Noncommutative symplectic manifolds for self-injective Nakayama algebras

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Abstract

In the paper, we study noncommutative symplectic manifolds $(A, \llbracket \omega \rrbracket)$, where A is a self-injective Nakayama algebra. There is given a full description of exact symplectic manifolds for A being the trivial extension of an hereditary Nakayama algebra.

Keywords Path algebra of a quiver · Karoubi-de Rham complex · Noncommutative symplectic manifold

Mathematics Subject Classification 14A22 · 58B34 · 83C65

1 Introduction

This paper is devoted and motivated by noncommutative differential geometry, to be more precise, symplectic geometry. Noncommutative symplectic geometry was introduced by Kontsevich [9].

If we want to consider a structure of noncommutative symplectic manifold linked to an associative algebra A , we have to yield a differential calculus on the algebra A . Following Dubois-Violette [5], a *differential calculus* on the algebra A is a differential algebra (Ω, d) such that $\Omega = \bigoplus_{n=0}^{\infty} \Omega^n$ is a graded algebra, $\Omega^0 = A$, and d is a super-derivation of degree 1 on Ω with $d^2 = 0$.

In the paper, we fix an algebraically closed field K whose characteristic is zero. By an algebra, we always mean an associative (in general noncommutative) unitary K -algebra A that is finite-dimensional. For such an algebra, we usually consider the Karoubi-de Rham complex $(DR_R^\bullet(A), d)$ for a fixed subalgebra R of A .

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For a given algebra A , any element $[\omega] \in DR_R^n(A)$ is called an R -relative differential n -form on A . If this form is non-degenerate and closed, then the pair $(A, [\omega])$ is said to be a *symplectic manifold* for the differential calculus $(DR_R^\bullet(A), d)$. If the algebra is noncommutative, then the symplectic manifold $(A, [\omega])$ is said to be *noncommutative*.

A particular class of finite-dimensional K -algebras form path algebras of quivers with relations, introduced by Gabriel in [6]. These algebras provide a wide class of examples to test many problems. For the Karoubi-de Rham differential calculus $(DR_R^\bullet(A), d)$, the path algebra $A = KQ$ of a quiver Q satisfies that there are no symplectic manifolds $(A, [\omega])$. That fact is shown in [2, 8, 12]. The shortest proof of this fact rests on computing the cohomology groups of the Karoubi-de Rham complex. The same holds for canonical algebras of Ringel (see [11]). The proof of this fact is obtained with a use of combinatorics. It seems to be possible to repeat the same arguments for any quiver algebra with relations, when the quiver has no oriented cycles.

On the contrary, there are symplectic manifolds $(K\bar{Q}, [\omega])$ for the so-called preprojective algebras $K\bar{Q}$ of quivers Q (see [3]). These symplectic manifolds are not classified. Their existence is obtained with a help of deep tools that are developed in [3]. The preprojective algebras are self-injective. This is the reason why the self-injective Nakayama algebras are considered. The class of Nakayama algebras, studied in the paper, consists of symmetric algebras. The main aim for considering these algebras is to find direct methods to classify symplectic manifolds.

Our main goal is to give a full description of exact symplectic manifolds $(B_{r,l}, [\omega])$ for a class of self-injective Nakayama algebras $B_{r,l}$, $r \geq 0$, $l = r + 2$, (see [1, 10]). Exactness of the manifold means that the form $[\omega]$ is exact, i.e., it is contained in the image of d . The considered class of algebras $B_{r,r+2}$ coincides to the class of the trivial extensions of the path algebras KQ of the quivers $Q = 0 \xrightarrow{\alpha_0} 1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{r-1}} r$.

The main results of the paper are the following.

Theorem 1.1 *For any algebra $B_{r,r+2}$, $r \geq 0$, and the Karoubi-de Rham calculus $(DR_R^\bullet(B_{r,r+2}), d)$, the following conditions are satisfied:*

- (1) *There exist symplectic manifolds $(B_{r,r+2}, [\omega])$.*
- (2) *A pair $(B_{r,r+2}, [\omega])$ is an exact symplectic manifold if and only if $[\omega] = \sum_{i=0}^r k_i [e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i] \neq 0$, and there is at most one index $i_0 \in \{0, 1, \dots, r\}$ such that $k_{i_0} = 0$, where $k_i \in K$, $i = 0, 1, \dots, r$.*

Theorem 1.2 *For any algebra $B_{r,r+2}$, $r \geq 0$, every exact symplectic manifold $(B_{r,r+2}, [\omega])$ is symplectomorphic to a manifold $(B_{r,r+2}, [\omega_0])$ such that $[\omega_0] = \sum_{i=0}^r k_i [e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]$ and $k_0 \in \{0, 1\} \subset K$ and $0 \neq k_i \in K$, $i = 1, 2, \dots, r$.*

In the sequel, there are explained all notations that have been used above. Furthermore, all information on path algebras of quivers can be found in [1].

2 Preliminaries

For a fixed K -algebra A that is unitary and finite-dimensional, let $\Omega_K^1(A)$ be an A -bimodule of noncommutative differential 1-forms on A , i.e., $\Omega_K^1(A) = \ker(m)$, where $m : A \otimes_K A \rightarrow A$ is an epimorphism given by $m(a \otimes b) = ab$.

