Braided L_∞ -algebras, braided field theory and noncommutative gravity

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Abstract

We define a new homotopy algebraic structure, that we call a braided L_∞ -algebra, and use it to systematically construct a new class of noncommutative field theories, that we call braided field theories. Braided field theories have gauge symmetries which realize a braided Lie algebra, whose Noether identities are inhomogeneous extensions of the classical identities, and which do not act on the solutions of the field equations. We use Drinfel'd twist deformation quantization techniques to generate new noncommutative deformations of classical field theories with braided gauge symmetries, which we compare to the more conventional theories with star-gauge symmetries. We apply our formalism to introduce a braided version of general relativity without matter fields in the Einstein–Cartan–Palatini formalism. In the limit of vanishing deformation parameter, the braided theory of noncommutative gravity reduces to classical gravity without any extensions.

Keywords L_∞ -algebra · Drinfeld twist · Noncommutative gravity

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1 Introduction

In this paper, we will present a new perspective on symmetries of noncommutative field theories, and develop first principles which enable systematic constructions of new examples of field theories with noncommutative gauge symmetries. To achieve this we develop a new notion of homotopy algebra that is suited to organize the symmetries and dynamical content of (generalized) noncommutative gauge theories; we call this new mathematical object a ‘braided L_∞ -algebra’. Although our formalism applies broadly to any perturbative classical field theory with symmetries, in this paper we focus exclusively on examples of diffeomorphism-invariant field theories for simplicity, in order to illustrate clearly and conceptually how our novel constructions work. In particular, we derive new noncommutative extensions of the Einstein–Cartan–Palatini formulation of gravity in three and four spacetime dimensions.

In this section, we begin with a discussion of some background and context, before turning to a more detailed description of our results and the contents of this paper.

Classical field theories and L_∞ -algebras

A *classical* field theory on a manifold can for a large part be understood as a dynamical system, that is, a system of partial differential equations which encodes the dynamics of fields. The field theory may have symmetries, particularly local symmetries, and we refer to these broadly as ‘generalized gauge symmetries’ to accommodate more general systems whose fields and symmetries are not necessarily associated with connections on a principal bundle. In the setting of dynamical systems, by ‘symmetry’ we mean that the field equations transform covariantly under the symmetry transformations; the gauge symmetry then acts on the space of classical solutions. The main object of interest is therefore the space of classical physical states, defined to be the moduli space of solutions to the field equations modulo the local symmetries. The field equations may or not follow from a variational principle. For a Lagrangian field theory, the data may be accompanied by a gauge-invariant action functional; however for many (but not all) considerations the action functional is supplementary data which is not needed to analyse the space of classical physical states.

This perspective is naturally and systematically organized by strong homotopy Lie algebras [41], or L_∞ -algebras, which are generalizations of differential graded Lie algebras with infinitely many graded antisymmetric brackets satisfying higher

homotopy versions of the Jacobi identity. The data of any perturbative¹ classical field theory gives rise to an L_∞ -algebra. This may be seen in several ways. From a practical standpoint, the L_∞ -algebra underlying a field theory may be constructed directly through an explicit ‘bootstrapping’ algorithm [36]. They can also be obtained through the duality between L_∞ -algebras and differential graded commutative algebras, with the latter in this case being in the guise of the BV–BRST formalism [37,50]. These constructions can also be understood more geometrically in the setting of derived geometry, where the L_∞ -algebras which appear are the (derived) tangent complexes on the space of fields describing the deformation theory around critical points of an action functional [27]. For a Lagrangian field theory, the extra data of an action functional is captured by imposing an additional cyclic structure on the underlying L_∞ -algebra.

Noncommutative gauge theories and gravity

The dynamical systems perspective becomes particularly fruitful in situations where one is interested in a noncommutative deformation of a given field theory,² and in particular of gravity. For example, in the metric formulation of general relativity, it is often problematic to write down a variational principle which deforms the standard Einstein–Hilbert action functional, due to ambiguities in defining noncommutative deformations of volume forms and scalar curvatures, as well as reality properties and the inherent non-polynomial nature of the action (see e.g. [49] for a discussion of this). Yet consistent deformations of the field equations are straightforward (see e.g. [11,15]). It is then natural to seek L_∞ -algebras which organize symmetries and dynamics of noncommutative field theories.

The noncommutative extension of the bootstrap approach has been employed by [21,39] as a means for constructing new examples of noncommutative gauge theories. In this approach the L_∞ -algebra framework encodes not only the symmetries and dynamics of the field theory, but also the general failure of the deformed exterior algebra from forming a differential graded algebra and, in the case of nonassociative gauge symmetries, the failure of the Jacobi identities. This generally produces L_∞ -algebras with an infinite number of nonzero brackets, which has several limitations: explicit expressions for all brackets are difficult to obtain, the bootstrapping of brackets is particularly difficult and involved for nonassociative gauge symmetries, and so far this approach has only met with some success in the semi-classical limit of a noncommutative gauge theory. The approach based on symplectic embeddings [40] drastically improves the situation in that it allows for explicit and systematic constructions of all brackets of the L_∞ -algebra, but so far this approach is also restricted to the semi-classical limit and moreover limited to purely kinematic considerations.

¹ In the case of a non-polynomial theory or a nonlinear space of fields $\mathcal{M}$, this holds at the level of the perturbative expansion of the theory on the tangent space $T_\phi\mathcal{M}$ at a fixed solution $\phi \in \mathcal{M}$.

² For us, this will be a field theory where the underlying spacetime manifold M is replaced by a noncommutative deformation of the algebra of functions on M , along with the relevant differential geometrical objects as modules over this algebra. Being a deformation of a commutative theory, the noncommutative structures and dynamics must reduce to the classical spacetime and theory in the commutative limit.

In this paper, we discuss a new alternative approach, which arose from our attempts to understand a noncommutative version of the Einstein–Cartan–Palatini theory of gravity in this language; the L_∞ -algebra formalism for the classical theory was developed in detail by [30]. In the standard noncommutative extension of this theory [4], the (extended) local Lorentz symmetry is implemented by the usual star-gauge transformations, and in principle one may construct correspondingly an L_∞ -algebra structure as in the bootstrapping approach to noncommutative gauge theories. However, diffeomorphism invariance in this theory is implemented by the usual ‘twisted’ actions [15] (see [52] for a review), and the dichotomy between the actions of the two types of symmetries leads to severe difficulties in the implementation of an underlying L_∞ -algebra as an organizing principle for noncommutative gravity: The twisted diffeomorphism symmetry does not appear to fit naturally into the standard L_∞ -algebra picture. The situation is reminiscent of what happens in double field theory [36, Section 5.1], which requires extending the diffeomorphism symmetries by ‘higher’ symmetries, implementing reducible gauge symmetries of the theory. In the case of noncommutative gravity this quickly becomes very complicated, and it is not clear how to manage the higher brackets of the L_∞ -algebra. In other words, the standard L_∞ -algebra formalism does not seem to be an appropriate or useful tool for organizing the dynamics of noncommutative gravity. Our proposal then is to deform the L_∞ -algebra structure itself in such a way that it becomes compatible with the twisted diffeomorphisms. This leads to a new type of homotopy algebra and a new type of gauge symmetry for noncommutative field theories in general.

Braided L_∞ -algebras and braided field theories

Following the classical treatment, we employ the machinery of Hopf algebras and Drinfel’d twist deformation quantization³ [43] to produce *braided* noncommutative deformations of field theories.⁴ By twisting the enveloping Hopf algebra of vector fields on a manifold M to a non-cocommutative Hopf algebra and simultaneously deforming its category of modules, we deform the L_∞ -algebras into *braided L_∞ -algebras*, a structure that was sketched in [31]. In essence, one invokes the definition of an L_∞ -algebra in terms of morphisms in a symmetric monoidal category with a non-trivial braiding. In practice, the axioms of a classical L_∞ -algebra are supplemented with a non-trivial isomorphism $\mathcal{R} : V \otimes V' \rightarrow V \otimes V'$ whenever a transposition isomorphism between two modules V and V' occurs. For example, antisymmetry is now braided: $\ell_2(v \otimes v') = -\ell_2 \circ \mathcal{R}(v' \otimes v)$, and similarly for all higher homotopy Jacobi relations. Braided L_∞ -algebras may be regarded as a higher homotopy coherent extension of the notion of a braided Lie algebra, which was introduced by Woronowicz under the name ‘quantum Lie algebra’ [56], and later formulated in a more general categorical setting by Majid [42].

³ We use the phrase ‘deformation quantization’ colloquially to mean formal noncommutative deformation. Indeed, our theories will be classical in the physical sense, and no quantum physics will take place in this manuscript.

⁴ In this paper, we use the term ‘braided’ as a synonym for ‘twisted’, as the former is more natural in the context in which our field theories are defined.

When applied to the L_∞ -algebra underlying a classical dynamical system, this procedure deforms the collection of kinematical and dynamical data simultaneously and consistently, producing a genuinely new type of noncommutative field theory, that we call a *braided field theory*. These are field theories with braided gauge symmetries which close as braided Lie algebras. The spaces of fields, or more precisely their tangent spaces, now form braided representations of these braided Lie algebras. Various aspects of the kinematical part of braided gauge symmetries have appeared before in e.g. [23,49], but as far as we are aware a complete and general treatment of noncommutative dynamics with a variational principle has not been addressed before. From our perspective, the braided homotopy relations guarantee the braided representation properties, braided covariance of the field equations, and braided gauge invariance of the action functional. These symmetries differ from the star-gauge symmetries that are usually employed in noncommutative gauge theories, which close as classical Lie algebras. As a consequence, our braided deformations do not require the introduction of any new degrees of freedom or larger enveloping algebras to achieve gauge closure. We compare our braided gauge symmetries with the more conventional noncommutative gauge symmetries based on star-gauge transformations in Sect. 5.3.

We point out and resolve a conceptual issue concerning braided gauge symmetries: they leave the field equations covariant and action functional invariant, but they do not act on the space of solutions of the field equations, contrary to the classical case. This is a consequence of the braided Leibniz rule of braided Lie algebra representation theory. However, the braided L_∞ -algebra framework allows for the derivation of a braided version of the Noether identities, i.e. a set of differential identities among the field equations and the dynamical fields, which imply that not all degrees of freedom are independent. That is, some of them are gauge redundant, justifying the terminology ‘braided gauge symmetry’. The braided Noether identities are non-trivial deformations of their classical counterparts.

Braided noncommutative gravity

In this paper, we apply our formalism of braided L_∞ -algebras to produce new braided noncommutative theories of gravity in three and four spacetime dimensions. We consider only the case of parallelizable spacetime manifolds M , for which the twist deformations are straightforward to implement and exhibit the salient features of our constructions more clearly; see [30] for the general treatment in the classical case. In three dimensions, the noncommutative field equations and action functional are naive deformations of the classical ones (with braided commutators); however in four dimensions we uncover non-trivial deformations. Both theories are crucially different from previous approaches for several reasons.

Firstly, in our braided approach the two sets of infinitesimal gauge symmetries, diffeomorphisms and local coframe rotations, each deform consistently with one another and together combine as a semi-direct product of braided Lie algebras, contrary to earlier approaches where diffeomorphisms are deformed as a braided Lie algebra but rotations as a classical Lie algebra (see e.g. [4]). Thus no new fields or extended enveloping algebras of symmetries need be introduced, contrary to these

other approaches. Secondly, our approach formalizes the treatment of braided gauge transformations and extends earlier treatments of twisted diffeomorphism symmetry of gravity to *all* gauge transformations, pointing out and clarifying several novel aspects of the space of solutions to the field equations compared to the unbraided case. Finally, one can in principle study solutions of the braided field equations, and compare our first order noncommutative theory of gravity in four dimensions with the existing metric formulation for noncommutative gravity (see e.g. [49]). We expect that these two theories are drastically different, but our approach enjoys an explicit action functional and the coupling to matter fields (such as fermions) is simply a straightforward calculation within the braided L_∞ -algebra framework. We stress that the classical solution space of earlier treatments of the metric formulation of noncommutative gravity with twisted diffeomorphism symmetry also suffers from the same interpretational problem mentioned above, while our framework provides an explanation of the meaning of these symmetries in this sense. We compare our braided theories of gravity with other noncommutative theories of gravity in Sect. 6.1.

Summary and outline of the paper

We begin in the next two sections by giving some background to the main developments of this paper. In Sect. 2, we review the basics of L_∞ -algebras, together with their cyclic versions, and how they are used to organize the symmetries and dynamics of classical field theories. We illustrate the formalism on the two main classes of examples of this paper: Chern–Simons gauge theory and the Einstein–Cartan–Palatini theory of gravity. In Sect. 3, we review the main ideas of Drinfel’d twist deformation quantization, using the enveloping Hopf algebra of vector fields on a manifold M , and look at the two particular examples of Hopf modules which form the backbone of this paper: the exterior algebra of differential forms on M and the algebras of vector fields on M themselves; these two modules also furnish our first examples of braided Lie algebras. At times we appeal to the categorical picture which interprets twist deformation quantization as a functorial equivalence of symmetric monoidal categories. This language is not strictly necessary to understand the main ideas or computations of this paper, but it does provide a convenient conceptual framework and at times simplifies what would otherwise be unwieldy technical computations.

In Sect. 4, we come to the main mathematical definitions and constructions of this paper: we develop in detail the notion of a braided L_∞ -algebra. It is here that the categorical language is particularly powerful, as it enables a straightforward formulation of the braided homotopy relations, which would be otherwise cumbersome and difficult to write out in general. Our definition makes sense in any symmetric monoidal category, and parallels the notion of a braided Lie algebra as a Lie algebra in such a category. When a classical L_∞ -algebra sits in the category of modules over the enveloping Hopf algebra of vector fields on M , we describe in detail how its Drinfel’d twist deformation defines a braided L_∞ -algebra (see Proposition 4.8). This result is very useful in applications, as once we are given a classical L_∞ -algebra, we know immediately that its twist quantization is a braided L_∞ -algebra, without the need of having to go through lengthy and technically involved checks of the braided homo-

topology relations. We also consider twist quantization of cyclic L_∞ -algebras via Drinfel'd twists which are compatible with the cyclic structure in a suitable sense, and show that they produce strictly cyclic braided L_∞ -algebras (see Proposition 4.15), a result which is important for applications to noncommutative field theories.

In Sect. 5, we turn to the main physical applications of the present paper. We formulate the notion of a braided field theory, taking the perspective that it defines a formal dynamical system. We give a general account of braided field theories starting from a braided L_∞ -algebra as input. After a detailed general discussion of braided gauge symmetries, we proceed to explain how to build the kinematics and dynamics of a noncommutative field theory in the braided L_∞ -algebra formalism, paralleling the constructions of the classical case. In general, the proofs of various properties such as gauge closure, braided covariance of the field equations, braided Noether identities and braided gauge invariance of an action functional are much more complicated compared to the classical case, as the trade-off of strict graded antisymmetry to its braided version no longer allows simplification of the necessary homotopy relations. In particular, we find that braided covariance is realized non-trivially and results in the peculiar features of the space of classical physical states discussed above. The Noether identities also assume a very different form and contain inhomogeneous field-dependent terms compared to their classical versions.

Our construction gives a straightforward algorithm for producing braided field theories as noncommutative deformations of classical field theories:

1. Given a classical field theory on a manifold M , work out its corresponding L_∞ -algebra, either by bootstrapping the initial brackets of its gauge transformations, field equations and Noether identities, or by using its dual BV–BRST formulation.
2. Given an enveloping Hopf algebra of vector fields on M which generate symmetries of the classical field theory, and over which the L_∞ -algebra is a module, choose a corresponding Drinfel'd twist element and quantize the L_∞ -algebra to a braided L_∞ -algebra.
3. Run this braided L_∞ -algebra through the formulas of Sect. 5 to produce a noncommutative field theory with braided gauge symmetries.

We apply this algorithm to the example of Chern–Simons gauge theory in Sect. 5.9. While the resulting braided field theory formally looks like a simple deformation of its classical version, it is quite different from the usual discussions of noncommutative Chern–Simons theory: in particular, it possesses braided gauge symmetries and its Noether identity is a deformation of the usual Bianchi identity. In general, the field equations and action functional of the resulting braided field theory will not simply be the ‘naive’ deformations of those of the classical field theory, to which they nevertheless reduce in the classical limit: our algorithm constructs genuinely new examples of noncommutative gauge theories. We illustrate this in Sect. 6 by applying our formalism to construct new braided theories of noncommutative gravity in three dimensions, whose field equations are given by (6.3) and (6.7), and in four dimensions, whose field equations are given by (6.23) and (6.27).

Outlook

This paper arose as part of an ongoing programme to formulate a nonassociative theory of gravity that is inspired by scenarios in closed non-geometric string theory, see e.g. [11,18,20]. We argued in [30] that the L_∞ -algebra formalism should provide the natural receptacle to capture the failure of closure and covariance of field equations under nonassociative gauge transformations. To accommodate these instances, one may in principle deform the classical L_∞ -algebra using cochain twists of the enveloping algebra of vector fields, rather than cocycle twists, which results in a quasi-Hopf algebra. The end result will be an L_∞ -algebra in a symmetric monoidal category with non-trivial braiding and associator isomorphisms. Using the corresponding deformed homotopy identities, nonassociative gauge transformations ought to have a straightforward definition, and the corresponding field equations and action functionals may be in reach within this formalism. In this paper, we focus on associative theories of noncommutative gravity, leaving these more complicated extensions for future investigations.

Although the present paper focuses on examples of diffeomorphism-invariant field theories, it is straightforward to apply our formalism to other field theories with less symmetry by restricting to a Hopf subalgebra of vector fields over which the classical L_∞ -algebra is a module. Our approach also offers a straightforward route to ‘noncommutative higher gauge theory’ through k -term braided L_∞ -algebras with $k > 4$. Along these lines, it is natural to wonder if our braided noncommutative field theories arise as suitable low energy limits of string theory, similarly to the conventional models based on star-gauge symmetry (see e.g. [51] for a review). A promising direction is to look for connections to braided symmetries of string amplitudes, for instance through the Hopf algebraic symmetries of worldsheet string correlation functions discussed in [1,2]. Such a connection further has the potential to connect noncommutative gravity with string theory for the first time. All of these interesting extensions of the present work are left for future investigations.

In this paper, we treat only the case of classical field theories. Quantization of our braided field theories should be made possible by firstly finding a braided version of the Chevalley–Eilenberg algebra of an L_∞ -algebra, which would then produce a braided version of the BV–BRST formalism for field theories. This should then be compared to Oeckl’s ‘braided quantum field theory’ framework [45,48], which however has so far been restricted to field theories without gauge symmetries. Some steps in this direction have been discussed recently in [44]. The formalism of [44] also suggests a natural way to handle the problem of defining the braided moduli space of classical solutions, which in the dual setting of the braided BV formalism may be completely characterized by its braided differential graded algebra of functionals. These possibilities and their physical relevance will be explored in future investigations.

2 L_∞ -algebras and classical field theory

2.1 L_∞ -algebras and cyclic L_∞ -algebras

An L_∞ -algebra is a $\mathbb{Z}$ -graded real vector space $V = \bigoplus_{k \in \mathbb{Z}} V_k$ equipped with graded antisymmetric multilinear maps

$$\ell_n : \bigotimes^n V \longrightarrow V, \quad v_1 \otimes \cdots \otimes v_n \longmapsto \ell_n(v_1, \dots, v_n)$$

for each $n \geq 1$, which we call n -brackets. The graded antisymmetry translates to

$$\ell_n(\dots, v, v', \dots) = -(-1)^{|v||v'|} \ell_n(\dots, v', v, \dots), \quad (2.1)$$

where we denote the degree of a homogeneous element $v \in V$ by $|v|$. The n -bracket is a map of degree $|\ell_n| = 2 - n$, that is

$$|\ell_n(v_1, \dots, v_n)| = 2 - n + \sum_{j=1}^n |v_j|.$$

The n -brackets ℓ_n are required to fulfill infinitely many identities $\mathcal{J}_n(v_1, \dots, v_n) = 0$ for each $n \geq 1$, called homotopy relations, with

$$\begin{aligned} \mathcal{J}_n(v_1, \dots, v_n) &:= \sum_{i=1}^n (-1)^{i(n-i)} \sum_{\sigma \in \text{Sh}_{i, n-i}} \chi(\sigma; v_1, \dots, v_n) \\ &\quad \times \ell_{n+1-i}(\ell_i(v_{\sigma(1)}, \dots, v_{\sigma(i)}), v_{\sigma(i+1)}, \dots, v_{\sigma(n)}), \end{aligned} \quad (2.2)$$

where, for each $i = 1, \dots, n$, the second sum runs over $(i, n-i)$ -shuffled permutations $\sigma \in S_n$ of degree n which are restricted as

$$\sigma(1) < \cdots < \sigma(i) \quad \text{and} \quad \sigma(i+1) < \cdots < \sigma(n).$$

The Koszul sign $\chi(\sigma; v_1, \dots, v_n) = \pm 1$ is determined from the grading by

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = \chi(\sigma; v_1, \dots, v_n) v_1 \wedge \cdots \wedge v_n.$$

Let us look explicitly at the first three identities. For $n = 1$ the identity $\mathcal{J}_1(v) = 0$ is

$$\ell_1(\ell_1(v)) = 0,$$

which states that the map $\ell_1 : V \rightarrow V$ is a differential making (V, ℓ_1) into a cochain complex. For $n = 2$ the identity $\mathcal{J}_2(v_1, v_2) = 0$ reads

$$\ell_1(\ell_2(v_1, v_2)) = \ell_2(\ell_1(v_1), v_2) + (-1)^{|v_1|} \ell_2(v_1, \ell_1(v_2)),$$

which states that ℓ_1 is a graded derivation with respect to the 2-bracket $\ell_2 : V \otimes V \rightarrow V$. For $n = 3$ the identity $\mathcal{J}_3(v_1, v_2, v_3) = 0$ expands into

$$\begin{aligned} & \ell_2(\ell_2(v_1, v_2), v_3) - (-1)^{|v_2||v_3|} \ell_2(\ell_2(v_1, v_3), v_2) \\ & + (-1)^{(|v_2|+|v_3|)|v_1|} \ell_2(\ell_2(v_2, v_3), v_1) \\ & = -\ell_3(\ell_1(v_1), v_2, v_3) - (-1)^{|v_1|} \ell_3(v_1, \ell_1(v_2), v_3) \\ & - (-1)^{|v_1|+|v_2|} \ell_3(v_1, v_2, \ell_1(v_3)) \\ & - \ell_1(\ell_3(v_1, v_2, v_3)), \end{aligned} \quad (2.3)$$

which for $\ell_3 = 0$ is just the graded Jacobi identity for the 2-bracket ℓ_2 ; in general it expresses the violation of the Jacobi identity by a cochain homotopy determined by ℓ_1 . In this sense L_∞ -algebras are (strong) homotopy deformations of differential graded Lie algebras which are the special cases where the ternary and all higher brackets vanish: $\ell_n = 0$ for all $n \geq 3$. In general, the homotopy relations for $n \geq 3$ are generalized Jacobi identities.

We are sometimes interested in the case where an L_∞ -algebra $(V, \{\ell_n\})$ is further endowed with a graded symmetric non-degenerate bilinear pairing $\langle -, - \rangle : V \otimes V \rightarrow \mathbb{R}$ which is cyclic in the sense that

$$\langle v_0, \ell_n(v_1, v_2, \dots, v_n) \rangle = (-1)^{n+(|v_0|+|v_n|)n+|v_n|\sum_{i=0}^{n-1}|v_i|} \langle v_n, \ell_n(v_0, v_1, \dots, v_{n-1}) \rangle$$

for all $n \geq 1$. This is the natural notion of an invariant inner product on an L_∞ -algebra, and if such a pairing exists the resulting algebraic structure is called a cyclic L_∞ -algebra; this generalizes the notion of a quadratic Lie algebra. If in addition the pairing is odd, say of degree $-2p - 1$, then the only non-vanishing pairings are

$$\langle -, - \rangle : V_k \otimes V_{2p-k+1} \longrightarrow \mathbb{R} \quad \text{for } k \leq p,$$

so that the pairing is strictly symmetric and the cyclicity condition simplifies to

$$\langle v_0, \ell_n(v_1, v_2, \dots, v_n) \rangle = (-1)^{(|v_0|+1)n} \langle v_n, \ell_n(v_0, v_1, \dots, v_{n-1}) \rangle.$$

2.2 Classical field theories in the L_∞ -algebra formalism

In this paper, by a classical field theory we will mean a polynomial (or perturbative) field theory with irreducible gauge symmetries. They can be completely encoded in 4-term L_∞ -algebras concentrated in degrees 0, 1, 2, and 3:

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3.$$

The kinematical data is encoded in the vector space V_0 of gauge parameters and in the vector space V_1 of fields, while the dynamical data is encoded in V_2 and V_3 . These vector spaces are typically infinite-dimensional spaces of sections of vector bundles over a manifold M .

The full symmetries and dynamics of field theories are expanded in terms of the brackets as follows. Given $\lambda \in V_0$ and $A \in V_1$, the gauge variations are encoded as the maps $A \mapsto A + \delta_\lambda A$ where

$$\delta_\lambda A = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \ell_{n+1}(\lambda, A, \dots, A), \quad (2.4)$$

where the brackets involve n insertions of the field A . The field equations $F_A = 0$ are encoded through

$$F_A = \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \ell_n(A, \dots, A), \quad (2.5)$$

with the covariant gauge variations

$$\delta_\lambda F_A = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \ell_{n+2}(\lambda, F_A, A, \dots, A). \quad (2.6)$$

If the distribution in TV_1 spanned by the gauge parameters is involutive off-shell, that is, when $F_A \neq 0$, then the homotopy relations imply that the closure relation for the gauge algebra has the form

$$[\delta_{\lambda_1}, \delta_{\lambda_2}]_\circ A = \delta_{\llbracket \lambda_1, \lambda_2 \rrbracket_A} A, \quad (2.7)$$

where $[\delta_{\lambda_1}, \delta_{\lambda_2}]_\circ := \delta_{\lambda_1} \circ \delta_{\lambda_2} - \delta_{\lambda_2} \circ \delta_{\lambda_1}$, and

$$\llbracket \lambda_1, \lambda_2 \rrbracket_A = - \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \ell_{n+2}(\lambda_1, \lambda_2, A, \dots, A).$$

The homotopy relations guarantee that the Jacobi identity is generally satisfied for any triple of maps δ_{λ_1} , δ_{λ_2} and δ_{λ_3} . The Noether identities are encoded by

$$\mathbf{d}_A F_A := \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \ell_{n+1}(F_A, A, \dots, A) = 0, \quad (2.8)$$

which vanishes identically as a consequence of the homotopy relations $\mathcal{J}_n(A, \dots, A) = 0$, for all $n \geq 1$, of the L_∞ -algebra [37].

For a field theory with a Lagrangian formulation, the action functional is encoded via a symmetric non-degenerate bilinear pairing $\langle -, - \rangle : V \otimes V \rightarrow \mathbb{R}$ of degree -3 , which makes $(V, \langle -, - \rangle)$ into a cyclic L_∞ -algebra. Then, it is easy to see that the field equations $F_A = 0$ follow from varying the action functional defined as

$$S(A) := \sum_{n=1}^{\infty} \frac{1}{(n+1)!} (-1)^{\frac{1}{2}n(n-1)} \langle A, \ell_n(A, \dots, A) \rangle, \quad (2.9)$$

since then cyclicity implies $\delta S(A) = \langle F_A, \delta A \rangle$. Cyclicity also implies

$$\delta_\lambda S(A) = \langle F_A, \delta_\lambda A \rangle = -\langle d_A F_A, \lambda \rangle, \quad (2.10)$$

so that gauge invariance of the action functional $\delta_\lambda S(A) = 0$ is then equivalent to the Noether identities $d_A F_A = 0$. For further details and the connections to the BV–BRST formalism see [37], while details of the aspects covered in this paper are found in [30].

2.3 Example I: Chern–Simons gauge theory

Let us illustrate how to reconstruct Chern–Simons theory in three spacetime dimensions in the L_∞ -algebra framework, which is the prototypical example of a gauge theory with a simple bracket structure, see e.g. [30,37]. Let $\mathfrak{g}$ be a real Lie algebra with Lie bracket $[-, -]_{\mathfrak{g}}$ and let M be a three-manifold. Then the 4-term L_∞ -algebra which organizes Chern–Simons gauge theory on M is simply the differential graded Lie algebra whose underlying cochain complex is the de Rham complex $(\Omega^\bullet(M, \mathfrak{g}), d)$ in three dimensions with values in the Lie algebra $\mathfrak{g}$, and whose underlying graded Lie algebra $(\Omega^\bullet(M, \mathfrak{g}), [-, -]_{\mathfrak{g}})$ is given by extending the Lie bracket of $\mathfrak{g}$ to $\Omega^\bullet(M, \mathfrak{g}) := \Omega^\bullet(M) \otimes \mathfrak{g}$ by the tensor product with the exterior multiplication of forms; that is,

$$\ell_1(\alpha) = d\alpha \quad \text{and} \quad \ell_2(\alpha, \beta) = -[\alpha, \beta]_{\mathfrak{g}}$$

for $\alpha, \beta \in V = \Omega^\bullet(M, \mathfrak{g})$.

It is easy to check that these brackets reproduce the kinematical and dynamical content of classical Chern–Simons gauge theory via the prescription of Sect. 2.2. The transformation of a gauge field $A \in \Omega^1(M, \mathfrak{g})$ by a gauge parameter $\lambda \in \Omega^0(M, \mathfrak{g})$ is given by

$$\delta_\lambda A = \ell_1(\lambda) + \ell_2(\lambda, A) = d\lambda + [A, \lambda]_{\mathfrak{g}}.$$

As for any differential graded Lie algebra, these transformations close a (field-independent) gauge algebra $\llbracket \lambda_1, \lambda_2 \rrbracket_A = -\ell_2(\lambda_1, \lambda_2) = [\lambda_1, \lambda_2]_{\mathfrak{g}}$, which in this case is just the Lie algebra $\Omega^0(M, \mathfrak{g})$. The field equations $F_A = 0$ are encoded by

$$F_A = \ell_1(A) - \frac{1}{2} \ell_2(A, A) = dA + \frac{1}{2} [A, A]_{\mathfrak{g}},$$

with

$$\delta_\lambda F_A = \ell_2(\lambda, F_A) = [F_A, \lambda]_{\mathfrak{g}}$$

expressing the covariance of the curvature 2-forms $F_A \in \Omega^2(M, \mathfrak{g})$ under local gauge transformations. Thus, the field equations $F_A = 0$ are solved by flat connections of a trivial principal bundle on the three-manifold M , which are locally gauge equivalent to the trivial connection $A = 0$. Finally,

$$d_A F_A = \ell_1(F_A) - \ell_2(A, F_A) = dF_A + [A, F_A]_{\mathfrak{g}} = d^A F_A,$$

where d^A denotes the usual covariant derivative. Thus, the Noether identities $d_A F_A = 0$ are equivalent to the Bianchi identities $d^A F_A = 0$ in $\Omega^3(M, \mathfrak{g})$. In this way, we reproduce the usual space of classical physical states of Chern–Simons theory as the moduli space of flat connections on M .

To reproduce the Chern–Simons action functional in this formulation, we need further structure: we assume that M is a closed oriented three-manifold and that $\mathfrak{g}$ is a quadratic Lie algebra. We may then construct a cyclic L_∞ -algebra structure on V by defining the pairing of $\mathfrak{g}$ -valued forms in complementary degrees:

$$\langle \alpha, \beta \rangle := \int_M \text{Tr}_{\mathfrak{g}}(\alpha \wedge \beta), \quad (2.11)$$

where $|\alpha| + |\beta| = 3$. This defines a cyclic non-degenerate pairing of degree -3 , where cyclicity follows from the invariance of the quadratic form $\text{Tr}_{\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$ on $\mathfrak{g}$. Then the Chern–Simons action functional is reproduced according to the general prescription of Sect. 2.2:

$$S(A) = \frac{1}{2} \langle A, \ell_1(A) \rangle - \frac{1}{6} \langle A, \ell_2(A, A) \rangle = \frac{1}{2} \int_M \text{Tr}_{\mathfrak{g}} \left(A \wedge dA + \frac{1}{3} A \wedge [A, A]_{\mathfrak{g}} \right). \quad (2.12)$$

2.4 Example II: Einstein–Cartan–Palatini gravity

We shall now review the reconstruction of the first-order formulation of gravity in the L_∞ -algebra framework, from the perspective of the Einstein–Cartan–Palatini (ECP) formalism for general relativity, following [30]. To set up notation and to illustrate the ideas, here we only consider the L_∞ -algebras in three spacetime dimensions and in Euclidean signature. The general cases of arbitrary dimensionality and signature are treated in detail in [30], and later we shall also deal with the four-dimensional case. We comment below on the differences when one extends these constructions to higher dimensions (independently of the signature of spacetime, see [30]).

Let M be a parallelizable three-manifold; for this it suffices to suppose that M is orientable, a condition we will need anyway later on. The graded vector space V underlying the L_∞ -algebra which organizes ECP gravity on M is given by

$$V := V_0 \oplus V_1 \oplus V_2 \oplus V_3 \quad (2.13)$$

where

$$\begin{aligned} V_0 &= \Gamma(TM) \times \Omega^0(M, \mathfrak{so}(3)), \\ V_1 &= \Omega^1(M, \mathbb{R}^3) \times \Omega^1(M, \mathfrak{so}(3)), \\ V_2 &= \Omega^2(M, \wedge^2 \mathbb{R}^3) \times \Omega^2(M, \mathbb{R}^3), \\ V_3 &= \Omega^1(M, \Omega^3(M)) \times \Omega^3(M, \mathbb{R}^3). \end{aligned}$$

In the following we denote elements in degrees 0, 1, 2 and 3, respectively, by $(\xi, \rho) \in V_0$, $(e, \omega) \in V_1$, $(E, \Omega) \in V_2$, and $(\mathcal{X}, \mathcal{P}) \in V_3$. This yields a 4-term L_∞ -algebra with differential defined by

$$\ell_1(\xi, \rho) = (0, d\rho), \quad \ell_1(e, \omega) = (d\omega, de) \quad \text{and} \quad \ell_1(E, \Omega) = (0, d\Omega).$$

The 2-brackets are defined by

$$\begin{aligned} \ell_2((\xi_1, \rho_1), (\xi_2, \rho_2)) &= ([\xi_1, \xi_2]_{\mathfrak{so}(3)}, -[\rho_1, \rho_2]_{\mathfrak{so}(3)} + L_{\xi_1}\rho_2 - L_{\xi_2}\rho_1), \\ \ell_2((\xi, \rho), (e, \omega)) &= (-\rho \cdot e + L_\xi e, -[\rho, \omega]_{\mathfrak{so}(3)} + L_\xi \omega), \\ \ell_2((\xi, \rho), (E, \Omega)) &= (-[\rho, E]_{\mathfrak{so}(3)} + L_\xi E, -\rho \cdot \Omega + L_\xi \Omega), \\ \ell_2((\xi, \rho), (\mathcal{X}, \mathcal{P})) &= (dx^\mu \otimes \text{Tr}(\iota_\mu d\rho \wedge \mathcal{P}) + L_\xi \mathcal{X}, -\rho \cdot \mathcal{P} + L_\xi \mathcal{P}), \\ \ell_2((e_1, \omega_1), (e_2, \omega_2)) &= -([\omega_1, \omega_2]_{\mathfrak{so}(3)}, \omega_1 \wedge e_2 + \omega_2 \wedge e_1), \\ \ell_2((e, \omega), (E, \Omega)) &= (dx^\mu \otimes \text{Tr}(\iota_\mu de \wedge E + \iota_\mu d\omega \wedge \Omega - \iota_\mu e \wedge dE \\ &\quad - \iota_\mu \omega \wedge d\Omega), E \wedge e - \omega \wedge \Omega), \end{aligned} \quad (2.14)$$

while all the rest of the brackets vanish. As shown in [30], this defines a differential graded Lie algebra.

