as the logarithm of the product.)

By working out the values of the primitive function in Eq. (3.22) at $s = 0$ and $s = a$, and writing the result symmetrically in $e_1 \leftrightarrow e_2$, one arrives at Eq. (3.18). For the relevant properties of the dilogarithm function, see [21, Chap. I.2].

The behaviour at the branch cuts of the dilogarithm function, determined by the imaginary part of x^0, can be worked out explicitly in both limits $\gamma \to 0$ and $\gamma \to \pi$. The singularity at $\gamma = \pi$ is like $\frac{O(\log(\pi-\gamma))}{\pi-\gamma}$ and hence integrable.

When $x^\perp$ is not spacelike, the variables and functions are defined as analytic functions in the forward tube $x - i\varepsilon u$ with the $i\varepsilon$ prescriptions of $\det_{x,e_i}$ and $\det_{x,e_1,e_2}$ as before. This defines the distribution as the boundary value $\varepsilon \downarrow 0$, invoking [14, Thm. IX.16] together with the mild growth properties of the logarithm and dilogarithm [21] functions. □

The regular behaviour at $\gamma = 0$ ($e_1 = e_2$) is expected because $(I_e^2 F)(x) = \int_0^\infty s\, \mathrm{d}s\, F(x + se)$ is well defined when regularized as in Eq. (3.7). The singular behaviour at $\gamma = \pi$ ($e_1 = -e_2$) is expected because $I_{-e} I_e$ is never defined. But because the singularity is integrable, one can extend the distributions $f(e_1, e_2)$ and $H(x, e_1, e_2)$ to $e_1 = -e_2$:

Corollary 3.7 *Let $\widetilde{f}(e, e') := f(e, -e')$ and $\widetilde{H}(x - x'; e, e') := H(x - x'; e, -e')$, which are symmetric under $e \leftrightarrow -e'$. The regularized two-point function of the escort field*

$$\begin{aligned}(2\pi)^2 \langle \phi(x, e)\phi(x', e')\rangle_v \\ = \frac{(ee')}{2} \widetilde{f}(e, e') \log\big(-\mu_v^2 \cdot (x - x' - i\varepsilon u)^2\big) + \widetilde{H}(x - x'; e, e') \quad (3.23)\end{aligned}$$

is a distribution on $\mathbb{R}^4 \times S^2 \times S^2$.

When smeared with the constant function $c_0(e_i) = \frac{1}{4\pi}$, Eq. (3.23) simplifies drastically. The averaging with c_0 can be done already in the momentum space representation Eq. (3.7):

Lemma 3.8 (see [12, Eq. (A.3)]) *For purely spatial e, u_0 the standard timelike unit vector as above, and $k^2 = 0$ one has*

$$\int_{S^2} \mathrm{d}\sigma(\vec{e})\, c_0(\vec{e})\, \frac{e}{(ek)_\pm} = \frac{u_0}{(u_0 k)} - \frac{k}{(u_0 k)^2}. \quad (3.24)$$

Because $(\frac{u_0}{(u_0k)} - \frac{k}{(u_0k)^2})^2 = -\frac{1}{(u_0k)^2}$, which is (minus) the denominator for two string integrations in the direction u_0, one obtains the desired result by computing $(I_{u_0} I_{u_0} W_0)_v$ along the same lines as $(I_{e_2} I_{e_1} W_0)_v$ before. The result is

Lemma 3.9 *For* $c_0(\vec{e}) = \frac{1}{4\pi}$, *one has*

$$(2\pi)^2 \langle \phi(x, c_0)\phi(x', c_0)\rangle_v = -\frac{1}{2} \log\big(-\widetilde{\mu}_v^2 \cdot ((x-x')^2)_-\big) + \widetilde{H}(x-x'; c_0, c_0), \tag{3.25}$$

$$\textit{with} \qquad \widetilde{H}(x; c_0, c_0) = \frac{x^0}{2r} \log \frac{x^0 - r - i\varepsilon}{x^0 + r - i\varepsilon} \qquad (r = |\vec{x}|). \tag{3.26}$$

Sketch of proof By a direct computation of the cut-off integral Eq. (3.8), which for two parallel strings becomes an elementary integral. The claim for the regularized integral Eq. (3.7) follows by the argument preceding Eq. (3.10), which leaves only the constant $\widetilde{\mu}_v^2$ unspecified. □