Now, we denote by $\bar{A}$ the quotient linear space $\bar{A} = A/(K \cdot 1_A)$ and let $A \rightarrow \bar{A}$ denote the projection. Then, the mapping $d_A : A \rightarrow \Omega_K^1(A)$ given by $x \mapsto d_A x = x \otimes 1_A - 1_A \otimes x$ is a derivation. Moreover, the mapping $A \otimes \bar{A} \rightarrow \Omega_K^1(A)$ given by $x \otimes \bar{y} \mapsto x \otimes y - xy \otimes 1_A = x d_A y$ is well-defined and it is an A -bimodule isomorphism (comp. [7]; Proposition 10.1.3).

Following Ginzburg, we can proceed more generally (comp. [4]). For a given K -algebra A and its subalgebra R , we can consider

$$\Omega_R^n(A) = A \otimes_R \underbrace{\bar{A} \otimes_R \cdots \otimes_R \bar{A}}_{n \times},$$

where now $\bar{A} = A/R$ as a linear K -space. Then,

$$\Omega_R^\bullet(A) = \bigoplus_{i=0}^{\infty} \Omega_R^i(A).$$

Further, we can define a differential $d_n : \Omega_R^n(A) \rightarrow \Omega_R^{n+1}(A)$ given by:

$$d_n(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) = 1_A \otimes \bar{a}_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n.$$

In this manner, we obtain a complex $(\Omega_R^\bullet(A), d)$. We define a product for this complex that is compatible with grading. We do it determining a map

$$\Omega_R^n(A) \times \Omega_R^{m-1-n}(A) \rightarrow \Omega_R^m(A)$$

as follows:

$$(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) \cdot (a_{n+1} \otimes \bar{a}_{n+2} \otimes \cdots \otimes \bar{a}_m) = (-1)^n a_0 a_1 \otimes \bar{a}_2 \otimes \cdots \otimes \bar{a}_m +$$

$$\sum_{i=1}^n (-1)^{n-i} a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_i \bar{a}_{i+1} \otimes \cdots \otimes \bar{a}_m.$$

Now, let

$$DR_R^\bullet(A) = \Omega_R^\bullet(A)/[\Omega_R^\bullet(A), \Omega_R^\bullet(A)]_{super}$$

be the Karoubi-de Rham complex, where $[-, -]_{super}$ denotes the K -linear subspace spanned by the super-commutators. The differential $d : \Omega_R^\bullet(A) \rightarrow \Omega_R^{\bullet+1}(A)$ induces a well-defined differential $d : DR_R^\bullet(A) \rightarrow DR_R^{\bullet+1}(A)$.

Fixing an R -derivation $\theta : A \rightarrow A$, we obtain in the unique way a super-derivation of degree (-1)

$$i_\theta : DR_R^\bullet(A) \rightarrow DR_R^{\bullet-1}(A)$$

of the algebra $DR_R^\bullet(A)$ that is given by the following formula:

$$i_\theta([a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_n}]) = \sum_{j=1}^n (-1)^{j-1} [a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{j-1}\theta(a_j)} \otimes \overline{a_{j+1}} \otimes \cdots \otimes \overline{a_n}],$$

where $[a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_n}]$ is the coset of the element $a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_n} \in \Omega_R^n(A)$ modulo $[\Omega_R^n(A), \Omega_R^n(A)]_{super}$.

An element $[\omega] \in DR_R^n(A)$ is said to be an R -relative differential n -form on A . For a fixed $[\omega] \in DR_R^2(A)$, the map $i_{[\omega]} : \text{Der}_R(A) \rightarrow DR_R^1(A)$ is a K -linear transformation given by $\theta \mapsto i_\theta([\omega])$, where $\text{Der}_R(A)$ is the space of the R -derivations on A . An R -relative differential 2-form $[\omega]$ is called *non-degenerate* provided that $i_{[\omega]}$ is a K -linear isomorphism and is called *degenerate* otherwise. A 2-form $[\omega]$ is *exact* if there is an R -relative differential 1-form $[\kappa]$ such that $d([\kappa]) = [\omega]$. Moreover, $[\omega]$ is said to be *closed* provided that $d([\omega]) = 0$ in $DR_R^3(A)$. Since $(DR_R^\bullet(A), d)$ is a complex, every exact 2-form is closed. Finally, a pair $(A, [\omega])$ is said to be a *symplectic manifold* if $[\omega] \in DR_R^2(A)$ is non-degenerate and closed. This manifold is called *exact* if $[\omega]$ is exact. For a symplectic manifold $(A, [\omega])$, we call the algebra A its *support*.

3 Nakayama algebras

A finite-dimensional associative unitary K -algebra A is called *elementary* (see [1]) if $A/\text{rad}(A) \cong K \times \cdots \times K$, where $\text{rad}(A)$ is the Jacobson radical of the algebra A . P. Gabriel [6] attached a bound quiver (Q_A, I_A) to every elementary K -algebra A in such a way that $A \cong KQ_A/I_A$, where Q_A is a finite quiver, KQ_A is the path algebra of the quiver Q_A , and I_A is an admissible two-sided ideal in KQ_A . Now, many algebras are defined by their bound quivers. The same can be done for algebras introduced by Nakayama [10].