Let us explain the notation used, as well as the content of these brackets following from the general prescription of Sect. 2.2. The dynamical fields of ECP gravity on M are pairs (e, ω) consisting of a coframe field $e \in \Omega^1(M, \mathbb{R}^3)$ and a spin connection $\omega \in \Omega^1(M, \mathfrak{so}(3))$. Using the isomorphism $\mathfrak{so}(3) \cong \wedge^2 \mathbb{R}^3$, one identifies the spin connection as an element $\omega \in \Omega^1(M, \wedge^2 \mathbb{R}^3)$; we may then expand $e = e^a E_a$ and $\omega = \omega^{ab} E_a \wedge E_b$, where e^a and $\omega^{ab} = -\omega^{ba}$ are 1-forms on M , and E_a for $a = 1, 2, 3$ form the standard orthonormal basis of $\mathbb{R}^3$. The $\wedge$ -product separately takes the exterior products of the ‘curved’ spacetime differential forms on the manifold M and ‘flat’ spacetime multivectors on the vector space $\mathbb{R}^3$; in particular, $\omega \wedge e = \omega^{ab} \wedge e^c E_a \wedge E_b \wedge E_c$ while $\omega \wedge e = \delta_{bc} \omega^{ab} \wedge e^c E_a$. Then, $\text{Tr} : \wedge^3 \mathbb{R}^3 \rightarrow \mathbb{R}$ is the Hodge duality operator on $\mathbb{R}^3$ endowed with the standard Euclidean metric, defined by $\text{Tr}(E_a \wedge E_b \wedge E_c) = \epsilon_{abc}$.

The gauge parameters are pairs (ξ, ρ) consisting of a vector field $\xi \in \Gamma(TM)$ and a local rotation $\rho \in \Omega^0(M, \mathfrak{so}(3))$. Following the prescription of Sect. 2.2, their action on fields is given by

$$\begin{aligned} \delta_{(\xi, \rho)}(e, \omega) &= \ell_1(\xi, \rho) + \ell_2((\xi, \rho), (e, \omega)) \\ &= (-\rho \cdot e + L_\xi e, d\rho - [\rho, \omega]_{\mathfrak{so}(3)} + L_\xi \omega) \end{aligned}$$

where L_ξ is the Lie derivative along the vector field ξ , and $\rho \cdot e = \rho^a{}_b e^b E_a$ denotes matrix multiplication in the fundamental representation of $\mathfrak{so}(3)$. Thus ξ generates the standard diffeomorphism symmetry of general relativity, while ρ generates changes of orthonormal coframes for a given metric. The gauge algebra

$$\begin{aligned} \llbracket (\xi_1, \rho_1), (\xi_2, \rho_2) \rrbracket_{(e, \omega)} &= -\ell_2((\xi_1, \rho_1), (\xi_2, \rho_2)) \\ &= (-[\xi_1, \xi_2]_{\mathfrak{v}}, [\rho_1, \rho_2]_{\mathfrak{so}(3)} - L_{\xi_1} \rho_2 + L_{\xi_2} \rho_1) \end{aligned}$$

is the semi-direct product of the Lie algebras of vector fields on M and of local rotations:

$$\Gamma(TM) \ltimes \Omega^0(M, \mathfrak{so}(3)). \quad (2.15)$$

We denote the Lie bracket of vector fields on $\mathfrak{v} := \Gamma(TM)$ by $[-, -]_{\mathfrak{v}}$. The field equations are given by

$$\begin{aligned} F_{(e, \omega)} &= \ell_1(e, \omega) - \frac{1}{2} \ell_2((e, \omega), (e, \omega)) \\ &= (d\omega, de) + \frac{1}{2} ([\omega, \omega]_{\mathfrak{so}(3)}, 2\omega \wedge e) = (R, T), \end{aligned}$$

where

$$R := d\omega + \frac{1}{2} [\omega, \omega]_{\mathfrak{so}(3)} \in \Omega^2(M, \mathfrak{so}(3))$$

is the curvature 2-form of the spin connection ω , while the covariant derivative of the coframe field

$$T := d^\omega e = de + \omega \wedge e \in \Omega^2(M, \mathbb{R}^3)$$

is the torsion 2-form. The field equations $F_{(e, \omega)} = (0, 0)$ thus imply two conditions: Firstly, that the torsion vanishes, $T = 0$, which for non-degenerate coframe fields e can be solved to identify ω with the Levi-Civita connection of the metric $\delta^{ab} e_a \otimes e_b$. Secondly, that the curvature vanishes, $R = 0$, which is just the vacuum Einstein field equation in three dimensions, whose solutions correspond to flat spacetimes which are locally isometric to Euclidean space. In this way we can recover the usual space of classical physical states of three-dimensional general relativity (in the absence of matter fields). However, the ECP theory makes sense for degenerate coframes e , and this extension of general relativity is necessary for its L_∞ -algebra formulation where the space of fields is required to be a vector (or an affine) space. The covariance of the field equations is expressed through

$$\delta_{(\xi, \rho)} F_{(e, \omega)} = \ell_2((\xi, \rho), F_{(e, \omega)}) = (-[\rho, R]_{\mathfrak{so}(3)} + L_\xi R, -\rho \cdot T + L_\xi T).$$

Finally,

$$\begin{aligned} d_{(e, \omega)} F_{(e, \omega)} &= \ell_1(F_{(e, \omega)}) - \ell_2((e, \omega), F_{(e, \omega)}) \\ &= (dx^\mu \otimes \text{Tr}(\iota_\mu e \wedge dR - \iota_\mu de \wedge R + \iota_\mu \omega \wedge dT \\ &\quad - \iota_\mu d\omega \wedge T), d^\omega T - R \wedge e), \end{aligned}$$

where ι_μ denotes the contraction with vectors $\partial_\mu = \frac{\partial}{\partial x^\mu}$ of the local holonomic frame dual to the basis $\{dx^\mu\}$ of 1-forms in a local coordinate chart on M , and we identify the vector space of 1-forms valued in 3-forms $\Omega^1(M, \Omega^3(M))$ with $\Omega^1(M) \otimes \Omega^3(M)$. The Noether identities $d_{(e,\omega)} F_{(e,\omega)} = (0, 0)$ thus impose two differential identities among the field equations, the second of which is equivalent to the first Bianchi identity $d^\omega T = R \wedge e$.

The action functional can be derived in this language by assuming M is closed⁵ and defining a cyclic pairing $\langle -, - \rangle : V \otimes V \rightarrow \mathbb{R}$ of degree -3 as [30]

$$\langle (e, \omega), (E, \Omega) \rangle := \int_M \text{Tr}(e \lrcorner E + \Omega \lrcorner \omega), \quad (2.16)$$

on $V_1 \otimes V_2$, and as

$$\langle (\xi, \rho), (\mathcal{X}, \mathcal{P}) \rangle := \int_M \iota_\xi \mathcal{X} + \int_M \text{Tr}(\rho \lrcorner \mathcal{P}), \quad (2.17)$$

on $V_0 \otimes V_3$, where ι_ξ denotes contraction with the vector field ξ . Then, the prescription of Sect. 2.2 yields the standard ECP action functional in three dimensions:

$$\begin{aligned} S(e, \omega) &= \frac{1}{2} \langle (e, \omega), \ell_1(e, \omega) \rangle - \frac{1}{6} \langle (e, \omega), \ell_2((e, \omega), (e, \omega)) \rangle \\ &= \int_M \text{Tr} \left(e \lrcorner \left(d\omega + \frac{1}{2} [\omega, \omega]_{\mathfrak{so}(3)} \right) \right) \\ &= \int_M \text{Tr}(e \lrcorner R) = \int_M \epsilon_{abc} e^a \wedge R^{bc}. \end{aligned} \quad (2.18)$$

The first Noether identity from $d_{(e,\omega)} F_{(e,\omega)} = (0, 0)$, in $\Omega^1(M, \Omega^3(M))$, corresponds to local diffeomorphism invariance $\delta_{(\xi,0)} S(e, \omega) = 0$, while the second Noether identity, in $\Omega^3(M, \mathbb{R}^3)$, corresponds to the local gauge symmetry $\delta_{(0,\rho)} S(e, \omega) = 0$. This construction can be easily extended to incorporate a cosmological constant [30], and we will do so later on when we study the braided version of this theory.

In higher dimensions $d \geq 4$, the L_∞ -algebras which organize ECP gravity contain higher brackets and are no longer simply differential graded Lie algebras, owing to the higher degree polynomial nature of the theory in the coframes. We will consider the four-dimensional case in detail later on, where the theory is again classically equivalent to general relativity, but now with propagating degrees of freedom.

3 Drinfel'd twist deformation quantization

3.1 Drinfel'd twists on manifolds

We briefly summarize the basic theory of Drinfel'd twists [43], specialized from the outset to the Lie algebra of vector fields $\mathfrak{v} := \Gamma(TM)$ on a manifold M , which

⁵ Alternatively, we may work with fields of suitable asymptotic decay.

generate infinitesimal diffeomorphisms of M ; here we regard $\mathfrak{v}$ as a Lie algebra over $\mathbb{C}$ (or more precisely its complexification $\mathfrak{v} \otimes \mathbb{C}$). The main object of interest in this theory is the universal enveloping algebra $U\mathfrak{v}$ of $\mathfrak{v}$, which is the tensor algebra (over $\mathbb{C}$) of $\mathfrak{v}$, regarded as the free unital algebra generated by $\mathfrak{v}$, modulo the two-sided ideal generated by $\xi_1 \xi_2 - \xi_2 \xi_1 - [\xi_1, \xi_2]_{\mathfrak{v}}$ for all $\xi_1, \xi_2 \in \mathfrak{v}$.

The enveloping algebra $U\mathfrak{v}$ is naturally a cocommutative Hopf algebra with coproduct $\Delta : U\mathfrak{v} \rightarrow U\mathfrak{v} \otimes U\mathfrak{v}$, counit $\varepsilon : U\mathfrak{v} \rightarrow \mathbb{C}$ and antipode $S : U\mathfrak{v} \rightarrow U\mathfrak{v}$ defined on generators by

$$\begin{aligned}\Delta(\xi) &= \xi \otimes 1 + 1 \otimes \xi & \text{and} & & \Delta(1) &= 1 \otimes 1, \\ \varepsilon(\xi) &= 0 & \text{and} & & \varepsilon(1) &= 1, \\ S(\xi) &= -\xi & \text{and} & & S(1) &= 1,\end{aligned}$$

for all $\xi \in \mathfrak{v}$. The maps Δ and ε are extended as algebra homomorphisms, and S as an algebra antihomomorphism to all of $U\mathfrak{v}$. We adopt the standard Sweedler notation $\Delta(X) =: X_{(1)} \otimes X_{(2)}$ (with summations understood) to abbreviate the coproduct of $X \in U\mathfrak{v}$.

From the perspective of field theory, the Hopf algebra $U\mathfrak{v}$ is generally ‘too big’ for the purposes of deformation quantization, as the ensuing constructions would then only work for diffeomorphism-invariant field theories. This will actually be the case for the examples treated in this paper, but to handle other field theories with a smaller set of spacetime symmetries, one could simply replace $\mathfrak{v}$ with any Lie subalgebra and apply all of the following constructions to the corresponding enveloping Hopf subalgebra $\mathcal{H} \subseteq U\mathfrak{v}$. This can be interpreted as performing the deformation quantization along the symmetries of M which leave the field theory on M invariant, see for instance [22,25,33,55] for examples.

To treat the typical examples of twists which arise in physics, we shall need to introduce formal power series extensions in a deformation parameter $\hbar$. If V is a complex vector space, we denote by $V[[\hbar]]$ the vector space of formal power series in $\hbar$ with coefficients in V ; it is naturally a module over $\mathbb{C}[[\hbar]]$. If V and W are complex vector spaces, then $V[[\hbar]] \otimes W[[\hbar]] \cong (V \otimes W)[[\hbar]]$, where on the left-hand side we use the appropriate topological tensor product (see e.g. [16] for further details), which for simplicity we do not distinguish from the usual tensor product of vector spaces over $\mathbb{C}$. With these conventions, we denote by $U\mathfrak{v}[[\hbar]]$ the formal power series extension of the cocommutative Hopf algebra $U\mathfrak{v}$, with operations applied term by term to the coefficients of series.

A *Drinfel’d twist* is a normalized 2-cocycle of the Hopf algebra $U\mathfrak{v}[[\hbar]]$. By this we mean an invertible element $\mathcal{F} \in U\mathfrak{v}[[\hbar]] \otimes U\mathfrak{v}[[\hbar]]$ satisfying the cocycle condition

$$\mathcal{F}_{12} (\Delta \otimes \text{id}) \mathcal{F} = \mathcal{F}_{23} (\text{id} \otimes \Delta) \mathcal{F},$$

where $\mathcal{F}_{12} = \mathcal{F} \otimes 1$ and $\mathcal{F}_{23} = 1 \otimes \mathcal{F}$, together with the normalization condition

$$(\varepsilon \otimes \text{id}) \mathcal{F} = 1 = (\text{id} \otimes \varepsilon) \mathcal{F}.$$

We write the power series expansion of the twist as $\mathcal{F} =: f^k \otimes f_k \in U\mathfrak{v}[[\hbar]] \otimes U\mathfrak{v}[[\hbar]]$, with the sum over k understood. Then the cocycle condition may be written in Sweedler notation as

$$f^k f_{(1)}^l \otimes f_k f_{(2)}^l \otimes f_l = f^l \otimes f^k f_{l(1)} \otimes f_k f_{l(2)}, \quad (3.1)$$

and the normalization condition as

$$\varepsilon(f^k) f_k = 1 = f^k \varepsilon(f_k).$$

As a consequence, the inverse twist $\mathcal{F}^{-1} =: \bar{f}^k \otimes \bar{f}_k \in U\mathfrak{v}[[\hbar]] \otimes U\mathfrak{v}[[\hbar]]$ satisfies similar conditions.

A Drinfel'd twist $\mathcal{F}$ defines a new Hopf algebra structure on the universal enveloping algebra $U\mathfrak{v}[[\hbar]]$, which we denote by $U_{\mathcal{F}}\mathfrak{v}$. As algebras, $U_{\mathcal{F}}\mathfrak{v} = U\mathfrak{v}[[\hbar]]$ and also the counit of $U_{\mathcal{F}}\mathfrak{v}$ is the same as the counit ε of $U\mathfrak{v}[[\hbar]]$. The new coproduct $\Delta_{\mathcal{F}}$ and antipode $S_{\mathcal{F}}$ of $U_{\mathcal{F}}\mathfrak{v}$ are given by

$$\Delta_{\mathcal{F}}(X) := \mathcal{F} \Delta(X) \mathcal{F}^{-1} \quad \text{and} \quad S_{\mathcal{F}}(X) := f^k S(f_k) S(X) S(\bar{f}^l) \bar{f}_l,$$

for all $X \in U\mathfrak{v}[[\hbar]]$. For $X \in U_{\mathcal{F}}\mathfrak{v}$, we adopt the Sweedler notation $\Delta_{\mathcal{F}}(X) =: X_{(\bar{1})} \otimes X_{(\bar{2})}$ to distinguish the twisted and untwisted coproducts.

This new Hopf algebra is not cocommutative in general, i.e. $\Delta_{\mathcal{F}} \neq \Delta_{\mathcal{F}}^{\text{op}} := \tau \circ \Delta_{\mathcal{F}}$, where τ is the transposition which interchanges the factors in a tensor product. However, the cocommutativity is controlled up to a *braiding* given by the invertible $\mathcal{R}$ -matrix $\mathcal{R} \in U\mathfrak{v}[[\hbar]] \otimes U\mathfrak{v}[[\hbar]]$ induced by the twist as

$$\mathcal{R} = \mathcal{F}_{21} \mathcal{F}^{-1} =: R^k \otimes R_k, \quad (3.2)$$

where $\mathcal{F}_{21} = \tau(\mathcal{F}) = f_k \otimes f^k$ is the twist with its legs swapped. Explicitly

$$\Delta_{\mathcal{F}}^{\text{op}}(X) = \mathcal{R} \Delta_{\mathcal{F}}(X) \mathcal{R}^{-1}.$$

It is easy to see that the $\mathcal{R}$ -matrix is triangular, that is

$$\mathcal{R}_{21} = \mathcal{R}^{-1} = R_k \otimes R^k,$$

and moreover that

$$(\Delta_{\mathcal{F}} \otimes \text{id})\mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23} \quad \text{and} \quad (\text{id} \otimes \Delta_{\mathcal{F}})\mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12},$$

where $\mathcal{R}_{13} = R^k \otimes 1 \otimes R_k$, or in Sweedler notation

$$R_{(\bar{1})}^k \otimes R_{(\bar{2})}^k \otimes R_k = R^l \otimes R^k \otimes R_l R_k \quad \text{and} \quad R^k \otimes R_{k(\bar{1})} \otimes R_{k(\bar{2})} = R^l R^k \otimes R_k \otimes R_l. \quad (3.3)$$

The $\mathcal{R}$ -matrix also satisfies the Yang–Baxter equation

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}. \quad (3.4)$$

Example 3.5 The standard example on $M = \mathbb{R}^d$ is the abelian Hermitian Moyal–Weyl twist

$$\mathcal{F}_\theta = \exp \left(-\frac{i\hbar}{2} \theta^{\mu\nu} \partial_\mu \otimes \partial_\nu \right) =: f_\theta^k \otimes f_{\theta k}, \quad (3.6)$$

where $(\theta^{\mu\nu})$ is a $d \times d$ antisymmetric real-valued matrix. This twist is based on the enveloping Hopf algebra $\mathcal{H}$ of the abelian Lie algebra of infinitesimal translations. In this case the twisted Hopf algebra is cocommutative (in fact $\Delta_{\mathcal{F}_\theta} = \Delta$), and the $\mathcal{R}$ -matrix is given by

$$\mathcal{R}_\theta = \mathcal{F}_\theta^{-2} = \exp \left(i\hbar \theta^{\mu\nu} \partial_\mu \otimes \partial_\nu \right).$$

However, our considerations in the following apply to a more general class of Drinfel’d twists that we shall specify more precisely later on.

3.2 Modules

Drinfel’d twist deformation quantization consists in twisting the enveloping Hopf algebra $U\mathfrak{v}$ to a non-cocommutative Hopf algebra $U_{\mathcal{F}}\mathfrak{v}$, while simultaneously twisting all of its modules [13]. There is a symmetric monoidal category $\mathcal{M}$ whose objects are (left) $U\mathfrak{v}$ -modules and whose morphisms are equivariant maps (see e.g. [16]). Since $U\mathfrak{v}$ is a cocommutative Hopf algebra (equivalently it has a triangular structure with trivial $\mathcal{R}$ -matrix $1 \otimes 1$), the braiding isomorphism of $\mathcal{M}$ is just the trivial transposition τ .

A $U\mathfrak{v}$ -module algebra is an algebra in the category $\mathcal{M}$. By this we mean an algebra $(\mathcal{A}, \mu)$ with a $U\mathfrak{v}$ -action $\triangleright : U\mathfrak{v} \otimes \mathcal{A} \rightarrow \mathcal{A}$ which is compatible with the algebra multiplication via the coproduct Δ , that is

$$X \triangleright \mu(a \otimes b) = \mu(\Delta(X) \triangleright (a \otimes b))$$

for all $X \in U\mathfrak{v}$ and $a, b \in \mathcal{A}$, where $\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is the product on $\mathcal{A}$. We will usually drop the symbol $\triangleright$ to simplify the notation. This condition means in particular that vector fields $\xi \in \mathfrak{v}$ act on $(\mathcal{A}, \mu)$ as derivations:

$$\xi(\mu(a \otimes b)) = \mu(\xi(a) \otimes b) + \mu(a \otimes \xi(b)).$$

A Drinfel’d twist $\mathcal{F}$ defines a functorially equivalent symmetric monoidal category ${}_{\mathcal{F}}\mathcal{M}$ of left $U_{\mathcal{F}}\mathfrak{v}$ -modules [16]. The braiding isomorphism of ${}_{\mathcal{F}}\mathcal{M}$ is now non-trivial and given by composing the transposition τ with the action of the inverse of the $\mathcal{R}$ -matrix. Since $\mathcal{R}$ is triangular, $\mathcal{R}_{21} = \mathcal{R}^{-1}$, the braiding is symmetric, i.e. the braiding

isomorphism squares to the identity morphism. The action of the inverse $\mathcal{R}$ -matrix on arbitrary tensor products of $U\mathcal{F}\mathfrak{v}$ -modules $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$ can be computed using identities (3.3) for $\mathcal{R}_{21} = \mathcal{R}^{-1}$ and their iterations to get

$$\begin{aligned} R_k(v_1 \otimes \cdots \otimes v_{n-1}) \otimes R^k(v_n) &= R_{k_1}(v_1) \otimes \cdots \otimes R_{k_{n-1}}(v_{n-1}) \otimes R^{k_{n-1}} \cdots R^{k_1}(v_n), \\ R_k(v_1) \otimes R^k(v_2 \otimes \cdots \otimes v_n) &= R_{k_1} \cdots R_{k_{n-1}}(v_1) \otimes R^{k_1}(v_2) \otimes \cdots \otimes R^{k_{n-1}}(v_n), \end{aligned} \quad (3.7)$$

for $n \geq 2$ and $v_i \in \mathcal{V}_i$. Intuitively, identities (3.7) tell us that passing an element ‘at once’ over many elements v_i is the same as passing successively over ‘each one’ individually.

If $(\mathcal{A}, \mu)$ is a (left) $U\mathfrak{v}$ -module algebra, then we can deform the product μ on $\mathcal{A}$ by precomposing it with the inverse of the twist $\mathcal{F}$ to get a new product

$$\mu_\star(a \otimes b) = \mu \circ \mathcal{F}^{-1}(a \otimes b) = \mu(\bar{f}^k(a) \otimes \bar{f}_k(b)), \quad (3.8)$$

for $a, b \in \mathcal{A}$, where on the right-hand side we extend μ to $\mathcal{A}[[\hbar]] \otimes \mathcal{A}[[\hbar]] \cong (\mathcal{A} \otimes \mathcal{A})[[\hbar]]$ by applying it term by term to the coefficients of a formal power series. The cocycle condition on $\mathcal{F}$ guarantees that this produces an associative star-product $\mu_\star$ on $\mathcal{A}[[\hbar]]$, and it generally defines a noncommutative $U\mathcal{F}\mathfrak{v}$ -module algebra $(\mathcal{A}[[\hbar]], \mu_\star)$, that is, an algebra in the category of $U\mathcal{F}\mathfrak{v}$ -modules:

$$X(\mu_\star(a \otimes b)) = \mu_\star(\Delta_{\mathcal{F}}(X)(a \otimes b)),$$

for all $X \in U\mathfrak{v}$ and $a, b \in \mathcal{A}$. In the following we denote $(\mathcal{A}, \mu)$ by $\mathcal{A}$ and $(\mathcal{A}[[\hbar]], \mu_\star)$ by $\mathcal{A}_\star$ for brevity.

If the algebra $\mathcal{A}$ is commutative, then $\mathcal{A}_\star$ is braided commutative: the noncommutativity of $\mathcal{A}_\star$ is controlled by the $\mathcal{R}$ -matrix as

$$\mu_\star(a \otimes b) = \mu_\star(R_k(b) \otimes R^k(a)),$$

which is easily proven by recalling that $\mathcal{R} = \mathcal{F}_{21} \mathcal{F}^{-1}$.

Example 3.9 We apply this construction to Example 3.5. Let $\mathcal{A}$ be the commutative algebra of smooth complex-valued functions on $\mathbb{R}^d$, with the usual pointwise multiplication $\mu(f \otimes g) = f g$. Then we obtain the noncommutative Moyal–Weyl star-product

$$\begin{aligned} f \star_\theta g &= \mu \circ \mathcal{F}_\theta^{-1}(f \otimes g) \\ &= \bar{f}_\theta^k(f) \bar{f}_{\theta k}(g) \\ &= f g + \sum_{k=1}^{\infty} \frac{(i \hbar)^k}{2^k k!} \theta^{\mu_1 \nu_1} \cdots \theta^{\mu_k \nu_k} \partial_{\mu_1} \cdots \partial_{\mu_k} f \partial_{\nu_1} \cdots \partial_{\nu_k} g \end{aligned}$$

of smooth functions f and g on $\mathbb{R}^d$.

The tensor algebra of differential forms and vector fields on the manifold M is covariant under the action of the universal enveloping algebra of infinitesimal diffeomorphisms $U\mathfrak{v}$. We can construct a noncommutative differential geometry on M by requiring it to be covariant with respect to the twisted Hopf algebra $U_{\mathcal{F}}\mathfrak{v}$ [11, 13]. We illustrate this below by considering the deformation quantization of the exterior algebra of differential forms and the (Lie and universal enveloping) algebra of vector fields. Drinfel'd twist deformation of differential and Cartan calculus on a manifold M has been studied in [3, 11, 12, 54].

3.3 Differential forms

Let $\mathcal{A} = (\Omega^\bullet(M), \wedge)$ be the exterior algebra of differential forms on M (or more precisely its complexification). This is a module over the Lie algebra of vector fields $\mathfrak{v}$, where the (left) action is given by the Lie derivative:

$$\xi(\alpha) := L_\xi \alpha$$

for $\xi \in \mathfrak{v}$ and $\alpha \in \Omega^\bullet(M)$. This action is extended to a $U\mathfrak{v}$ -action via successive applications of the Lie derivative, viewed as higher order differential operators:

$$\xi_1 \cdots \xi_n(\alpha) = (L_{\xi_1} \circ \cdots \circ L_{\xi_n})\alpha$$

for $\xi_1, \dots, \xi_n \in \mathfrak{v}$. This makes $(\Omega^\bullet(M), \wedge)$ into a $U\mathfrak{v}$ -module algebra, since

$$\xi(\alpha \wedge \beta) = L_\xi(\alpha \wedge \beta) = L_\xi \alpha \wedge \beta + \alpha \wedge L_\xi \beta = \wedge \circ \Delta(\xi)(\alpha \otimes \beta)$$

for all $\xi \in \mathfrak{v}$ and $\alpha, \beta \in \Omega^\bullet(M)$.

The twisted exterior algebra $\mathcal{A}_\star = (\Omega^\bullet(M)[[\hbar]], \wedge_\star)$ is a $U_{\mathcal{F}}\mathfrak{v}$ -module algebra. Following prescription (3.8), for $\alpha, \beta \in \Omega^\bullet(M)$ we set

$$\alpha \wedge_\star \beta := \bar{f}^k(\alpha) \wedge \bar{f}_k(\beta),$$

with (graded) braided commutativity controlled by the $\mathcal{R}$ -matrix as

$$\alpha \wedge_\star \beta = (-1)^{|\alpha||\beta|} R_k(\beta) \wedge_\star R^k(\alpha).$$

The standard (undeformed) exterior derivative $d : \Omega^\bullet(M) \rightarrow \Omega^\bullet(M)$ is then an ordinary (not braided!) graded derivation of the twisted exterior algebra $\mathcal{A}_\star$:

$$d(\alpha \wedge_\star \beta) = d\alpha \wedge_\star \beta + (-1)^{|\alpha|} \alpha \wedge_\star d\beta.$$

This property is fulfilled because the usual exterior derivative commutes with the Lie derivatives that enter in the definition of the star-exterior product. It is also still a differential on $\mathcal{A}_\star$, that is, $d^2 = 0$. This twists the de Rham complex $(\Omega^\bullet(M), d)$ of the manifold M to a noncommutative differential graded algebra $(\mathcal{A}_\star, d)$.

In our applications to field theory, we shall be interested in a special class of Drinfel'd twists. If M is a closed oriented manifold of dimension d , we will require that the usual integral over M is graded cyclic under the deformed exterior product $\wedge_\star$:

$$\int_M \alpha \wedge_\star \beta = (-1)^{|\alpha||\beta|} \int_M \beta \wedge_\star \alpha, \quad (3.10)$$

where $|\alpha| + |\beta| = d$. This is guaranteed if the twist $\mathcal{F}$ satisfies the condition $S(\bar{f}^k) \bar{f}_k = 1$, where S is the antipode of $U\mathfrak{v}$ [4]; for example, this holds for abelian twists⁶, whereby $\mathcal{F}_{21} = \mathcal{F}^{-1}$. We shall further require that the twist $\mathcal{F}$ is Hermitian, that is, it defines a Hermitian star-product $\wedge_\star$:

$$(\alpha \wedge_\star \beta)^* = (-1)^{|\alpha||\beta|} \beta^* \wedge_\star \alpha^*,$$

where $*$ denotes complex conjugation. This is guaranteed if $\mathcal{F}$ satisfies the reality condition given by $\bar{f}^{k*} \otimes \bar{f}_k^* = S(\bar{f}_k) \otimes S(\bar{f}^k)$.

All of this has an extension to the graded Lie algebra $\mathcal{A} = (\Omega^\bullet(M, \mathfrak{g}), [-, -]_\mathfrak{g})$ of differential forms valued in a Lie algebra $\mathfrak{g}$, where $[-, -]_\mathfrak{g}$ denotes the tensor product of exterior multiplication of forms with the Lie bracket of $\mathfrak{g}$. With the trivial $U\mathfrak{v}$ -action on $\mathfrak{g}$, we use (3.8) to define

$$[\alpha, \beta]_\mathfrak{g}^\star := [\bar{f}^k(\alpha), \bar{f}_k(\beta)]_\mathfrak{g},$$

for $\alpha, \beta \in \Omega^\bullet(M, \mathfrak{g})$. This makes $\mathcal{A}_\star = (\Omega^\bullet(M, \mathfrak{g}), [[\hbar]], [-, -]_\mathfrak{g}^\star)$ into a (graded) *braided Lie algebra*, that is, a graded Lie algebra in the category $\mathcal{FM}$ [17]. The new bracket is now (graded) braided antisymmetric:

$$[\alpha_1, \alpha_2]_\mathfrak{g}^\star = -(-1)^{|\alpha_1||\alpha_2|} [R_k(\alpha_2), R^k(\alpha_1)]_\mathfrak{g}^\star,$$

and satisfies the (graded) braided Jacobi identity:

$$[\alpha_1, [\alpha_2, \alpha_3]_\mathfrak{g}^\star]_\mathfrak{g}^\star = [[\alpha_1, \alpha_2]_\mathfrak{g}^\star, \alpha_3]_\mathfrak{g}^\star + (-1)^{|\alpha_1||\alpha_2|} [R_k(\alpha_2), [R^k(\alpha_1), \alpha_3]_\mathfrak{g}^\star]_\mathfrak{g}^\star, \quad (3.11)$$

for all $\alpha_1, \alpha_2, \alpha_3 \in \Omega^\bullet(M, \mathfrak{g})$; this will follow as a special example of Proposition 4.8. Graded cyclicity can then be formulated as above by precomposing the integral over M with an invariant quadratic form $\text{Tr}_\mathfrak{g}$ on the Lie algebra $\mathfrak{g}$.

3.4 Diffeomorphisms

Since we are also interested in diffeomorphism symmetry in our applications to gravity, we have to consider the Drinfel'd twist deformation of the algebra of vector fields

⁶ That is, a Drinfel'd twist generated by an abelian subalgebra of $\mathfrak{v}$, such as the Moyal–Weyl twist of Example 3.5.

$\mathfrak{v} = \Gamma(TM)$. Deforming the Lie bracket following the general prescription (3.8), we obtain the star-bracket of two vector fields $\xi_1, \xi_2 \in \Gamma(TM)$ as

$$[\xi_1, \xi_2]_{\mathfrak{v}}^* = [\bar{f}^k(\xi_1), \bar{f}_k(\xi_2)]_{\mathfrak{v}} = \xi_1 \star \xi_2 - R_k(\xi_2) \star R^k(\xi_1),$$

where the second equality takes place in the twisting of the universal enveloping algebra of vector fields $U\mathfrak{v}$. The star-bracket makes $(\Gamma(TM)[[\hbar]], [-, -]_{\mathfrak{v}}^*)$ into the braided Lie algebra of vector fields. The star-bracket is braided antisymmetric:

$$[\xi_1, \xi_2]_{\mathfrak{v}}^* = -[R_k(\xi_2), R^k(\xi_1)]_{\mathfrak{v}}^*,$$

and it also fulfils the braided Jacobi identity

$$[\xi_1, [\xi_2, \xi_3]_{\mathfrak{v}}^*]_{\mathfrak{v}}^* = [[\xi_1, \xi_2]_{\mathfrak{v}}^*, \xi_3]_{\mathfrak{v}}^* + [R_k(\xi_2), [R^k(\xi_1), \xi_3]_{\mathfrak{v}}^*]_{\mathfrak{v}}^*,$$

for all $\xi_1, \xi_2, \xi_3 \in \Gamma(TM)$.

Braided diffeomorphisms act via the star-Lie derivative, which is defined following (3.8) as

$$L_{\xi}^* T = L_{\bar{f}^k(\xi)} \bar{f}_k(T),$$

for any vector field ξ and any tensor field T on M . It provides a representation of the braided Lie algebra of vector fields on differential forms and tensor fields, since it satisfies

$$[L_{\xi_1}^*, L_{\xi_2}^*]_{\circ}^* := L_{\xi_1}^* \circ L_{\xi_2}^* - L_{R_k(\xi_2)}^* \circ L_{R^k(\xi_1)}^* = L_{[\xi_1, \xi_2]_{\mathfrak{v}}^*}^*.$$

It is also a *braided derivation*, that is

$$L_{\xi}^*(T_1 \otimes_{\star} T_2) = L_{\xi}^* T_1 \otimes_{\star} T_2 + R_k(T_1) \otimes_{\star} L_{R^k(\xi)}^* T_2,$$

for any vector field ξ and any two tensors T_1, T_2 on M , where $T_1 \otimes_{\star} T_2 := \bar{f}^k(T_1) \otimes_{C^\infty(M)} \bar{f}_k(T_2)$. These properties of the star-Lie derivative will be generalized to more general braided Lie algebra actions in Sect. 5.