3.3 Vertex operator correlations and commutation relations

We briefly sketch the definition of operators $:e^{i\phi(g\otimes c)}:_v$ (smeared in both x and e) through a massless limit [12, Sect. 3.1]:

$$:e^{i\phi(g\otimes c)}:_v := \lim_{m\to 0} e^{-\frac{\widehat{g}(0)^2}{2} d_{m,v}(c,c)} \cdot :e^{i\phi_m(g\otimes c)}: \equiv \frac{e^{i\phi(g\otimes c)}}{e^{-\frac{1}{2}w_v(g\otimes c, g\otimes c)}}, \tag{3.27}$$

where ϕ_m is the IR-regular massive escort field whose two-point function diverges in the limit $m \to 0$, and, with the Lorentz invariant measure $d\mu_m(k) = (2\pi)^{-3} d^4k\, \delta(k^2 - m^2)\theta(k^0)$,

$$d_{m,v}(e, e') = -(ee') \int \frac{d\mu_m(k)\, v(k)}{((ke) - i\varepsilon)((ke') + i\varepsilon)}, \tag{3.28}$$

smeared over e and e' is the divergent part. The second writing in Eq. (3.27) is the normal-ordering w.r.t. the non-positive two-point function w_v defined as $-(ee')$ times Eq. (3.7). But because the denominator is positive, correlations of Eq. (3.27) evaluated in the limit of massive vacuum states define a positive functional. The correlations can be worked out by the Weyl formula. The crucial feature is that they contain the IR divergent parts in the combination

$$e^{-\frac{1}{2}d_{m,v}(C,C)} \tag{3.29}$$

where $C(e) = \sum_i \widehat{g}_i(0)c_i(e)$. Because $d_{m,v}(C, C)$ diverges to $+\infty$ unless $C = 0$, one obtains in the limit a Kronecker delta $\delta_{C,0}$.

The coefficient $\delta_{C,0}$ defines an uncountable superselection rule: the GNS Hilbert space splits into an uncountable direct sum of subspaces $\mathcal{H}_C$ which carry inequivalent representations of the Weyl subalgebra generated by $e^{i\phi(g\otimes c)}$ with $\widehat{g}(0) = 0$.

Therefore, correlation functions of $:e^{i\phi(g\otimes c)}:_v$ are given by

$$\big\langle :e^{i\phi(g_1\otimes c_1)}:_v \dots :e^{i\phi(g_n\otimes c_n)}:_v \big\rangle = \delta_{C,0} \cdot \prod\nolimits_{i<j} e^{-w_v(g_i\otimes c_i, g_j\otimes c_j)}. \tag{3.30}$$

When w_v as computed in Eq. (3.23) is inserted, the (unspecified) terms $\log \mu_v^2(e, -e')$ in the exponent contribute only as c-dependent but irrelevant normalizations $N_v(c)$ of the fields, as follows. Collect these terms in Eq. (3.30) as

$$\exp \sum_{i<j} \frac{\widehat{g}_i(0)\widehat{g}_j(0)}{8\pi^2} \lambda_v(c_i, c_j) \tag{3.31}$$

with the symmetric bilinear form

$$\lambda_v(c, c') := \frac{1}{2} \int d\sigma(\vec{e})\, c(\vec{e}) \int d\sigma(\vec{e}\,')c'(\vec{e}\,')\, (\vec{e} \cdot \vec{e}\,')\, \widetilde{f}(e, e') \log \mu_v^2(e, -e'). \tag{3.32}$$

Because $\sum_i \widehat{g}_i(0)c_i = 0$, the exponential of the sum factorizes:

$$\exp \sum_{i<j} \frac{\widehat{g}_i(0)\widehat{g}_j(0)}{8\pi^2} \lambda_v(c_i, c_j) = \prod_i \exp\left(-\frac{\widehat{g}_i(0)^2}{16\pi^2} \lambda_v(c_i, c_i)\right) =: \prod_i N_v(c_i)^{-1}. \tag{3.33}$$

Vertex operators are defined by choosing $g(y) = q\delta_x(y)$, hence $\widehat{g}(0) = q$, and normalizing as

$$V_{qc}(x) := N_v(c) \cdot :e^{iq\phi(x,c)}:_v \qquad \left(N_v(c) = e^{\frac{q^2}{16\pi^2}\lambda_v(c,c)}\right). \tag{3.34}$$

By inspection of their correlation functions, displayed in Cor. 3.10, together with (a version of) [14, Thm. IX.16] and the mild growth properties of the logarithm and dilogarithm functions, one can conclude that the vertex operators are distributions.