We shall consider finite-dimensional K -algebras of the form $B_{r,l} = KQ_r/I_l$, where Q_r is a quiver of the form

$$\begin{array}{ccccc} 0 & \xrightarrow{\alpha_0} & 1 & \xrightarrow{\alpha_1} & 2 \\ \alpha_r \uparrow & & & & \downarrow \alpha_2 \\ r & \xleftarrow{\alpha_{r-1}} & \cdots & & \end{array}$$

where $r = 0, 1, 2, \dots$ and I_l is the two-sided ideal in the path algebra KQ_r generated by the paths of length l , $l = 2, 3, \dots$

In the algebra $B_{r,l}$, every element $b \in B_{r,l}$ is said to be a path of length $d(b) = n \geq 1$ provided that $b = \beta_n \dots \beta_1 + I_l$, where β_j is an arrow in the quiver Q_r , $j = 1, 2, \dots, n$, and the source $s(\beta_{j+1})$ of the arrow β_{j+1} coincides to the sink $t(\beta_j)$ of the arrow β_j for each $j = 1, 2, \dots, n-1$. Further, $s(\beta_1)$ is said to be the source of the path b and is denoted by $s(b)$, and $t(\beta_n)$ is said to be the sink of the path b and it is denoted by $t(b)$. Moreover, each vertex $x \in \{0, 1, \dots, r\}$ of the quiver Q_r is identified to the trivial path $e_x + I_l$ of length 0. It is obvious that $s(e_x) = t(e_x) = x$. We shall usually write elements of an algebra $B_{r,l}$ as K -linear combinations of paths in the quiver Q_r , keeping in mind that if such a combination belongs to I_l , then we have a zero element of the algebra.

We shall denote by R the semi-simple subalgebra of $B_{r,l}$ that is generated by the trivial paths (orthogonal idempotents) e_i , $i = 0, 1, \dots, r$. Then, $R \cong K^{r+1}$.

For a fixed algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, an element $u_0 \otimes \bar{u}_1 \otimes \dots \otimes \bar{u}_n \in \Omega_R^n(B_{r,l})$ is defined to be a *generalized path* in $\Omega_R^n(B_{r,l})$ if $u_0, u_1, \dots, u_n$ are paths in the quiver Q_r and the following conditions are satisfied:

- (1) $u_0 \otimes \bar{u}_1 \otimes \dots \otimes \bar{u}_n \neq 0$ in $\Omega_R^n(B_{r,l})$.
- (2) For each $i = 0, 1, \dots, n-1$, we have $s(u_i) = t(u_{i+1})$ in Q_r .

A path u in the quiver Q_r is called an *oriented cycle* provided that $s(u) = t(u)$ and u is non-trivial. That is why a generalized path in $\Omega_R^n(B_{r,l})$ of the form $u_0 \otimes \bar{u}_1 \otimes \dots \otimes \bar{u}_n$ is said to be a *generalized cycle* if $t(u_0) = s(u_n)$ in the quiver Q_r . Generalized cycles will play a prominent role in our further considerations.

For a fixed algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, and any integer $n \geq 1$ let $\mathcal{Y}^{(n)}(B_{r,l}) = \{u_0 \otimes \bar{u}_1 \otimes \dots \otimes \bar{u}_n \in \Omega_R^n(B_{r,l}); u_0 \otimes \bar{u}_1 \otimes \dots \otimes \bar{u}_n \text{ is a generalized path that is not a generalized cycle}\}$.

Lemma 3.1 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, and any integer $n \geq 1$ we have $\mathcal{Y}^{(n)}(B_{r,l}) \subset [B_{r,l}, \Omega_R^n(B_{r,l})]$.*

Proof We have that $[B_{r,l}, \Omega_R^n(B_{r,l})]$ is the K -linear subspace of $\Omega_R^n(B_{r,l})$ spanned by the commutators $[b, v]$ with $b \in B_{r,l}$ and $v \in \Omega_R^n(B_{r,l})$. If $v_0 \otimes \bar{v}_1 \otimes \dots \otimes \bar{v}_n \in \mathcal{Y}^{(n)}(B_{r,l})$, then $[e_j, v_0 \otimes \bar{v}_1 \otimes \dots \otimes \bar{v}_n] = e_j v_0 \otimes \bar{v}_1 \otimes \dots \otimes \bar{v}_n - (-1)^n v_0 v_1 \otimes \bar{v}_2 \otimes \dots \otimes \bar{v}_n \otimes \bar{e}_j - (-1)^{n-1} v_0 \otimes \bar{v}_1 v_2 \otimes \bar{v}_3 \otimes \dots \otimes \bar{v}_n \otimes \bar{e}_j - \dots - v_0 \otimes \bar{v}_1 \otimes \dots \otimes \bar{v}_n e_j = v_0 \otimes \bar{v}_1 \otimes \dots \otimes \bar{v}_n$ for $j = t(v_0)$, because $w_0 \otimes \bar{w}_1 \otimes \dots \otimes \bar{w}_{n-1} \otimes \bar{e}_j = 0$ in $\Omega_R^n(B_{r,l})$ and $v_n e_j = 0$ since $t(v_0) \neq s(v_n)$. Therefore, the required condition holds. $\square$

Lemma 3.2 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, and any integer $n \geq 1$, we have: if $u_0 \otimes \bar{u}_1 \otimes \dots \otimes \bar{u}_n$ is a generalized path in $\Omega_R^n(B_{r,l})$ and z is a nonzero path in $B_{r,l}$ such that $zu_0 u_1 \dots u_n$ is an oriented cycle in the quiver Q_r , then the commutator $[z, u_0 \otimes \bar{u}_1 \otimes \dots \otimes \bar{u}_n]$ is either a sum of generalized cycles or zero.*