4 Braided L_∞ -algebras

4.1 L_∞ -algebras in braided representation categories

Let $(V, \{\ell_n\})$ be an L_∞ -algebra as defined in Sect. 2.1. The key to defining our notion of a braided L_∞ -algebra consists of reinterpreting all the defining relations in terms of morphisms in a suitable category, before acting on elements of V . This reformulation also has the advantage of absorbing the cumbersome sign factors which arise from gradings in the homotopy relations of an L_∞ -algebra.

To illustrate the idea, let us begin by considering the simplest case where only the 2-bracket ℓ_2 is non-vanishing. Let $\mathcal{V}$ be the symmetric monoidal category of vector spaces (sitting in degree 0), whose braiding isomorphism is given by transposition τ which exchanges factors of a tensor product of two vector spaces. Then, symmetric and antisymmetric morphisms in $\mathcal{V}$ are defined by acting with the symmetric group S_n of degree n via the braiding τ . As an object in $\mathcal{V}$, the vector space V carries the structure of a Lie algebra: the morphism $\ell_2 : V \otimes V \rightarrow V$ is antisymmetric:

$$\ell_2 = -\ell_2 \circ \sigma_{(12)} = -\ell_2 \circ \tau_{V,V},$$

where $\sigma_{(12)} \in S_2$ is the non-identity permutation of degree 2; on elements this reads

$$\ell_2(v_1 \otimes v_2) = -\ell_2(\tau_{V,V}(v_1 \otimes v_2)) = -\ell_2(v_2 \otimes v_1).$$

The Jacobi identity for ℓ_2 in this framework is $\mathcal{J}_3 = 0$, where the Jacobiator is the morphism $\mathcal{J}_3 : V \otimes V \otimes V \rightarrow V$ in $\mathcal{V}$ defined via elements $\sigma \in S_3$ permuting the entries in $V \otimes V \otimes V$ by

$$\begin{aligned} \mathcal{J}_3 &:= \ell_2 \circ (\ell_2 \otimes \text{id}) \circ (\sigma_{\text{id}} + \sigma_{(132)} - \sigma_{(23)}) \\ &= \ell_2 \circ (\ell_2 \otimes \text{id}) \circ (\text{id}^{\otimes 3} + \tau_{V,V \otimes V} - \text{id} \otimes \tau_{V,V}). \end{aligned} \quad (4.1)$$

It is easy to check on elements of V that $\mathcal{J}_3(v_1, v_2, v_3) = 0$ is just the standard Jacobi identity for the 2-bracket ℓ_2 .

Let now $\mathcal{V}^\sharp$ be the symmetric monoidal category of $\mathbb{Z}$ -graded vector spaces, whose braiding isomorphism is given by transposition times the degree swap multiplication, which we denote by $\tau^\sharp$. Now symmetric and antisymmetric morphisms in $\mathcal{V}^\sharp$ are defined using the braiding $\tau^\sharp$ to act with the symmetric group S_n , and yield graded versions. As an object in $\mathcal{V}^\sharp$, the vector space V thus carries the structure of a graded Lie algebra: the morphism $\ell_2 : V \otimes V \rightarrow V$ of degree 0 is (graded) antisymmetric:

$$\ell_2 = -\ell_2 \circ \sigma_{(12)} = -\ell_2 \circ \tau_{V,V}^\sharp,$$

which on homogeneous elements reads

$$\ell_2(v_1 \otimes v_2) = -\ell_2(\tau_{V,V}^\sharp(v_1 \otimes v_2)) = -(-1)^{|v_1||v_2|} \ell_2(v_2 \otimes v_1).$$

The graded Jacobi identity for ℓ_2 in this language is analogous to the vanishing of the Jacobiator (4.1), with τ replaced by $\tau^\sharp$. We have already encountered a concrete example in Sect. 3.3: in the present context, $(\Omega^\bullet(M, \mathfrak{g}), [-, -]_{\mathfrak{g}})$ is a Lie algebra in the representation category $\mathcal{M}^\sharp$ of $\mathbb{Z}$ -graded $U\mathfrak{v}$ -modules, with $U\mathfrak{v}$ itself regarded as sitting in degree 0.

It should now be clear how to pass to the definition of a braided Lie algebra: one simply requires a permutation $\sigma \in S_n$ to act via a (graded) braided transposition in a braided monoidal category. This can be defined in any symmetric monoidal category, but for concreteness, and with any eye to the applications we are interested in later on,

we specialize immediately to the twisted representation category $\mathcal{F}\mathcal{M}^\sharp$ of $\mathbb{Z}$ -graded left $U\mathcal{F}\mathfrak{v}$ -modules. Similarly to Sect. 3.3, the braiding isomorphism in this category is determined by the $\mathcal{R}$ -matrix as ${}_{\mathcal{F}}\tau^\sharp := \mathcal{R}^{-1} \circ \tau^\sharp$. Then as an object in $\mathcal{F}\mathcal{M}^\sharp$, the vector space V carries the structure of a braided Lie algebra: the morphism $\ell_2 : V \otimes V \rightarrow V$ satisfies

$$\ell_2 = -\ell_2 \circ \sigma_{(12)} = -\ell_2 \circ {}_{\mathcal{F}}\tau_{V,V}^\sharp,$$

which on homogeneous elements reads

$$\ell_2(v_1 \otimes v_2) = -\ell_2({}_{\mathcal{F}}\tau_{V,V}^\sharp(v_1 \otimes v_2)) = -(-1)^{|v_1||v_2|} \ell_2(\mathbf{R}_k(v_2) \otimes \mathbf{R}^k(v_1)).$$

The braided Jacobi identity can be expressed as in (4.1) by replacing τ with ${}_{\mathcal{F}}\tau^\sharp$. In Sect. 3.3, we already encountered the concrete example of the Lie algebra $(\Omega^\bullet(M, \mathfrak{g})[[\hbar]], [-, -]_{\mathfrak{g}}^*)$ in the symmetric monoidal category $\mathcal{F}\mathcal{M}^\sharp$.

At this stage it is straightforward to pass from Lie algebras to L_∞ -algebras, allowing other brackets ℓ_n to be nonzero for $n \neq 2$. An L_∞ -algebra consists of an object V in the category $\mathcal{V}^\sharp$ together with morphisms $\ell_n : \bigotimes^n V \rightarrow V$ which are antisymmetric in $\mathcal{V}^\sharp$. The key point is that the homotopy relations (2.2) can now be expressed, without acting explicitly on elements of V , as the vanishing of morphisms $\mathcal{J}_n : \bigotimes^n V \rightarrow V$ in $\mathcal{V}^\sharp$ defined by

$$\mathcal{J}_n := \sum_{i=1}^n (-1)^{i(n-i)} \sum_{\sigma \in \text{Sh}_{i,n-i}} \text{sgn}(\sigma) \ell_{n+1-i} \circ (\ell_i \otimes \text{id}^{\otimes n-i}) \circ \sigma, \quad (4.2)$$

where the permutations σ act via the graded transposition $\tau^\sharp$ on $\bigotimes^n V$, and $\text{sgn}(\sigma)$ is the sign of σ . It is easy to check that on elements $v_1, \dots, v_n \in V$ this coincides with (2.2).

This leads us to the central mathematical concept of this paper.

Definition 4.3 A *braided L_∞ -algebra* is an L_∞ -algebra $(V, \{\ell_n\})$ in the symmetric monoidal category $\mathcal{F}\mathcal{M}^\sharp$.

This definition simply amounts to letting the symmetric group act in all relations of the L_∞ -algebra, written in terms of morphisms, through the braiding isomorphism ${}_{\mathcal{F}}\tau^\sharp$ of $\mathcal{F}\mathcal{M}^\sharp$; we write ${}_{\mathcal{F}}\sigma$ to distinguish the action of a permutation σ via the braided transposition from the usual one. Explicitly, as in the case of Lie algebras, this gives the braided antisymmetry of the morphisms $\ell_n : \bigotimes^n V \rightarrow V$ in $\mathcal{F}\mathcal{M}^\sharp$:

$$\ell_n(\dots, v, v', \dots) = -(-1)^{|v||v'|} \ell_n(\dots, \mathbf{R}_k(v'), \mathbf{R}^k(v), \dots), \quad (4.4)$$

together with *braided homotopy relations* $\mathcal{J}_n(v_1, \dots, v_n) = 0$ in $\mathcal{F}\mathcal{M}^\sharp$ which follow from (4.2). The first two relations $\mathcal{J}_1(v) = 0$ and $\mathcal{J}_2(v_1, v_2) = 0$ are unchanged from the classical case of Sect. 2.1:

$$\begin{aligned}\ell_1(\ell_1(v)) &= 0, \\ \ell_1(\ell_2(v_1, v_2)) &= \ell_2(\ell_1(v_1), v_2) + (-1)^{|v_1|} \ell_2(v_1, \ell_1(v_2)),\end{aligned}\quad (4.5)$$

hence the map $\ell_1 : V \rightarrow V$ is still a differential which is a graded (but not braided!) derivation of the 2-bracket $\ell_2 : V \otimes V \rightarrow V$. The first homotopy relation which differs from the classical case is $\mathcal{J}_3(v_1, v_2, v_3) = 0$, which is a deformation of the classical identity (2.3) in accordance with the non-trivial braiding (4.4):

$$\begin{aligned}& \ell_2(\ell_2(v_1, v_2), v_3) - (-1)^{|v_2||v_3|} \ell_2(\ell_2(v_1, R_k(v_3)), R^k(v_2)) \\ & + (-1)^{(|v_2|+|v_3|)|v_1|} \ell_2(\ell_2(R_k(v_2), R_l(v_3)), R^l R^k(v_1)) \\ & = -\ell_3(\ell_1(v_1), v_2, v_3) - (-1)^{|v_1|} \ell_3(v_1, \ell_1(v_2), v_3) \\ & - (-1)^{|v_1|+|v_2|} \ell_3(v_1, v_2, \ell_1(v_3)) \\ & - \ell_1(\ell_3(v_1, v_2, v_3)).\end{aligned}\quad (4.6)$$

4.2 Drinfel'd twist deformations of L_∞ -algebras

A large class of examples of braided L_∞ -algebras can be obtained from Drinfel'd twist deformation quantization, starting from suitable L_∞ -algebras in the classical sense. Let $(V, \{\ell_n\})$ be an L_∞ -algebra in the category $\mathcal{M}^\sharp$ of $U\mathfrak{v}$ -modules. Concretely, this means that $V = \bigoplus_{k \in \mathbb{Z}} V_k$ is a $\mathbb{Z}$ -graded (left) $U\mathfrak{v}$ -module and the n -brackets $\ell_n : \bigotimes^n V \rightarrow V$ are equivariant maps, that is, they all commute with the action of $\mathfrak{v} = \Gamma(TM)$ on V via the trivial coproduct Δ . Given any Drinfel'd twist $\mathcal{F} \in U\mathfrak{v}[[\hbar]] \otimes U\mathfrak{v}[[\hbar]]$, we can deform the brackets ℓ_n to twisted brackets ℓ_n^* which are morphisms in the twisted representation category ${}_{\mathcal{F}}\mathcal{M}^\sharp$ of $\mathbb{Z}$ -graded $U_{\mathcal{F}}\mathfrak{v}$ -modules, that is, they commute with the action of $\Gamma(TM)$ on $V[[\hbar]]$ via the twisted coproduct $\Delta_{\mathcal{F}}$. Following the standard prescription (3.8), we set $\ell_1^* := \ell_1$ and

$$\ell_n^*(v_1 \otimes \cdots \otimes v_n) := \ell_n(v_1 \otimes_\star \cdots \otimes_\star v_n) \quad (4.7)$$

for $n \geq 2$, where $v \otimes_\star v' := \mathcal{F}^{-1}(v \otimes v') = \bar{f}^k(v) \otimes \bar{f}_k(v')$ for $v, v' \in V$.

Proposition 4.8 *If $(V, \{\ell_n\})$ is an L_∞ -algebra in the category $\mathcal{M}^\sharp$, then $(V[[\hbar]], \{\ell_n^*\})$ is a braided L_∞ -algebra.*

Proof Braided antisymmetry of the n -brackets ℓ_n^* follows by definition (4.7), and the only thing we need to check is that the braided homotopy relations for ℓ_n^* follow from the (undeformed) homotopy relations for ℓ_n . The proof follows straightforwardly from well-known abstract arguments, see e.g. [16, 17], and should be obvious to experts, but we spell out the details here for the benefit of the non-expert reader.

Drinfel'd twist deformation quantization produces a symmetric monoidal functor $\mathcal{F} : (\mathcal{M}^\sharp, \otimes) \rightarrow ({}_{\mathcal{F}}\mathcal{M}^\sharp, \otimes_{\mathcal{F}})$ which is an equivalence, where $\otimes_{\mathcal{F}}$ denotes the tensor product of $U_{\mathcal{F}}\mathfrak{v}$ -modules acted upon using the twisted coproduct $\Delta_{\mathcal{F}}$. The

functor $\mathcal{F}$ acts on objects and morphisms as the identity,⁷ with the coherence maps $\varphi : \mathcal{F}(V) \otimes_{\mathcal{F}} \mathcal{F}(W) \rightarrow \mathcal{F}(V \otimes W)$ given by

$$\varphi(v \otimes w) = \bar{f}^k(v) \otimes \bar{f}_k(w) = v \otimes_{\star} w,$$

where V, W are $U\mathfrak{v}$ -modules with $v \in V$ and $w \in W$ (the inverse is given by acting with the twist $\mathcal{F} = f^k \otimes f_k$). These fit into the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(V) \otimes_{\mathcal{F}} \mathcal{F}(W) & \xrightarrow{\mathcal{F}\tau^{\sharp}} & \mathcal{F}(W) \otimes_{\mathcal{F}} \mathcal{F}(V) \\ \varphi \downarrow & & \downarrow \varphi \\ \mathcal{F}(V \otimes W) & \xrightarrow{\mathcal{F}(\tau^{\sharp})} & \mathcal{F}(W \otimes V) \end{array} \quad (4.9)$$

in $\mathcal{F}\mathcal{M}^{\sharp}$. The coherence maps are iterated to arbitrary n -fold tensor products of $U\mathfrak{v}$ -modules V_i to get maps $\varphi_n : \mathcal{F}(V_1) \otimes_{\mathcal{F}} \cdots \otimes_{\mathcal{F}} \mathcal{F}(V_n) \rightarrow \mathcal{F}(V_1 \otimes \cdots \otimes V_n)$ for $n \geq 1$ defined by

$$\varphi_n(v_1 \otimes \cdots \otimes v_n) = v_1 \otimes_{\star} \cdots \otimes_{\star} v_n,$$

for $v_i \in V_i$, with $\varphi_2 = \varphi$ and $\varphi_1 = \text{id}$.

Morphisms (4.7) are thus defined by precomposing $\mathcal{F}(\ell_n)$ with the coherence maps φ_n . We will show that applying this to morphisms (4.2) in $\mathcal{M}^{\sharp}$ gives morphisms $\mathcal{J}_n^{\star} = \mathcal{F}(\mathcal{J}_n) \circ \varphi_n$ in $\mathcal{F}\mathcal{M}^{\sharp}$, where

$$\mathcal{J}_n^{\star} := \sum_{i=1}^n (-1)^i \binom{n-i}{i} \sum_{\sigma \in \text{Sh}_{i, n-i}} \text{sgn}(\sigma) \ell_{n+1-i}^{\star} \circ (\ell_i^{\star} \otimes \text{id}^{\otimes n-i}) \circ \mathcal{F}\sigma. \quad (4.10)$$

Since the functor $\mathcal{F}$ preserves the braidings by diagram (4.9), these are just the braided homotopy relations, i.e. $\mathcal{J}_n(v_1 \otimes \cdots \otimes v_n) = 0$ in $\mathcal{M}^{\sharp}$ implies

$$\mathcal{J}_n^{\star}(v_1 \otimes \cdots \otimes v_n) = \mathcal{J}_n(v_1 \otimes_{\star} \cdots \otimes_{\star} v_n) = 0$$

in $\mathcal{F}\mathcal{M}^{\sharp}$.

Let us now demonstrate that $\mathcal{J}_n^{\star} = \mathcal{J}_n \circ \varphi_n$ explicitly. Note that this is trivially satisfied for $n = 1$, as $\ell_1^{\star} = \ell_1$ and $\varphi_1 = \text{id}$. We will show that

$$\ell_{n+1-i}^{\star} \circ (\ell_i^{\star} \otimes \text{id}^{\otimes n-i}) \circ \mathcal{F}\sigma = \ell_{n+1-i} \circ (\ell_i \otimes \text{id}^{\otimes n-i}) \circ \sigma \circ \varphi_n$$

⁷ To be precise, we need to precompose $\mathcal{F}$ with the symmetric monoidal functor which sends an object V in $\mathcal{M}^{\sharp}$ to the object $V[[h]]$ in the symmetric monoidal category of (left) $U\mathfrak{v}[[h]]$ -modules over $\mathbb{C}[[h]]$, see [16] for details. We suppress this extra step here for clarity of exposition and because we will not need it in our applications later on.

for each $n \geq 2$, $i = 1, \dots, n$ and $\sigma \in \text{Sh}_{i, n-i}$. A simple expansion shows

$$\begin{aligned} \ell_{n+1-i}^* \circ (\ell_i^* \otimes \text{id}^{\otimes n-i}) &= \ell_{n+1-i} \circ \varphi_{n+1-i} \circ ((\ell_i \circ \varphi_i) \otimes \text{id}^{\otimes n-i}) \\ &= \ell_{n+1-i} \circ \varphi_{n+1-i} \circ (\ell_i \otimes \text{id}^{\otimes n-i}) \circ (\varphi_i \otimes \text{id}^{\otimes n-i}) \\ &= \ell_{n+1-i} \circ (\ell_i \otimes \text{id}^{\otimes n-i}) \circ \varphi^{i, n-i} \circ (\text{id}^{\otimes i} \otimes \varphi_{n-i}) \circ (\varphi_i \otimes \text{id}^{\otimes n-i}), \end{aligned}$$

where $\varphi^{i, n-i}(v_1 \otimes \dots \otimes v_n) := (v_1 \otimes \dots \otimes v_i) \otimes_\star (v_{i+1} \otimes \dots \otimes v_n)$, and the third equality follows from

$$\begin{aligned} \varphi_{n+1-i} \circ (\ell_i \otimes \text{id}^{\otimes n-i})(v_1 \otimes \dots \otimes v_n) &= \bar{f}^k(\ell_i(v_1 \otimes \dots \otimes v_i)) \otimes \bar{f}_k(v_{i+1} \otimes_\star \dots \otimes_\star v_n) \\ &= \ell_i(\bar{f}^k(v_1 \otimes \dots \otimes v_i)) \otimes \bar{f}_k(v_{i+1} \otimes_\star \dots \otimes_\star v_n) \\ &= (\ell_i \otimes \text{id}^{\otimes n-i}) \circ \varphi^{i, n-i} \circ (\text{id}^{\otimes i} \otimes \varphi_{n-i})(v_1 \otimes \dots \otimes v_n), \end{aligned}$$

where ℓ_i commutes with $\bar{f}^k$ since it is a morphism in $\mathcal{M}^\sharp$. Since $v_1 \otimes_\star \dots \otimes_\star v_n = (v_1 \otimes_\star \dots \otimes_\star v_i) \otimes_\star (v_{i+1} \otimes_\star \dots \otimes_\star v_n)$, the map φ_n may be equivalently written as

$$\varphi_n = \varphi^{i, n-i} \circ (\text{id}^{\otimes i} \otimes \varphi_{n-i}) \circ (\varphi_i \otimes \text{id}^{\otimes n-i}),$$

and hence

$$\ell_{n+1-i}^* \circ (\ell_i^* \otimes \text{id}^{\otimes n-i}) = \ell_{n+1-i} \circ (\ell_i \otimes \text{id}^{\otimes n-i}) \circ \varphi_n.$$

Next we show that φ_n commutes with any transposition via the braiding isomorphism $\mathcal{F}\tau^\sharp$. That is, for any $n \geq 2$ and $j = 0, 1, \dots, n-2$ one has

$$\varphi_n \circ (\text{id}^{\otimes j} \otimes \mathcal{F}\tau^\sharp \otimes \text{id}^{\otimes n-j-2}) = (\text{id}^{\otimes j} \otimes \tau^\sharp \otimes \text{id}^{\otimes n-j-2}) \circ \varphi_n.$$

Since $v_1 \otimes_\star \dots \otimes_\star v_n = (v_1 \otimes_\star \dots \otimes_\star v_j) \otimes_\star ((v_{j+1} \otimes_\star v_{j+2}) \otimes_\star (v_{j+3} \otimes_\star \dots \otimes_\star v_n))$, the map φ_n may be also written as

$$\varphi_n = \varphi^{j, n-j} \circ (\text{id}^{\otimes j} \otimes \varphi^{2, n-j-2}) \circ (\varphi_j \otimes \text{id}^{\otimes 2} \otimes \varphi_{n-j-2}) \circ (\text{id}^{\otimes j} \otimes \varphi \otimes \text{id}^{\otimes n-j-2}).$$

Since $\mathcal{F}\tau^\sharp = \varphi^{-1} \circ \tau^\sharp \circ \varphi$ by (4.9), we have

$$\begin{aligned} \varphi_n \circ (\text{id}^{\otimes j} \otimes \mathcal{F}\tau^\sharp \otimes \text{id}^{\otimes n-j-2}) &= \varphi^{j, n-j} \circ (\text{id}^{\otimes j} \otimes \varphi^{2, n-j-2}) \circ (\varphi_j \otimes \text{id}^{\otimes 2} \otimes \varphi_{n-j-2}) \\ &\quad \circ (\text{id}^{\otimes j} \otimes \tau^\sharp \otimes \text{id}^{\otimes n-j-2}) \circ (\text{id}^{\otimes j} \otimes \varphi \otimes \text{id}^{\otimes n-j-2}) \\ &= \varphi^{j, n-j} \circ (\text{id}^{\otimes j} \otimes \varphi^{2, n-j-2}) \circ (\text{id}^{\otimes j} \otimes \tau^\sharp \otimes \text{id}^{\otimes n-j-2}) \\ &\quad \circ (\varphi_j \otimes \text{id}^{\otimes 2} \otimes \varphi_{n-j-2}) \circ (\text{id}^{\otimes j} \otimes \varphi \otimes \text{id}^{\otimes n-j-2}). \end{aligned}$$

The map involving the transposition commutes with the remaining maps since

$$\begin{aligned}
 & (\mathrm{id}^{\otimes j} \otimes \varphi^{2,n-j-2}) \circ (\mathrm{id}^{\otimes j} \otimes \tau^{\sharp} \otimes \mathrm{id}^{\otimes n-j-2})(v_1 \otimes \cdots \otimes v_n) \\
 &= v_1 \otimes \cdots \otimes v_j \otimes (\Delta(\bar{f}^k) \circ \tau^{\sharp})(v_{j+1} \otimes v_{j+2}) \otimes \bar{f}_k(v_{j+3} \otimes \cdots \otimes v_n) \\
 &= v_1 \otimes \cdots \otimes v_j \otimes (\tau^{\sharp} \circ \Delta(\bar{f}^k))(v_{j+1} \otimes v_{j+2}) \otimes \bar{f}_k(v_{j+3} \otimes \cdots \otimes v_n) \\
 &= (\mathrm{id}^{\otimes j} \otimes \tau^{\sharp} \otimes \mathrm{id}^{\otimes n-j-2}) \circ (\mathrm{id}^{\otimes j} \otimes \varphi^{2,n-j-2})(v_1 \otimes \cdots \otimes v_n),
 \end{aligned}$$

where the second equality follows since the trivial coproduct Δ is cocommutative, $\Delta^{\mathrm{op}} = \Delta$, and so $\Delta \circ \tau^{\sharp} = \tau^{\sharp} \circ \Delta^{\mathrm{op}} = \tau^{\sharp} \circ \Delta$. Similarly one has

$$\begin{aligned}
 & \varphi^{j,n-j} \circ (\mathrm{id}^{\otimes j} \otimes \tau^{\sharp} \otimes \mathrm{id}^{\otimes n-j-2})(v_1 \otimes \cdots \otimes v_n) \\
 &= \bar{f}^k(v_1 \otimes \cdots \otimes v_j) \otimes \bar{f}_k(\tau^{\sharp}(v_{j+1} \otimes v_{j+2}) \otimes v_{j+3} \otimes \cdots \otimes v_n) \\
 &= \bar{f}^k(v_1 \otimes \cdots \otimes v_j) \otimes \bar{f}_{k(1)}(\tau^{\sharp}(v_{j+1} \otimes v_{j+2})) \otimes \bar{f}_{k(2)}(v_{j+3} \otimes \cdots \otimes v_n) \\
 &= \bar{f}^k(v_1 \otimes \cdots \otimes v_j) \otimes (\Delta(\bar{f}_{k(1)}) \circ \tau^{\sharp})(v_{j+1} \otimes v_{j+2}) \otimes \bar{f}_{k(2)}(v_{j+3} \otimes \cdots \otimes v_n) \\
 &= \bar{f}^k(v_1 \otimes \cdots \otimes v_j) \otimes (\tau^{\sharp} \circ \Delta(\bar{f}_{k(1)}))(v_{j+1} \otimes v_{j+2}) \otimes \bar{f}_{k(2)}(v_{j+3} \otimes \cdots \otimes v_n) \\
 &= (\mathrm{id}^{\otimes j} \otimes \tau^{\sharp} \otimes \mathrm{id}^{\otimes n-j-2}) \circ \varphi^{j,n-j}(v_1 \otimes \cdots \otimes v_n).
 \end{aligned}$$

Collecting everything together, we have shown that

$$\begin{aligned}
 & \ell_{n+1-i}^* \circ (\ell_i^* \otimes \mathrm{id}^{\otimes n-i}) \circ (\mathrm{id}^{\otimes j} \otimes \mathcal{F}\tau^{\sharp} \otimes \mathrm{id}^{\otimes n-j-2}) \\
 &= \ell_{n+1-i} \circ (\ell_i \otimes \mathrm{id}^{\otimes n-i}) \circ (\mathrm{id}^{\otimes j} \otimes \tau^{\sharp} \otimes \mathrm{id}^{\otimes n-j-2}) \circ \varphi_n
 \end{aligned}$$

for all $n \geq 2$, $i = 1, \dots, n$ and $j = 0, 1, \dots, n-2$. Since any permutation can be written as a composition of transpositions, the result follows immediately. $\square$

Example 4.11 If $\ell_n = 0$ for all $n \geq 3$, then Proposition 4.8 is just the statement that a differential graded Lie algebra twist quantizes to a differential (graded) braided Lie algebra. This is precisely what we saw in Sect. 3.3 for the special case of exterior differential forms valued in a Lie algebra.

4.3 Cyclic structures

So far we have not discussed cyclic structures on our braided L_{∞} -infinity algebras, which as we saw in Sect. 2.2 is a crucial ingredient for a classical field theory to have a Lagrangian formulation in the L_{∞} -algebra framework. Let $(V, \{\ell_n\}, \langle -, - \rangle)$ be a cyclic L_{∞} -algebra in the category $\mathcal{M}^{\sharp}$ of $\mathbb{Z}$ -graded $U\mathfrak{v}$ -modules. This means that the cyclic pairing $\langle -, - \rangle : V \otimes V \rightarrow \mathbb{R}$ has to additionally be $U\mathfrak{v}$ -invariant:

$$\langle X_{(1)}(v_1), X_{(2)}(v_2) \rangle = 0 \quad (4.12)$$

for all $X \in U\mathfrak{v}$ and $v_1, v_2 \in V$; for vector fields $\xi \in \mathfrak{v} = \Gamma(TM)$ this reads $\langle \xi(v_1), v_2 \rangle = -\langle v_1, \xi(v_2) \rangle$, which is the natural requirement of diffeomorphism invariance of the pairing.

Following our standard prescription (3.8), we can twist deform the pairing $\langle -, - \rangle$ to a new pairing $\langle -, - \rangle_\star$ defined by

$$\langle v_1, v_2 \rangle_\star := \langle \tilde{f}^k(v_1), \tilde{f}_k(v_2) \rangle. \quad (4.13)$$

Non-degeneracy of the pairing $\langle -, - \rangle_\star : V[[\hbar]] \otimes V[[\hbar]] \rightarrow \mathbb{R}[[\hbar]]$ follows from the untwisted non-degeneracy and an order by order argument in the formal deformation parameter $\hbar$. In general, graded symmetry of the cyclic pairing $\langle -, - \rangle$ implies that the twisted pairing $\langle -, - \rangle_\star$ is naturally (graded) braided symmetric, and the natural notion of cyclicity for a braided L_∞ -algebra would of course be that of ‘braided cyclicity’ defined by inserting suitable factors of the $\mathcal{R}$ -matrix in the obvious way. However, this level of generality is not suitable for applications to field theory, as the loss of *strict* graded symmetry and cyclicity would lead to problems with the variational principle for the corresponding action functionals. We therefore restrict from the outset the types of Drinfel’d twists that we will use for deformation quantization of a given cyclic L_∞ -algebra.

Definition 4.14 A Drinfel’d twist $\mathcal{F} \in U\mathfrak{v}[[\hbar]] \otimes U\mathfrak{v}[[\hbar]]$ is *compatible* with a cyclic structure $\langle -, - \rangle : V \otimes V \rightarrow \mathbb{R}$ on an L_∞ -algebra in $\mathcal{M}^\sharp$ if

$$\langle R_k(v_1), R^k(v_2) \rangle_\star = \langle v_1, v_2 \rangle_\star$$

for all $v_1, v_2 \in V$.

Given the concrete examples of cyclic L_∞ -algebra structures that we considered in Sects. 2.3 and 2.4, it should come as no surprise that this definition is an abstraction of the graded cyclicity property (3.10) for integration of twisted differential forms which we already discussed in Sect. 3.3, and with it we have

Proposition 4.15 Let $(V, \{\ell_n\}, \langle -, - \rangle)$ be a cyclic L_∞ -algebra in the category $\mathcal{M}^\sharp$, and let $\mathcal{F}$ be a compatible Drinfel’d twist. Then $(V[[\hbar]], \{\ell_n^\star\}, \langle -, - \rangle_\star)$ is a strictly cyclic braided L_∞ -algebra.

Proof For clarity of exposition, in the following we drop the signs arising from the gradings in the cyclicity conditions of Sect. 2.1, as they play no role in the proof since they are the same for the original and the twist deformed L_∞ -algebra. We then need to show that

$$\langle v_0, \ell_n^\star(v_1, v_2, \dots, v_n) \rangle_\star = \langle v_n, \ell_n^\star(v_0, v_1, \dots, v_{n-1}) \rangle_\star.$$

For $n = 1$, since $\ell_1^\star = \ell_1$ is not deformed, the left-hand side reads

$$\begin{aligned} \langle v_0, \ell_1(v_1) \rangle_\star &= \langle \tilde{f}^k(v_0), \ell_1(\tilde{f}_k(v_1)) \rangle \\ &= \langle \tilde{f}_k(v_1), \tilde{f}^k(\ell_1(v_0)) \rangle \end{aligned}$$

$$\begin{aligned}
&= \langle \bar{f}^k R_l(v_1), \bar{f}_k R^l(\ell_1(v_0)) \rangle \\
&= \langle R_l(v_1), R^l(\ell_1(v_0)) \rangle_\star \\
&= \langle v_1, \ell_1(v_0) \rangle_\star
\end{aligned}$$

where in the first and fourth equalities we use the definition of the star-pairing, in the first and second equalities we use commutativity of ℓ_1 with the $U\mathfrak{v}$ -action, in the second equality we use classical cyclicity, in the third equality the definition of the $\mathcal{R}$ -matrix, and in the fifth equality the compatibility of the twist.

For $n \geq 2$, this calculation extends to

$$\begin{aligned}
&\langle v_0, \ell_n(v_1 \otimes_\star \cdots \otimes_\star v_n) \rangle_\star \\
&= \langle \bar{f}^k(v_0), \bar{f}_k(\ell_n(\bar{f}^l(v_1 \otimes_\star \cdots \otimes_\star v_{n-1}) \otimes \bar{f}_l(v_n))) \rangle \\
&= \langle \bar{f}^k(v_0), \ell_n(\bar{f}_{k(1)} \bar{f}^l(v_1 \otimes_\star \cdots \otimes_\star v_{n-1}) \otimes \bar{f}_{k(2)} \bar{f}_l(v_n)) \rangle \\
&= \langle \bar{f}_{(1)}^k \bar{f}^l(v_0), \ell_n(\bar{f}_{(2)}^k \bar{f}_l(v_1 \otimes_\star \cdots \otimes_\star v_{n-1}) \otimes \bar{f}_k(v_n)) \rangle \\
&= \langle \bar{f}_k(v_n), \ell_n(\bar{f}_{(1)}^k \bar{f}^l(v_0) \otimes \bar{f}_{(2)}^k \bar{f}_l(v_1 \otimes_\star \cdots \otimes_\star v_{n-1})) \rangle \\
&= \langle \bar{f}_k(v_n), \bar{f}^k(\ell_n(\bar{f}^l(v_0) \otimes \bar{f}_l(v_1 \otimes_\star \cdots \otimes_\star v_{n-1}))) \rangle \\
&= \langle \bar{f}^k R_m(v_n), \bar{f}_k R^m(\ell_n(\bar{f}^l(v_0) \otimes \bar{f}_l(v_1 \otimes_\star \cdots \otimes_\star v_{n-1}))) \rangle \\
&= \langle R_m(v_n), R^m(\ell_n(v_0 \otimes_\star v_1 \otimes_\star \cdots \otimes_\star v_{n-1})) \rangle_\star \\
&= \langle v_n, \ell_n(v_0 \otimes_\star v_1 \otimes_\star \cdots \otimes_\star v_{n-1}) \rangle_\star
\end{aligned}$$

where we used the same steps as in the $n = 1$ case above, and now also the cocycle property (3.1) for the inverse of the Drinfel'd twist and the $U\mathfrak{v}$ -invariance (4.12) of the classical pairing in the third equality. $\square$

5 Braided field theory

5.1 Noncommutative field theories in the braided L_∞ -algebra formalism

We can now build up large classes of noncommutative field theories by employing a braided version of the ‘bottoms-up’ approach to classical field theories which we discussed in Sect. 2.2, whereby a classical field theory is defined as a dynamical system by its 4-term L_∞ -algebra as the initial input. These can be constructed starting from any suitable braided L_∞ -algebra in the sense of Sect. 4.1; we call a field theory obtained in this way a *braided field theory*. That is, in this paper, a braided field theory is defined as a (formal) dynamical system by its 4-term braided L_∞ -algebra as initial input.⁸

Usually these theories will come from deforming a classical field theory on a manifold M through Drinfel'd twist deformation quantization of its corresponding

⁸ Our formalism may however also be used to treat more general field theories with reducible gauge symmetries, which are captured by k -term braided L_∞ -algebras with $k > 4$.