Corollary 3.10 (see [12, Sect. 3.1]) *The correlation functions of vertex operators are*

$$\begin{aligned} &\left\langle V_{q_1c_1}(x_1)\dots V_{q_nc_n}(x_n)\right\rangle \\ &\quad = \delta_{\sum_i q_ic_i,0} \cdot \prod_{i<j} \left(\frac{-1}{(x_i - x_j)_-^2}\right)^{-\frac{q_iq_j}{8\pi^2}\langle c_i,c_j\rangle} e^{-\frac{q_iq_j}{4\pi^2}\widetilde{H}(x_i-x_j;c_i,c_j)}, \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} &\langle c, c'\rangle := \int d\sigma(\vec{e})\, c(\vec{e}) \int d\sigma(\vec{e}\,')c'(\vec{e}\,')\, (\vec{e} \cdot \vec{e}\,')\, \widetilde{f}(e, e') \\ &\left(\widetilde{f}(e, e') = \frac{\pi - \angle(\vec{e}, \vec{e}\,')}{\sin(\angle(\vec{e}, \vec{e}\,'))}\right). \end{aligned} \tag{3.36}$$

Proof The statement follows by combining the definition Eq. (3.34) with Eq. (3.30), where w_v is specified by Eq. (3.23) and Proposition 3.6. The formula Eq. (3.36) uses Eq. (3.16). □

Remark 3.11 Among the smearing functions of unit weight, c_0 is a stationary point of the functional $\langle c, c\rangle$, and $\langle c_0, c_0\rangle = 1$. It is presumably a minimum, so that correlations with $c \neq c_0$ would decay faster than with c_0.

We conclude this section with some miscellaneous results about correlations of vertex operators, with only indications of proofs.

3.3.1 Commutation relations

Proposition 3.12 *The vertex operators satisfy anyonic commutation relations*

$$V_{qc}(x)V_{q'c'}(x') = e^{iqq'\beta(x-x';c,c')} \cdot V_{q'c'}(x')V_{qc}(x), \tag{3.37}$$

where $\beta(x-x'; c, c')$ arises by smearing with $c(e)$ and $c'(e')$ the escort commutator function

$$\beta(x-x'; e, e') = i[\phi(x, e), \phi(x', e')] = -(ee')(I_{-e'}I_eC_0)(x-x'). \tag{3.38}$$

Proof The claim follows from the fact that vertex operators are defined as limits of multiples of Weyl operators, Eqs. (3.27) and (3.34). The escort commutator function does not suffer from the IR divergence because the Fourier transform of C_0 vanishes at $k = 0$. □

For $x^0 = x'^0$, Eq. (3.38) vanishes because the equal-time commutator vanishes and e, e' are purely spatial. Otherwise it is a rather simple geometric quantity in terms of the intersection of the null-cone with the planar wedge $x - x' + \mathbb{R}_+e - \mathbb{R}_+e'$, which is a subset of a circle. It can be written in a symmetric form, by writing $e_1 = e$ and $e_2 = -e'$ as before, so that $\gamma = \angle(\vec{e}_1, \vec{e}_2)$:

Lemma 3.13 (see [15]) *Denote by A the total arc-length of the intersection of the null-cone with the planar wedge $x + \mathbb{R}_+e_1 + \mathbb{R}_+e_2$ of opening angle γ. Then,*

$$\beta(x; e_1, e_2) = -\mathrm{sign}(x^0)\frac{A}{4\pi \tan\gamma}. \tag{3.39}$$

Sketch of proof Perform the double string-integral in Eq. (3.38) in Euclidean polar coordinates (r, φ) of the plane $x + \mathbb{R}e_1 + \mathbb{R}e_2$. The change of coordinates contributes the Jacobi determinant $\frac{1}{\sin\gamma}$. Using Eq. (3.19), the relevant integral can be written as

$$\int r\,\mathrm{d}r\,\mathrm{d}\varphi\,\chi_W(r, \varphi)\delta(r^2 - x^{\perp 2}) = \frac{1}{2}A, \tag{3.40}$$

where χ_W is the characteristic function of the wedge, and $x^\perp$ is the component of x perpendicular to the plane. Finally, $(e_1e_2) = -\cos\gamma$. Collecting all factors, yields Eq. (3.39). □

The intersection of the wedge with the null-cone is empty if the two strings are spacelike separated, in particular when $x^{\perp 2} < 0$. In this case, $A = 0$ and the vertex operators commute.

3.3.2 Spectrum

One would like to know the Fourier transform of the two-point function of vertex operators because its c-dependent energy–momentum distribution supported in the interior of V^+ reveals the energy–momentum spectrum of the state created by the vertex operator [12, Sect. 3.1]. It has to be added to the mass-shell energy–momentum of the free Dirac particle. The sum is the spectrum of the infraparticle state generated by the dressed Dirac field Eq. (1.4).