Proof If $zu_0 u_1 \dots u_n$ is an oriented cycle, then $[z, u_0 \otimes \bar{u}_1 \otimes \dots \otimes \bar{u}_n] = zu_0 \otimes \bar{u}_1 \otimes \dots \otimes \bar{u}_n - (-1)^n u_0 u_1 \otimes \bar{u}_2 \otimes \dots \otimes \bar{u}_n \otimes \bar{z} - \dots - u_0 \otimes \bar{u}_1 \otimes \dots \otimes \bar{u}_n z$, where each of the summands can be zero or a generalized cycle. Thus, the required condition holds. $\square$

Lemma 3.3 For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, and any integer $n \geq 1$ we have: if $u_0 \otimes \bar{u}_1 \otimes \cdots \otimes \bar{u}_n$ is a generalized path in $\Omega_R^n(B_{r,l})$ and z is a nonzero path in $B_{r,l}$ such that $zu_0u_1 \cdots u_n$ is not an oriented cycle in Q_r , then the commutator $[z, u_0 \otimes \bar{u}_1 \otimes \cdots \otimes \bar{u}_n]$ is either zero or a sum of generalized paths that are not generalized cycles.

Proof We have $[z, u_0 \otimes \bar{u}_1 \otimes \cdots \otimes \bar{u}_n] = zu_0 \otimes \bar{u}_1 \otimes \cdots \otimes \bar{u}_n - (-1)^n u_0 u_1 \otimes \bar{u}_2 \otimes \cdots \otimes \bar{u}_n \otimes \bar{z} - \cdots - u_0 \otimes \bar{u}_1 \otimes \cdots \otimes \bar{u}_n z$. Notice that if $zu_0u_1 \cdots u_n$ is not an oriented cycle in Q_r , then $u_0u_1 \cdots u_n z$ is not an oriented cycle, either. Thus, the above sum is either zero or is a sum of generalized paths that are not generalized cycles. $\square$

Lemma 3.4 For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, we have $[B_{r,l}, B_{r,l}]$ is spanned by the noncyclic paths and differences of cyclic permutations of the cyclic paths.

Proof We have by definition that $[B_{r,l}, B_{r,l}]$ is the K -linear subspace of $B_{r,l}$ that is spanned by the elements of the form $[u, v] = uv - (-1)^{0 \cdot 0} vu$, $u, v \in B_{r,l}$. It is clear that if $w \in I_l$, then $w = 0$ in $B_{r,l}$ and the required condition holds.

Further notice that for any non-trivial path w in the quiver Q_r , we have $e_{t(w)}w - we_{t(w)} = e_{t(w)}w = w$ when $t(w) \neq s(w)$. Thus, for any non-trivial path w that is not an oriented cycle we obtain that $w \in [B_{r,l}, B_{r,l}]$.

If c is an oriented cycle, then $c = vu$ and we have $[u, v] = uv - vu = c - vu$, and vu is a cyclic permutation of c . $\square$

4 R -derivations of the algebras $B_{r,l}$

First, we introduce a notion that will be used in the sequel. We shall denote by c_i , $i = 0, 1, \dots, r$, the oriented cycle $c_i = \alpha_{i-1} \cdots \alpha_1 \alpha_0 \cdots \alpha_i$ in the quiver Q_r .

We shall denote by $\text{Der}_R(B_{r,l})$ the K -linear subspace of $\text{Der}_K(B_{r,l})$ spanned by the derivations of $B_{r,l}$ vanishing on the subalgebra R .

Lemma 4.1 If $\theta \in \text{Der}_R(B_{r,l})$, $r \geq 0$, $l \geq 2$, then for any arrow α_i in Q_r , $i = 0, 1, \dots, r$, we have $\theta(\alpha_i) \in e_{i+1}B_{r,l}e_i$, where $i = s(\alpha_i)$, $i+1 = t(\alpha_i)$.

Proof We have the following series of equalities: $\theta(\alpha_i) = \theta(e_{i+1}(\alpha_i e_i)) = \theta(e_{i+1})\alpha_i e_i + e_{i+1}\theta(\alpha_i e_i) = e_{i+1}\theta(\alpha_i)e_i + e_{i+1}\alpha_i\theta(e_i) = e_{i+1}\theta(\alpha_i)e_i \in e_{i+1}B_{r,l}e_i$. $\square$

Lemma 4.2 For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, and any derivation $\theta \in \text{Der}_R(B_{r,l})$, the following conditions are satisfied:

- (1) If $l \leq r+1$, then for any arrow α_i , $i = 0, 1, \dots, r$, there exists $k_i \in K$ such that $\theta(\alpha_i) = k_i \alpha_i$.
- (2) If $l \geq r+2$, then for any arrow α_i , $i = 0, 1, \dots, r$, there exist $k_{i,0}, k_{i,1}, \dots, k_{i,s} \in K$ such that $\theta(\alpha_i) = \sum_{j=0}^s k_{i,j} \alpha_i c_i^j$, where $l = s(r+1) + l_0$ and $1 \leq l_0 \leq r+1$.