L_∞ -algebra, as described in Sect. 4.2, and indeed this will be the case for the examples treated in this paper. If $(V, \{\ell_n\})$ is a 4-term L_∞ -algebra with underlying graded vector space

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3$$

in the representation category $\mathcal{M}^\sharp$, then, following the prescription of Sect. 4.2, we construct the corresponding 4-term braided L_∞ -algebra $(V[[\hbar]], \{\ell_n^\star\})$ in the twisted representation category $\mathcal{F}\mathcal{M}^\sharp$. In simple instances, a braided field theory obtained in this way through Drinfel'd twist deformation of a classical Lagrangian field theory will be the 'naive' deformation, obtained by twisting products of fields in the equations of motion and in the action functional to star-products in the standard way. However, we shall see that this is not always the case, a notable example being the four-dimensional braided ECP theory of gravity that we define later on. But even in simple cases the notion of braided gauge symmetry brings with it several novelties and complexities that we explain in detail below.

We shall now run through the various ingredients involved in the definition of a noncommutative field theory in this way for a general 4-term braided L_∞ -algebra $(V, \{\ell_n\})$, following the standard classical prescription of Sect. 2.2.

5.2 Braided Lie algebras of gauge symmetries

Let us first discuss what should generally be meant by a 'gauge symmetry' in this setting, and then later pass to its realization in the L_∞ -algebra formalism.

Braided gauge transformations

In the context of generalized gauge symmetries, wherein one considers general non-linear field-dependent transformations, a *braided gauge transformation* is a map $V_0 \times V_1 \rightarrow V_1$, denoted as $(\lambda, A) \mapsto \delta_\lambda^L A$, which is linear in the gauge parameters $\lambda \in V_0$ and has a potentially arbitrary polynomial dependence on fields $A \in V_1$. This means that the map may be realized as the diagonal of a map $V_0 \otimes \odot^\bullet V_1 \rightarrow V_1$ as an arrow in the category $\mathcal{F}\mathcal{M}^\sharp$, where the symmetric tensor product is taken with respect to the braided transposition $\mathcal{F}\tau^\sharp$. Explicitly, a braided gauge transformation $\delta_\lambda^L A$ may be expanded in tensor powers of the fields $A^{\otimes n}$ as

$$\delta_\lambda^L A = \delta_0^L(\lambda) + \sum_{n=1}^{\infty} \delta_{\lambda, n}^L(A^{\otimes n}),$$

where each polynomial order is now a linear braided symmetric map on V_1 in $\mathcal{F}\mathcal{M}^\sharp$. Similarly to the classical case, the collection of transformations on fields V_1 are required to close (in a suitable sense to be made precise later on) under the braided commutator bracket

$$[\delta_{\lambda_1}^L, \delta_{\lambda_2}^L]_\circ^\star := \delta_{\lambda_1}^L \circ \delta_{\lambda_2}^L - \delta_{R_k(\lambda_2)}^L \circ \delta_{R^k(\lambda_1)}^L,$$

for $\lambda_1, \lambda_2 \in V_0$. The braided commutator is a braided Lie bracket, that is, it is braided antisymmetric and satisfies the braided Jacobi identity; this is a special instance of the braided commutator of endomorphisms in the category $\mathcal{F}\mathcal{M}^\sharp$ which makes the braided derivations of a $U_{\mathcal{F}\mathfrak{v}}$ -module algebra into a braided Lie algebra [17]. The simplest prototypical case, and in fact the case most relevant for the examples of this paper, is that of a linear braided Lie algebra action, which we discuss in Example 5.6.

By setting $\delta_\lambda^L|_{V_0} = 0$, a braided gauge transformation is extended to a map $\delta_\lambda^L : V \rightarrow V$ of degree 0 as a braided derivation of operations defined on $V \otimes V$. It is convenient to formalize the braided derivation property by stating that the action of a braided gauge transformation δ_λ^L on a tensor product is through the twisted coproduct

$$\Delta_{\mathcal{F}}(\delta_\lambda^L) = \delta_\lambda^L \otimes \text{id} + R_k \otimes \delta_{R^k(\lambda)}^L.$$

This is also a map $\Delta_{\mathcal{F}}(\delta_\lambda^L) : V \otimes V \rightarrow V \otimes V$ of degree 0, which encodes the braided Leibniz rule. This map is extended inductively to higher tensor products $\Delta_{\mathcal{F}}^n(\delta_\lambda^L) : \bigotimes^n V \rightarrow \bigotimes^n V$ for $n \geq 3$ by

$$\Delta_{\mathcal{F}}^n(\delta_\lambda^L) = \delta_\lambda^L \otimes \text{id}^{\otimes n-1} + \sum_{i=1}^{n-1} R_{k_1} \otimes \cdots \otimes R_{k_i} \otimes \delta_{R^{k_i} \cdots R^{k_1}(\lambda)}^L \otimes \text{id}^{\otimes n-i-1}, \quad (5.1)$$

and we set $\Delta_{\mathcal{F}}^1(\delta_\lambda^L) := \delta_\lambda^L$ and $\Delta_{\mathcal{F}}^2(\delta_\lambda^L) := \Delta_{\mathcal{F}}(\delta_\lambda^L)$. Thus if μ_n is any map on $\bigotimes^n V$ in the category $\mathcal{F}\mathcal{M}^\sharp$, we define its braided gauge transformation by

$$\delta_\lambda^L \mu_n := \mu_n \circ \Delta_{\mathcal{F}}^n(\delta_\lambda^L). \quad (5.2)$$

For example, with $n = 2$ one has

$$\delta_\lambda^L \mu_2(v_1 \otimes v_2) = \mu_2(\delta_\lambda^L v_1 \otimes v_2) + \mu_2(R_k(v_1) \otimes \delta_{R^k(\lambda)}^L v_2) \quad (5.3)$$

for all $v_1, v_2 \in V$.

The superscript L is used to indicate that this map is regarded as a *left* braided gauge transformation which ‘acts from the left’. Correspondingly, there is a *right* braided gauge transformation $V_1 \times V_0 \rightarrow V_1$, $(A, \lambda) \mapsto \delta_\lambda^R A$ which is regarded as ‘acting from the right’ and is defined by

$$\delta_\lambda^R A := \delta_{R^k(\lambda)}^L R^k(A) = \delta_0^L(\lambda) + \sum_{n=1}^{\infty} \delta_{R^k(\lambda), n}^L (R^k(A^{\otimes n})).$$

In the classical case, where $\mathcal{R} = 1 \otimes 1$, the two transformations of course coincide. The action of a right braided gauge transformation δ_λ^R on tensor products is through the twisted coproduct

$$\Delta_{\mathcal{F}}(\delta_\lambda^R) = \text{id} \otimes \delta_\lambda^R + \delta_{R^k(\lambda)}^R \otimes R^k,$$

encoding the braided Leibniz rule from the right, and its higher iterations

$$\Delta_{\mathcal{F}}^n(\delta_\lambda^R) = \text{id}^{\otimes n-1} \otimes \delta_\lambda^R + \sum_{i=1}^{n-1} \text{id}^{\otimes n-i-1} \otimes \delta_{R_{k_i} \dots R_{k_1}(\lambda)}^R \otimes R^{k_1} \otimes \dots \otimes R^{k_i}. \quad (5.4)$$

Suppose now that the map μ_2 is braided symmetric, that is, the maps

$$\mu_2 = \mu_2 \circ \mathcal{F}\tau_{V,V}^\sharp$$

are equal as morphisms in the category $\mathcal{F}\mathcal{M}^\sharp$, where $\mathcal{F}\tau_{V,V}^\sharp : V \otimes V \rightarrow V \otimes V$ is the braiding given by $\mathcal{F}\tau^\sharp := \mathcal{R}^{-1} \circ \tau^\sharp = \tau^\sharp \circ \mathcal{R}$ with $\tau^\sharp$ the graded transposition. On homogeneous elements this reads

$$\mu_2(v_1 \otimes v_2) = (-1)^{|v_1||v_2|} \mu_2(R_k(v_2) \otimes R^k(v_1)). \quad (5.5)$$

We check that definition (5.2) is compatible with this braided symmetry, that is, that the left braided gauge transformation of the right-hand side of (5.5) is equal to (5.3):

$$\begin{aligned} \delta_\lambda^L(\mu_2 \circ \mathcal{F}\tau_{V,V}^\sharp)(v_1 \otimes v_2) &:= (\mu_2 \circ \mathcal{R}^{-1} \circ \tau_{V,V}^\sharp \circ \Delta_{\mathcal{F}}(\delta_\lambda^L))(v_1 \otimes v_2) \\ &= (-1)^{|v_1||v_2|} (\mu_2 \circ \mathcal{R}^{-1})(v_2 \otimes \delta_\lambda^L v_1 \\ &\quad + \delta_{R^k(\lambda)}^L v_2 \otimes R_k(v_1)) \\ &= (-1)^{|v_1||v_2|} \mu_2(R_l(v_2) \otimes R^l(\delta_\lambda^L v_1) \\ &\quad + R_l(\delta_{R^k(\lambda)}^L v_2) \otimes R^l R_k(v_1)) \\ &= \mu_2(\delta_\lambda^L v_1 \otimes v_2 + R_k(v_1) \otimes \delta_{R^k(\lambda)}^L v_2) \\ &= (\mu_2 \circ \Delta_{\mathcal{F}}(\delta_\lambda^L))(v_1 \otimes v_2) \\ &=: \delta_\lambda^L \mu_2(v_1 \otimes v_2), \end{aligned}$$

where in the fourth equality we used braided symmetry (5.5). This consistency check also obviously holds when μ_2 is braided antisymmetric, and it easily extends to braided (anti)symmetric maps μ_n on higher tensor products $\bigotimes^n V$ for $n \geq 3$. Similarly, one verifies consistency with right braided gauge transformations.

Note that here and in the following the ordering between the braiding isomorphism and gauge transformations is important: one must write a map as a composition of morphisms in $\mathcal{F}\mathcal{M}^\sharp$ and precompose them with $\Delta_{\mathcal{F}}(\delta_\lambda^{L,R})$ as in definition (5.2), including $\mu_2 \circ \mathcal{F}\tau_{V,V}^\sharp$, because in general the braiding and the twisted coproduct do not commute: $\mathcal{F}\tau_{V,V}^\sharp \circ \Delta_{\mathcal{F}}(\delta_\lambda^{L,R}) \neq \Delta_{\mathcal{F}}(\delta_\lambda^{L,R}) \circ \mathcal{F}\tau_{V,V}^\sharp$. In other words

$$\Delta_{\mathcal{F}}(\delta_\lambda^L)(R_l(v_2) \otimes R^l(v_1)) \neq R_l(\delta_{R^k(\lambda)}^L v_2) \otimes R^l R_k(v_1) + R_l(v_2) \otimes R^l(\delta_\lambda^L v_1),$$

and similarly for right gauge transformations.

Drinfel'd twist deformations of gauge symmetries

If our noncommutative field theory arises from Drinfel'd twist quantization of a classical field theory on a manifold M , then a left braided gauge transformation $\delta_\lambda^{\star L} : V[[\hbar]] \rightarrow V[[\hbar]]$ is induced from a classical gauge transformation $\delta_\lambda : V \rightarrow V$ through the usual prescription (3.8):

$$\delta_\lambda^{\star L} v := \delta_{\bar{f}^k(\lambda)} \bar{f}_k(v)$$

for $\lambda \in V_0$ and $v \in V$. The corresponding right braided gauge transformation $\delta_\lambda^{\star R} : V[[\hbar]] \rightarrow V[[\hbar]]$ is given by

$$\delta_\lambda^{\star R} v = \delta_{R_k(\lambda)}^{\star L} R^k(v) = \delta_{\bar{f}_k(\lambda)} \bar{f}^k(v),$$

where in the second equality we used $\mathcal{F}^{-1} \mathcal{R}_{21} = \mathcal{F}^{-1} \mathcal{R}^{-1} = \mathcal{F}_{21}^{-1}$.

Example 5.6 Let us look at the prototypical case that will apply to all examples in this paper. Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[-, -]_{\mathfrak{g}}$, and let W be a linear representation of $\mathfrak{g}$; we denote the action of $\mathfrak{g}$ on W by $\triangleright : \mathfrak{g} \otimes W \rightarrow W$. The action of a gauge parameter $\lambda \in \Omega^0(M, \mathfrak{g})$ on a p -form field $\phi \in \Omega^p(M, W)$ valued in W is given by⁹

$$\delta_\lambda \phi = -\lambda \triangleright \phi. \quad (5.7)$$

In this definition we make no distinction between left and right actions.

To pass to the twist deformed field theory, we regard the representation spaces W as trivial $U\mathfrak{v}$ -modules and $\Omega^p(M, W)$ as the tensor product of $U\mathfrak{v}$ -modules, similarly to the graded Lie algebra $\Omega^\bullet(M, \mathfrak{g})$ (cf. Sect. 3.3). We may then define both a left braided representation

$$\delta_\lambda^{\star L} \phi = -\lambda \triangleright_\star \phi := -\bar{f}^k(\lambda) \triangleright \bar{f}_k(\phi) \quad (5.8)$$

and a right braided representation

$$\delta_\lambda^{\star R} \phi = -\phi \star \lhd \lambda := -R_k(\lambda) \triangleright_\star R^k(\phi) = -\bar{f}^l R_k(\lambda) \triangleright \bar{f}_l R^k(\phi) = -\bar{f}_k(\lambda) \triangleright \bar{f}^k(\phi), \quad (5.9)$$

where the first expression encodes the intuition of swapping factors and then acting on the representation. The twisted coproducts $\Delta_{\mathcal{F}}^n(\delta_\lambda^{\star L})$ and $\Delta_{\mathcal{F}}^n(\delta_\lambda^{\star R})$ for $n \geq 2$ then define the tensor products of left and right braided representations, respectively.

⁹ The minus sign is the standard convention in gauge theory, so that the action of the corresponding Lie group is a right action.

Both left and right gauge transformations provide a representation of the braided Lie algebra $(\Omega^0(M, \mathfrak{g})[[\hbar]], [-, -]_{\mathfrak{g}}^*)$ on $\Omega^p(M, W)[[\hbar]]$:

$$[\delta_{\lambda_1}^{\star L, R}, \delta_{\lambda_2}^{\star L, R}]_{\circ}^{\star} = \delta_{[\lambda_1, \lambda_2]_{\mathfrak{g}}}^{\star L, R}, \quad (5.10)$$

for all $\lambda_1, \lambda_2 \in \Omega^0(M, \mathfrak{g})$. For this, we compute

$$\begin{aligned} \delta_{\lambda_1}^{\star L} \delta_{\lambda_2}^{\star L} \phi &= \delta_{\lambda_1}^{\star L} (-\lambda_2 \triangleright_{\star} \phi) \\ &= -R_k(\lambda_2) \triangleright_{\star} \delta_{R^k(\lambda_1)}^{\star L} \phi \\ &= \bar{f}_m^m R_k(\lambda_2) \triangleright (\bar{f}_{m(1)}^l \bar{f}_l^l R^k(\lambda_1) \triangleright \bar{f}_{m(2)} \bar{f}_l(\phi)) \\ &= \bar{f}_{(1)}^m \bar{f}_l^l R_k(\lambda_2) \triangleright (\bar{f}_{(2)}^m \bar{f}_l R^k(\lambda_1) \triangleright \bar{f}_m(\phi)) \\ &= \bar{f}_{(1)}^m \bar{f}_l^l f^p \bar{f}_k(\lambda_2) \triangleright (\bar{f}_{(2)}^m \bar{f}_l f_p \bar{f}^k(\lambda_1) \triangleright \bar{f}_m(\phi)) \\ &= \bar{f}_{(2)}^m \bar{f}_k(\lambda_2) \triangleright (\bar{f}_{(1)}^m \bar{f}^k(\lambda_1) \triangleright \bar{f}_m(\phi)) \end{aligned}$$

where in the fourth line we used the cocycle property (3.1) for the inverse of the Drinfel'd twist, in the fifth line we used $\mathcal{R}_{21} = \mathcal{R}^{-1} = \mathcal{F} \mathcal{F}_{21}^{-1}$, and in the last step we used $\mathcal{F}^{-1} \mathcal{F} = 1 \otimes 1$ along with cocommutativity of the trivial coproduct Δ . With analogous calculations, this correspondingly gives

$$\begin{aligned} \delta_{R_l(\lambda_2)}^{\star L} \delta_{R^l(\lambda_1)}^{\star L} \phi &= \bar{f}_{(2)}^m \bar{f}_k^l R^l(\lambda_1) \triangleright (\bar{f}_{(1)}^m \bar{f}_l^k R_l(\lambda_2) \triangleright \bar{f}_m(\phi)) \\ &= \bar{f}_{(1)}^m \bar{f}_l^l(\lambda_1) \triangleright (\bar{f}_{(2)}^m \bar{f}_l(\lambda_2) \triangleright \bar{f}_m(\phi)). \end{aligned}$$

Thus we get

$$\begin{aligned} [\delta_{\lambda_1}^{\star L}, \delta_{\lambda_2}^{\star L}]_{\circ}^{\star} \phi &= \bar{f}_{(2)}^m \bar{f}_l(\lambda_2) \triangleright (\bar{f}_{(1)}^m \bar{f}^l(\lambda_1) \triangleright \bar{f}_m(\phi)) \\ &\quad - \bar{f}_{(1)}^m \bar{f}^l(\lambda_1) \triangleright (\bar{f}_{(2)}^m \bar{f}_l(\lambda_2) \triangleright \bar{f}_m(\phi)) \\ &= -[\bar{f}_{(1)}^m \bar{f}^l(\lambda_1), \bar{f}_{(2)}^m \bar{f}_l(\lambda_2)]_{\mathfrak{g}} \triangleright \bar{f}_m(\phi) \\ &= \bar{f}^m(-[\lambda_1, \lambda_2]_{\mathfrak{g}}) \triangleright \bar{f}_m(\phi) \\ &= \delta_{[\lambda_1, \lambda_2]_{\mathfrak{g}}}^{\star L} \phi, \end{aligned}$$

where in the second line we used the fact that (5.7) is a representation of $\mathfrak{g}$ on $\Omega^p(M, W)$. The proof for right gauge transformations is completely analogous.

For the special case when the field ϕ is a gauge connection $a \in \Omega^1(M, \mathfrak{g})$, W is the adjoint representation of $\mathfrak{g}$, and the left and right braided adjoint actions are given by

$$\lambda \triangleright_{\star}^{\text{ad}} a := [\lambda, a]_{\mathfrak{g}}^{\star} \quad \text{and} \quad a \triangleleft_{\star}^{\text{ad}} \lambda := [R_k(\lambda), R^k(a)]_{\mathfrak{g}}^{\star} = -[a, \lambda]_{\mathfrak{g}}^{\star},$$

where in the last equality we used braided antisymmetry of the braided Lie bracket involving gauge parameters. For a matrix Lie algebra $\mathfrak{g}$ one can write

$$[\lambda, a]_{\mathfrak{g}}^* = \lambda \star a - R_k(a) \star R^k(\lambda),$$

where the star-product here is the tensor product of matrix multiplication with the twisted product of functions and forms. The left and right braided adjoint representations induce the left and right braided gauge transformations

$$\delta_{\lambda}^{\star L} a := d\lambda - [\lambda, a]_{\mathfrak{g}}^* \quad \text{and} \quad \delta_{\lambda}^{\star R} a := d\lambda + [a, \lambda]_{\mathfrak{g}}^*.$$

These give representations (5.10) on $\Omega^1(M, \mathfrak{g})[[\hbar]]$. This can be shown by using braided antisymmetry and the braided Jacobi identity for the bracket $[-, -]_{\mathfrak{g}}^*$, along with the undeformed Leibniz rule for the exterior derivative d (cf. Sect. 3.3); it will follow as a special case of a more general result in Sect. 5.5 (see Example 5.24).

The braiding of symmetries illustrated by Example 5.6 is independent of the particular form of a (classical) Lie algebra V_0 of gauge parameters: the only necessary condition is that V_0 is a Lie algebra in the category of $U\mathfrak{v}$ -modules.

5.3 Braided versus star-gauge symmetry

Example 5.6 illustrates an important distinction between braided gauge symmetries and the well-known star-gauge symmetries. These latter gauge transformations are the most prominent in the noncommutative field theory literature, with applications ranging from the Standard Model to gravity, and they are the ones which are consistent with the embeddings of these theories into string theory. In these more conventional settings, one works with a matrix Lie algebra $\mathfrak{g}$ and introduces a star-commutator bracket

$$[\lambda_1 \star \lambda_2]_{\mathfrak{g}} := \lambda_1 \star \lambda_2 - \lambda_2 \star \lambda_1, \quad (5.11)$$

where here the product is the tensor product of star-multiplication with matrix multiplication of $\lambda_1, \lambda_2 \in \Omega^0(M, \mathfrak{g})$. Demanding closure as an ordinary (not braided) Lie algebra is then not possible for gauge algebras beyond the unitary Lie algebras $\mathfrak{g} = \mathfrak{u}(N)$ in the fundamental representation, unless one extends the space of gauge parameters $\Omega^0(M, \mathfrak{g})$ [4]. This correspondingly introduces new dynamical degrees of freedom into the noncommutative field theory, which are hard to justify or motivate from a physical standpoint, and moreover cause problems when considering commutative limits of the theories. One can avoid the introduction of new fields by instead regarding (5.11) as an element of the enveloping algebra $U\mathfrak{g}[[\hbar]]$ and using the Seiberg–Witten map [38]; however, the Seiberg–Witten map is not generally known in closed form and so this approach is limited to a description of star-gauge transformations to only the first few orders in the deformation parameter $\hbar$.

In contrast, in the present setting, the Lie algebra $(\Omega^0(M, \mathfrak{g}), [-, -]_{\mathfrak{g}})$ of gauge parameters is deformed to the braided Lie algebra $(\Omega^0(M, \mathfrak{g})[[\hbar]], [-, -]_{\mathfrak{g}}^*)$ without

the need of extending the vector space $\Omega^0(M, \mathfrak{g})$ (beyond the usual formal power series extension), for *arbitrary* Lie algebras. Even for a matrix Lie algebra $\mathfrak{g}$, using $\mathcal{F}_{21}^{-1} = \mathcal{F}^{-1} \mathcal{R}^{-1} = \mathcal{F}^{-1} \mathcal{R}_{21}$ one finds that the braided commutator

$$[\lambda_1, \lambda_2]_{\mathfrak{g}}^* = \lambda_1 \star \lambda_2 - R_k(\lambda_2) \star R^k(\lambda_1) \quad (5.12)$$

is not equal to the star-commutator (5.11), even for the unitary Lie algebras and for simple twists like the Moyal–Weyl star-product of Example 3.9; a striking example is the Lie algebra $\mathfrak{g} = \mathfrak{u}(1)$ (or any abelian Lie algebra), where the braided commutator (5.12) always vanishes but the star-commutator (5.11) does not. In our approach the twisting of the gauge algebra and its representations themselves avoids the introduction of new fields altogether. The dynamical sector of the noncommutative field theory is then correspondingly constructed by twist deforming the entire underlying L_∞ -algebra structure.

A hybrid definition of ‘twisted gauge symmetry’, which in a sense lies in between the standard star-gauge transformations and our braided gauge transformations, was suggested in [14, 46, 53]. In our setting, this would follow from twist deforming the coproduct of gauge transformations $\Delta(\delta_\lambda)$ by regarding the twist $\mathcal{F}$ in the universal enveloping algebra of the semi-direct product Lie algebra $\Gamma(TM) \ltimes \Omega^0(M, \mathfrak{g})$. Then the gauge transformations are just the usual classical gauge transformations, but they act as braided derivations similarly to our braided gauge transformations. Since the actions of $\Omega^0(M, \mathfrak{g})$ themselves are not deformed, the closure of gauge transformations is the same as in the classical case: the gauge algebra closes in the classical Lie algebra for all Lie algebras $\mathfrak{g}$ and their representations. However, this approach inevitably also involves star-commutators, rather than braided commutators, and so also introduces additional dynamical degrees of freedom [14], since the star-commutator does not generally close in the (matrix) Lie algebra $\mathfrak{g}$. This mixing between braided and star-gauge transformations also does not appear to be consistent or natural within the category $\mathcal{F}\mathcal{M}^\sharp$.

5.4 Braided noncommutative kinematics

Continuing Example 5.6, we will now define the natural geometric objects which are covariant under braided gauge transformations. These will form the building blocks for the braided gauge-invariant noncommutative field theories that we study in detail later on. Aspects of this story have been studied extensively at the kinematical level by [23], and also alluded to by [49]. We will see how this extends to include dynamics by systematically defining the full theory using the braided L_∞ -algebra formalism, which constructs braided field theories in a model-independent way.

Braided covariant derivatives

Given a (classical) connection 1-form $a \in \Omega^1(M, \mathfrak{g})$, the covariant derivative $d^a \phi \in \Omega^{p+1}(M, W)$ of any field $\phi \in \Omega^p(M, W)$ is given classically by

$$d^a \phi = d\phi + a \triangleright \phi,$$

where the action of $\Omega^1(M, \mathfrak{g})$ on $\Omega^p(M, W)$ is the tensor product of exterior multiplication of forms with the action of $\mathfrak{g}$ on the representation W . In the twist deformed field theory, the transformation of ϕ under left gauge transformations (5.8) suggests that the correct definition of the *left* covariant derivative is simply by acting with the corresponding left braided representation:

Definition 5.13 Given a (braided) connection 1-form $a \in \Omega^1(M, \mathfrak{g})$, the *left braided covariant derivative* of a field $\phi \in \Omega^p(M, W)$ is

$$d_{\star L}^a \phi := d\phi + a \triangleright_{\star} \phi.$$

This is indeed covariant under left braided gauge transformations:

$$\begin{aligned} \delta_{\lambda}^{\star L}(d_{\star L}^a \phi) &= \delta_{\lambda}^{\star L}(d\phi + a \triangleright_{\star} \phi) \\ &= d\delta_{\lambda}^{\star L} \phi + \delta_{\lambda}^{\star L} a \triangleright_{\star} \phi + R_k(a) \triangleright_{\star} \delta_{R^k(\lambda)}^{\star L} \phi \\ &= -d(\lambda \triangleright_{\star} \phi) + (d\lambda - [\lambda, a]_{\mathfrak{g}}^{\star}) \triangleright_{\star} \phi - R_k(a) \triangleright_{\star} (R^k(\lambda) \triangleright_{\star} \phi) \\ &= -d\lambda \triangleright_{\star} \phi - \lambda \triangleright_{\star} d\phi + d\lambda \triangleright_{\star} \phi - [\lambda, a]_{\mathfrak{g}}^{\star} \triangleright_{\star} \phi \\ &\quad - R_k(a) \triangleright_{\star} (R^k(\lambda) \triangleright_{\star} \phi) \\ &= -\lambda \triangleright_{\star} d\phi - \lambda \triangleright_{\star} (a \triangleright_{\star} \phi) \\ &= -\lambda \triangleright_{\star} (d_{\star L}^a \phi), \end{aligned}$$

where in the second and fifth lines we used the twisted coproduct $\Delta_{\mathcal{F}}(\delta_{\lambda}^{\star L})$ to implement the Leibniz rule from the left.

Similarly, we can show that the left covariant derivative is also covariant under *right* braided gauge transformations (5.9):

$$\begin{aligned} \delta_{\lambda}^{\star R}(d_{\star L}^a \phi) &= \delta_{\lambda}^{\star R}(d\phi + a \triangleright_{\star} \phi) \\ &= d\delta_{\lambda}^{\star R} \phi + a \triangleright_{\star} \delta_{\lambda}^{\star R} \phi + \delta_{R^k(\lambda)}^{\star R} a \triangleright_{\star} R^k(\phi) \\ &= d(-\phi \triangleleft \lambda) + a \triangleright_{\star} (-\phi \triangleleft \lambda) + (dR_k(\lambda) + [a, R_k(\lambda)]_{\mathfrak{g}}^{\star}) \triangleright_{\star} R^k(\phi) \\ &= -d\phi \triangleleft \lambda - \phi \triangleleft d\lambda - a \triangleright_{\star} (R_k(\lambda) \triangleright_{\star} R^k(\phi)) + \phi \triangleleft d\lambda \\ &\quad + [a, R_k(\lambda)]_{\mathfrak{g}}^{\star} \triangleright_{\star} R^k(\phi) \\ &= -d\phi \triangleleft \lambda - [a, R_k(\lambda)]_{\mathfrak{g}}^{\star} \triangleright_{\star} R^k(\phi) - R_l R_k(\lambda) \triangleright_{\star} (R^l(a) \triangleright_{\star} R^k(\phi)) \\ &\quad + [a, R_k(\lambda)]_{\mathfrak{g}}^{\star} \triangleright_{\star} R^k(\phi) \\ &= -d\phi \triangleleft \lambda - R_k(\lambda) \triangleright_{\star} R^k(a \triangleright_{\star} \phi) \end{aligned}$$

$$\begin{aligned}
&= -(\mathrm{d}\phi + a \triangleright_\star \phi) \star \triangleleft \lambda \\
&= -(\mathrm{d}_\star^a \phi) \star \triangleleft \lambda,
\end{aligned}$$

where in the second line we used the twisted coproduct $\Delta_{\mathcal{F}}(\delta_\lambda^{\star R})$ to implement the braided Leibniz rule from the right, in the fifth line we used the twisted coproduct $\Delta_{\mathcal{F}}(\delta_\lambda^{\star L})$, and in the sixth line we used the second $\mathcal{R}$ -matrix identity of (3.7).

As one might have anticipated, we can define analogously the *right* covariant derivative of $\phi \in \Omega^p(M, W)$ by

$$\mathrm{d}_\star^a \mathrm{R} \phi := \mathrm{d}\phi + (-1)^p \phi \star \triangleleft a = \mathrm{d}\phi + \mathrm{R}_k(a) \triangleright_\star \mathrm{R}^k(\phi).$$

Similar calculations to those of the left covariant derivative establish that the right covariant derivative is covariant under both left and right braided gauge transformations.

What distinguishes the left and right braided covariant derivatives is their behaviour with respect to the action of the noncommutative algebra of functions on M . The vector space of braided fields $\Omega^p(M, W)[[\hbar]]$ carrying a braided representation of $\mathfrak{g}$ is a bimodule over the algebra $\mathcal{A}_\star = (C^\infty(M)[[\hbar]], \star)$: one can star-multiply with functions on the left and on the right to produce p -forms which transform identically under left and right braided gauge transformations, because functions transform in the trivial representation of the Lie algebra $\mathfrak{g}$. For example, for a function $f \in \mathcal{A}_\star$ we have

$$\delta_\lambda^{\star L}(f \star \phi) = \mathrm{R}_k(f) \star \delta_{\mathrm{R}^k(\lambda)} \phi = \mathrm{R}_k(f) \star (-\mathrm{R}^k(\lambda) \triangleright_\star \phi) = -\lambda \triangleright_\star (f \star \phi),$$

where in the last equality we used the left braided Leibniz rule with λ acting via the trivial representation on f . Similar results hold for right star-multiplication and right gauge transformations.

However, the left and right covariant derivatives do not satisfy the expected (undeformed) Leibniz rules when viewing $\Omega^p(M, W)[[\hbar]]$ as a bimodule over $\mathcal{A}_\star$, but only as right and left $\mathcal{A}_\star$ -modules, respectively: $\mathrm{d}_\star^a \mathrm{L}$ is a derivation with respect to right star-multiplication by $\mathcal{A}_\star$ and analogously $\mathrm{d}_\star^a \mathrm{R}$ is a left $\mathcal{A}_\star$ -module map. This is where potential confusion for our left/right definitions may arise. Explicitly, it is easy to check the derivation properties

$$\begin{aligned}
\mathrm{d}_\star^a \mathrm{L}(\phi \star f) &= \mathrm{d}_\star^a \mathrm{L} \phi \star f + (-1)^p \phi \wedge_\star \mathrm{d}f \quad \text{and} \\
\mathrm{d}_\star^a \mathrm{R}(f \star \phi) &= \mathrm{d}f \wedge_\star \phi + f \star \mathrm{d}_\star^a \mathrm{R} \phi,
\end{aligned}$$

but the Leibniz rule does not hold for the other two $\mathcal{A}_\star$ -module actions (here $\wedge_\star$ is simply the twisted exterior product of the form parts).

In classical differential geometry one can define a connection directly as the covariant derivative map. In noncommutative differential geometry, one has two options which are *a priori* different: a left module connection and a right module connection (see e.g. [3, 49]). In contrast, here we have a single gauge connection a which induces both covariant derivatives.

Braided curvature

The left curvature of the gauge field a is now similarly defined with W the adjoint representation and the usual factor of $\frac{1}{2}$:

Definition 5.14 Given a braided gauge field $a \in \Omega^1(M, \mathfrak{g})$, the *left braided curvature* is

$$F_a^{\star L} := da + \frac{1}{2} [a, a]_{\mathfrak{g}}^{\star}. \quad (5.15)$$

The right curvature is correspondingly

$$F_a^{\star R} := da + \frac{1}{2} [R_k(a), R^k(a)]_{\mathfrak{g}}^{\star} = da + \frac{1}{2} [a, a]_{\mathfrak{g}}^{\star} = F_a^{\star L},$$

where we used braided symmetry of the bracket $[-, -]_{\mathfrak{g}}^{\star}$ on 1-forms. Hence, there is only one curvature $F_a^{\star} := F_a^{\star L} = F_a^{\star R}$. The proof of left and right covariance of the curvature follows from analogous calculations to those above:

$$\delta_{\lambda}^{\star L} F_a^{\star} = -[\lambda, F_a^{\star}]_{\mathfrak{g}}^{\star} \quad \text{and} \quad \delta_{\lambda}^{\star R} F_a^{\star} = [F_a^{\star}, \lambda]_{\mathfrak{g}}^{\star}.$$

For a matrix Lie algebra $\mathfrak{g}$, one can write

$$F_a^{\star} = da + \frac{1}{2} (a \wedge_{\star} a + R_k(a) \wedge_{\star} R^k(a)),$$

where here the star-product is the tensor product of matrix multiplication with the twisted exterior product; this differs from the usual definition in conventional non-commutative gauge theories, where one would instead write $F_a^{\star} = da + a \wedge_{\star} a$.