We do not know how to compute this Fourier transform for general c, other than by working out an exponential series of convolution products of Eq. (3.23) (which is impractical). However, the case of the constant smearing function $c(\vec{e}) = c_0 = \frac{1}{4\pi}$, for which the two-point function drastically simplifies, see Lemma 3.9, allows to quantify the ensuing dissolution of the mass shell in a special case.

By Eq. (3.25), the two-point function of the vertex operator V_{qc_0} is

$$\left\langle V^*_{qc_0}(x_1)V_{qc_0}(x_2)\right\rangle = \left[\frac{\left(\frac{x^0-r-i\varepsilon}{x^0+r-i\varepsilon}\right)^{\frac{x^0}{r}}}{-(x^2-i\varepsilon x^0)}\right]^{\frac{\alpha}{2\pi}}, \qquad (x = x_1 - x_2,\ r = |\vec{x}|). \tag{3.41}$$

Here, $\alpha := \frac{q^2}{4\pi}$, which in the application to QED [12] is the fine structure constant.

Equation (3.41) equals $(ix^0 + \varepsilon)^{-\frac{\alpha}{\pi}}$ multiplied with a power series in $\frac{r^2}{x_0^2}$. This structure allows to extract quantitative details of the Fourier transform, and hence of the rotationally invariant energy–momentum distribution $\rho(\omega, \vec{k})$ in the state created by V_{qc_0} [15]. By putting $r = 0$, one concludes that the distribution $\rho(\omega) = \int d^3k\, \rho(\omega, \vec{k})$ of energies decays like $\omega^{\frac{\alpha}{\pi}-1}$. By applying powers of the Laplacian before putting $r = 0$, one can compute averages of powers of $|\vec{k}|^2$ at fixed energy ω. For example, the average of the invariant masses $\omega^2-|\vec{k}|^2$ at given energy ω is found to be $\frac{\alpha}{\pi}\cdot\omega^2+O(\alpha^2)$ with variance $\frac{4}{9}\frac{\alpha}{\pi}\cdot\omega^4 + O(\alpha^2)$. These data are roughly compatible with an inverse power-law distribution $\sim (\omega^2 - |\vec{k}|^2)^{1-\frac{\alpha}{\pi}}$ near the mass shell $|\vec{k}| = \omega^2$ [12, Fig. 1].

3.3.3 Scattering

Vertex operator correlation functions Eq. (3.35) involve exponentials of the distribution $\widetilde{H}$ (smeared in the strings), and the same is true for correlations of the dressed Dirac field Eq. (1.4). Asymptotic properties of $\widetilde{H}$ at large times will become relevant in a future scattering theory. At this moment, a scattering theory for infrafields like the dressed Dirac field has not yet been formulated. The LSZ method fails because of the absence of a sharp mass shell, and for the same reason, "asymptotic creation operators" needed in the Haag–Ruelle theory have not been found. Apart from the lack of the mass shell, a major obstacle is the product structure of vertex operator correlations that is very different from that of free correlations. It probably entails that "two-infraparticle states" are not tensor products as in the Fock space.

A toy model without a mass gap would be a scattering theory for vertex operators without the Dirac field. A candidate would be a version of Buchholz' scattering theory for massless waves in two dimensions [1], in which the scattering amplitude is a limit of four-point correlations relative to two-point correlations, when the positions go to infinity like $x \pm t\ell$, $x' \pm t\ell'$ in future and past lightlike directions. Although its assumption that $m^2 = 0$ be an eigenvalue of the mass operator is not fulfilled, this method is also successful in the two-dimensional vertex operator model (no $\widetilde{H}$-terms) where it yields an S-matrix that is a complex phase [6]. Tentatively applying the same prescription in the four-dimensional case at hand, one would consider the quantity

$$\lim_{t\to\infty} \frac{\big\langle V_c(f_{t\ell})^* V_{c'}(f'_{t\ell'})^* V_{c'}(f'_{-t\ell'}) V_c(f_{-t\ell})\big\rangle}{\big\langle V_c(f_{t\ell})^* V_c(f_{-t\ell})\big\rangle \big\langle V_{c'}(f'_{t\ell'})^* V_{c'}(f'_{-t\ell'})\big\rangle}, \tag{3.42}$$

where ℓ, ℓ' are two non-parallel future-directed lightlike vectors, and $f_{t\ell}(x) = f(x - t\ell)$ etc. are smearing functions shifted in lightlike directions. To simplify matters, we absorb the charge factor q in the string smearing functions c which may therefore have arbitrary total weight $q \in \mathbb{R}$.

The following result illustrates how features of $\widetilde{H}$ have an impact on scattering theory, and in particular shows that the prescription Eq. (3.42) is certainly too naive to define a scattering. The challenge remains to understand which modification of Eq. (3.42), properly taking into account the directional dependence arising through $\widetilde{H}$, would possibly define an S-matrix in our four-dimensional model without zero mass eigenstates, or whether a variant of the more complicated strategy in [2], that was formulated for massless particles in 4D, should be developed.