Proof The required conditions can be obtained straightforwardly by Lemma 4.1, because the system $\{\alpha_i, \alpha_i c_i, \dots, \alpha_i c_i^t\}$ forms a basis of the K -linear space $e_{i+1}B_{r,l}e_i$,

where $t = \begin{cases} 0 & \text{if } l \leq r+1 \\ s & \text{if } l = s(r+1) + l_0 \text{ and } 1 \leq l_0 \leq r+1. \end{cases}$ $\square$

Lemma 4.3 For any algebra $B_{r,l}$, $r \geq 0$, $2 \leq l \leq r + 2$, and any derivation $\theta \in \text{Der}_R(B_{r,l})$, it holds: if $w = \beta_z \dots \beta_1 \notin I_l$, β_j is an arrow for $j = 1, \dots, z$, then $\theta(w) = \sum_{j=1}^z k_j w$, where $\theta(\beta_j) = k_j \beta_j$, $j = 1, \dots, z$.

Proof We shall proceed by induction on z . If $z = 1$, then the required condition is clear by Lemma 4.2. Assume that the required condition holds for the paths of length $\leq z_0$ and consider a path $w' = \beta_{z_0+1} w \notin I_l$. Then, we have $\theta(w') = \theta(\beta_{z_0+1})w + \beta_{z_0+1}\theta(w) = k_{z_0+1}w' + \sum_{j=1}^{z_0} k_j w'$. $\square$

Now, we define a family of derivations in $\text{Der}_R(B_{r,l})$ for a fixed algebra $B_{r,l}$. We assume that $l = s(r + 1) + l_0$, $s \geq 0$, $1 \leq l_0 \leq r + 1$. Let $\theta_{(\alpha_i)}^j \in \text{Der}_R(B_{r,l})$, $i = 0, 1, \dots, r$, $j = 0, 1, \dots, s$, be as follows:

$$\theta_{(\alpha_i)}^j(\alpha_n) = \begin{cases} \alpha_i c_i^j & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.4 For any derivation $\theta_{(\alpha_i)}^0$, $i = 0, 1, \dots, r$, and any path w in Q_r that does not belong to I_l , we have

$$\theta_{(\alpha_i)}^0(w) = \begin{cases} pw_1 \alpha_i w_2 & \text{if } w = w_1 \alpha_i w_2 \\ 0 & \text{otherwise,} \end{cases}$$

where p is the multiplicity of appearance of the arrow α_i in the path w .

Proof We shall prove the lemma inductively on the length n of the path w . If $n = 1$, then $w = \alpha_i$ and the required condition holds for each derivation $\theta_{(\alpha_i)}^0$, $i = 0, 1, \dots, r$.

Assume that for the paths w of length $\leq n_0$ the required condition holds for each derivation $\theta_{(\alpha_i)}^0$, $i = 0, 1, \dots, r$. Now, consider a path $w = \beta_{n_0+1} \beta_{n_0} \dots \beta_1$, β_j is an arrow for each $j = 1, 2, \dots, n_0 + 1$. Then, for a fixed $i_0 \in \{0, 1, \dots, r\}$, we have $\theta_{(\alpha_{i_0})}^0(w) = \theta_{(\alpha_{i_0})}^0(\beta_{n_0+1} \beta_{n_0} \dots \beta_1) = \theta_{(\alpha_{i_0})}^0(\beta_{n_0+1}) \beta_{n_0} \dots \beta_1 + \beta_{n_0+1} \theta_{(\alpha_{i_0})}^0(\beta_{n_0} \dots \beta_1) =$

$$\begin{cases} \beta_{n_0+1} \beta_{n_0} \dots \beta_1 + p \beta_{n_0+1} \beta_{n_0} \dots \beta_1 & \text{if } \alpha_{i_0} = \beta_{n_0+1} \\ p \beta_{n_0+1} \beta_{n_0} \dots \beta_1 & \text{if } \alpha_{i_0} \neq \beta_{n_0+1}, \end{cases}$$

where the arrow α_{i_0} appears p times in the path $\beta_{n_0} \dots \beta_1$. Hence, the required condition holds. $\square$

Proposition 4.5 For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, $l = s(r + 1) + l_0$, $1 \leq l_0 \leq r + 1$, the system $\{\theta_{(\alpha_i)}^j\}_{i=0,1,\dots,r; j=0,1,\dots,s}$ is a basis of the K -linear space $\text{Der}_R(B_{r,l})$.

Proof We start our proof with checking whether the system $\{\theta_{(\alpha_i)}^j\}_{i=0,1,\dots,r; j=0,1,\dots,s}$ is linearly independent. In order to do this, we consider a linear combination $\sum_{j=0}^s \sum_{i=0}^r k_{j,i} \theta_{(\alpha_i)}^j = 0$. Then, for a fixed arrow α_{i_0} we have $\sum_{j=0}^s \sum_{i=0}^r k_{j,i} \theta_{(\alpha_i)}^j(\alpha_{i_0}) = \sum_{j=0}^s k_{j,i_0} \theta_{(\alpha_{i_0})}^j(\alpha_{i_0}) = k_{0,i_0} \alpha_{i_0} + k_{1,i_0} \alpha_{i_0} c_{i_0} + \dots + k_{s,i_0} \alpha_{i_0} c_{i_0}^s = 0$. Hence, $k_{j,i_0} = 0$ for $j = 0, 1, \dots, s$. Since i_0 can be arbitrary, we obtain that all $k_{j,i} = 0$, $i = 0, 1, \dots, r$, $j = 0, 1, \dots, s$. Therefore, the considered system is linearly independent.