One of the novelties of the braided curvature 2-form is that it generally violates the Bianchi identity, that is, it is no longer covariantly constant. The braided deformations of the Bianchi identity are the three-forms in $\Omega^3(M, \mathfrak{g})[[\hbar]]$ given by

$$\begin{aligned} d_{\star L}^a F_a &= \frac{1}{2} [F_a, a]_{\mathfrak{g}}^{\star} + \frac{1}{2} [a, F_a]_{\mathfrak{g}}^{\star} - \frac{1}{4} [R_k(a), [R^k(a), a]_{\mathfrak{g}}^{\star}]_{\mathfrak{g}}^{\star}, \\ d_{\star R}^a F_a &= -\frac{1}{2} [F_a, a]_{\mathfrak{g}}^{\star} - \frac{1}{2} [a, F_a]_{\mathfrak{g}}^{\star} - \frac{1}{4} [R_k(a), [R^k(a), a]_{\mathfrak{g}}^{\star}]_{\mathfrak{g}}^{\star}. \end{aligned} \quad (5.16)$$

Both of these relations simply follow by taking the exterior derivative of expression (5.15) and rewriting it in terms of F_a to generate the differential identity

$$dF_a = \frac{1}{2} [F_a, a]_{\mathfrak{g}}^{\star} - \frac{1}{2} [a, F_a]_{\mathfrak{g}}^{\star} - \frac{1}{4} ([[a, a]_{\mathfrak{g}}^{\star}, a]_{\mathfrak{g}}^{\star} - [a, [a, a]_{\mathfrak{g}}^{\star}]_{\mathfrak{g}}^{\star}),$$

and using the (odd) braided Jacobi identity (3.11) on 1-forms $\Omega^1(M, \mathfrak{g})[[\hbar]]$ with $\alpha_1 = \alpha_2 = \alpha_3 = a$; contrary to the classical case, this differential identity also involves the gauge field a itself. This violation is well-known in twisted noncommutative differential geometry, see e.g. [3, 11, 18]. For trivial $\mathcal{R}$ -matrix, the right-hand sides of these covariant derivative equations vanish identically and we recover the usual Bianchi identity in classical geometry.

One of the remarkable features of our braided field theory formalism will be that such kinematical violations are simply natural consequences of the braided homotopy relations for the underlying L_∞ -algebra, as we discuss further later on. In other words, the violations of the Bianchi identity here will be controlled by a braided L_∞ -algebra. Let us finally turn to this formalism.

5.5 Braided gauge transformations and L_∞ -algebras

Generalizing Example 5.6, let us now formulate a braided version of the L_∞ -algebra gauge transformations (2.4). Let $(V, \{\ell_n\})$ be a 4-term braided L_∞ -algebra in the category $\mathcal{FM}^\sharp$ of graded left $U\mathcal{FV}$ -modules. Following the classical case, we have

Definition 5.17 For a gauge parameter $\lambda \in V_0$, the *left braided gauge transformation* of a field $A \in V_1$ is

$$\delta_\lambda^L A := \ell_1(\lambda) + \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \ell_{n+1}(\lambda, A, \dots, A). \quad (5.18)$$

The right braided gauge transformation is similarly defined by

$$\begin{aligned} \delta_\lambda^R A &:= \delta_{R_k(\lambda)}^L R^k(A) = \ell_1(\lambda) + \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \ell_{n+1}(R_k(\lambda), R^k(A \otimes \dots \otimes A)) \\ &= \ell_1(\lambda) + \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n+1)} \ell_{n+1}(A, \dots, A, \lambda), \end{aligned} \quad (5.19)$$

where in the last equality we used braided antisymmetry of the $n+1$ -brackets under interchange of gauge parameters and fields, together with the $\mathcal{R}$ -matrix identities in (3.7) (and their inverses) to write

$$\begin{aligned} &\ell_{n+1}(R_k(\lambda), R^k(A \otimes \dots \otimes A)) \\ &= \ell_{n+1}(R_{k_1} \dots R_{k_n}(\lambda), R^{k_1}(A), \dots, R^{k_n}(A)) \\ &= (-1)^n \ell_{n+1}(R_{l_1} R^{k_1}(A), \dots, R_{l_n} R^{k_n}(A), R^{l_1} \dots R^{l_n} R_{k_1} \dots R_{k_n}(\lambda)) \\ &= (-1)^n \ell_{n+1}(R_l R^k(A \otimes \dots \otimes A), R^l R_k(\lambda)) \\ &= (-1)^n \ell_{n+1}(A, \dots, A, \lambda). \end{aligned} \quad (5.20)$$

Braided Lie algebras of gauge transformations

Whereas a closure statement for the braided gauge L_∞ -algebra should follow as in the classical case (2.7), the general proof is complicated by the complex explicit forms of the braided homotopy relations. For the field theories that we study in this paper, the gauge algebra is always field-independent and closes off-shell, that is, without the need to impose any field equations. In these instances, it is relatively straightforward to establish closure of these left and right braided gauge transformations. For this,

as suggested from the theories we shall study, we require that the 4-term braided L_∞ -algebra $(V, \{\ell_n\})$ has the vanishing sets of brackets

$$\ell_{n+2}(\lambda_1, \lambda_2, A_1, \dots, A_n) = 0 \quad \text{and} \quad \ell_{n+2}(\lambda_1, \lambda_2, F, A_1, \dots, A_{n-1}) = 0 \quad (5.21)$$

for all $n \geq 1$, $\lambda_1, \lambda_2 \in V_0$, $A_1, \dots, A_n \in V_1$ and $F \in V_2$. If our L_∞ -algebra is further equipped with a (braided or strictly) cyclic structure of degree -3 , as will be the case for all field theories considered in this paper, then the two vanishing conditions in (5.21) are equivalent. In this case both left and right braided gauge transformations (5.18) and (5.19) on fields V_1 close the braided Lie algebra

$$[\delta_{\lambda_1}^{L,R}, \delta_{\lambda_2}^{L,R}]_\circ^\star = \delta_{-\ell_2(\lambda_1, \lambda_2)}^{L,R}. \quad (5.22)$$

Let us indicate why the braided commutation relations (5.22) hold. Using (5.18) and the braided Leibniz rule (5.1) we compute the composition of two left braided gauge transformations evaluated on a field $A \in V_1$ to get

$$\begin{aligned} \delta_{\lambda_1}^L \delta_{\lambda_2}^L A &= \sum_{n=1}^{\infty} \frac{(-1)^{\frac{1}{2}n(n-1)}}{n!} \ell_{n+1}(\mathbf{R}_l(\lambda_2), \Delta_{\mathcal{F}}^n(\delta_{\mathbf{R}'(\lambda_1)}^L)(A \otimes \dots \otimes A)) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{\frac{1}{2}n(n-1)}}{n!} \left(\ell_{n+1}(\mathbf{R}_l(\lambda_2), \delta_{\mathbf{R}'(\lambda_1)}^L A, A^{\otimes n-1}) \right. \\ &\quad \left. + \sum_{i=1}^{n-1} \ell_{n+1}(\mathbf{R}_l(\lambda_2), \mathbf{R}_{k_1}(A), \dots, \mathbf{R}_{k_i}(A), \delta_{\mathbf{R}^{k_1} \dots \mathbf{R}^{k_i} \mathbf{R}'(\lambda_1)}^L A, A^{\otimes n-i-1}) \right) \\ &= \sum_{n,m=0}^{\infty} \frac{(-1)^{\frac{1}{2}n(n+1) + \frac{1}{2}m(m-1)}}{(n+1)!m!} \left(\ell_{n+2}(\mathbf{R}_l(\lambda_2), \ell_{m+1}(\mathbf{R}'(\lambda_1), A^{\otimes m}), A^{\otimes n}) \right. \\ &\quad \left. + \sum_{i=1}^n \ell_{n+2}(\mathbf{R}_l(\lambda_2), \mathbf{R}_{k_1}(A), \dots, \mathbf{R}_{k_i}(A), \ell_{m+1}(\mathbf{R}^{k_i} \dots \mathbf{R}^{k_1} \mathbf{R}'(\lambda_1), A^{\otimes m}), A^{\otimes n-i}) \right) \\ &= \sum_{n,m=0}^{\infty} \frac{(-1)^{\frac{1}{2}n(n+1) + \frac{1}{2}m(m-1)}}{(n+1)!m!} \sum_{i=0}^n \ell_{n+2}(\mathbf{R}_l(\lambda_2), \mathbf{R}_k(A^{\otimes i}), \ell_{m+1}(\mathbf{R}^k \mathbf{R}'(\lambda_1), A^{\otimes m}), A^{\otimes n-i}) \end{aligned}$$

where in the last step we used the first identity from (3.7). Reordering the double sum we obtain

$$\begin{aligned} \delta_{\lambda_1}^L \delta_{\lambda_2}^L A &= \sum_{n=0}^{\infty} \frac{(-1)^{\frac{1}{2}n(n-1)}}{n!} \sum_{j=0}^n \frac{(-1)^{(j-1)(n-j)}}{n-j+1} \binom{n}{j} \\ &\quad \times \sum_{i=0}^{n-j} \ell_{n-j+2}(\mathbf{R}_l(\lambda_2), \mathbf{R}_k(A^{\otimes i}), \ell_{j+1}(\mathbf{R}^k \mathbf{R}'(\lambda_1), A^{\otimes j}), A^{\otimes n-i-j}). \end{aligned}$$

For the corresponding composition $\delta_{R^p(\lambda_2)}^L \delta_{R^p(\lambda_1)}^L A$, we use

$$\begin{aligned} & \ell_{n-j+2}(R_l R^p(\lambda_1), R_k(A^{\otimes i}), \ell_{j+1}(R^k R^l R_p(\lambda_2), A^{\otimes j}), A^{\otimes n-i-j}) \\ &= \ell_{n-j+2}(\lambda_1, R_k(A^{\otimes i}), \ell_{j+1}(R^k(\lambda_2), A^{\otimes j}), A^{\otimes n-i-j}). \end{aligned}$$

Collecting terms order by order in tensor powers $A^{\otimes n}$, we see that the closure formula (5.22) for left gauge transformations evaluated on a field $A \in V_1$ is satisfied if

$$\begin{aligned} & \sum_{j=0}^n \sum_{i=0}^{n-j} \frac{(-1)^{(j-1)(n-j)}}{n-j+1} \binom{n}{j} \left(\ell_{n-j+2}(\lambda_1, R_k(A^{\otimes i}), \ell_{j+1}(R^k(\lambda_2), A^{\otimes j}), A^{\otimes n-i-j}) \right. \\ & \quad \left. - \ell_{n-j+2}(R_l(\lambda_2), R_k(A^{\otimes i}), \ell_{j+1}(R^k R^l(\lambda_1), A^{\otimes j}), A^{\otimes n-i-j}) \right) \\ &= \ell_{n+1}(\ell_2(\lambda_1, \lambda_2), A^{\otimes n}) \end{aligned} \quad (5.23)$$

for each $n \geq 0$. This is a consequence of the braided homotopy relations $\mathcal{J}_{n+2}(\lambda_1, \lambda_2, A^{\otimes n}) = 0$ from (4.2) and the imposed restrictions (5.21). To illustrate the complicated $\mathcal{R}$ -matrix gymnastics that are involved in the proof, even with our simplifying assumptions (5.21), we demonstrate this explicitly for the first two orders.

For $n = 0$ the left-hand side of (5.23) evaluates to

$$\begin{aligned} \ell_2(\lambda_1, \ell_1(\lambda_2)) - \ell_2(R_l(\lambda_2), \ell_1(R^l(\lambda_1))) &= \ell_2(\lambda_1, \ell_1(\lambda_2)) - \ell_2(R_l(\lambda_2), R^l(\ell_1(\lambda_1))) \\ &= \ell_2(\lambda_1, \ell_1(\lambda_2)) + \ell_2(\ell_1(\lambda_1), \lambda_2) \\ &= \ell_1(\ell_2(\lambda_1, \lambda_2)) \end{aligned}$$

as required, where in the first equality we used $U\mathfrak{v}$ -equivariance of the map ℓ_1 , in the second equality we used braided antisymmetry of the 2-bracket ℓ_2 , and in the last step we used the derivation property of the homotopy identities (4.5).

For $n = 1$ the left-hand side of (5.23) is

$$\begin{aligned} & -\frac{1}{2} \ell_3(\lambda_1, \ell_1(\lambda_2), A) - \frac{1}{2} \ell_3(\lambda_1, R_k(A), \ell_1(R^k(\lambda_2))) + \ell_2(\lambda_1, \ell_2(\lambda_2, A)) \\ & \quad + \frac{1}{2} \ell_3(R_l(\lambda_2), \ell_1(R^l(\lambda_1)), A) + \frac{1}{2} \ell_3(R_l(\lambda_2), R_k(A), \ell_1(R^k R^l(\lambda_1))) \\ & \quad - \ell_2(R_l(\lambda_2), \ell_2(R^l(\lambda_1), A)) \\ &= -\frac{1}{2} \ell_3(\lambda_1, \ell_1(\lambda_2), A) - \frac{1}{2} \ell_3(\lambda_1, R_k(A), R^k(\ell_1(\lambda_2))) \\ & \quad - \ell_2(\ell_2(R_k(\bar{1})(\lambda_2), R_k(\bar{2})(A)), R^k(\lambda_1)) \\ & \quad + \frac{1}{2} \ell_3(R_l(\lambda_2), R^l(\ell_1(\lambda_1)), A) \\ & \quad + \frac{1}{2} \ell_3(R_l(\lambda_2), R_k(A), R^k R^l(\ell_1(\lambda_1))) \\ & \quad + \ell_2(\ell_2(R_k(\bar{1})R^l(\lambda_1), R_k(\bar{2})(A)), R^k R_l(\lambda_2)) \\ &= -\ell_3(\lambda_1, \ell_1(\lambda_2), A) - \ell_2(\ell_2(R_k(\lambda_2), R_l(A)), R^l R^k(\lambda_1)) - \frac{1}{2} \ell_3(\ell_1(\lambda_1), \lambda_2, A) \\ & \quad + \frac{1}{2} \ell_3(R_k(\bar{1})(\lambda_2), R_k(\bar{2})(A), R^k(\ell_1(\lambda_1))) \end{aligned}$$

$$\begin{aligned}
& + \ell_2(\ell_2(R_k R^l(\lambda_1), R_m(A)), R^m R^k R_l(\lambda_2)) \\
& = -\ell_3(\lambda_1, \ell_1(\lambda_2), A) - \frac{1}{2} \ell_3(\ell_1(\lambda_1), \lambda_2, A) \\
& \quad - \frac{1}{2} \ell_3(R_m R_l R^k(\ell_1(\lambda_1)), R^m R_k(\bar{1})(\lambda_2), R^l R_k(\bar{2})(A)) \\
& \quad - \ell_2(\ell_2(R_k(\lambda_2), R_l(A)), R^l R^k(\lambda_1)) + \ell_2(\ell_2(\lambda_1, R_m(A)), R^m(\lambda_2)) \\
& = -\ell_3(\lambda_1, \ell_1(\lambda_2), A) - \frac{1}{2} \ell_3(\ell_1(\lambda_1), \lambda_2, A) \\
& \quad - \frac{1}{2} \ell_3(R_l R^k(\ell_1(\lambda_1)), R_{(\bar{1})}^l R_k(\bar{1})(\lambda_2), R_{(\bar{2})}^l R_k(\bar{1})(A)) \\
& \quad - \ell_2(\ell_2(R_k(\lambda_2), R_l(A)), R^l R^k(\lambda_1)) + \ell_2(\ell_2(\lambda_1, R_m(A)), R^m(\lambda_2)) \\
& = -\ell_3(\lambda_1, \ell_1(\lambda_2), A) - \ell_3(\ell_1(\lambda_1), \lambda_2, A) \\
& \quad - \ell_2(\ell_2(R_k(\lambda_2), R_l(A)), R^l R^k(\lambda_1)) + \ell_2(\ell_2(\lambda_1, R_m(A)), R^m(\lambda_2)) \\
& = \ell_2(\ell_2(\lambda_1, \lambda_2), A)
\end{aligned}$$

as required. In the first equality we used $U\mathfrak{v}$ -equivariance and braided antisymmetry of ℓ_2 . In the second equality we used braided antisymmetry of ℓ_3 and the second $\mathcal{R}$ -matrix identity (3.7). In the third equality we used braided symmetry on degree 1 elements followed by braided antisymmetry involving degree 0 elements. In the fourth equality we used the first $\mathcal{R}$ -matrix identity from (3.7). In the final step we used the braided homotopy relation (4.6) together with the vanishing brackets $\ell_3(\lambda_1, \lambda_2, A) = 0$ and $\ell_3(\lambda_1, \lambda_2, \ell_1(A)) = 0$ by our requirements (5.21). Similarly, one checks explicitly the leading orders of the closure relation (5.22) evaluated on a field $A \in V_1$ for right braided gauge transformations (5.19). A complete proof, including the general statement of the closure relation, is beyond the scope of this paper.

Example 5.24 Returning to Example 5.6, let us now consider a 4-term braided L_∞ -algebra whose kinematical sector is given by the homogeneous subspaces

$$V_0 = \Omega^0(M, \mathfrak{g}) \quad \text{and} \quad V_1 = \Omega^p(M, W) \times \Omega^1(M, \mathfrak{g}).$$

Comparing the left and right braided gauge transformations with prescriptions (5.18) and (5.19) we read off the differential as

$$\ell_1^*(\lambda) = (0, d\lambda),$$

while the only nonzero brackets involving gauge parameters and fields are given by

$$\ell_2^*(\lambda, (\phi, a)) = (-\lambda \triangleright_\star \phi, -[\lambda, a]_\mathfrak{g}^*) \quad \text{and} \quad \ell_2^*((\phi, a), \lambda) = (\phi \star \triangleleft \lambda, -[a, \lambda]_\mathfrak{g}^*).$$

The brackets in this sector satisfy (5.21), and comparing (5.10) with (5.22) we find the nonzero 2-bracket among gauge parameters

$$\ell_2^*(\lambda_1, \lambda_2) = -[\lambda_1, \lambda_2]_\mathfrak{g}^*.$$

Example 5.24 illustrates the typical features of braided gauge symmetries in most field theories of interest in physics, and in particular those treated in the present paper.

Given the complexity of the calculations illustrated above using braided homotopy relations even at the lowest orders, for simplicity and ease of exposition, in the following we will impose the further vanishing conditions

$$\ell_{n+1}(\lambda, A_1, \dots, A_n) = 0 \quad \text{and} \quad \ell_{n+1}(\lambda, F, A_1, \dots, A_{n-1}) = 0, \quad (5.25)$$

for all $n \geq 2$, $\lambda \in V_0$, $A_1, \dots, A_n \in V_1$ and $F \in V_2$. The first condition implies that the braided gauge transformations are at most linear in the field A , that is, $\delta_\lambda^L A = \ell_1(\lambda) + \ell_2(\lambda, A)$ and $\delta_\lambda^R A = \ell_1(\lambda) - \ell_2(A, \lambda)$. The second condition is again equivalent to the first for (braided or strictly) cyclic L_∞ -algebras. With these assumptions, the lowest orders calculations presented above constitute a complete proof of the closure condition (5.22).

5.6 Braided covariant dynamics

For a field $A \in V_1$, the braided analog of (2.5) which dictates the field equations follows exactly as in the classical case.

Definition 5.26 For $A \in V_1$, the *braided field equations* $F_A = 0$ in V_2 are given by

$$F_A := \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \ell_n(A, \dots, A). \quad (5.27)$$

Left gauge covariance

Expression (5.27) is covariant under the left gauge transformations discussed in Sect. 5.5, namely

$$\delta_\lambda^L F_A = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (-1)^{\frac{1}{2}n(n-1)} \sum_{i=0}^n (-1)^i \ell_{n+2}(\lambda, A^{\otimes i}, F_A, A^{\otimes n-i}), \quad (5.28)$$

for all gauge parameters $\lambda \in V_0$. In the classical case, where $\mathcal{R} = 1 \otimes 1$, expression (5.28) agrees with (2.6).

Let us indicate why (5.28) is true. We compute the gauge transformation of (5.27) using (5.18) and similar calculations to those in Sect. 5.5:

$$\begin{aligned} \delta_\lambda^L F_A &= \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \ell_n(\Delta_{\mathcal{F}}^n(\delta_\lambda^L)(A \otimes \dots \otimes A)) \\ &= \sum_{n,m=0}^{\infty} \frac{(-1)^{\frac{1}{2}n(n+1) + \frac{1}{2}m(m-1)}}{(n+1)!m!} \\ &\quad \sum_{i=0}^n \ell_{n+1}(\mathcal{R}_k(A^{\otimes i}), \ell_{m+1}(\mathcal{R}^k(\lambda), A^{\otimes m}), A^{\otimes n-i}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-1)^{\frac{1}{2}n(n-1)}}{n!} \sum_{j=0}^n \sum_{i=0}^{n-j} \frac{(-1)^{(j-1)(n-j)}}{n-j+1} \binom{n}{j} \ell_{n-j+1}(\mathbf{R}_k(A^{\otimes i}), \ell_{j+1} \\
&\quad (\mathbf{R}^k(\lambda), A^{\otimes j}), A^{\otimes n-i-j})
\end{aligned} \quad (5.29)$$

where in the last step we reorganized the double sum. On the other hand, using (5.27) the right-hand side of (5.28) is equal to

$$\begin{aligned}
&\sum_{n,m=0}^{\infty} \frac{(-1)^{\frac{1}{2}n(n-1)+\frac{1}{2}m(m-1)}}{(n+1)!m!} \sum_{i=0}^n (-1)^i \ell_{n+2}(\lambda, A^{\otimes i}, \ell_m(A^{\otimes m}), A^{\otimes n-i}) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{\frac{1}{2}n(n-1)}}{n!} \sum_{j=0}^n \sum_{i=0}^{n-j} \frac{(-1)^{j(n-j)+i}}{n-j+1} \binom{n}{j} \ell_{n-j+2}(\lambda, A^{\otimes i}, \ell_j(A^{\otimes j}), A^{\otimes n-i-j})
\end{aligned}$$

where we set $\ell_0 := 0$.

Thus, collecting terms order by order in tensor powers of the fields $A^{\otimes n}$, we see that (5.28) is satisfied if

$$\begin{aligned}
&\sum_{j=0}^n \sum_{i=0}^{n-j} \frac{(-1)^{j(n-j)}}{n-j+1} \binom{n}{j} \left((-1)^{n-j} \ell_{n-j+1}(\mathbf{R}_k(A^{\otimes i}), \ell_{j+1}(\mathbf{R}^k(\lambda), A^{\otimes j}), A^{\otimes n-i-j}) \right. \\
&\quad \left. - (-1)^i \ell_{n-j+2}(\lambda, A^{\otimes i}, \ell_j(A^{\otimes j}), A^{\otimes n-i-j}) \right) = 0
\end{aligned} \quad (5.30)$$

for each $n \geq 0$. This now follows by the braided homotopy relations $\mathcal{J}_{n+1}(\lambda, A^{\otimes n}) = 0$ from (4.2). For $n = 0$ expression (5.30) reads $\ell_1(\ell_1(\lambda)) = 0$, which is the differential identity in (4.5). For $n = 1$ the left-hand side of expression (5.30) reads

$$\begin{aligned}
&-\frac{1}{2} \ell_2(\ell_1(\lambda), A) - \frac{1}{2} \ell_2(\mathbf{R}_k(A), \ell_1(\mathbf{R}^k(\lambda))) + \ell_1(\ell_2(\lambda, A)) - \ell_2(\lambda, \ell_1(A)) \\
&= \ell_1(\ell_2(\lambda, A)) - \ell_2(\ell_1(\lambda), A) - \ell_2(\lambda, \ell_1(A)) = 0
\end{aligned}$$

as required, where we used $U\mathfrak{v}$ -equivariance of ℓ_1 and braided symmetry of ℓ_2 on degree 1 elements in the first step, and the derivation identity of (4.5) in the last step. For $n = 2$, with similar manipulations one can reduce the left-hand side of (5.30) to

$$\begin{aligned}
&\frac{1}{3} \ell_3(\ell_1(\lambda), A, A) + \frac{1}{3} \ell_3(\mathbf{R}_k(A), \ell_1(\mathbf{R}^k(\lambda)), A) \\
&\quad + \frac{1}{3} \ell_3(\mathbf{R}_k(\bar{1})(A), \mathbf{R}_k(\bar{2})(A), \ell_1(\mathbf{R}^k(\lambda))) \\
&\quad + \ell_2(\ell_2(\lambda, A), A) - \ell_2(\mathbf{R}_k(A), \ell_2(\mathbf{R}^k(\lambda), A)) + \ell_3(\lambda, \ell_1(A), A) \\
&\quad - \ell_3(\lambda, A, \ell_1(A)) \\
&\quad + \ell_1(\ell_3(\lambda, A, A)) - \ell_2(\lambda, \ell_2(A, A))
\end{aligned}$$

$$\begin{aligned}
&= \ell_3(\ell_1(\lambda), A, A) + \ell_3(\lambda, \ell_1(A), A) - \ell_3(A, A, \ell_1(A)) + \ell_1(\ell_3(\lambda, A, A)) \\
&\quad + \ell_2(\ell_2(\lambda, A), A) + \ell_2(\ell_2(\lambda, R_k(A)), R^k(A)) \\
&\quad + \ell_2(\ell_2(R_k(A), R_l(A)), R^l R^k(\lambda)) = 0
\end{aligned}$$

as required, where in the final step we used (4.6).

For $n \geq 3$, we will only give the proof under the vanishing conditions (5.25) that we motivated in Sect. 5.5; that is, we show $\delta_\lambda^\perp F_A = \ell_2(\lambda, F_A)$. In this case, all summands of the second line of (5.30) vanish except for the term with $j = n$, which is $-\ell_2(\lambda, \ell_n(A^{\otimes n}))$. In the first line, only the summands with $j = 0, 1$ are nonzero. For $j = 0$ the contributions to the sum over i are

$$\ell_{n+1}(R_k(A^{\otimes i}), \ell_1(R^k(\lambda)), A^{\otimes n-i}) = \ell_{n+1}(\ell_1(\lambda), A^{\otimes n})$$

for each $i = 0, 1, \dots, n$, where we used $U\mathfrak{v}$ -equivariance of the bracket ℓ_1 and braided symmetry of the brackets ℓ_{n+1} on degree 1 elements. For $j = 1$ the contributions to the sum over i can be written in the form

$$\begin{aligned}
\ell_n(R_k(A^{\otimes i}), \ell_2(R^k(\lambda), A), A^{\otimes n-1-i}) &= \ell_n(R_l(\ell_2(R^k(\lambda), A), R^l R_k(A^{\otimes i}), A^{\otimes n-1-i})) \\
&= \ell_n(\ell_2(\lambda, R_k(A)), R^k(A^{\otimes i}), A^{\otimes n-1-i})
\end{aligned}$$

for each $i = 0, 1, \dots, n-1$, where in the first equality we used braided symmetry of ℓ_n on degree 1 elements, and in the last step we used the first $\mathcal{R}$ -matrix identity from (3.7) when multiplied on the right by $R^k \otimes 1 \otimes R_k$ and acted on by the braided antisymmetric map ℓ_2 of the first two factors.

Collecting everything together, condition (5.30) in this way reads

$$\begin{aligned}
(-1)^n \ell_{n+1}(\ell_1(\lambda), A^{\otimes n}) &= \ell_2(\lambda, \ell_n(A^{\otimes n})) \\
&\quad - \sum_{i=1}^n \ell_n(\ell_2(\lambda, R_k(A)), R^k(A^{\otimes i-1}), A^{\otimes n-i}).
\end{aligned} \tag{5.31}$$

But (5.31) is just the braided homotopy relation $\mathcal{J}_{n+1}(\lambda, A^{\otimes n}) = 0$ from (4.2) in this case: terms in the sum over i in $\mathcal{J}_{n+1}(\lambda, A^{\otimes n})$ with $i \neq 1, 2, n$ all vanish because each one involves a bracket from (5.25). After removing the terms involving the vanishing brackets from (5.25), the remaining terms in $\mathcal{J}_{n+1}(\lambda, A^{\otimes n}) = 0$ with $i = 1, 2, n$ are precisely relations (5.31) for all $n \geq 3$.

Right gauge covariance

The covariance of (5.27) under right gauge transformations is similarly given by

$$\delta_{\lambda}^R F_A = \delta_{R_k(\lambda)}^L R^k(F_A) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (-1)^{\frac{1}{2}n(n+1)} \sum_{i=0}^n (-1)^{i+1} \ell_{n+2}(A^{\otimes i}, F_A, A^{\otimes n-i}, \lambda).$$

The proof is completely analogous to that above and will not be repeated.

Because of this, in the remainder of this paper we will mostly only mention left gauge transformations δ_{λ}^L explicitly, with the corresponding statements for right gauge transformations being the obvious modifications involving suitable rearrangements of arguments and sign changes. Indeed left and right braided gauge transformations are not independent, as the right transformations are defined in terms of left transformations (cf. (5.19)). This will become apparent in Sect. 5.7, where we will also see that they both induce the same interdependence among the field equations.

Space of physical states

One of the novelties of braided field theory is the extent to which its symmetries can be regarded as genuine gauge symmetries from a dynamical perspective. Recall that in the classical case of an ordinary field theory on a manifold M , by a (infinitesimal) gauge symmetry one usually means an infinite-dimensional symmetry parametrized by sections of a vector bundle over M ; in the case of Lagrangian field theories these are symmetries of the action functional. As such, they induce redundancies in the degrees of freedom which are solutions of the field equations. For example, suppose that M is a globally hyperbolic spacetime. Choosing a Cauchy surface $\Sigma \subset M$, along with values of the fields and their jets on Σ representing initial data, for any solution of the field equations which extends the values of the fields on M , one may use the gauge symmetries to produce new solutions with the same initial data. Generally, by Noether's second theorem, gauge symmetries are in a one-to-one correspondence with differential identities among the field equations, which further exhibits that not all degrees of freedom are independent: the space of *physical* states is the moduli space of classical solutions modulo gauge transformations.

In a braided field theory, the situation is markedly different. In this case there is no properly defined moduli space of classical solutions, as braided gauge transformations do not generally act on the subspace of solutions $A \in V_1$ to the field equations $F_A = 0$ and one cannot take the quotient by braided gauge transformations. In other words, braided gauge symmetries *do not* produce new solutions of the field equations. To understand this point, let us consider the covariance of the field equations. Let $A \in V_1$ be a solution of

$$F_A = 0.$$

From covariance (5.28) it then follows that

$$\delta_\lambda^L F_A = 0$$

on solutions of $F_A = 0$, for any gauge parameter $\lambda \in V_0$. However, computing the first order variation in the infinitesimal gauge parameter we find

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} F_{A+t\delta_\lambda^L A} &= \frac{d}{dt} \Big|_{t=0} \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \ell_n(A + t\delta_\lambda^L A, \dots, A + t\delta_\lambda^L A) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \sum_{i=0}^n \ell_n(A^{\otimes i}, \delta_\lambda^L A, A^{\otimes n-1-i}) \\ &= \sum_{n,m=0}^{\infty} \frac{(-1)^{\frac{1}{2}n(n+1) + \frac{1}{2}m(m-1)}}{(n+1)!m!} \sum_{i=0}^n \ell_{n+1}(A^{\otimes i}, \ell_{m+1}(\lambda, A^{\otimes m}), A^{\otimes n-i}). \end{aligned}$$

This differs from the second line of (5.29) by the $\mathcal{R}$ -matrix factors in the sum over i , and we have shown that in general

$$F_{A+\delta_\lambda^L A} \neq F_A + \delta_\lambda^L F_A,$$

even at order $\mathcal{O}(\lambda^1)$, except in the classical case where $\mathcal{R} = 1 \otimes 1$ and the expected equality holds. This implies that a braided gauge transformation of a solution A to the field equations $F_A = 0$ may result in a field with $F_{A+\delta_\lambda^L A} \neq 0$, even at order $\mathcal{O}(\lambda^1)$.

What is then the meaning of a braided gauge symmetry, and why should one even speak of them as ‘gauge’ transformations? The answer comes from a braided version of Noether’s second theorem: Even though the solution generating aspect of gauge transformations is lost, they still induce interdependence of the field equations, and hence gauge redundancies in this sense. These identities follow, analogously to the classical case, from our braided L_∞ -algebra formalism, and is our next step in the construction of a braided field theory in this formulation.

5.7 Braided Noether identities

The most complicated part of the story, and yet the most important in light of our discussion about the significance of braided gauge symmetries from Sect. 5.6, concerns the braided version of the Noether identities. They are relations for the field equations F_A which hold off-shell, that is, when $F_A \neq 0$, and are a consequence of the braided homotopy relations $\mathcal{J}_n(A^{\otimes n}) = 0$ for all $n \geq 1$. In Sect. 5.8, we will discuss how they correspond to braided gauge symmetries.

Working under the same assumptions (5.25), we will prove the general vanishing identity

$$\begin{aligned}
d_A F_A &:= \ell_1(F_A) + \frac{1}{2} (\ell_2(F_A, A) - \ell_2(A, F_A)) \\
&\quad - \sum_{n=3}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \ell_1(\ell_n(A, \dots, A)) \\
&\quad - \sum_{n=2}^{\infty} \frac{1}{2n!} (-1)^{\frac{1}{2}n(n-1)} \left(\ell_2(\ell_n(A, \dots, A), A) - \ell_2(A, \ell_n(A, \dots, A)) \right) \\
&= 0.
\end{aligned} \tag{5.32}$$

We comment on the general case below. Note that, in contrast to the classical case (2.8), the operation $d_A F_A$ is no longer linear in the field equations F_A and contains inhomogeneous terms involving brackets of the fields A themselves. This is related to the violations of the Bianchi identities which we discussed at the end of Sect. 5.4, and we will come back to this point in our explicit examples later on.

To prove identity (5.32), we first note that, for any 4-term braided L_∞ -algebra with a (braided or strict) cyclic structure of degree -3 , the second of the conditions in (5.25) is equivalent to

$$\ell_{n+1}(F, A_1, \dots, A_n) = 0, \tag{5.33}$$

for all $n \geq 2$, $A_1, \dots, A_n \in V_1$ and $F \in V_2$. This vanishing condition significantly simplifies the braided homotopy relation $\mathcal{J}_n(A^{\otimes n}) = 0$ from (4.2): in this case the terms with ℓ_{n+1-i} acting then vanish for $i \leq n-2$, and the sum over i receives nonzero contributions only from terms with $i = n-1, n$.