Lemma 3.14 (see [19]) *Let ℓ, ℓ' be future directed lightlike vectors, and $c, c' \in C^\infty(S^2)$. Then in the limit $t \to \infty$, Eq. (3.42) converges to*

$$e^{-i\frac{\langle c,c'\rangle}{4\pi}} \cdot e^{\frac{1}{4\pi^2}\left(\widetilde{H}(\ell+\ell';c,c')+\widetilde{H}(\ell+\ell';c',c)-\widetilde{H}(\ell-\ell';c,c')-\widetilde{H}(\ell-\ell';c',c)\right)}. \tag{3.43}$$

The second factor is, in general, not a complex phase. Replacing c by $-c$, if necessary, its modulus may be > 1.

Sketch of proof By the structure Eq. (3.35) of vertex operator correlations, the four-point function is a product of six factors. One can convince oneself that in the limit the smearing functions in x can be neglected, so that two of the six factors in the numerator cancel against the denominator. This essentially leaves the four factors

$$\frac{e^{-\frac{1}{4\pi^2}\widetilde{H}(x+t\ell-x'-t\ell';c,c')}}{(-(x+t\ell-x'-t\ell'-i\varepsilon u)^2)^{-\frac{\langle c,c'\rangle}{8\pi^2}}} \cdot \frac{e^{-\frac{1}{4\pi^2}\widetilde{H}(x'-t\ell'-x+t\ell;c',c)}}{(-(x'-t\ell'-x+t\ell-i\varepsilon u)^2)^{-\frac{\langle c,c'\rangle}{8\pi^2}}} \cdot \\ \frac{e^{\frac{1}{4\pi^2}\widetilde{H}(x+t\ell-x'+t\ell';c,c')}}{(-(x+t\ell-x'+t\ell'-i\varepsilon u)^2)^{+\frac{\langle c,c'\rangle}{8\pi^2}}} \cdot \frac{e^{\frac{1}{4\pi^2}\widetilde{H}(x'+t\ell'-x+t\ell;c',c)}}{(-(x'+t\ell'-x+t\ell-i\varepsilon u)^2)^{+\frac{\langle c,c'\rangle}{8\pi^2}}}. \tag{3.44}$$

Because $\ell \pm \ell'$ are timelike resp. spacelike, the denominators in Eq. (3.44) are dominated by powers of $-t^2((\ell\pm\ell')-i\varepsilon u)^2 = 2t^2[\mp(\ell\ell')+i\varepsilon(u(\ell\pm\ell'))]$. Since $(\ell\ell') > 0$, the spacelike cases in the first line give $[2t^2(\ell\ell')]^{\frac{\langle c,c'\rangle}{8\pi^2}}$ each. The timelike cases in the second line give $[2t^2(\ell\ell')(-1+i\varepsilon)]^{-\frac{\langle c,c'\rangle}{8\pi^2}}$ each. Together, they yield the first factor in Eq. (3.43). Because H is a homogeneous function, the numerators in Eq. (3.44) yield the second factor in Eq. (3.43). Now, one may regard $L_\pm = \ell \pm \ell'$ as a pair of orthogonal vectors, one timelike and one spacelike, and otherwise independent. For L_- purely spacelike, $\widetilde{H}(L_-; c, c') + \widetilde{H}(L_-; c', c)$ is a real function with a non-trivial dependence on L_-. It cannot be cancelled by the real part of $\widetilde{H}(L_+; c, c') + \widetilde{H}(L_+; c', c)$. □

The first factor in Eq. (3.43) is a phase, as in the two-dimensional model [6]. But because the second factor may have modulus > 1, Eq. (3.43) cannot be interpreted as an S-matrix element.

The conclusion is only avoided if all strings are orthogonal to both ℓ and ℓ', in which case $H(\ell \pm \ell'; e_1, e_2)$ are independent of ℓ, ℓ'. Namely, in this case $\zeta(\ell \pm \ell'; e_1, e_2)$ are independent of ℓ, ℓ'. But this would require a smearing in the intersection of the sphere S^2 with a plane (i.e. a circle S^1). This is not an option because, by Remark 3.5, the singularity at $e = e'$ would no longer be integrable, and, e.g. the exponents Eq. (3.36) would be ill defined.

3.4 Derivative formula

The derivative of Eq. (3.11) w.r.t. x is surprisingly simple to compute.