Now, consider a derivation $\theta \in \text{Der}_R(B_{r,l})$. We infer by Lemma 4.2 that for every arrow α_i , $i = 0, 1, \dots, r$, there is $k_i \in K$, $i = 0, 1, \dots, r$ such that $\theta(\alpha_i) = k_i \alpha_i$ or there are $k_{i,0}, k_{i,1}, \dots, k_{i,s} \in K$ such that $\theta(\alpha_i) = \sum_{j=0}^s k_{i,j} \alpha_i c_i^j$. Then, we get $\theta = \sum_{i=0}^r k_i \theta_{(\alpha_i)}^0$ or $\theta = \sum_{j=0}^s \sum_{i=0}^r k_{i,j} \theta_{(\alpha_i)}^j$, respectively. Indeed, for any arrow α_{i_0} we have $\theta(\alpha_{i_0}) = k_{i_0} \alpha_{i_0}$ and $\sum_{i=0}^r k_i \theta_{(\alpha_i)}^0(\alpha_{i_0}) = k_{i_0} \alpha_{i_0}$ in the first case. In the second case, similarly we have $\theta(\alpha_{i_0}) = \sum_{j=0}^s k_{i_0,j} \alpha_{i_0} c_{i_0}^j$ and $\sum_{j=0}^s \sum_{i=0}^r k_{i,j} \theta_{(\alpha_i)}^j(\alpha_{i_0}) = \sum_{j=0}^s k_{i_0,j} \alpha_{i_0} c_{i_0}^j$. Then, θ has the required form. $\square$

Corollary 4.6 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, and $l = s(r+1) + l_0$, $1 \leq l_0 \leq r+1$, we have $\dim_K \text{Der}_R(B_{r,l}) = s(r+1)$.*

Proof Our corollary is an obvious consequence of Proposition 4.5.

5 Description of $DR_R^0(B_{r,l})$

In this section, we shall prove that $DR_R^0(B_{r,l})$ is always nonzero and indicate its basis.

Lemma 5.1 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, the following conditions are satisfied:*

- (1) *Any trivial path e_i , $i = 0, 1, \dots, r$, does not belong to $[B_{r,l}, B_{r,l}]$.*
- (2) *Any nonzero oriented cycle c does not belong to $[B_{r,l}, B_{r,l}]$.*

Proof In order to prove condition (1), it is enough to check whether $e_i = [u, v]$ for some $u, v \in B_{r,l}$. Notice that if $u = e_{i_1}$, $v = e_{i_2}$, then $[e_{i_1}, e_{i_2}] = 0$. If $u = e_{i_1}$ and v is a non-trivial path, then $[e_{i_1}, v] \in \{v, -v, 0\}$. Hence, it cannot happen $e_i = [e_{i_1}, v]$.

Now, consider the case when u, v are both non-trivial paths. Then, the commutator $[u, v]$ is zero or a non-trivial path or else a nonzero difference of non-trivial paths. Therefore, it cannot be $e_i = [u, v]$ in the case, either. Consequently, condition (1) follows.

For a proof of condition (2) notice that any cycle c_i , $i = 0, 1, \dots, r$, cannot be obtained as a commutator $[u, v]$ for some paths u, v that satisfy $s(u) \neq t(v)$ or $s(v) \neq t(u)$. Hence, we have to consider only paths u, v such that $s(u) = t(v)$ and $s(v) = t(u)$. Then, we have $[u, v] = c_{s(v)} - c_{s(u)}$ and $[u, v] = 0$ if $s(v) = s(u)$, and $[u, v] = c_{s(v)} - c_{s(u)} \neq 0$ when $s(u) \neq s(v)$.

Now, suppose to the contrary that, say c_0 , belongs to $[B_{r,l}, B_{r,l}]$. Then, we infer by the above considerations that $c_0 = \sum_j k'_j [u_j, v_j] = \sum_{s,i} k_{i,s} (c_i - c_j)$, where $k_{i,s} \in K$ and $k_{i,i} = 0$, $i = 0, 1, \dots, r$. Then, we have $c_0 = \sum_{i=0}^r (k_{0,i} - k_{i,0}) c_0 + \sum_{i=0}^r (k_{1,i} - k_{i,1}) c_1 + \dots + \sum_{i=0}^r (k_{r,i} - k_{i,r}) c_r$. This equality implies the following system of equalities in the field K : $\sum_{i=0}^r k_{0,i} - \sum_{i=0}^r k_{i,0} = 1$, $\sum_{i=0}^r k_{1,i} - \sum_{i=0}^r k_{i,1} = 0$, $\sum_{i=0}^r k_{2,i} - \sum_{i=0}^r k_{i,2} = 0, \dots, \sum_{i=0}^r k_{r,i} - \sum_{i=0}^r k_{i,r} = 0$. Further, we have that the equality $\sum_{i=0}^r k_{1,i} - \sum_{i=0}^r k_{i,1} = 0$ implies that $k_{0,1} - k_{1,0} = (k_{1,2} - k_{2,1}) + (k_{1,3} - k_{3,1}) + \dots + (k_{1,r} - k_{r,1})$. The equality $\sum_{i=0}^r k_{2,i} - \sum_{i=0}^r k_{i,2} = 0$ implies that $k_{0,2} - k_{2,0} = (k_{2,1} - k_{1,2}) + (k_{2,3} - k_{3,2}) + \dots + (k_{2,r} - k_{r,2})$.