The derivation of (5.32) follows by considering a particular weighted sum of braided homotopy relations $\mathcal{J}_n(A^{\otimes n}) = 0$, which vanishes term by term, expanding each $\mathcal{J}_n(A^{\otimes n})$ in brackets, and manipulating the expressions algebraically to make the field equations F_A appear ‘maximally’, as many times as possible. We start from

$$\begin{aligned}
0 &= \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \mathcal{J}_n(A^{\otimes n}) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \sum_{i=1}^n (-1)^{i(n-i)} \\
&\quad \sum_{\sigma \in \text{Sh}_{i, n-i}} \text{sgn}(\sigma) \ell_{n+1-i} \circ (\ell_i \otimes \text{id}^{\otimes n-i}) \circ \mathcal{F}\sigma(A^{\otimes n}) \\
&= \ell_1(\ell_1(A)) - \frac{1}{2} \left(\ell_1(\ell_2(A, A)) - \ell_2(\ell_1(A), A) + \ell_2(A, \ell_1(A)) \right) \\
&\quad + \sum_{n=3}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \left((-1)^{n-1} \sum_{\sigma \in \text{Sh}_{n-1, 1}} \text{sgn}(\sigma) \ell_2 \circ (\ell_{n-1} \otimes \text{id}) \circ \mathcal{F}\sigma \right. \\
&\quad \left. + \sum_{\sigma \in \text{Sh}_{n, 0}} \text{sgn}(\sigma) \ell_1 \circ \ell_n \circ \mathcal{F}\sigma \right) (A^{\otimes n}),
\end{aligned} \tag{5.34}$$

where in the third equality we expanded out the $n = 1, 2$ terms, while using the simplification of $\mathcal{J}_n(A^{\otimes n})$ for $n \geq 3$. Next we notice the last sum over $(n, 0)$ -shuffled permutations contains only the identity permutation, thus combining the first term, the second term and the $\ell_1 \circ \ell_n$ terms of the sum:

$$\begin{aligned} 0 &= \ell_1 \left(\ell_1(A) - \frac{1}{2} \ell_2(A, A) + \sum_{n=3}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)} \ell_n(A^{\otimes n}) \right) \\ &\quad + \frac{1}{2} \left(\ell_2(\ell_1(A), A) - \ell_2(A, \ell_1(A)) \right) \\ &\quad + \sum_{n=3}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2}n(n-1)+n-1} \sum_{\sigma \in \text{Sh}_{n-1,1}} \text{sgn}(\sigma) \ell_2 \circ (\ell_{n-1} \otimes \text{id}) \circ \mathcal{F}\sigma(A^{\otimes n}) \\ &= \ell_1(F_A) + \frac{1}{2} \left(\ell_2(\ell_1(A), A) - \ell_2(A, \ell_1(A)) \right) \\ &\quad + \sum_{n=2}^{\infty} \frac{1}{(n+1)!} (-1)^{\frac{1}{2}n(n-1)} \sum_{i=0}^n \ell_2(\ell_n(A^{\otimes n-i}, R_k(A^{\otimes i})), R^k(A)), \end{aligned}$$

where we cancelled the sign factor $\text{sgn}(\sigma)$ with the same sign coming out of the graded (braided) permutation after acting with $\mathcal{F}\sigma$ on $A^{\otimes n}$, since A is of degree 1. Next we separate out the terms with $i = 0, n$ from the sum over i , write $\frac{1}{(n+1)!} = \frac{1}{2n!} - \frac{n-1}{2(n+1)!}$, and use braided antisymmetry for the $i = n$ term to get

$$\begin{aligned} 0 &= \ell_1(F_A) + \frac{1}{2} \left(\ell_2(\ell_1(A), A) - \ell_2(A, \ell_1(A)) \right) \\ &\quad + \sum_{n=2}^{\infty} \left(\frac{1}{2n!} - \frac{n-1}{2(n+1)!} \right) (-1)^{\frac{1}{2}n(n-1)} \left(\ell_2(\ell_n(A^{\otimes n}), A) - \ell_2(A, \ell_n(A^{\otimes n})) \right) \\ &\quad + \sum_{n=2}^{\infty} \frac{1}{(n+1)!} (-1)^{\frac{1}{2}n(n-1)} \sum_{i=1}^{n-1} \ell_2(\ell_n(A^{\otimes n-i}, R_k(A^{\otimes i})), R^k(A)) \\ &= \ell_1(F_A) + \frac{1}{2} \left(\ell_2(F_A, A) - \ell_2(A, F_A) \right) \\ &\quad - \frac{1}{2} \sum_{n=2}^{\infty} \frac{n-1}{(n+1)!} \left(\ell_2(\ell_n(A^{\otimes n}), A) - \ell_2(A, \ell_n(A^{\otimes n})) \right) \\ &\quad + \sum_{n=2}^{\infty} \frac{1}{(n+1)!} (-1)^{\frac{1}{2}n(n-1)} \sum_{i=1}^{n-1} \ell_2(\ell_n(A^{\otimes n-i}, R_k(A^{\otimes i})), R^k(A)), \quad (5.35) \end{aligned}$$

where in the second equality we combine the second and third terms with the $\frac{1}{2n!}$ terms of the first sum.

Expression (5.35) is one form of the braided Noether identities. Written in this way, the reduction to the classical case is immediate: when $\mathcal{R} = 1 \otimes 1$ the terms within the sum over i are all equal and so the two sums cancel each other by strict antisymmetry of

bracket ℓ_2 , while the second and third terms likewise combine, thus reducing identity (5.35) to the classical Noether identity (2.8) (when (5.33) is applied), as expected. Identity (5.35) can be written in a slightly simpler fashion by using $\mathcal{J}_{n+1}(A^{\otimes n+1}) = 0$ to rewrite the sum over i as

$$\begin{aligned} \sum_{i=1}^{n-1} \ell_2(\ell_n(A^{\otimes n-i}, R_k(A^{\otimes i})), R^k(A)) &= \ell_2(A, \ell_n(A^{\otimes n})) - \ell_2(\ell_n(A^{\otimes n}), A) \\ &\quad - (-1)^n \ell_1(\ell_{n+1}(A^{\otimes n+1})). \end{aligned}$$

Substituting this in (5.35), after a short manipulation of the prefactors we arrive at (5.32).

It is clear from this calculation how the braided Noether identities follow for any braided field theory. One simply adds up all the braided homotopy relations $\mathcal{J}_n(A^{\otimes n}) = 0$ and makes the field equations appear within the other brackets. The field equations F_A will appear within brackets up to as high order as the brackets which appear in the braided gauge transformation formula (5.18), like they did from restriction (5.33) in the present case; many more extra brackets involving solely the fields A will appear in the most general case. The details of the general case are beyond the scope of this paper and will be left for future investigations, as the present result is sufficient for all the theories we shall consider.

The correspondence with braided gauge symmetries is immediately apparent in an action formalism for the braided field theory, where the braided Noether identities $d_A F_A$ are ‘dual’ to the braided gauge transformations $\delta_\lambda^L A$ and $\delta_\lambda^R A$ via a cyclic structure on the braided L_∞ -algebra, as in the classical case from Sect. 2.2. Indeed this is how Noether identities are usually derived in classical field theories, and we will discuss this further in Sect. 5.8. But we stress that this is unnecessary and the Noether identities follow from the known braided homotopy relations in the L_∞ -algebra approach to field theories.

5.8 Action formulation

The variational formulation of the braided covariant field equations helps to elucidate the meaning of braided gauge symmetries as discussed previously, while also demonstrating some unexpected subtleties and differences in the action principle as compared to the classical case. Although all of the dynamics of a braided field theory can be encoded in the braided L_∞ -algebra structure as we have thus far described, the action formalism will make more clear to what extent particular examples of braided field theories differ from their classical counterparts.

The field equations $F_A = 0$, given by (5.27), can be described as the stationary locus of an action functional $S : V_1 \rightarrow \mathbb{R}$ which is constructed by endowing our 4-term braided L_∞ -algebra $(V, \{\ell_n\})$ with a non-degenerate *strictly* cyclic pairing $\langle -, - \rangle : V \otimes V \rightarrow \mathbb{R}$ of degree -3 , that is

$$\langle v_0, v_1 \rangle = \langle R_k(v_0), R^k(v_1) \rangle = \langle v_1, v_0 \rangle, \quad (5.36)$$

and

$$\langle v_0, \ell_n(v_1, v_2, \dots, v_n) \rangle = (-1)^{(|v_0|+1)n} \langle v_n, \ell_n(v_0, v_1, \dots, v_{n-1}) \rangle, \quad (5.37)$$

for all $n \geq 1$ and $v_0, v_1, \dots, v_n \in V$. We can interpret the strict symmetry (5.36) as saying that we focus on pairings which are invariant under the natural action of the twisted Hopf algebra. Then the action functional is given exactly as in the classical case.

Definition 5.38 The *braided action functional* $S : V_1 \rightarrow \mathbb{R}$ is

$$S(A) := \sum_{n=1}^{\infty} \frac{1}{(n+1)!} (-1)^{\frac{1}{2}n(n-1)} \langle A, \ell_n(A, \dots, A) \rangle, \quad (5.39)$$

on dynamical fields $A \in V_1$.

Variational principle

The strictness requirement on the cyclic structure ensures that the critical locus of the action functional (5.39) coincides with solutions of the field equations from Sect. 5.6. For a general field variation δA on the tangent space TV_1 to field space, the variation of the action functional (or any other functional of $A \in V_1$) is defined in the usual way by

$$\delta S(A) := \left. \frac{d}{dt} \right|_{t=0} S(A + t \delta A).$$

This defines δ as a strict derivation of degree 0, that is, it obeys an undeformed Leibniz rule with the trivial coproduct $\Delta(\delta) = \delta \otimes \text{id} + \text{id} \otimes \delta$. We further require that δ commutes with the $\mathcal{R}$ -matrices of the braiding, analogously to the action of the exterior derivative d on twisted differential forms, but in contrast to braided gauge transformations. Similarly to Sect. 5.2, we iterate this coproduct to act on maps μ_n defined on $\bigotimes^n V$ by $\delta \mu_n := \mu_n \circ \Delta^n(\delta)$; in particular, we define $\delta \langle -, - \rangle := \langle -, - \rangle \circ \Delta(\delta)$.

Then, the variation δS of the action functional (5.39) proceeds identically to the classical case, where both the cyclicity requirement and the derivation properties of δ are the same: we use

$$\begin{aligned} \delta \langle A, \ell_n(A^{\otimes n}) \rangle &= \langle \delta A, \ell_n(A^{\otimes n}) \rangle + \sum_{i=0}^{n-1} \langle A, \ell_n(A^{\otimes i}, \delta A, A^{\otimes n-i-1}) \rangle \\ &= (n+1) \langle \delta A, \ell_n(A^{\otimes n}) \rangle, \end{aligned}$$

where in the last step we used strict cyclicity to set all terms in the sum equal to the first term. Using (5.27) this shows that

$$\delta S(A) = \langle \delta A, F_A \rangle \quad (5.40)$$

for arbitrary variations δA of the fields. Hence, by non-degeneracy of the pairing, the extrema of the action functional (5.39) are precisely the solutions $A \in V_1$ of the field equations $F_A = 0$.

Braided gauge invariance

The problem of invariance of the action functional (5.39) under braided gauge transformations (5.18) is much more involved and does not simply follow the classical route. A potentially problematic new feature which immediately arises is the following. Recalling our discussion from Sect. 5.2 regarding the consistency with braided symmetric maps, the braided Leibniz rule is not generally compatible with the strict symmetry (5.36) of the cyclic pairing $\langle -, - \rangle$. Strict symmetry means $\langle -, - \rangle = \langle -, - \rangle \circ \tau_{V,V}^\sharp$, but this is not the consistent notion of symmetry for a morphism in the category $\mathcal{F}\mathcal{M}^\sharp$, because it only involves the graded transposition $\tau^\sharp$ which is *not* the braiding isomorphism in $\mathcal{F}\mathcal{M}^\sharp$ unless $\mathcal{R} = 1 \otimes 1$. As a result $\delta_\lambda^L \langle -, - \rangle \neq \delta_\lambda^L (\langle -, - \rangle \circ \tau^\sharp)$; explicitly

$$\begin{aligned} \delta_\lambda^L \langle v_1, v_2 \rangle &= \langle \delta_\lambda^L v_1, v_2 \rangle + \langle R_k(v_1), \delta_{R^k(\lambda)}^L v_2 \rangle = \langle \delta_{R^k(\lambda)}^L v_2, R_k(v_1) \rangle \\ &+ \langle v_2, \delta_\lambda^L v_1 \rangle \neq \delta_\lambda^L \langle v_2, v_1 \rangle, \end{aligned} \quad (5.41)$$

which is not generally consistent with $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$ unless $\mathcal{R} = 1 \otimes 1$ in which case equality holds.¹⁰

Despite this evident inconsistency, the gauge transformation of the action functional (5.39) is nevertheless well-defined, because the cyclicity condition (5.37) ensures compatibility with strict symmetry of the pairing order by order in tensor powers of the fields $A^{\otimes n}$ for $n \geq 1$: Using (5.18), and repeatedly applying strict symmetry (5.36), cyclicity (5.37) and the first $\mathcal{R}$ -matrix identity from (3.7), we compute

$$\begin{aligned} \delta_\lambda^L \langle A, \ell_n(A^{\otimes n}) \rangle &= \langle \delta_\lambda^L A, \ell_n(A^{\otimes n}) \rangle + \langle R_l(A), \ell_n(\Delta_{\mathcal{F}}^n(\delta_{R^l(\lambda)}^L)(A^{\otimes n})) \rangle \\ &= \langle A, \ell_n(\delta_\lambda^L A, A^{\otimes n-1}) \rangle \\ &\quad + \sum_{i=0}^{n-2} \langle R_l(A), \ell_n(R_k(A^{\otimes i}), \delta_{R^k R^l(\lambda)}^L A, A^{\otimes n-i-1}) \rangle \\ &\quad + \langle R_l(A), \ell_n(R_k(A^{\otimes n-1}), \delta_{R^k R^l(\lambda)}^L A) \rangle \\ &= \langle \ell_n(\delta_\lambda^L A, A^{\otimes n-1}), A \rangle \\ &\quad + \sum_{i=0}^{n-2} \langle \ell_n(R_l(A), R_k(A^{\otimes i}), \delta_{R^k R^l(\lambda)}^L A, A^{\otimes n-i-2}), A \rangle \\ &\quad + \langle \ell_n(R_l(A), R_k(A^{\otimes n-1})), \delta_{R^k R^l(\lambda)}^L A \rangle \\ &= \langle \ell_n(\delta_\lambda^L A, A^{\otimes n-1}), A \rangle \end{aligned}$$

¹⁰ Strictly speaking, the gauge transformations δ_λ^L are not even defined on maps which do not live in $\mathcal{F}\mathcal{M}^\sharp$, so formally this discussion is not even well-defined from the categorical point of view.

$$\begin{aligned}
& + \sum_{i=0}^{n-2} \langle \ell_n(R_k(A^{\otimes i+1}), \delta_{R^k(\lambda)}^L A, A^{\otimes n-i-2}), A \rangle \\
& + \langle \ell_n(R_k(A^{\otimes n})), \delta_{R^k(\lambda)}^L A \rangle \\
& = \langle \ell_n(\Delta_{\mathcal{F}}^n(\delta_\lambda^L)(A^{\otimes n})), A \rangle + \langle R_k(\ell_n(A^{\otimes n})), \delta_{R^k(\lambda)}^L A \rangle \\
& = \delta_\lambda^L \langle \ell_n(A^{\otimes n}), A \rangle,
\end{aligned} \tag{5.42}$$

where in the fourth equality we used

$$\begin{aligned}
& \ell_n(R_l(A), R_k(A^{\otimes i}), \ell_{m+1}(R^k R^l(\lambda), A^{\otimes m}), A^{\otimes n-i-2}) \\
& = \ell_n(R_k(A^{\otimes i+1}), \ell_{m+1}(R^k(\lambda), A^{\otimes m}), A^{\otimes n-i-2})
\end{aligned}$$

and similarly

$$\langle \ell_n(R_l(A), R_k(A^{\otimes n-1})), \ell_{m+1}(R^k R^l(\lambda), A^{\otimes m}) \rangle = \langle \ell_n(R_k(A^{\otimes n})), \ell_{m+1}(R^k(\lambda), A^{\otimes m}) \rangle,$$

for all $m \geq 0$. Hence, $\delta_\lambda^L S(A)$ is well-defined.

Recall that in the classical case, the exact same cyclicity argument as that used to arrive at (5.40) would show that a general gauge transformation of the action functional can be written as $\delta_\lambda S(A) = \langle \delta_\lambda A, F_A \rangle$ (cf. (2.10)). In this case, a classical gauge transformation can be identified with a specific direction in the tangent space TV_1 to field space, as a special instance of a general variation of the action functional. However, while this identification is essentially tautological in the classical case, it ceases to be true in our braided setting, because general variations and braided gauge transformations are treated differently: a transformation $\delta_\lambda^L A$ of a field $A \in V_1$ along a gauge parameter $\lambda \in V_0$ is no longer a special case of a general variation δA of the field, because the gauge transformations are now defined as *braided* derivations, in contrast to the general variations. In other words, in the braided case the first equality in (2.10) simply no longer holds in general:

$$\delta_\lambda^L S(A) \neq \langle \delta_\lambda^L A, F_A \rangle. \tag{5.43}$$

This is explicitly evident in the calculation of (5.42).

With these preliminary considerations out of the way, we will now establish braided gauge invariance of the action functional:

$$\delta_\lambda^L S(A) = 0 \tag{5.44}$$

for all $\lambda \in V_0$ and $A \in V_1$. Because of (5.43), we have to work much harder than in the classical case. By using steps analogous to those in (5.29), we compute the gauge variation of the action functional (5.39) using (5.18) to write

$$\delta_\lambda^L S(A) = \sum_{n,m=0}^{\infty} \frac{(-1)^{\frac{1}{2}n(n+1) + \frac{1}{2}m(m-1)}}{(n+2)!m!} \left(\langle \ell_{m+1}(\lambda, A^{\otimes m}), \ell_{n+1}(A^{\otimes n+1}) \rangle \right)$$

$$+ \sum_{i=0}^n \langle R_l(A), \ell_{n+1}(R_k(A^{\otimes i}), \ell_{m+1}(R^k R^l(\lambda), A^{\otimes m}), A^{\otimes n-i}) \rangle.$$

Next we use cyclicity as well as the by now familiar manipulations with the $\mathcal{R}$ -matrix identities (3.7) to isolate the gauge parameter λ in the left entries of the pairings. This yields

$$\langle \ell_{m+1}(\lambda, A^{\otimes m}), \ell_{n+1}(A^{\otimes n+1}) \rangle = (-1)^{m+1} \langle \lambda, \ell_{m+1}(A^{\otimes m}, \ell_{n+1}(A^{\otimes n+1})) \rangle,$$

and

$$\begin{aligned} & \langle R_l(A), \ell_{n+1}(R_k(A^{\otimes i}), \ell_{m+1}(R^k R^l(\lambda), A^{\otimes m}), A^{\otimes n-i}) \rangle \\ &= \langle A, \ell_{n+1}(R_l(A), R_k(A^{\otimes i}), \ell_{m+1}(R^k R^l(\lambda), A^{\otimes m}), A^{\otimes n-i-1}) \rangle \\ &= \langle A, \ell_{n+1}(R_l(A^{\otimes i+1}), \ell_{m+1}(R^l(\lambda), A^{\otimes m}), A^{\otimes n-i-1}) \rangle \\ &= \langle A, \ell_{n+1}(R_k(\ell_{m+1}(R^l(\lambda), A^{\otimes m})), R^k R_l(A^{\otimes i+1}), A^{\otimes n-i-1}) \rangle \\ &= \langle A, \ell_{n+1}(\ell_{m+1}(\lambda, R_k(A^{\otimes m})), R^k(A^{\otimes i+1}), A^{\otimes n-i-1}) \rangle \\ &= \langle \ell_{m+1}(\lambda, R_k(A^{\otimes m})), \ell_{n+1}(R^k(A^{\otimes i+1}), A^{\otimes n-i}) \rangle \\ &= (-1)^{m+1} \langle \lambda, \ell_{m+1}(R_k(A^{\otimes m}), \ell_{n+1}(R^k(A^{\otimes i+1}), A^{\otimes n-i})) \rangle, \end{aligned}$$

for $i = 0, 1, \dots, n$, where in the first equality we used cyclicity, in the second equality we used the first $\mathcal{R}$ -matrix identity from (3.7), in the third equality we used braided symmetry of the bracket ℓ_{n+1} , in the fourth equality we used the first $\mathcal{R}$ -matrix identity from (3.7) again, and in the last two equalities we applied cyclicity again. Thus the gauge transformation is given by

$$\begin{aligned} \delta_\lambda^L S(A) &= - \sum_{n,m=0}^{\infty} \frac{(-1)^{\frac{1}{2}n(n+1) + \frac{1}{2}m(m+1)}}{(n+2)!m!} \\ & \quad \sum_{i=0}^{n+1} \langle \lambda, \ell_{m+1}(R_k(A^{\otimes m}), \ell_{n+1}(R^k(A^{\otimes i}), A^{\otimes n-i+1})) \rangle \\ &= - \sum_{n=0}^{\infty} \frac{(-1)^{\frac{1}{2}n(n+1)}}{n!} \sum_{j=0}^n \sum_{i=0}^{n-j+1} \frac{(-1)^j (n-j)}{(n-j+1)(n-j+2)} \binom{n}{j} \\ & \quad \times \langle \lambda, \ell_{j+1}(R_k(A^{\otimes j}), \ell_{n-j+1}(R^k(A^{\otimes i}), A^{\otimes n-i-j+1})) \rangle, \end{aligned} \quad (5.45)$$

for all $\lambda \in V_0$.

By collecting terms order by order in tensor powers $A^{\otimes n+1}$, and using non-degeneracy of the cyclic pairing, we see that (5.44) is satisfied if

$$\sum_{j=0}^n \sum_{i=0}^{n-j+1} \frac{(-1)^j (n-j)}{(n-j+1)(n-j+2)}$$

$$\binom{n}{j} \ell_{j+1}(\mathbf{R}_k(A^{\otimes j}), \ell_{n-j+1}(\mathbf{R}^k(A^{\otimes i}), A^{\otimes n-i-j+1})) = 0 \quad (5.46)$$

for each $n \geq 0$. This is a consequence of the fact that this expression can be brought to the form of the braided homotopy relations $\mathcal{J}_{n+1}(A^{\otimes n+1}) = 0$ from (4.2). The proof follows in exactly the same vein as the proof of braided covariance of the field equations from Sect. 5.6, and will not be repeated here; the details are left for the interested reader to fill in.

Braided Noether identities

We can now make a more precise statement about the correspondence between the braided Noether identities discussed in Sect. 5.7 and braided gauge symmetries of our field theory. Despite the fact that the first equality of (2.10) does not carry over to the braided case (cf. (5.43)), one can still write the gauge transformation of the action functional in the form

$$\delta_\lambda^L S(A) = -\langle \lambda, \mathbf{d}_A F_A \rangle,$$

by using cyclicity to isolate the gauge parameter λ , along with the braided homotopy relations, while making the field equations F_A appear. In this sense the operation $\mathbf{d}_A$ is the ‘braided adjoint’ to δ_λ^L with respect to the cyclic pairing, and the braided Noether identities $\mathbf{d}_A F_A = 0$ then follow from braided gauge invariance $\delta_\lambda^L S(A) = 0$ of the action functional (5.39), as discussed above. This is the way that Noether identities of field theories are usually derived in the classical case.

The isolation of λ was done in calculation (5.45), from which an explicit formula for $\mathbf{d}_A F_A$ can be extracted: the weighted sum of braided homotopy relations (5.34) is exactly the combination that appears in (5.45). What remains to be done is to massage the expression into a form involving F_A explicitly. However, this is not necessary as we saw in Sect. 5.7, as the calculation follows from the braided homotopy relations: the most effective and concise way to derive the braided Noether identities is through the braided L_∞ -algebra framework which we have developed in this section. Similarly, cyclicity of the pairing implies that right braided gauge invariance $\delta_\lambda^R S(A) = 0$ induces the same braided Noether identities $\mathbf{d}_A F_A = 0$.

Drinfel’d twist deformations of action functionals

In our concrete examples later on, the pairing in a braided field theory used to define an action functional will be obtained by twist deformation quantization of the pairing used in a classical field theory. Note, however, that the resulting action functional from the braided L_∞ -algebra formulation is not generally the same as that obtained by naively twist deforming the Lagrangian of a classical field theory, rather we twist only the structure maps of the cyclic L_∞ -algebra structure used for the construction of the classical field theory. We will see explicitly in Sect. 6.3 that the two prescriptions do not generally define the same noncommutative field theory.

5.9 Example: Braided Chern–Simons theory

In the remainder of this paper we apply the general framework developed in this section to concrete examples of (diffeomorphism-invariant) braided field theories. We start with the simplest illustrative example to close off the present section. Similarly to the classical case, our formalism provides a vast generalization of braided noncommutative Chern–Simons gauge theory on a three-dimensional manifold M based on a Lie algebra $\mathfrak{g}$, which is the prototypical example of a braided field theory. Starting from the differential graded Lie algebra of $\mathfrak{g}$ -valued exterior differential forms on M from Sect. 2.3, we use Proposition 4.8 to construct the 4-term differential (graded) braided Lie algebra $(\Omega^\bullet(M, \mathfrak{g})[[\hbar]], \ell_1^*, \ell_2^*)$ from Sect. 3.3 with the brackets

$$\ell_1^*(\alpha) = d\alpha \quad \text{and} \quad \ell_2^*(\alpha, \beta) = -[\alpha, \beta]_{\mathfrak{g}}^*,$$

for $\alpha, \beta \in V = \Omega^\bullet(M, \mathfrak{g})$.

Using (5.18) we define the (left) braided gauge transformation of a gauge field $A \in \Omega^1(M, \mathfrak{g})$ by a gauge parameter $\lambda \in \Omega^0(M, \mathfrak{g})$ as

$$\delta_\lambda^* L A := \ell_1^*(\lambda) + \ell_2^*(\lambda, A) = d\lambda - [\lambda, A]_{\mathfrak{g}}^*. \quad (5.47)$$

This is just the braided gauge transformation of a connection from Examples 5.6 and 5.24; in particular, they close the braided Lie algebra (5.22) on $\Omega^0(M, \mathfrak{g})[[\hbar]]$ with bracket $\ell_2^*(\lambda_1, \lambda_2) = -[\lambda_1, \lambda_2]_{\mathfrak{g}}^*$. Using (5.27), the field equations $F_A^* = 0$ in $\Omega^2(M, \mathfrak{g})[[\hbar]]$ are encoded by

$$F_A^* := \ell_1^*(A) - \frac{1}{2} \ell_2^*(A, A) = dA + \frac{1}{2} [A, A]_{\mathfrak{g}}^*, \quad (5.48)$$

which is just the braided curvature of the gauge field A from Sect. 5.4; in particular, from (5.28) we obtain the anticipated braided gauge covariance

$$\delta_\lambda^* L F_A^* = \ell_2^*(\lambda, F_A^*) = -[\lambda, F_A^*]_{\mathfrak{g}}^*.$$

Thus the braided Chern–Simons field equations state that the braided connection A is flat, which is just the obvious deformation of the corresponding statement from the classical case.

At this point, however, the classical and braided field theories diverge. Recalling from Sect. 5.6 that $F_{A+\delta_\lambda^* L A}^* \neq F_A^* + \delta_\lambda^* L F_A^*$, there is no moduli space of flat connections modulo braided gauge transformations, and the space of physical states cannot be described as in the classical case. This deviation is accounted for by the different Noether identity in the braided case, which is no longer simply the Bianchi identity for the braided curvature, which is generally violated as we discussed in Sect. 5.4. Using (5.32) and the braided homotopy relations, the braided Noether identity in $\Omega^3(M, \mathfrak{g})[[\hbar]]$ is given by

$$d_A^* F_A^* := \ell_1^*(F_A^*) - \frac{1}{2} (\ell_2^*(A, F_A^*) - \ell_2^*(F_A^*, A)) + \frac{1}{4} \ell_2^*(R_k(A), \ell_2^*(R^k(A), A))$$

$$\begin{aligned}
&= dF_A^\star + \frac{1}{2} [A, F_A^\star]_{\mathfrak{g}}^\star - \frac{1}{2} [F_A^\star, A]_{\mathfrak{g}}^\star + \frac{1}{4} [R_k(A), [R^k(A), A]_{\mathfrak{g}}^\star]_{\mathfrak{g}}^\star \\
&= \frac{1}{2} (d_{\star L}^A F_A^\star + d_{\star R}^A F_A^\star) + \frac{1}{4} [R_k(A), [R^k(A), A]_{\mathfrak{g}}^\star]_{\mathfrak{g}}^\star = 0,
\end{aligned} \tag{5.49}$$

in agreement with the deformed Bianchi identities (5.16). Thus $d_A^\star$ does not coincide with the symmetrized braided covariant derivative $\frac{1}{2} (d_{\star L}^A + d_{\star R}^A)$.

Finally, to formulate the braided Chern–Simons action functional, we assume that the three-manifold M is compact and oriented, that $\mathfrak{g}$ is a quadratic Lie algebra, and that the Drinfel'd twist $\mathcal{F} \in U\mathfrak{b}[[\hbar]] \otimes U\mathfrak{b}[[\hbar]]$ is Hermitian and compatible with the cyclic structure (2.11); in this case the compatibility conditions were discussed in Sect. 3.3. By Proposition 4.15, the pairing defined by

$$\langle \alpha, \beta \rangle_\star := \int_M \mathrm{Tr}_{\mathfrak{g}}(\alpha \wedge_\star \beta),$$

with $|\alpha| + |\beta| = 3$, thus determines a strictly cyclic structure on the underlying braided L_∞ -algebra $(\Omega^\bullet(M, \mathfrak{g})[[\hbar]], \ell_1^\star, \ell_2^\star)$.

Then, we can construct a braided gauge invariant action functional $S_\star : \Omega^1(M, \mathfrak{g}) \rightarrow \mathbb{R}[[\hbar]]$, for which the braided Chern–Simons field equation $F_A^\star = 0$ describes its critical locus, according to the general prescription of (5.39):

$$\begin{aligned}
S_\star(A) &= \frac{1}{2} \langle A, \ell_1^\star(A) \rangle_\star - \frac{1}{6} \langle A, \ell_2^\star(A, A) \rangle_\star \\
&= \frac{1}{2} \int_M \mathrm{Tr}_{\mathfrak{g}} \left(A \wedge_\star dA + \frac{1}{3} A \wedge_\star [A, A]_{\mathfrak{g}}^\star \right).
\end{aligned} \tag{5.50}$$

This is just the expected braided deformation of the classical Chern–Simons action functional (2.12) (with the braided Lie bracket). Using graded cyclicity of the integral over M with respect to the twisted exterior product $\wedge_\star$, in the sense of Sect. 3.3, along with invariance of the quadratic form $\mathrm{Tr}_{\mathfrak{g}}$ on $\mathfrak{g}$ and the usual $\mathcal{R}$ -matrix identities (3.7), one may check explicitly that this action functional is invariant under the braided gauge transformations (5.47), that it gives the field equations defined by (5.48) under arbitrary variations of the gauge fields δA , and that it yields the braided Noether identity (5.49) along the lines explained in Sect. 5.8 using braided gauge invariance of $S_\star(A)$ and cyclicity of the pairing.

We stress that the constraints imposed in this last step on the manifold M , the Lie algebra $\mathfrak{g}$, and the twist $\mathcal{F}$ are only needed if one wishes to view the field equations as the stationary locus of an action functional. The braided gauge symmetries and dynamics of the braided field theory can be formulated entirely without these restrictions if one is satisfied with regarding it as a non-Lagrangian field theory, which for many (but not all) considerations is sufficient at the classical level.

6 Noncommutative gravity

6.1 Noncommutative deformations of general relativity

Over the last 20 years there has been an ongoing effort to construct consistent noncommutative theories of gravity, motivated from several research programmes. On one hand, it is argued that they should serve as low-energy limits of any fully fledged theory of quantum gravity, retaining certain aspects of quantum-induced noncommutativity. On the other hand, T-duality and the conjectural noncommutativity of certain string theoretic backgrounds suggest that gravity ought to be formulated in such terms as well. These noncommutative theories of gravity rely on the notions of noncommutative spacetime and noncommutative geometry, and in appropriate limits they reduce to classical general relativity. One of the main problems encountered in these approaches is the breaking of the diffeomorphism symmetry of general relativity. In most theories the diffeomorphism symmetry, or at least a part of it, is broken and one needs to make sense of this breaking and of the remaining symmetries, if any survive. To put our theories into proper context with previous literature on the subject, we start by briefly reviewing some previous theories of noncommutative gravity based on deformation quantization in the various flavours that are similar in spirit to the approach taken in the present paper: geometric quantization, star-product deformations (equivalently quantization of Poisson structures), and Drinfel'd twist deformations.

In the second order or metric formalism, noncommutative theories of gravity were originally introduced via Drinfel'd twist deformation quantization in the seminal works [12,15]. In these theories one starts from the Hopf algebra of the classical diffeomorphism symmetry of general relativity and applies a Drinfel'd twist to generate the twisted Hopf algebra of diffeomorphisms, exactly as we have done in the present paper. The noncommutative differential geometry is then covariant with respect to the twisted diffeomorphism symmetry [8]. Consistent noncommutative deformations of the (vacuum) Einstein equations can be written down, and some particular solutions based on twists constructed from Killing or semi-Killing vector fields have been obtained in [5,47], as twist deformations of classical solutions and their symmetries.

In the first order or Cartan formalism, noncommutative gravity can be formulated as a noncommutative gauge theory of the Lorentz or (A)dS group using the enveloping algebra approach and Seiberg–Witten maps [38], as discussed in Sect. 5.3. A noncommutative deformation of the Einstein–Cartan–Palatini action was discussed in [4,10,24,26], and the coupling to matter fields (fermion and gauge fields) was discussed in [6,7,32,35]; a noncommutative extension of Palatini–Holst theory was considered by [28]. A common feature of these models is that the first order correction to the classical theory (in the deformation parameter $\hbar$) vanishes if the corresponding action functional is real. The first non-vanishing correction is then at order $\hbar^2$. A solution of the field equations corresponding to a deformation of Minkowski spacetime via the Moyal–Weyl twist was found in [29]. Some of these theories also have a twisted diffeomorphism symmetry realized via the action of the star-Lie derivative from Sect. 3.4 (see [6,7]). A relation between the Seiberg–Witten map approach and noncommutative theories of gravity via Fedosov deformation quantization of endomorphism bundles

was developed in [34]. Deformed dispersion relations in a κ -deformation of Minkowski space defined via a Jordanian twist were derived in [9].