Lemma 3.15 *Let* $F(x) = \frac{1}{x^2}$. *For* x^2, e_i^2, $\det_{e_1,e_2}$, $\det_{x,e_i}$ *and* $\det_{x,e_1,e_2}$ *all nonzero, it holds*

$$(I_{e_2} I_{e_1} \partial F)(x) = \frac{1}{2}\big[f(e_1, e_2)\partial_x + f(x, e_2)\partial_{e_1} + f(x, e_1)\partial_{e_2}\big] \log \det{}_{x,e_1,e_2}. \tag{3.45}$$

Proof Let $P := (I_{e_2} I_{e_1} \partial F)(x)$. By definition as a convergent integral over $\partial_x \frac{1}{(x+s_1e_1+s_2e_2)^2}$, it is a linear combination $b_1e_1 + b_2e_2 + b_3x$. From Eq. (2.12), we know that $(e_i P) = f(x, e_j)$ for $i, j \in \{1, 2\}$ pairwise distinct. From Eq. (3.11), we know that $(xP) = f(e_1, e_2)$. Then

$$\begin{pmatrix} (e_1 P) \\ (e_2 P) \\ (xP) \end{pmatrix} = \begin{pmatrix} f(x, e_2) \\ f(x, e_1) \\ f(e_1, e_2) \end{pmatrix} = G \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{3.46}$$

where G is the Gram matrix, is solved for the coefficient functions b_i by using Lemma 3.2. □

A remarkable feature of Eq. (3.45) is its formal symmetry in the three vector variables, despite the unsymmetric definition. The symmetry can be understood by a change of variables $(s_1, s_2) \to (\frac{1}{s_1}, \frac{s_2}{s_1})$ in the defining double string-integral.

Given Lemma 3.15, to specify $(I_{e_2} I_{e_1} \partial W_0)(x)$ for $e_1, e_2 \in H_1$ purely spatial, it suffices to specify the $i\varepsilon$-prescriptions at the possibly singular configurations. For $e_1, e_2 \in H_1$ purely spatial and x replaced by $x - i\varepsilon u$, the distributions $f(x, e_i)$

and $f(e_1, e_2)$ are well defined. It remains to consider the reciprocal of $\det_{x,e_1,e_2}$ in Eq. (3.45) when $\det_{x,e_1,e_2} = 0$. Because $i\varepsilon$ appears only in the term $(x - i\varepsilon u)^2 \det_{e_1,e_2}$, the distribution is well defined unless $\det_{e_1,e_2} = 0$, i.e. when $e_1 \neq \pm e_2$. The case $e_1 = e_2$ is regular by Proposition 3.6. Because the singularity at $e_1 = -e_2$ is integrable w.r.t. the invariant measure on $S^2 \times S^2$, we conclude

Proposition 3.16

$$
\begin{aligned}
&-(2\pi)^2 (I_{e_2} I_{e_1} \partial W_0)(x) \\
&\quad = \frac{1}{2}\big[f(e_1, e_2)\partial_x + f(x, e_2)\partial_{e_1} + f(x, e_1)\partial_{e_2}\big] \log \det{}_{x,e_1,e_2}, \qquad (3.47)
\end{aligned}
$$

with the distributions $f(x, e_i)$ as specified in Sect. 3.1 and $f(e_1, e_2)$ given by Eq. (3.16)*, is well defined as a distribution on $\mathbb{R}^4 \times S^2 \times S^2$.*

The result can also be obtained with a cumbersome computation of the derivative of $(I_{e_2} I_{e_1} \partial W_0)(x)$ given by Eqs. (3.11) and (3.18). This may be taken as a non-trivial check of Eq. (3.18).

4 Propagators

While the IR-regularized *two-point function* of the escort field is needed for the non-perturbative construction of the dressed Dirac field in [12], *propagators* are needed in perturbation theory. Defining QED as a perturbative expansion with interaction density $q\, A_\mu(e) j^\mu$ in the Wigner Hilbert space, where the two-point function of $A_\mu(e)$ is

$$
\begin{aligned}
&\langle A_\mu(x, e) A_\nu(x', e')\rangle \\
&\quad = -\big[\eta_{\mu\nu} + e_\nu I_e^x \partial_\mu^x + e'_\mu I_{e'}^{x'} \partial_\nu^{x'} + (ee') I_{e'}^{x'} I_e^x \partial_\mu^x \partial_\nu^{x'}\big] W_0(x - x'), \qquad (4.1)
\end{aligned}
$$

one needs only string integrations over *derivatives* of the massless two-point function and propagator. The string integrations over derivatives are IR-regular. We therefore focus on the analogues of the derivative of Eq. (3.2) and of Eq. (3.47) for the propagator, whose existence has been established in momentum space with methods of microlocal analysis in [9].