Generally, the equality $\sum_{i=0}^r k_{j,i} - \sum_{i=0}^r k_{i,j} = 0$, $j = 1, 2, \dots, r$ implies that $k_{0,j} - k_{j,0} = (k_{j,1} - k_{1,j}) + (k_{j,2} - k_{2,j}) + \dots + (k_{j,r} - k_{r,j})$.

Now, notice that the difference $k_{i,j} - k_{j,i}$ appears in the above presentation of $k_{0,i} - k_{i,0}$ if and only if the difference $k_{j,i} - k_{i,j}$ appears in the above presentation of $k_{0,j} - k_{j,0}$. Therefore, $\sum_{i=0}^r k_{0,i} - \sum_{i=0}^r k_{i,0} = \sum_{i=0}^r (k_{0,i} - k_{i,0}) = 0$ which contradicts to the equality $\sum_{i=0}^r k_{0,i} - \sum_{i=0}^r k_{i,0} = 1$, because $\text{char}(K) = 0$.

Finally, we obtain that $c_0 \notin [B_{r,l}, B_{r,l}]$ and we infer by Lemma 3.4 that every oriented cycle $c_i \notin [B_{r,l}, B_{r,l}]$, $i = 0, 1, \dots, r$.

Since every nonzero oriented cycle c in $B_{r,l}$ is of the form c_i^j for some $i = 0, 1, \dots, r$, and some integer $j \geq 1$, repeating the above arguments for c_0^j we obtain similarly that $c_i^j \notin [B_{r,l}, B_{r,l}]$, which shows condition (2). $\square$

Lemma 5.2 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, the K -linear space $B_{r,l}/[B_{r,l}, B_{r,l}]$ has a basis of the form $\{\llbracket e_i \rrbracket\}_{i=0,1,\dots,r} \cup \{\llbracket c_0^s \rrbracket\}_{s=1,2,\dots,n}$, where $l = n(r+1) + l_0$, $1 \leq l_0 \leq r+1$ and $\llbracket b \rrbracket$ denotes the coset of $b \in B_{r,l}$ modulo $[B_{r,l}, B_{r,l}]$.*

Proof First notice that the elements of the considered system are nonzero in view of Lemma 5.1. Further, we have that for any $j \in \{1, 2, \dots, r\}$ there is a path $\alpha_{j-1} \cdots \alpha_0 = u$ and a path $\alpha_r \cdots \alpha_j = v$. If c_0 is a nonzero oriented cycle, then the paths u, v are nonzero and $[u, v] = c_j - c_0$. Hence, $\llbracket c_j \rrbracket = \llbracket c_0 \rrbracket$. Similarly, $\llbracket c_j^s \rrbracket = \llbracket c_0^s \rrbracket$ for $s = 2, \dots, n$. Therefore, we infer by Lemma 3.4 that the system $\{\llbracket e_i \rrbracket\}_{i=0,1,\dots,r} \cup \{\llbracket c_0^s \rrbracket\}_{s=1,2,\dots,n}$ is indeed a set of generators of the K -linear space $B_{r,l}/[B_{r,l}, B_{r,l}]$.

Now, consider a linear combination $\sum_{i=0}^r k_i \llbracket e_i \rrbracket + \sum_{s=1}^n k'_s \llbracket c_0^s \rrbracket = 0$ for some $k_0, k_1, \dots, k_r \in K$ and some $k'_1, k'_2, \dots, k'_n \in K$. Then, we have $\sum_{i=0}^r k_i e_i + \sum_{s=1}^n k'_s c_0^s \in [B_{r,l}, B_{r,l}]$. Repeating the arguments from the proof of Lemma 5.1, we obtain that $\sum_{i=0}^r k_i e_i + \sum_{s=1}^n k'_s c_0^s = \sum_{s=1}^n \sum_{a,b=0}^r k_{a,b,s} (c_a^s - c_b^s)$. Thus, it is clear that $k_i = 0$ for $i = 0, 1, \dots, r$.

If we have $k'_{s_0} \neq 0$ for some $s_0 \in \{1, \dots, n\}$, then $\sum_{a,b=0}^r k_{a,b,s_0} (c_a^{s_0} - c_b^{s_0}) = k'_{s_0} c_0^{s_0}$ and we obtain a contradiction as in the proof of Lemma 5.1. Therefore, all $k'_s = 0$, $s = 1, 2, \dots, n$. Consequently, the considered system is linearly independent and forms a basis of $B_{r,l}/[B_{r,l}, B_{r,l}]$. $\square$

6 Description of $DR_R^1(B_{r,l})$

Our main aim of this section is a computation of the dimension of $DR_R^1(B_{r,l})$ for some $B_{r,l}$.