The main goal of this final section is to apply the formalism of braided field theory that we have developed in Sect. 5 to construct new braided theories of noncommutative gravity. Our approach has many advantages compared to previous treatments. In particular, our theories are symmetric under braided diffeomorphisms, like the theories of [6, 7, 12, 15]. However, in contrast to the first order formalism of [6, 7], our approach retains braided versions of all symmetries with good classical limits, without the need of introducing new spurious degrees of freedom that have no interpretation in the classical world (see the discussion in Sect. 5.3). Our theories have an action formulation and the coupling to fermion fields is straightforward. In both the metric and Cartan approaches, the failure of twisted diffeomorphisms in generating new classical solutions has always been prevalent (though seemingly not noticed before), and our framework clarifies this unconventional feature (see the discussion in Sect. 5.6). We proceed now to our new braided versions of noncommutative gravity, treating separately the three-dimensional and four-dimensional theories in turn.

6.2 Braided gravity in three dimensions

The braided noncommutative deformation of Einstein–Cartan–Palatini gravity in three dimensions follows a similar path to the braided Chern–Simons theory constructed in Sect. 5.9. It serves to introduce as well some of the ingredients that will appear later in the more complicated four-dimensional version of the theory. Our first observation is that the differential graded Lie algebra of Sect. 2.4 is an L_∞ -algebra in the category $\mathcal{M}^\sharp$, that is, the vector space (2.13) is a graded $U\mathfrak{v}$ -module and the brackets ℓ_1, ℓ_2 are $U\mathfrak{v}$ -equivariant; this stems from the diffeomorphism symmetry of general relativity. We proceed to twist deform it using Proposition 4.8 to a differential (graded) braided Lie algebra, starting from the same graded vector space (2.13) and with the same notation. We use the same nonzero brackets ℓ_1 and ℓ_2 , except that now we include an extra term which will accommodate the presence of a cosmological constant $\Lambda \in \mathbb{R}$; this modifies only the 2-brackets $\ell_2((e_1, \omega_1), (e_2, \omega_2))$ between fields in (2.14) by shifting the first slot with the term $2\Lambda e_1 \wedge e_2$ [30].

Braided L_∞ -algebra

The 4-term braided L_∞ -algebra underlying three-dimensional noncommutative gravity has differential given by

$$\ell_1^*(\xi, \rho) = (0, d\rho), \quad \ell_1^*(e, \omega) = (d\omega, de) \quad \text{and} \quad \ell_1^*(E, \Omega) = (0, d\Omega),$$

which as always coincides with the classical differential. The 2-brackets are modified non-trivially from the classical case to

$$\begin{aligned} \ell_2^*((\xi_1, \rho_1), (\xi_2, \rho_2)) &= ([\xi_1, \xi_2]_{\mathfrak{v}}^*, -[\rho_1, \rho_2]_{\mathfrak{so}(3)}^* + L_{\xi_1}^* \rho_2 - L_{R_k(\xi_2)}^* R^k(\rho_1)), \\ \ell_2^*((\xi, \rho), (e, \omega)) &= (-\rho \star e + L_\xi^* e, -[\rho, \omega]_{\mathfrak{so}(3)}^* + L_\xi^* \omega), \end{aligned}$$

$$\begin{aligned}
\ell_2^*((\xi, \rho), (E, \Omega)) &= (-[\rho, E]_{\mathfrak{so}(3)}^* + L_\xi^* E, -\rho \star \Omega + L_\xi^* \Omega), \\
\ell_2^*((\xi, \rho), (\mathcal{X}, \mathcal{P})) &= (dx^\mu \otimes \text{Tr}(\iota_\mu d\bar{f}^k(\rho) \wedge \bar{f}_k(\mathcal{P})) + L_\xi^* \mathcal{X}, -\rho \star \mathcal{P} + L_\xi^* \mathcal{P}), \\
\ell_2^*((e_1, \omega_1), (e_2, \omega_2)) &= -([\omega_1, \omega_2]_{\mathfrak{so}(3)}^* + 2 \wedge e_1 \wedge_\star e_2, \omega_1 \wedge_\star \omega_2 \\
&\quad + R_k(\omega_2) \wedge_\star R^k(e_1)), \\
\ell_2^*((e, \omega), (E, \Omega)) &= (dx^\mu \otimes \text{Tr}(\iota_\mu d\bar{f}^k(e) \wedge \bar{f}_k(E) + \iota_\mu d\bar{f}^k(\omega) \wedge \bar{f}_k(\Omega) \\
&\quad - \iota_\mu \bar{f}^k(e) \wedge d\bar{f}_k(E) \\
&\quad - \iota_\mu \bar{f}^k(\omega) \wedge d\bar{f}_k(\Omega)), R_k(E) \wedge_\star R^k(e) - \omega \wedge_\star \Omega).
\end{aligned} \tag{6.1}$$

In (6.1), the notation $\rho \star e := \rho \triangleright_\star e = \rho^a{}_b \star e^b \mathbb{E}_a$ is shorthand for the left braided fundamental representation of $\rho \in \Omega^0(M, \mathfrak{so}(3))$ on $e \in \Omega^1(M, \mathbb{R}^3)$, as a special instance of the general definition in (5.8); similarly the action of $\omega \in \Omega^1(M, \mathfrak{so}(3))$ on $e \in \Omega^1(M, \mathbb{R}^3)$ is $\omega \wedge_\star e := \omega \triangleright_\star e$, where here the star-exterior product includes matrix multiplication in the fundamental representation of $\mathfrak{so}(3)$. For a vector field $\xi \in \Gamma(TM)$, the star-Lie derivative L_ξ^* is defined in Sect. 3.4. The twisted product $\wedge_\star$ is the tensor product of the star-exterior product of differential forms on M with the undeformed exterior product of multivectors on the vector space $\mathbb{R}^3$; in particular $e_1 \wedge_\star e_2 = R_k(e_2) \wedge_\star R^k(e_1)$. Note that in the fourth and sixth brackets, for generic twists it is not possible to write the first slots entirely in terms of twisted products $\wedge_\star$ or $\otimes_\star$, as the various terms do not commute appropriately; we will come back to this point below.

In the following, we use some calculational identities which are deformations of those used in [30]. One has

$$\begin{aligned}
(\rho \star e) \wedge_\star E &= -\rho \wedge_\star (R_k(E) \wedge_\star R^k(e)), \\
(\rho \star e) \wedge_\star e \wedge_\star e &= -\rho \wedge_\star (R_k(e \wedge_\star e) \wedge_\star R^k(e)).
\end{aligned}$$

In the classical case, the second line vanishes, but in the noncommutative theory

$$(e \wedge_\star e) \wedge_\star e \neq 0 \tag{6.2}$$

because 1-forms no longer strictly anticommute with each other.

Braided Lie algebra of gauge symmetries

From (5.18), the left braided gauge transformation of a field $A = (e, \omega)$ by a gauge parameter $\lambda = (\xi, \rho)$ is encoded by

$$\begin{aligned}
\delta_{(\xi, \rho)}^{\star L}(e, \omega) &= \ell_1^*(\xi, \rho) + \ell_2^*((\xi, \rho), (e, \omega)) \\
&= (-\rho \star e + L_\xi^* e, d\rho - [\rho, \omega]_{\mathfrak{so}(3)}^* + L_\xi^* \omega).
\end{aligned}$$

This combines the braided gauge transformations from Examples 5.6 and 5.24, for $\mathfrak{g} = \mathfrak{so}(3)$, $W = \mathbb{R}^3$ and $p = 1$, with the braided diffeomorphisms from Sect. 3.4 which abide by the rules of braided gauge transformations spelled out in Sect. 5.2.¹¹

According to (5.22), these two individual braided Lie algebras of gauge symmetries combine into the single braided Lie algebra with bracket given by

$$-\ell_2^*((\xi_1, \rho_1), (\xi_2, \rho_2)) = (-[\xi_1, \xi_2]_{\mathfrak{so}(3)}^*, [\rho_1, \rho_2]_{\mathfrak{so}(3)}^* + L_{R_k(\xi_2)}^* R^k(\rho_1) - L_{\xi_1}^* \rho_2).$$

We identify this combined gauge algebra as the braided semi-direct product

$$\Gamma(TM)[[\hbar]] \ltimes_{\star} \Omega^0(M, \mathfrak{so}(3))[[\hbar]]$$

of the braided Lie algebra of vector fields from Sect. 3.4, and the braided Lie algebra from Examples 5.6 and 5.24 with $\mathfrak{g} = \mathfrak{so}(3)$, $W = \mathbb{R}^3$ and $p = 1$. Thus our twisting of both the diffeomorphism and local rotational symmetries in a democratic manner preserves the semi-direct product structure of the classical Lie algebra (2.15), and simultaneously avoids the introduction of new degrees of freedom into the theory, in contrast to some previous approaches to noncommutative gravity (cf. Sect. 5.3).

Braided curvature and torsion

From (5.27), the field equations $F_{(e, \omega)}^* = (0, 0)$ for the three-dimensional braided ECP theory are encoded through

$$F_{(e, \omega)}^* = \ell_1^*(e, \omega) - \frac{1}{2} \ell_2^*((e, \omega), (e, \omega)) =: (F_e^*, F_\omega^*).$$

The first entry is given by

$$F_e^* = R^* + \Lambda e \wedge_{\star} e \quad (6.3)$$

where

$$R^* = d\omega + \frac{1}{2} [\omega, \omega]_{\mathfrak{so}(3)}^* = d\omega + \frac{1}{2} (\omega \wedge_{\star} \omega + R_k(\omega) \wedge_{\star} R^k(\omega)) \quad (6.4)$$

is the braided curvature 2-form of the spin connection ω , as an example of braided curvature from Sect. 5.4. The second entry is given by

$$F_\omega^* = de + \frac{1}{2} \omega \wedge_{\star} e + \frac{1}{2} R_k(\omega) \wedge_{\star} R^k(e).$$

¹¹ Analogously to our discussion of left and right braided gauge transformations from Sect. 5.2, there are correspondingly a left and a right star-Lie derivative. The star-Lie derivative appearing here acts from the left, but for simplicity we do not indicate this explicitly in the notation L_{ξ}^* .

To understand the meaning of this expression, recall from Sect. 5.4 that there are two separate definitions of braided covariant derivative, and hence there are correspondingly two separate consistent definitions of torsion: A *left* torsion 2-form

$$T_L^\star := d_\star^\omega e = de + \omega \wedge_\star e, \quad (6.5)$$

and a *right* torsion 2-form

$$T_R^\star := d_\star^\omega e = de + R_k(\omega) \wedge_\star R^k(e). \quad (6.6)$$

Hence, the corresponding field equation is given by the symmetrized torsion

$$\boxed{F_\omega^\star = \frac{1}{2} (T_L^\star + T_R^\star)} \quad (6.7)$$

According to (5.30), the braided covariance of the field equations is encoded by

$$\delta_{(\xi, \rho)}^{\star L} F_{(e, \omega)}^\star = (\delta_{(\xi, \rho)}^{\star L} F_e^\star, \delta_{(\xi, \rho)}^{\star L} F_\omega^\star) = \ell_2^\star((\xi, \rho), (F_e^\star, F_\omega^\star)), \quad (6.8)$$

where

$$\delta_{(\xi, \rho)}^{\star L} F_e^\star = -[\rho, F_e^\star]_{\mathfrak{so}(3)}^\star + L_\xi^\star F_e^\star \quad \text{and} \quad \delta_{(\xi, \rho)}^{\star L} F_\omega^\star = -\rho \star F_\omega^\star + L_\xi^\star F_\omega^\star. \quad (6.9)$$

This is in complete harmony with the discussion from Sect. 5.4, which shows that the braided curvature is covariant, and that the left and right torsions are both covariant with respect to left braided gauge transformations.

Thus, the geometrical meaning of the braided field equations is just a deformation of that from the classical case. The equation $F_\omega^\star = 0$ is simply the balanced torsion-free condition¹² $T_L^\star + T_R^\star = 0$, and it would be interesting to see if this can be solved uniquely and algebraically for the spin connection ω in terms of the coframe e , as in the classical case where it would yield the Levi–Civita connection. Conditions for retrieving unique connections in similar noncommutative geometry settings have been suggested before (see e.g. [3, 19]), and it would be interesting to compare these conditions with our balanced torsion-free condition. The equation $F_e^\star = 0$ is the braided analog of the classical Einstein equation with cosmological constant in our formalism (before inserting the Levi–Civita connection). We stress that our field equations are different from those previously suggested for the first order formulation of noncommutative gravity [4], already at the kinematic level: in our approach the internal degrees of freedom are braided and hence there are no new dynamical fields introduced from having to extend the underlying Lie algebra $\mathfrak{so}(3)$.

Our remarks from Sect. 5.6 concerning the space of classical solutions of course also apply here. In particular, a braided diffeomorphism $\delta_{(\xi, 0)}^{\star L}(e, \omega)$ can map a classical solution into a field configuration which no longer solves the field equations; for example, for $\Lambda = 0$, a braided flat noncommutative spacetime can be braided

¹² This improves the interpretation of the torsion equation from [31], where only the left torsion 2-form was considered.

diffeomorphic to a braided curved noncommutative spacetime. This peculiar feature appears not only in our formalism, but also in the many earlier treatments of noncommutative gravity which used an underlying symmetry principle based on twisted diffeomorphisms (cf. Sect. 6.1), which was discussed in Sect. 3.4. As far as we are aware, this salient point has not been mentioned or appreciated before. According to the general framework of braided field theory that we have developed in the present paper, the meaning of the local symmetries in noncommutative gravity are captured by braided Noether identities.

Braided Noether identities

By (5.32), the gauge redundancies of braided gravity in three dimensions are encoded by the braided Noether identities

$$\begin{aligned} d_{(e,\omega)}^* F_{(e,\omega)}^* &= \ell_1^*(F_e^*, F_\omega^*) - \frac{1}{2} \left(\ell_2^*((e, \omega), (F_e^*, F_\omega^*)) - \ell_2^*((F_e^*, F_\omega^*), (e, \omega)) \right) \\ &\quad + \frac{1}{4} \ell_2^*(R_k(e, \omega), \ell_2^*(R^k(e, \omega), (e, \omega))) = (0, 0), \end{aligned} \quad (6.10)$$

where $R_k(e, \omega) := (R_k(e), R_k(\omega))$, and we used the slightly simpler form (5.35). They lead to a pair of differential identities among the field equations (6.3) and (6.7).

The Noether identity corresponding to braided local $\mathfrak{so}(3)$ rotations is given by the second slot of (6.10). Compared to the classical case [30], it now acquires a contribution from the cosmological constant due to the braiding, in addition to the usual inhomogeneous terms:

$$\begin{aligned} \frac{1}{2} (d_{\star L}^\omega F_\omega^* + d_{\star R}^\omega F_\omega^* - R_k(F_e^*) \wedge_\star R^k(e) - F_e^* \wedge_\star e) \\ + \frac{1}{4} (-[\omega, R_k(\omega)]_{\mathfrak{so}(3)}^* \wedge_\star R^k(e) + R_k(\omega) \wedge_\star R^k(\omega) \wedge_\star e + R_k(\omega \wedge_\star \omega) \wedge_\star R^k(e) \\ - 2 \Lambda (e \wedge_\star e) \wedge_\star e - 2 \Lambda R_k(e \wedge_\star e) \wedge_\star R^k(e)) = 0. \end{aligned} \quad (6.11)$$

The last two terms here vanish in the classical case, but they are generally non-vanishing in the noncommutative case due to (6.2). This identity can also be derived directly from the braided deformations of the first Bianchi identity:

$$\begin{aligned} d_{\star L}^\omega T_L^* &= R^* \wedge_\star e + \frac{1}{2} \omega \wedge_\star \omega \wedge_\star e - \frac{1}{2} R_k(\omega) \wedge_\star R^k(\omega) \wedge_\star e, \\ d_{\star R}^\omega T_R^* &= R_k(R^*) \wedge_\star R^k(e) - \frac{1}{2} R_k(\omega) \wedge_\star R_l(\omega) \wedge_\star R^l R^k(e) \\ &\quad + \frac{1}{2} R_m R_l(\omega) \wedge_\star R^m R_k(\omega) \wedge_\star R^l R^k(e), \end{aligned}$$

which follow easily by taking the exterior derivatives of (6.5) and (6.6), respectively. In the classical case, the last two terms on the right-hand sides of both equations cancel and we recover the expected identity $d^\omega T = R \wedge e$, which is equivalent to (6.11) in that case.

The Noether identity corresponding to braided diffeomorphisms is given by the first slot of (6.10). Given the form of the classical map $\ell_2 : V_1 \otimes V_2 \rightarrow$

V_3 from (2.14), we have to use the defining expression $\ell_2^*((e, \omega), (\mathcal{F}_e, \mathcal{F}_\omega)) := \ell_2(\bar{f}^k(e, \omega), \bar{f}_k(\mathcal{F}_e, \mathcal{F}_\omega))$ directly in terms of the undeformed classical bracket. Expanding (6.10) using classical antisymmetry of brackets in the third line, we find

$$dx^\mu \otimes \text{Tr}(N_\mu^*(F_e^*, F_\omega^*, e, \omega)) = 0, \quad (6.12)$$

where the 3-form $N_\mu^*(F_e^*, F_\omega^*, e, \omega) \in \Omega^3(M, \wedge^3 \mathbb{R}^3)[[\hbar]]$ is given by

$$\begin{aligned} N_\mu^* = & \frac{1}{2} (\iota_\mu d \bar{f}^k(e) \wedge \bar{f}_k(F_e^*) + \iota_\mu d \bar{f}^k(\omega) \wedge \bar{f}_k(F_\omega^*) - \iota_\mu \bar{f}^k(e) \wedge d \bar{f}_k(F_e^*) \\ & - \iota_\mu \bar{f}^k(\omega) \wedge d \bar{f}_k(F_\omega^*) \\ & + \iota_\mu d \bar{f}_k(e) \wedge \bar{f}^k(F_e^*) + \iota_\mu d \bar{f}_k(\omega) \wedge \bar{f}^k(F_\omega^*) - \iota_\mu \bar{f}_k(e) \wedge d \bar{f}^k(F_e^*) \\ & - \iota_\mu \bar{f}_k(\omega) \wedge d \bar{f}^k(F_\omega^*)) \\ & - \frac{1}{4} \left(\iota_\mu d \bar{f}^k R_l(e) \wedge \bar{f}_k([R^l(\omega), \omega]_{\mathfrak{so}(3)}^* + 2 \wedge R^l(e) \wedge_\star e) \right. \\ & + \iota_\mu d \bar{f}^k R_l(\omega) \wedge \bar{f}_k(R^l(\omega) \wedge_\star e - R_m(\omega) \wedge_\star R^m R^l(e)) \\ & - \iota_\mu \bar{f}^k R_l(e) \wedge d \bar{f}_k([R^l(\omega), \omega]_{\mathfrak{so}(3)}^* + 2 \wedge R^l(e) \wedge_\star e) \\ & \left. - \iota_\mu \bar{f}^k R_l(\omega) \wedge d \bar{f}_k(R^l(\omega) \wedge_\star e - R_m(\omega) \wedge_\star R^m R^l(e)) \right). \end{aligned}$$

This expression does not immediately involve the star-contraction defined by $\iota_\xi^* \alpha = \iota_{\bar{f}^k(\xi)} \bar{f}_k(\alpha)$ for $\xi \in \Gamma(TM)$ and $\alpha \in \Omega^\bullet(M)$. Nevertheless, expression (6.12) is globally well-defined on M as an element of $\Omega^1(M) \otimes \Omega^3(M)[[\hbar]]$. In some special cases where the legs of the twist $\mathcal{F}$ commute with the local basis vector fields, we may write the braided diffeomorphism Noether identity in a slightly more compact form; for example, this holds for abelian twists locally around every point of M outside a set of measure zero [5, 49].

Example 6.13 Consider the Moyal–Weyl twist $\mathcal{F}_\theta = f_\theta^k \otimes f_{\theta k}$ on $M = \mathbb{R}^3$ from Examples 3.5 and 3.9. Then, $\iota_\mu \circ \bar{f}_\theta^k = \bar{f}_\theta^k \circ \iota_\mu$ since $\iota_\mu \circ L_{\partial_v} = L_{\partial_v} \circ \iota_\mu + \iota_{[\partial_\mu, \partial_v]} \mathbf{v} = L_{\partial_v} \circ \iota_\mu$ by the Cartan structure equations, and $\iota_\mu^{\star\theta} = \iota_\mu$ since $L_{\partial_v} \partial_\mu = [\partial_v, \partial_\mu] \mathbf{v} = 0$ in a holonomic basis. In this case we have

$$\iota_\mu d \bar{f}_\theta^k(e) \wedge \bar{f}_{\theta k}(F_e^{\star\theta}) = \iota_\mu^{\star\theta} de \wedge_{\star_\theta} F_e^{\star\theta}$$

and

$$\iota_\mu^{\star\theta} d \bar{f}_{\theta k}(e) \wedge \bar{f}_\theta^k(F_e^{\star\theta}) = \bar{f}_\theta^k(F_e^{\star\theta}) \wedge \bar{f}_{\theta k}(\iota_\mu^{\star\theta} de) = F_e^{\star\theta} \wedge_{\star_\theta} \iota_\mu^{\star\theta} de,$$

with analogous expressions for the rest of the terms. Since also $L_{\partial_v} dx^\mu = 0$, it follows that the expression for the braided diffeomorphism Noether identity (6.12) simplifies

slightly to an expression in $\Omega^1(\mathbb{R}^3) \otimes_{\star_\theta} \Omega^3(\mathbb{R}^3)[[\hbar]]$ given by

$$dx^\mu \otimes_{\star_\theta} \text{Tr}(N_\mu^{\star_\theta}(F_e^\star, F_\omega^\star, e, \omega)) = 0 \quad (6.14)$$

with

$$\begin{aligned} N_\mu^{\star_\theta} = & \frac{1}{2} (\iota_\mu^{\star_\theta} de \wedge_{\star_\theta} F_e^{\star_\theta} + \iota_\mu^{\star_\theta} d\omega \wedge_{\star_\theta} F_\omega^{\star_\theta} - \iota_\mu^{\star_\theta} e \wedge_{\star_\theta} dF_e^{\star_\theta} - \iota_\mu^{\star_\theta} \omega \wedge_{\star_\theta} dF_\omega^{\star_\theta} \\ & + F_e^{\star_\theta} \wedge_{\star_\theta} \iota_\mu^{\star_\theta} de + F_\omega^{\star_\theta} \wedge_{\star_\theta} \iota_\mu^{\star_\theta} d\omega - dF_e^{\star_\theta} \wedge_{\star_\theta} \iota_\mu^{\star_\theta} e - dF_\omega^{\star_\theta} \wedge_{\star_\theta} \iota_\mu^{\star_\theta} \omega) \\ & - \frac{1}{4} \left(\iota_\mu^{\star_\theta} dR_l(e) \wedge_{\star_\theta} [R^l(\omega), \omega]_{\mathfrak{so}(3)}^{\star_\theta} + 2 \wedge e \wedge_{\star_\theta} \iota_\mu^{\star_\theta} de \wedge_{\star_\theta} e \right. \\ & + \iota_\mu^{\star_\theta} dR_l(\omega) \wedge_{\star_\theta} (R^l(\omega) \wedge_{\star_\theta} e - R_k(\omega) \wedge_{\star_\theta} R^k R^l(e)) \\ & - \iota_\mu^{\star_\theta} R_l(e) \wedge_{\star_\theta} d[R^l(\omega), \omega]_{\mathfrak{so}(3)}^{\star_\theta} + 2 \wedge e \wedge_{\star_\theta} \iota_\mu^{\star_\theta} de \wedge_{\star_\theta} e \\ & \left. - \iota_\mu^{\star_\theta} R_l(\omega) \wedge_{\star_\theta} d(R^l(\omega) \wedge_{\star_\theta} e - R_k(\omega) \wedge_{\star_\theta} R^k R^l(e)) \right). \end{aligned}$$

Action functional

The variational principle for three-dimensional braided gravity follows, similarly to the braided Chern–Simons theory of Sect. 5.9, from choosing a Hermitian Drinfel'd twist $\mathcal{F} \in U\mathfrak{v}[[\hbar]] \otimes U\mathfrak{v}[[\hbar]]$ which is compatible with the cyclic structure of the classical ECP theory defined by (2.16) and (2.17). Using Proposition 4.15, we can then turn $(V[[\hbar]], \ell_1^\star, \ell_2^\star)$ into a strictly cyclic braided L_∞ -algebra with cyclic pairing defined by

$$\langle (e, \omega), (E, \Omega) \rangle_\star := \int_M \text{Tr}(e \wedge_\star E + \Omega \wedge_\star \omega), \quad (6.15)$$

on $V_1 \otimes V_2$ and

$$\langle (\xi, \rho), (\mathcal{X}, \mathcal{P}) \rangle_\star := \int_M \iota_\xi^\star \mathcal{X} + \int_M \text{Tr}(\rho \wedge_\star \mathcal{P}), \quad (6.16)$$

on $V_0 \otimes V_3$.

According to (5.39), the action functional is then given by

$$\begin{aligned} S_\star(e, \omega) &= \frac{1}{2} \langle (e, \omega), \ell_1^\star(e, \omega) \rangle_\star - \frac{1}{6} \langle (e, \omega), \ell_2^\star((e, \omega), (e, \omega)) \rangle_\star \\ &= \int_M \text{Tr} \left(e \wedge_\star R^\star + \frac{\Lambda}{3} e \wedge_\star e \wedge_\star e \right) \\ &= \int_M \epsilon_{abc} \left(e^a \wedge_\star R^{\star bc} + \frac{\Lambda}{3} e^a \wedge_\star e^b \wedge_\star e^c \right). \end{aligned}$$

In the second line we used invariance of a top-vector under the left and right braided actions by $\mathfrak{so}(3)$ -valued 1-forms, analogously to the classical case [30] order by order

in the formal power series expansion of the twist in $\hbar$; for example

$$0 = (\omega \star e) \star \lrcorner \omega = -\omega \star (R_k(\omega) \wedge_\star R^k(e)) + [R_k R_l(\omega), R^k(\omega)]_{\mathfrak{so}(3)}^\star \star R^l(e),$$

and so on. This is just the expected deformation of the classical action functional (2.18); it is the unique action functional constructed from the braided pairing, and the corresponding polynomial orders in the coframe field and the curvature, with the correct classical form when $\mathcal{R} = 1 \otimes 1$.

Using the $\mathcal{R}$ -matrix identities (3.7), the cyclicity of integration (3.10) with respect to the star-product $\wedge_\star$, and the invariance of a top-vector under braided $\mathfrak{so}(3)$ rotations, one can directly check that varying this action functional with respect to e gives the field equation $F_e^\star = 0$ defined in (6.3), while varying with respect to ω gives the field equation $F_\omega^\star = 0$ defined in (6.7), as was guaranteed by the general theory of Sect. 5.8. Also in agreement with the general theory, one checks invariance of the action functional $\delta_{(0,\rho)}^{\star L} S_\star(e, \omega) = 0$ under braided $\mathfrak{so}(3)$ rotations using braided covariance $\delta_\rho^{\star L} R^\star = -[\rho, R^\star]_{\mathfrak{so}(3)}^\star$ and $\mathfrak{so}(3)$ top-vector invariance such as

$$\rho \star (e \wedge_\star R^\star) = \bar{f}^k(\rho) \cdot \bar{f}_k(e \wedge_\star R^\star) = 0,$$

as well as invariance $\delta_{(\xi,0)}^{\star L} S_\star(e, \omega) = 0$ under braided diffeomorphisms using the classical Cartan formula for the Lie derivative and Stokes' theorem order by order in $\hbar$. These two braided gauge invariances further imply, respectively, the braided Noether identities (6.11) along the lines discussed in Sect. 5.8, though again the most effective and concise way to do this is through the braided L_∞ -algebra framework of this section.

6.3 Braided gravity in four dimensions

Both the braided Chern–Simons theory from Sect. 5.9 and the three-dimensional braided ECP theory from Sect. 6.2 are based on differential (graded) braided Lie algebras, while their field equations and variational principle could have been more or less guessed from the outset by simply replacing ordinary products of fields with star-products. This does not preclude, however, the different kinematical sector of these theories due to the braided Lie brackets, as well as the novel general nature of classical solutions in braided field theory and the corresponding modified forms of gauge redundancies through the braided Noether identities. We will now discuss a highly non-trivial example which is based on an L_∞ -algebra which is not simply a differential graded Lie algebra. It does not follow the naive deformations of the three-dimensional theories studied thus far, and it illustrates the novelty of our approach in generating new noncommutative field theories with interesting symmetry structures. Indeed, we shall now derive a new noncommutative theory of gravity in four dimensions by applying the braided L_∞ -algebra framework to the classical four-dimensional ECP theory.

Braided L_∞ -algebra

Let M be an oriented parallelizable four-manifold. The 4-term L_∞ -algebra underlying the classical Einstein–Cartan–Palatini formulation of general relativity on M with cosmological constant $\Lambda \in \mathbb{R}$ was developed in detail by [30] for an arbitrary signature pseudo-Riemannian structure on M ; for definiteness, we focus on the case of Euclidean signature, but everything we do holds identically for arbitrary signature (p, q) (with $p+q=4$): we simply replace the Lie algebra $\mathfrak{so}(4)$ and its fundamental representation $\mathbb{R}^4$ by their indefinite versions $\mathfrak{so}(p, q)$ and $\mathbb{R}^{p,q}$. As in the three-dimensional case, this defines an L_∞ -algebra in the category $\mathcal{M}^\sharp$ of $\mathbb{Z}$ -graded $U\mathfrak{v}$ -modules. Then the underlying graded vector space is given by

$$V := V_0 \oplus V_1 \oplus V_2 \oplus V_3$$

where

$$\begin{aligned} V_0 &= \Gamma(TM) \times \Omega^0(M, \mathfrak{so}(4)), \\ V_1 &= \Omega^1(M, \mathbb{R}^4) \times \Omega^1(M, \mathfrak{so}(4)), \\ V_2 &= \Omega^3(M, \wedge^3 \mathbb{R}^4) \times \Omega^3(M, \wedge^2 \mathbb{R}^4), \\ V_3 &= \Omega^1(M, \Omega^4(M)) \times \Omega^4(M, \wedge^2 \mathbb{R}^4). \end{aligned}$$

We use the same conventions and notation for fields as we did in the case of three-dimensional gravity from Sects. 2.4 and 6.2. Following the prescription of Sect. 4.2, we twist deform the classical brackets written down in [30, Section 8.2], which leads to a 4-term braided L_∞ -algebra $(V[[\hbar]], \ell_1^*, \ell_2^*, \ell_3^*)$ with the following non-vanishing braided brackets.

The differential ℓ_1^* is given by

$$\ell_1^*(\xi, \rho) = (0, d\rho), \quad \ell_1^*(e, \omega) = (0, 0) \quad \text{and} \quad \ell_1^*(E, \Omega) = (0, -d\Omega). \quad (6.17)$$

The 2-bracket ℓ_2^* is defined by

$$\begin{aligned} \ell_2^*((\xi_1, \rho_1), (\xi_2, \rho_2)) &= ([\xi_1, \xi_2]_{\mathfrak{v}}^*, -[\rho_1, \rho_2]_{\mathfrak{so}(4)}^* + L_{\xi_1}^* \rho_2 - L_{R_k(\xi_2)}^* R^k(\rho_1)), \\ \ell_2^*((\xi, \rho), (e, \omega)) &= (-\rho \star e + L_\xi^* e, -[\rho, \omega]_{\mathfrak{so}(4)}^* + L_\xi^* \omega), \\ \ell_2^*((\xi, \rho), (E, \Omega)) &= (-\rho \star E + L_\xi^* E, -[\rho, \Omega]_{\mathfrak{so}(4)}^* + L_\xi^* \Omega), \\ \ell_2^*((\xi, \rho), (\mathcal{X}, \mathcal{P})) &= (dx^\mu \otimes \text{Tr}(\iota_\mu d\tilde{f}^k(\rho) \wedge \tilde{f}_k(\mathcal{P})) + L_\xi^* \mathcal{X}, \\ &\quad -[\rho, \mathcal{P}]_{\mathfrak{so}(4)}^* + L_\xi^* \mathcal{P}), \\ \ell_2^*((e_1, \omega_1), (e_2, \omega_2)) &= -(e_1 \wedge_\star d\omega_2 + R_k(e_2) \wedge_\star dR^k(\omega_1), e_1 \wedge_\star de_2 \\ &\quad + R_k(e_2) \wedge_\star dR^k(e_1)), \\ \ell_2^*((e, \omega), (E, \Omega)) &= (dx^\mu \otimes \text{Tr}(\iota_\mu d\tilde{f}^k(e) \wedge \tilde{f}_k(E) - \iota_\mu d\tilde{f}^k(\omega) \wedge \tilde{f}_k(\Omega) \end{aligned}$$

$$\begin{aligned}
& -\iota_\mu \bar{f}^k(e) \wedge d\bar{f}_k(E) + \iota_\mu \bar{f}^k(\omega) \wedge d\bar{f}_k(\Omega)), \\
& \frac{3}{2} R_k(E) \wedge_\star R^k(e) + [\omega, \Omega]_{\mathfrak{so}(4)}^\star), \quad (6.18)
\end{aligned}$$

with all other 2-brackets equal to zero. Finally, there is a single nonzero 3-bracket $\ell_3^\star$ defined on fields by

$$\begin{aligned}
& \ell_3^\star((e_1, \omega_1), (e_2, \omega_2), (e_3, \omega_3)) \\
& = -\left(e_1 \wedge_\star [\omega_2, \omega_3]_{\mathfrak{so}(4)}^\star + R_k(e_2) \wedge_\star [R^k(\omega_1), \omega_3]_{\mathfrak{so}(4)}^\star \right. \\
& \quad + R_k(e_3) \wedge_\star R^k([\omega_1, \omega_2]_{\mathfrak{so}(4)}^\star) + 6 \wedge e_1 \wedge_\star e_2 \wedge_\star e_3, \\
& \quad e_1 \wedge_\star (\omega_2 \wedge_\star e_3) + e_1 \wedge_\star (R_k(\omega_3) \wedge_\star R^k(e_2)) \\
& \quad + R_k(e_2) \wedge_\star (R^k(\omega_1) \wedge_\star e_3) + R_k(e_2) \wedge_\star (R_l(\omega_3) \wedge_\star R^l R^k(e_1)) \\
& \quad \left. + R_k(e_3) \wedge_\star R^k(\omega_1 \wedge_\star e_2) + R_k R_l(e_3) \wedge_\star (R_m R^l(\omega_2) \wedge_\star R^m R^k(e_1)) \right). \quad (6.19)
\end{aligned}$$

Note that the differential of a generic field $A = (e, \omega) \in V_1$ vanishes, which will lead to some simplifications in the calculations involving the braided L_∞ -algebra brackets below.