At first sight, the case of the propagator should be very parallel to that of the two-point function except for a different $i\varepsilon$-prescription: $x^2 - i\varepsilon$ rather than $(x - i\varepsilon u)^2$. Instead of an "analytic continuation through the forward tube", one needs an analytic continuation in the variable x^2. For the single string-integral of the propagator, it suffices to define the derivative of $f(x, e)$ by simply substituting $(x - i\varepsilon u)^2$ in Eqs. (3.2) and (3.3) by $x^2 - i\varepsilon$.

For the double string-integral over the derivative of the propagator, however, this prescription is not sufficient because the reciprocal of $\det_{x,e_1,e_2}$ becomes

$$
\frac{1}{\det_{x,e_1,e_2} - i\varepsilon \det_{e_1,e_2}}. \qquad (4.2)
$$

Namely, in the Wigner Hilbert space approach to QED, the strings are not restricted to be purely spatial [12, Sect. 4.2], and in fact they should not, in order to evade the velocity superselection rule [12, Sect. 4.4]. But in Lorentzian metric, if $e_i \in H_1$ are not purely spatial, $\det_{e_1,e_2}$ may vanish in a larger submanifold of configurations, see Lemma 4.3. For such e_1, e_2, Eq. (4.2) does not define a distribution in x at $\det_{x,e_1,e_2} = 0$. The following Lemma, which is a corollary to Lemma 3.3, allows to reduce the set of configurations where Eq. (4.2) is not defined.

Lemma 4.1 *Suppose that* $\det_{y_1,y_2,y_3} = 0$. *Then, for* $i, j, k \in \{1, 2, 3\}$ *pairwise distinct, hold:*

(i) All "diagonal" cofactors $\det_{y_j,y_k}$ *have the same sign, or vanish.*
(ii) If $\det_{y_i,y_j} = 0$, *then also the cofactors* Λ_i *and* Λ_j *vanish.*

Proof (i) By Lemma 3.3, if $\det_{y_1,y_2,y_3} = 0$, then $\det_{y_i,y_j} \det_{y_i,y_k} = \Lambda_i^2 \geq 0$. Thus $\det_{y_i,y_j}$ and $\det_{y_i,y_k}$ cannot be nonzero with opposite sign.
(ii) is obvious from Lemma 3.3.

□

Because by Lemma 4.1(i) the three two-variable Gram determinants cannot have opposite signs when $\det_{x,e_1,e_2} = 0$, one may improve the definition the reciprocal of $\det_{x,e_1,e_2}$ as a pullback of a boundary value of an analytic function in the variables $x^2 - i\varepsilon$, $e_1^2 - i\varepsilon$, and $e_2^2 - i\varepsilon$, i.e.

$$\frac{1}{\det_{x,e_1,e_2} - i\varepsilon(\det_{e_1,e_2} + \det_{x,e_1} + \det_{x,e_2})}. \tag{4.3}$$

This prescription, applied to Eq. (3.45) in the case of the propagator, extends the previous definition Eq. (4.2) of the reciprocal of $\det_{x,e_1,e_2}$, and hence of the propagator of $A_\mu(e)$, as distributions everywhere on $\mathbb{R} \times H_1 \times H_1$, except on configurations where all four Gram determinants vanish simultaneously. When this happens, then by Lemma 3.3 (or Lemma 4.1(ii)), also the non-diagonal cofactors Λ_i of the Gram determinant vanish.

The following Lemmas allow to characterize configurations with vanishing Gram determinants in Lorentzian metric, and in particular the configurations where Eq. (4.3) is not defined.

Lemma 4.2 *Suppose that* $\det_{y_1,y_2} \neq 0$. *Then,* $\det_{y_1,y_2,y_3} = 0$ *if and only if there exist* $\alpha, \beta \in \mathbb{R}$ *and* ℓ *lightlike* ($\ell^2 = 0$) *such that*

$$y_3 = \alpha y_1 + \beta y_2 + \ell \quad \textit{and} \quad (\ell y_1) = (\ell y_2) = 0. \tag{4.4}$$

Proof Because $\det_{y_1,y_2} \neq 0$, one may define α, β by

$$\begin{pmatrix} y_1^2 & (y_1 y_2) \\ (y_2 y_1) & y_2^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} (y_3 y_1) \\ (y_3 y_2) \end{pmatrix}. \tag{4.5}$$

With $\ell := y_3 - \alpha y_1 - \beta y_2$, an elementary computation gives

$$\det{}_{y_1,y_2,y_3} = \det{}_{y_1,y_2,\ell} = \ell^2 \det{}_{y_1,y_2}. \tag{4.6}$$

Hence $\det{}_{y_1,y_2,y_3} = 0$ if and only if $\ell^2 = 0$. □

Analogous statements with analogous proofs hold for $n \times n$ Gram determinants (which are trivially zero in 4 dimensions for $n > 4$). The version for $n = 2$ is

Lemma 4.3 *Suppose that $y_1^2 \neq 0$. Then, $\det_{y_1,y_2} = 0$ if and only if there exists $\alpha \in \mathbb{R}$ and ℓ lightlike such that*

$$y_2 = \alpha y_1 + \ell \quad \textit{and} \quad (\ell y_1) = 0. \tag{4.7}$$

Lemma 4.4 *Let $y_1^2 \neq 0$, $y_2^2 \neq 0$. If $\det_{y_1,y_2,y_3} = \det_{y_1,y_2} = 0$ and either $\det_{y_1,y_3} = 0$ or $\det_{y_2,y_3} = 0$, then the set $\{y_1, y_2, y_3\}$ is linearly dependent.*

Proof Assume $\det_{y_1,y_3} = 0$. Define

$$v := y_2 - \frac{(y_1 y_2)}{y_1^2} y_1, \qquad w := y_3 - \frac{(y_2 y_3)}{y_2^2} y_2.$$

One computes that $v^2 = w^2 = 0$ because $\det_{y_1,y_2} = \det_{y_1,y_3} = 0$, and $(vw) = 0$ because $\Lambda_2 = 0$ by Lemma 4.1(ii). Because two lightlike vectors that are orthogonal must be linearly dependent, it follows that v and w, and consequently y_1, y_2, y_3 are linearly dependent. The case with $\det_{y_2,y_3} = 0$ is similar, replacing $y_1 \leftrightarrow y_2$. □

The conclusion of Lemma 4.4 is *not* true if $\det_{y_1,y_2} \neq 0$ but the other two 2×2 determinants are zero. It is easy to find counter examples.

We can now classify the configurations where the inverse Gram determinant is not defined by the prescription Eq. (4.3).

Lemma 4.5 *For $(x, e_1, e_2) \in \mathbb{R}^4 \times H_1 \times H_1$ it holds $\det_{x,e_1,e_2} = \det_{x,e_1} = \det_{x,e_2} = \det_{e_1,e_2} = 0$ if and only if either*

$$e_1 = \pm e_2, \qquad x = \alpha e_1 + \ell \tag{4.8}$$

for some $\alpha \in \mathbb{R}$ and lightlike ℓ with $(e_1 \ell) = 0$, or

$$e_2 = a' e_1 + \ell', \qquad x = \alpha_1 e_1 + \alpha_2 e_2 \tag{4.9}$$

for some $\alpha', \alpha_1, \alpha_2 \in \mathbb{R}$ and lightlike ℓ' with $(e_1 \ell') = (e_2 \ell') = 0$. In either case, the set $\{x, e_1, e_2\}$ is linearly dependent.

Proof The "If" statements are trivial by linear dependency. Conversely, $\det_{e_1,e_2} = 0$ implies $e_2 = a e_1 + \ell'$ with $\ell'^2 = 0$ and $(e_1 \ell') = 0$ by Lemma 4.3. It then trivially follows that also $(e_2 \ell') = 0$. If $\ell' = 0$, then $a = \pm 1$, and Lemma 4.3 applied to $\det_{x,e_i} = 0$ entails Eq. (4.8). If $\ell' \neq 0$, then e_1 and e_2 are not linearly dependent. Because the three vectors x, e_1, e_2 are linearly dependent by Lemma 4.4, x must be a linear combination of e_1 and e_2. This is Eq. (4.9). □

The configurations of type Eqs. (4.8) and (4.9) both have codimension 3 in $\mathbb{R}^4 \times H_1 \times H_1$. At this moment, we do not know how to naturally extend the propagator to these sets. See, however, [9] where it was proven with microlocal methods that the string-localized propagator of $A_\mu(x, e)$, when smeared in the strings, can be defined on all of $\mathbb{R}^4$. This suggests that one can find coordinates in

$$\mathbb{R}^4 \times H_1 \times H_1$$

with respect to which the singularities of the inverse determinant at configurations as in Lemma 4.5 are integrable, similarly to the integrability of $\frac{\alpha}{\sin \alpha}$ observed earlier. In [9], it is also shown that products of smeared string-localized propagators have natural extensions as distributions in x in $\mathbb{R}^4 \setminus \{0\}$, exactly as in point-localized QFT.

Acknowledgements We thank J. Mund for valuable discussions, and the referees for suggestions for improvement. CG acknowledges financial support by the Studienstiftung des Deutschen Volkes.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data Availability The manuscript has no associated data.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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