The derivation of the braided gauge symmetries of noncommutative gravity in four dimensions is identical to that in three dimensions which was presented in Sect. 6.2, and so will not be repeated here; the only change is that the Lie algebra of rotations $\mathfrak{so}(3)$ is replaced with its counterpart $\mathfrak{so}(4)$ appropriate to four dimensions. We shall therefore dive straight away into a description of the novel dynamics of the braided gravity theory, where we will uncover many unexpected and surprising features.

Field equations

By (5.27), the field equations $F_{(e,\omega)}^\star = (0, 0)$ of the braided ECP theory in four dimensions are given by

$$F_{(e,\omega)}^\star = -\frac{1}{2} \ell_2^\star((e, \omega), (e, \omega)) - \frac{1}{6} \ell_3^\star((e, \omega), (e, \omega), (e, \omega)) =: (F_e^\star, F_\omega^\star).$$

Expanding the brackets, the components are given by

$$\begin{aligned}
F_e^\star &= \frac{1}{2} (e \wedge_\star d\omega + R_k(e) \wedge_\star dR^k(\omega)) + \wedge e \wedge_\star e \wedge_\star e \\
&+ \frac{1}{6} (e \wedge_\star [\omega, \omega]_{\mathfrak{so}(4)}^\star + R_k(e) \wedge_\star [R^k(\omega), \omega]_{\mathfrak{so}(4)}^\star + R_k(e) \wedge_\star R^k([\omega, \omega]_{\mathfrak{so}(4)}^\star)), \quad (6.20)
\end{aligned}$$

and

$$\begin{aligned}
F_\omega^\star &= \frac{1}{2} (e \wedge_\star de + R_k(e) \wedge_\star dR^k(e)) \\
&+ \frac{1}{6} (e \wedge_\star (\omega \wedge_\star e) + e \wedge_\star (R_k(\omega) \wedge_\star R^k(e)) + R_k(e) \wedge_\star (R^k(\omega) \wedge_\star e)
\end{aligned}$$

$$\begin{aligned}
& + R_k(e) \lrcorner_\star (R_l(\omega) \wedge_\star R^l R^k(e)) + R_k(e) \lrcorner_\star R^k(\omega \wedge_\star e) \\
& + R_k R_l(e) \lrcorner_\star (R_m R^l(\omega) \wedge_\star R^m R^k(e)).
\end{aligned} \quad (6.21)$$

According to (5.30), the field equations transform under braided gauge transformations as in (6.8), with

$$\delta_{(\xi, \rho)}^{\star L} F_e^\star = -\rho \triangleright_\star F_e^\star + L_\xi^\star F_e^\star \quad \text{and} \quad \delta_{(\xi, \rho)}^{\star L} F_\omega^\star = -\rho \triangleright_\star F_\omega^\star + L_\xi^\star F_\omega^\star. \quad (6.22)$$

The goal is now to simplify these field equations using various $\mathcal{R}$ -matrix identities to make the terms appear in manifestly covariant forms, as would be naively expected from the covariance property that is guaranteed by the general theory of Sect. 5.6. This will introduce the curvature 2-form $R^\star$ defined analogously to (6.4), as well as the left and right torsion 2-forms $T_L^\star$ and $T_R^\star$ from (6.5) and (6.6).

Braided torsion condition

Let us start with the second component $F_\omega^\star$. We will show that

$$F_\omega^\star = \frac{1}{6} (e \lrcorner_\star T_L^\star - T_R^\star \lrcorner_\star e - d_{\star L}^\omega(e \lrcorner_\star e) - d_{\star R}^\omega(e \lrcorner_\star e)) \quad (6.23)$$

This expression is manifestly braided covariant, and it reduces exactly (with the correct numerical factor) to the expected torsion form $F_\omega = e \lrcorner T$ in the classical case.

For this, we use the Yang–Baxter equation (3.4) for the $\mathcal{R}$ -matrix to write the last term in (6.21) as

$$R_m R_l(e) \lrcorner_\star (R^m R_k(\omega) \wedge_\star R^l R^k(e)) = R_m(e) \lrcorner_\star R^m (R_k(\omega) \wedge_\star R^k(e)).$$

Using then braided antisymmetry of the $\lrcorner_\star$ -product on the latter, second and seventh terms, we get

$$\begin{aligned}
F_\omega^\star = & \frac{1}{6} \left(3 e \lrcorner_\star de - 3 de \lrcorner_\star e + e \lrcorner_\star (\omega \wedge_\star e) + e \lrcorner_\star (R_k(\omega) \wedge_\star R^k(e)) \right. \\
& + R_k(e) \lrcorner_\star (R^k(\omega) \wedge_\star e) \\
& \left. + R_k(e) \lrcorner_\star (R_l(\omega) \wedge_\star R^l R^k(e)) - (\omega \wedge_\star e) \lrcorner_\star e - (R_k(\omega) \wedge_\star R^k(e)) \lrcorner_\star e \right).
\end{aligned} \quad (6.24)$$

Now notice that

$$R_k(e) \lrcorner_\star (R^k(\omega) \wedge_\star e) = -\omega \triangleright_\star (e \lrcorner_\star e) + (\omega \wedge_\star e) \lrcorner_\star e \quad (6.25)$$

and

$$\begin{aligned}
 R_k(e) \lrcorner_\star (R_l(\omega) \wedge_\star R^l R^k(e)) &= -R_k(e) \lrcorner_\star (R^k(e) \star \lrcorner \omega) \\
 &= -(R_k(e) \lrcorner_\star R^k(e)) \star \lrcorner \omega \\
 &\quad - (R_k(e) \star \lrcorner R_l(\omega)) \lrcorner_\star R^l R^k(e) \\
 &= -(e \lrcorner_\star e) \star \lrcorner \omega - R_k(e \star \lrcorner \omega) \lrcorner_\star R^k(e) \\
 &= -(e \lrcorner_\star e) \star \lrcorner \omega + e \lrcorner_\star (e \star \lrcorner \omega). \tag{6.26}
 \end{aligned}$$

Substituting both (6.25) and (6.26) in (6.24), and recalling that

$$e \star \lrcorner \omega := -R_k(\omega) \triangleright_\star R^k(e) = -R_k(\omega) \wedge_\star R^k(e),$$

after cancelling terms we find

$$\begin{aligned}
 F_\omega^\star &= \frac{1}{6} \left(3 e \lrcorner_\star de - 3 de \lrcorner_\star e + e \lrcorner_\star (\omega \wedge_\star e) - \omega \triangleright_\star (e \lrcorner_\star e) - (e \lrcorner_\star e) \star \lrcorner \omega \right. \\
 &\quad \left. - (R_k(\omega) \wedge_\star R^k(e)) \lrcorner_\star e \right) \\
 &= \frac{1}{6} \left(e \lrcorner_\star de + e \lrcorner_\star (\omega \wedge_\star e) - de \lrcorner_\star e - (R_k(\omega) \wedge_\star R^k(e)) \lrcorner_\star e \right. \\
 &\quad \left. - d(e \lrcorner_\star e) - \omega \triangleright_\star (e \lrcorner_\star e) \right. \\
 &\quad \left. - d(e \lrcorner_\star e) - (e \lrcorner_\star e) \star \lrcorner \omega \right).
 \end{aligned}$$

After identifying left and right braided covariant derivatives, we finally arrive at (6.23).

Braided Einstein equation

Next consider $F_e^\star$, which is our braided noncommutative analog of the four-dimensional Einstein equation with cosmological constant (that is, before solving for the Levi–Civita connection using the classical torsion equation). We will show that

$$\begin{aligned}
 F_e^\star &= \frac{1}{6} \left(2 e \lrcorner_\star R^\star + 2 R^\star \lrcorner_\star e + 6 \Lambda e \lrcorner_\star e \lrcorner_\star e \right. \\
 &\quad \left. + e \lrcorner_\star d\omega + d\omega \lrcorner_\star e + R_k(e) \lrcorner_\star [R^k(\omega), \omega]_{\mathfrak{so}(4)}^\star \right) \tag{6.27}
 \end{aligned}$$

The first three terms are manifestly braided covariant, but the rest are not. However, a few lines of calculations shows that the last three terms form a braided covariant combination, as expected from the general covariance property which is guaranteed by the braided homotopy calculations of Sect. 5.6. In the classical case, this expression reduces exactly (with the correct numerical factors) to the expected curvature form $F_e = e \lrcorner R + \Lambda e \lrcorner e \lrcorner e$.

Starting from (6.20), we use braided antisymmetry on the second and last terms to write

$$\begin{aligned} F_e^* &= \frac{1}{2} (e \lrcorner_\star d\omega + \frac{1}{3} e \lrcorner_\star [\omega, \omega]_{\mathfrak{so}(4)}^* + d\omega \lrcorner_\star e \\ &\quad + \frac{1}{3} [\omega, \omega]_{\mathfrak{so}(4)}^* \lrcorner_\star e + 2 \Delta e \lrcorner_\star e \lrcorner_\star e \\ &\quad + \frac{1}{3} R_k(e) \lrcorner_\star [R^k(\omega), \omega]_{\mathfrak{so}(4)}^*) \\ &= \frac{1}{2} (e \lrcorner_\star R^* + R^* \lrcorner_\star e + 2 \Delta e \lrcorner_\star e \lrcorner_\star e \\ &\quad - \frac{1}{6} e \lrcorner_\star [\omega, \omega]_{\mathfrak{so}(4)}^* - \frac{1}{6} [\omega, \omega]_{\mathfrak{so}(4)}^* \lrcorner_\star e + \frac{1}{3} R_k(e) \lrcorner_\star [R^k(\omega), \omega]_{\mathfrak{so}(4)}^*), \end{aligned}$$

which is easily brought to form (6.27). It is somewhat surprising and remarkable that the second line of (6.27) is braided covariant, despite not being expressible in terms of manifestly covariant objects. It is instructive to explicitly demonstrate how this happens. For this, consider a left braided $\mathfrak{so}(4)$ gauge transformation of the first term there:

$$\begin{aligned} \delta_{(0,\rho)}^{*\text{L}}(e \lrcorner_\star d\omega) &= -(\rho \star e) \lrcorner_\star d\omega + R_k(e) \lrcorner_\star d(R^k(d\rho) - [R^k(\rho), \omega]_{\mathfrak{so}(4)}^*) \\ &= -(\rho \star e) \lrcorner_\star d\omega - R_k(e) \lrcorner_\star [R^k(d\rho), \omega]_{\mathfrak{so}(4)}^* \\ &\quad - R_k(e) \lrcorner_\star [R^k(\rho), d\omega]_{\mathfrak{so}(4)}^* \\ &= -\rho \triangleright_\star (e \lrcorner_\star d\omega) - R_k(e) \lrcorner_\star [R^k(d\rho), \omega]_{\mathfrak{so}(4)}^*, \end{aligned} \quad (6.28)$$

where in the first and last equalities we used the left braided Leibniz rule. Similarly, one calculates

$$\delta_{(0,\rho)}^{*\text{L}}(d\omega \lrcorner_\star e) = -\rho \triangleright_\star (d\omega \lrcorner_\star e) - [d\rho, \omega]_{\mathfrak{so}(4)}^* \lrcorner_\star e. \quad (6.29)$$

The non-covariant terms involving $d\rho$ in (6.28) and (6.29) cancel exactly with the corresponding non-covariant terms in the gauge transformation of the last term in (6.27): after a couple of lines of similar manipulations we find

$$\begin{aligned} \delta_{(0,\rho)}^{*\text{L}}(R_k(e) \lrcorner_\star [R^k(\omega), \omega]_{\mathfrak{so}(4)}^*) &= -\rho \triangleright_\star (R_k(e) \lrcorner_\star [R^k(\omega), \omega]_{\mathfrak{so}(4)}^*) \\ &\quad + R_k(e) \lrcorner_\star [R^k(d\rho), \omega]_{\mathfrak{so}(4)}^* \\ &\quad + R_k R_l(e) \lrcorner_\star [R^k R_m(\omega), R^l R^m(d\rho)]_{\mathfrak{so}(4)}^*. \end{aligned} \quad (6.30)$$

The second term in (6.30) cancels the non-covariant term in (6.28). Using the Yang–Baxter equation (3.4), the last term in (6.30) can be written as

$$\begin{aligned} R_l R_k(e) \lrcorner_\star [R_m R^k(\omega), R^m R^l(d\rho)]_{\mathfrak{so}(4)}^* &= R_l R_k(e) \lrcorner_\star [R^l(d\rho), R^k(\omega)]_{\mathfrak{so}(4)}^* \\ &= R_k(e) \lrcorner_\star R^k([d\rho, \omega]_{\mathfrak{so}(4)}^*) \\ &= [d\rho, \omega]_{\mathfrak{so}(4)}^* \lrcorner_\star e, \end{aligned}$$

which cancels the remaining non-covariant term in (6.29); here we use braided symmetry of the braided Lie bracket, then the second $\mathcal{R}$ -matrix identity from (3.7) followed by braided symmetry of the λ_* -product. Altogether, we have explicitly demonstrated (6.22), as expected on general grounds from the proof using the braided L_∞ -algebra structure in Sect. 5.6.

The field equations (6.23) and (6.27) of braided noncommutative gravity in four dimensions together are one of the main results of this paper. They are drastically different from the field equations previously suggested for the first order formulation of noncommutative gravity. Besides the universal (dimension-independent) differences that we discussed in the three-dimensional case of Sect. 6.2, where the braided field equations were relatively mild ‘naive’ deformations of the classical equations, here we see that the explicit forms of the braided field equations are very different and not simply ‘naive’ deformations: the properties of the field equations of braided gravity vary drastically with dimension, in contrast to the classical case. In particular, the torsion equation $F_\omega^* = 0$ now involves more braided covariant forms than just those constructed from the left and right torsion forms. The curvature equation $F_e^* = 0$ is even more striking, as in addition to the anticipated braided covariant forms, there is a further non-manifestly covariant combination whose individual forms are not braided covariant on their own. These latter forms could not have been guessed or even motivated solely by deformation of the classical equations. They are a necessary and natural consequence of twisting the kinematical and dynamical content simultaneously via the braided L_∞ -algebra approach to noncommutative field theory: in this approach braided gauge covariance is formulated for the doublet of fields $A = (e, \omega) \in V_1$ (and the corresponding doublet of field equations), which allows for non-trivial mixing of the component fields e and ω in non-manifestly covariant ways, but in such a way that the braided homotopy relations ensure that all combinations are covariant under braided gauge transformations.

Generally, we expect this complexity to increase as the number and orders of nonzero brackets ℓ_n^* for $n > 2$ increases. In this sense braided noncommutative field theories generally contain many more covariant forms than naively expected from classical considerations.

Braided Noether identities

By (5.32), the gauge redundancies of braided gravity in four dimensions are encoded by the braided Noether identities

$$\begin{aligned} d_{(e,\omega)}^* F_{(e,\omega)}^* &= \ell_1^*(F_e^*, F_\omega^*) - \frac{1}{12} \ell_1^*(\ell_3^*((e, \omega), (e, \omega), (e, \omega))) \\ &\quad + \frac{1}{4} \ell_2^*(R_k(e, \omega), \ell_2^*(R^k(e, \omega), (e, \omega))) \\ &\quad + \frac{1}{12} \left(\ell_2^*(\ell_3^*((e, \omega), (e, \omega), (e, \omega)), (e, \omega)) \right. \\ &\quad \left. - \ell_2^*((e, \omega), \ell_3^*((e, \omega), (e, \omega), (e, \omega))) \right) \end{aligned}$$

$$-\frac{1}{2} \left(\ell_2^*((e, \omega), (F_e^*, F_\omega^*)) - \ell_2^*((F_e^*, F_\omega^*), (e, \omega)) \right) = (0, 0), \quad (6.31)$$

where we again used the slightly simpler form (5.35), along with $\ell_1^*(e, \omega) = (0, 0)$. As previously, we rewrite these as differential identities among the field equations (6.23) and (6.27).

The braided Noether identity for local $\mathfrak{so}(4)$ rotations, in $\Omega^4(M, \wedge^2 \mathbb{R}^4)[[\hbar]]$, is given by the second entry of (6.31). Inserting the corresponding brackets from (6.17)–(6.19), using the usual $\mathcal{R}$ -matrix identities from (3.4) and (3.7), and collecting terms using (6.25) and (6.26), we find

$$\begin{aligned} & \frac{1}{2} \left(d_{\star L}^\omega F_\omega^* + d_{\star R}^\omega F_\omega^* + \frac{3}{2} R_k(F_e^*) \wedge_\star R^k(e) + \frac{3}{2} F_e^* \wedge_\star e \right) \\ & + \frac{1}{4} \left(\frac{1}{3} (d\Phi_{(e, \omega)}^* - [\Phi_{(e, \omega)}^*, \omega]_{\mathfrak{so}(4)}^* - [\omega, \Phi_{(e, \omega)}^*]_{\mathfrak{so}(4)}^*) - \frac{1}{2} (\tilde{\Phi}_{(e, \omega)}^* \wedge_\star e \right. \\ & + R_l(\tilde{\Phi}_{(e, \omega)}^*) \wedge_\star R^l(e)) \\ & - \frac{3}{2} (e \wedge_\star R_k(d\omega) + d\omega \wedge_\star R_k(e)) \wedge_\star R^k(e) + [d(e \wedge_\star R_k(e)), R^k(\omega)]_{\mathfrak{so}(4)}^* \\ & \left. - 3 \Lambda (e \wedge_\star e \wedge_\star e) \wedge_\star e - 3 \Lambda R_k(e \wedge_\star e \wedge_\star e) \wedge_\star R^k(e) \right) = 0, \quad (6.32) \end{aligned}$$

where we defined

$$\begin{aligned} \Phi_{(e, \omega)}^* &= e \wedge_\star (\omega \wedge_\star e) - (R_k(\omega) \wedge_\star R^k(e)) \wedge_\star e - \omega \triangleright_\star (e \wedge_\star e) - (e \wedge_\star e) \star \triangleleft \omega, \\ \tilde{\Phi}_{(e, \omega)}^* &= e \wedge_\star [\omega, \omega]_{\mathfrak{so}(4)}^* + R_k(e) \wedge_\star [R^k(\omega), \omega]_{\mathfrak{so}(4)}^* + [\omega, \omega]_{\mathfrak{so}(4)}^* \wedge_\star e. \end{aligned}$$

In the classical case, only the first line of (6.32) survives and reduces to the expected Noether identity $d^\omega F_\omega = -\frac{3}{2} F_e \wedge e$, which follows from (but is no longer equivalent to) the first Bianchi identity $d^\omega T = R \wedge e$ [30].

The Noether identity corresponding to the braided diffeomorphism invariance of the four-dimensional theory is valued in $\Omega^1(M) \otimes \Omega^4(M)[[\hbar]]$ and is given by the first entry of (6.31). The contributions from the first two terms vanish by definition of the differential ℓ_1^* , while the last line contributes terms of the same form as in the three-dimensional case, up to signs. The remaining terms can be expanded by commuting the exterior derivative d with the legs of the twist, along with identities (6.25) and (6.26). Altogether, the explicit form of the braided diffeomorphism Noether identity for four-dimensional gravity may be expressed similarly to (6.12), now with the 4-form $N_\mu^*(F_e^*, F_\omega^*, e, \omega) \in \Omega^4(M, \wedge^4 \mathbb{R}^4)[[\hbar]]$ given by

$$\begin{aligned} N_\mu^* &= \frac{1}{2} (\iota_\mu \bar{f}^k(d\omega) \wedge \bar{f}_k(F_e^*) - \iota_\mu \bar{f}^k(d\omega) \wedge \bar{f}_k(F_\omega^*) - \iota_\mu \bar{f}^k(e) \wedge \bar{f}_k(dF_e^*) \\ & + \iota_\mu \bar{f}^k(\omega) \wedge \bar{f}_k(dF_\omega^*) \\ & + \iota_\mu \bar{f}_k(d\omega) \wedge \bar{f}^k(F_e^*) - \iota_\mu \bar{f}_k(d\omega) \wedge \bar{f}^k(F_\omega^*) - \iota_\mu \bar{f}_k(e) \wedge \bar{f}^k(dF_e^*) \end{aligned}$$

$$\begin{aligned}
& + \iota_\mu \bar{f}_k(\omega) \wedge \bar{f}^k(dF_\omega^*) \\
& + \frac{1}{4} \left(\iota_\mu \bar{f}^k R_l(d\omega) \wedge \bar{f}_k(R^l(e) \wedge_\star d\omega + R^l(d\omega) \wedge_\star e) \right. \\
& + \iota_\mu \bar{f}^k R_l(d\omega) \wedge d\bar{f}_k(R^l(e) \wedge_\star e) \\
& \left. - \iota_\mu \bar{f}^k R_l(e) \wedge \bar{f}_k(R^l(de) \wedge_\star d\omega + R^l(d\omega) \wedge_\star de) \right) \\
& + \frac{1}{72} \left(\iota_\mu \bar{f}^k(de) \wedge \bar{f}_k(\Psi_{(e,\omega)}^*) - \iota_\mu \bar{f}^k(e) \wedge d\bar{f}_k(\Psi_{(e,\omega)}^*) \right. \\
& + \iota_\mu \bar{f}_k(de) \wedge \bar{f}^k(\Psi_{(e,\omega)}^*) - \iota_\mu \bar{f}_k(e) \wedge d\bar{f}^k(\Psi_{(e,\omega)}^*) \\
& - \iota_\mu \bar{f}^k(d\omega) \wedge \bar{f}_k(\tilde{\Psi}_{(e,\omega)}^*) + \iota_\mu \bar{f}^k(\omega) \wedge d\bar{f}_k(\tilde{\Psi}_{(e,\omega)}^*) \\
& \left. - \iota_\mu \bar{f}_k(d\omega) \wedge \bar{f}^k(\tilde{\Psi}_{(e,\omega)}^*) + \iota_\mu \bar{f}_k(\omega) \wedge d\bar{f}^k(\tilde{\Psi}_{(e,\omega)}^*) \right),
\end{aligned}$$

where we defined

$$\begin{aligned}
\Psi_{(e,\omega)}^* &= e \wedge_\star [\omega, \omega]_{\mathfrak{so}(4)}^* + [\omega, \omega]_{\mathfrak{so}(4)}^* \wedge_\star e + R_l(e) \wedge_\star [R^l(\omega), \omega]_{\mathfrak{so}(4)}^* \\
&+ 6 \Lambda e \wedge_\star e \wedge_\star e, \\
\tilde{\Psi}_{(e,\omega)}^* &= e \wedge_\star (\omega \wedge_\star e) - (R_l(\omega) \wedge_\star R^l(e)) \wedge_\star e - \omega \triangleright_\star (e \wedge_\star e) + (e \wedge_\star e) \star \lhd \omega.
\end{aligned}$$

We see that the braided Noether identities become extremely complicated in this case, even with the simplified twists that we discussed in the three-dimensional theory, as we see in Example 6.33. Nevertheless, the identities exist in explicit form and provide a highly non-trivial differential identification among the braided field equations, which captures the statement of braided gauge invariance of the field theory.

Example 6.33 Similarly to Example 6.13, the expression for the braided diffeomorphism Noether identity can be reduced slightly in the example of the Moyal–Weyl twist $\mathcal{F}_\theta$ on $M = \mathbb{R}^4$. In this case we can again write the Noether identity as in (6.14), now with the 4-form

$$\begin{aligned}
N_\mu^{\star\theta} &= \frac{1}{2} \left(\iota_\mu^{\star\theta} de \wedge_{\star\theta} F_e^{\star\theta} - \iota_\mu^{\star\theta} d\omega \wedge_{\star\theta} F_\omega^{\star\theta} - \iota_\mu^{\star\theta} e \wedge_{\star\theta} dF_e^{\star\theta} + \iota_\mu^{\star\theta} \omega \wedge_{\star\theta} dF_\omega^{\star\theta} \right. \\
&+ F_e^{\star\theta} \wedge_{\star\theta} \iota_\mu^{\star\theta} de - F_\omega^{\star\theta} \wedge_{\star\theta} \iota_\mu^{\star\theta} d\omega - dF_e^{\star\theta} \wedge_{\star\theta} \iota_\mu^{\star\theta} e + dF_\omega^{\star\theta} \wedge_{\star\theta} \iota_\mu^{\star\theta} \omega \\
&+ \frac{1}{4} \left(e \wedge_{\star\theta} \iota_\mu^{\star\theta} de \wedge_{\star\theta} d\omega + d\omega \wedge_{\star\theta} \iota_\mu^{\star\theta} de \wedge_{\star\theta} e + de \wedge_{\star\theta} \iota_\mu^{\star\theta} d\omega \wedge_{\star\theta} e \right. \\
&- e \wedge_{\star\theta} \iota_\mu^{\star\theta} d\omega \wedge_{\star\theta} de - de \wedge_{\star\theta} \iota_\mu^{\star\theta} e \wedge_{\star\theta} d\omega - d\omega \wedge_{\star\theta} \iota_\mu^{\star\theta} e \wedge_{\star\theta} de \\
&+ \frac{1}{72} \left(\iota_\mu^{\star\theta} de \wedge_{\star\theta} \Psi_{(e,\omega)}^{\star\theta} - \iota_\mu^{\star\theta} e \wedge_{\star\theta} d\Psi_{(e,\omega)}^{\star\theta} + \Psi_{(e,\omega)}^{\star\theta} \wedge_{\star\theta} \iota_\mu^{\star\theta} de \right. \\
&- d\Psi_{(e,\omega)}^{\star\theta} \wedge_{\star\theta} \iota_\mu^{\star\theta} e \\
&- \iota_\mu^{\star\theta} d\omega \wedge_{\star\theta} \tilde{\Psi}_{(e,\omega)}^{\star\theta} + \iota_\mu^{\star\theta} \omega \wedge_{\star\theta} d\tilde{\Psi}_{(e,\omega)}^{\star\theta} - \tilde{\Psi}_{(e,\omega)}^{\star\theta} \wedge_{\star\theta} \iota_\mu^{\star\theta} d\omega \\
&\left. + d\tilde{\Psi}_{(e,\omega)}^{\star\theta} \wedge_{\star\theta} \iota_\mu^{\star\theta} \omega \right).
\end{aligned}$$

Action functional

Twisting the four-dimensional classical ECP pairing from [30, Section 8.2] using an arbitrary Drinfel'd twist $\mathcal{F}$, we obtain the (braided) cyclic structure

$$\begin{aligned} \langle (e, \omega), (E, \Omega) \rangle_\star &:= \int_M \text{Tr}(e \star E - \omega \star \Omega), \\ \langle (\xi, \rho), (\mathcal{X}, \mathcal{P}) \rangle_\star &:= \int_M \iota_\xi^\star \mathcal{X} + \int_M \text{Tr}(\rho \star \mathcal{P}), \end{aligned} \quad (6.34)$$

on $V_1 \otimes V_2$ and $V_0 \otimes V_3$, respectively. We shall write out and simplify the four-dimensional braided noncommutative ECP functional in two steps. First we work with a general twist (or alternatively ignore strict cyclicity for a compatible twist). Then we shall employ a compatible twist, and use $\wedge_\star$ -cyclicity to simplify the corresponding action functional. The ECP action functional is defined via (5.39):

$$\begin{aligned} S_\star(e, \omega) &\asymp \frac{1}{2} \langle (e, \omega), \ell_1^\star(e, \omega) \rangle_\star - \frac{1}{6} \langle (e, \omega), \ell_2^\star((e, \omega), (e, \omega)) \rangle_\star \\ &\quad - \frac{1}{24} \langle (e, \omega), \ell_3^\star((e, \omega), (e, \omega), (e, \omega)) \rangle_\star \\ &= \int_M \frac{1}{6} \text{Tr}(e \star e \star d\omega + e \star d\omega \star e - \omega \star e \star de + \omega \star de \star e) \\ &\quad + \frac{1}{24} \text{Tr}(e \star e \star [\omega, \omega]_{\mathfrak{so}(4)}^\star + e \star R_k(e) \star [R^k(\omega), \omega]_{\mathfrak{so}(4)}^\star \\ &\quad + e \star [\omega, \omega]_{\mathfrak{so}(4)}^\star \star e - \omega \star e \star (\omega \wedge_\star e) \\ &\quad - \omega \star e \star (R_k(\omega) \wedge_\star R^k(e)) - \omega \star R_k(e) \star (R^k(\omega) \wedge_\star e) \\ &\quad - \omega \star R_k(e) \star (R_l(\omega) \wedge_\star R^l R^k(e)) + \omega \star (\omega \wedge_\star e) \star e \\ &\quad + \omega \star (R_k(\omega) \wedge_\star R^k(e)) \wedge_\star e + 6 \Lambda e \star e \star e \star e), \end{aligned} \quad (6.35)$$

where we used $\ell_1^\star(e, \omega) = (0, 0)$.

We note that the first, second, fifth, sixth, and seventh terms in (6.35) combine modulo exact forms as

$$\begin{aligned} &\frac{1}{24} (2e \star e \star R^\star + 2e \star R^\star \star e + e \star R_k(e) \star [R^k(\omega), \omega]_{\mathfrak{so}(4)}^\star \\ &\quad + 2e \star e \star d\omega + 2e \star d\omega \star e) \\ &= \frac{1}{24} (2e \star e \star R^\star + 2e \star R^\star \star e - (R_k(\omega) \wedge_\star R^k(e)) \star e \star \omega \\ &\quad + e \star (\omega \wedge_\star e) \star \omega - 2d(e \star e) \star \omega + 2e \star d\omega \star e) \\ &= \frac{1}{24} (2e \star e \star R^\star + 2e \star R^\star \star e - T_R^\star \star e \star \omega + e \star T_L^\star \star \omega \\ &\quad + e \star e \star d\omega + 2e \star d\omega \star e), \end{aligned}$$

where in the first equality we used invariance of top-vectors under braided local $\mathfrak{so}(4)$ transformations on the third term and integration by parts on the fourth term, while in the second equality we simply identified the left and right torsion 2-forms. The rest of the terms in (6.35) may be analogously manipulated into the expression

$$\begin{aligned} & \frac{1}{24} (\omega \wedge_\star T_R^\star \wedge_\star e - \omega \wedge_\star e \wedge_\star T_L^\star + d\omega \wedge_\star e \wedge_\star e + \omega \wedge_\star d_\star^\omega(e \wedge_\star e) \\ & + \omega \wedge_\star d_\star^\omega(e \wedge_\star e)). \end{aligned}$$

Summing these two expressions, we obtain the 4-dimensional ECP action functional in the simplified form

$$\begin{aligned} S_\star(e, \omega) = & \frac{1}{24} \int_M \text{Tr} \Big(2e \wedge_\star e \wedge_\star R^\star + 2e \wedge_\star R^\star \wedge_\star e \\ & - T_R^\star \wedge_\star e \wedge_\star \omega + e \wedge_\star T_L^\star \wedge_\star \omega \\ & + \omega \wedge_\star T_R^\star \wedge_\star e - \omega \wedge_\star e \wedge_\star T_L^\star + \omega \wedge_\star d_\star^\omega(e \wedge_\star e) \\ & + \omega \wedge_\star d_\star^\omega(e \wedge_\star e) \\ & + e \wedge_\star e \wedge_\star d\omega + 2e \wedge_\star d\omega \wedge_\star e + d\omega \wedge_\star e \wedge_\star e + 6\Lambda e \wedge_\star e \wedge_\star e \wedge_\star e \Big). \end{aligned} \quad (6.36)$$

The first two terms of (6.36) are manifestly braided covariant, while the rest are not individually covariant since the gauge field ω appears explicitly. However, one may check directly that they do form covariant combinations. Let us explicitly demonstrate that the eighth and the last term form a covariant combination. The potential non-covariance arises from the terms proportional to $d\rho$ in a braided $\mathfrak{so}(4)$ gauge transformation, since the covariant components proportional to ρ vanish by invariance of a top-vector. Hence, working modulo exact forms we find

$$\begin{aligned} & \delta_\rho^{\star, L} (\omega \wedge_\star d_\star^\omega(e \wedge_\star e) + d\omega \wedge_\star e \wedge_\star e) \\ & = d\rho \wedge_\star d_\star^\omega(e \wedge_\star e) - [d\rho, \omega]_{\mathfrak{so}(4)}^\star \wedge_\star e \wedge_\star e \\ & = d\rho \wedge_\star (\omega \triangleright_\star (e \wedge_\star e)) - d\rho \wedge_\star (\omega \triangleright_\star (e \wedge_\star e)) \\ & = 0, \end{aligned}$$

where in the second equality we dropped an exact top-form from the first term, and used invariance of top-vectors under the braided $\mathfrak{so}(4)$ action on the second term. The rest of the terms in (6.36) form covariant combinations via analogous calculations. Hence, the invariance of the action functional under braided $\mathfrak{so}(4)$ transformations is fulfilled, as it should be by the general discussion of Sect. 5.8.

By restricting to compatible twists as in Proposition 4.15, the pairing becomes a strictly cyclic structure on $(V[[\hbar]], \ell_1^\star, \ell_2^\star, \ell_3^\star)$. In this case, the four-dimensional braided ECP action functional simplifies further to

$$\begin{aligned} S_\star(e, \omega) = & \int_M \text{Tr} \Big(\frac{1}{2} e \wedge_\star e \wedge_\star R^\star + \frac{\Lambda}{4} e \wedge_\star e \wedge_\star e \wedge_\star e \Big) \\ & - \frac{1}{24} \int_M \text{Tr} \Big(\omega \wedge_\star (2e \wedge_\star T_L^\star - 2T_R^\star \wedge_\star e + d_\star^\omega(e \wedge_\star e) \\ & + d_\star^\omega(e \wedge_\star e)) \Big) \end{aligned} \quad (6.37)$$

which follows immediately from (6.36) by $\wedge_\star$ -cyclicity under the integral and the identity

$$\begin{aligned} \mathrm{Tr}(d\omega \wedge_\star e \wedge_\star e) &= \mathrm{Tr}\left(2 R^\star \wedge_\star e \wedge_\star e - \frac{1}{2} \omega \wedge_\star d_\star^\omega(e \wedge_\star e) \right. \\ &\quad \left. - \frac{1}{2} \omega \wedge_\star d_\star^\omega(e \wedge_\star e)\right), \end{aligned}$$

that may be checked by a simple expansion of the terms on the right-hand side. The first line of (6.37) is exactly the ‘naive’ deformation of the classical Einstein–Cartan–Palatini action functional in four dimensions, with the correct numerical prefactors; indeed, the second line vanishes in the classical case $\mathcal{R} = 1 \otimes 1$. Using similar identities to those employed before, one can explicitly check that the variation of this action functional with respect to the spin connection ω yields the field equations $\mathcal{F}_\omega^\star = 0$ from (6.23), while varying with respect to coframe field e yields the field equations $\mathcal{F}_e^\star = 0$ from (6.27), as expected on general grounds from Sect. 5.8.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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