Lemma 6.1 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, the following conditions are satisfied:*

- (1) *For every nonzero oriented cycle c_i^{n+m} , $i = 0, 1, \dots, r$; $n, m \geq 1$, we have $\llbracket c_i^n \otimes \overline{c_i^m} \rrbracket + \llbracket c_i^m \otimes \overline{c_i^n} \rrbracket = \llbracket e_i \otimes \overline{c_i^{n+m}} \rrbracket$ in $DR_R^1(B_{r,l})$.*
- (2) *If $c_i^s \notin I_l$ for some integer $s \geq 1$ and $vu = c_i^s$ in Q_r for non-trivial paths u, v , then it holds $\llbracket u \otimes \overline{v} \rrbracket + \llbracket v \otimes \overline{u} \rrbracket = \llbracket e_i \otimes \overline{c_i^s} \rrbracket$ in $DR_R^1(B_{r,l})$.*

Proof In order to prove condition (1) notice that for a nonzero oriented cycle c_i^{n+m} , we have $[c_i^n, e_i \otimes \overline{c_i^m}] = c_i^n (e_i \otimes \overline{c_i^m}) - (e_i \otimes \overline{c_i^m}) c_i^n = c_i^n \otimes \overline{c_i^m} - (-c_i^m \otimes \overline{c_i^n} + e_i \otimes \overline{c_i^{n+m}}) =$

$c_i^n \otimes \overline{c_i^m} + c_i^m \otimes \overline{c_i^n} - e_i \otimes \overline{c_i^{n+m}}$. Hence, $\llbracket c_i^n \otimes \overline{c_i^m} \rrbracket + \llbracket c_i^m \otimes \overline{c_i^n} \rrbracket = \llbracket e_i \otimes \overline{c_i^{n+m}} \rrbracket$ in $DR_R^1(B_{r,l})$.

In order to prove condition (2), we have $[u, e_i \otimes \overline{v}] = u(e_i \otimes \overline{v}) - (e_i \otimes \overline{v})u = u \otimes \overline{v} - (-v \otimes \overline{u} + e_i \otimes \overline{vu}) = u \otimes \overline{v} + v \otimes \overline{u} - e_i \otimes \overline{c_i^s}$, which shows (2). $\square$

Lemma 6.2 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, and any nonzero oriented cycle $uv\beta \in B_{r,l}$, where β is an arrow and u, v are non-trivial paths, the following equality $\llbracket u \otimes \overline{v\beta} \rrbracket = \llbracket \beta u \otimes \overline{v} \rrbracket + \llbracket uv \otimes \overline{\beta} \rrbracket$ holds in $DR_R^1(B_{r,l})$.*

Proof Notice that $[\beta, u \otimes \overline{v}] = \beta(u \otimes \overline{v}) - (u \otimes \overline{v})\beta = \beta u \otimes \overline{v} + uv \otimes \overline{\beta} - u \otimes \overline{v\beta}$. Therefore, we have $\llbracket u \otimes \overline{v\beta} \rrbracket = \llbracket \beta u \otimes \overline{v} \rrbracket + \llbracket uv \otimes \overline{\beta} \rrbracket$ in $DR_R^1(B_{r,l})$. $\square$

Lemma 6.3 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, and any oriented cycle $u\beta_n \cdots \beta_1$ in Q_r that is nonzero in $B_{r,l}$ and $\beta_1, \dots, \beta_n$ are arrows, $n \geq 2$, u is a non-trivial path, the following equality*

$$\llbracket u \otimes \overline{\beta_n \cdots \beta_1} \rrbracket = \llbracket u\beta_n \cdots \beta_2 \otimes \overline{\beta_1} \rrbracket + \llbracket \beta_1 u\beta_n \cdots \beta_3 \otimes \overline{\beta_2} \rrbracket + \cdots + \llbracket \beta_{n-1} \cdots \beta_1 u \otimes \overline{\beta_n} \rrbracket$$

holds in $DR_R^1(B_{r,l})$.

Proof We know from Lemma 6.2 that $\llbracket u \otimes \overline{\beta_n \cdots \beta_1} \rrbracket = \llbracket \beta_1 u \otimes \overline{\beta_n \cdots \beta_2} \rrbracket + \llbracket u\beta_n \cdots \beta_2 \otimes \overline{\beta_1} \rrbracket$. Further, applying Lemma 6.2, we obtain $\llbracket \beta_1 u \otimes \overline{\beta_n \cdots \beta_2} \rrbracket = \llbracket \beta_2 \beta_1 u \otimes \overline{\beta_n \cdots \beta_3} \rrbracket + \llbracket \beta_1 u\beta_n \cdots \beta_3 \otimes \overline{\beta_2} \rrbracket$. Hence, we get $\llbracket u \otimes \overline{\beta_n \cdots \beta_1} \rrbracket = \llbracket \beta_2 \beta_1 u \otimes \overline{\beta_n \cdots \beta_3} \rrbracket + \llbracket \beta_1 u\beta_n \cdots \beta_3 \otimes \overline{\beta_2} \rrbracket + \llbracket u\beta_n \cdots \beta_2 \otimes \overline{\beta_1} \rrbracket$. Continuing the above decompositions, we obtain the required equality. $\square$

Lemma 6.4 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, and any generalized path $v_0 \otimes \overline{v_1} \in \Omega_R^1(B_{r,l})$ and a nonzero path u in $B_{r,l}$ such that $uv_0, v_1u \notin I_l$, the following equivalence holds: $uv_0 \otimes \overline{v_1}$ is a generalized cycle if and only if $v_0 \otimes \overline{v_1u}$ is a generalized cycle.*

Proof Notice that if $uv_0 \otimes \overline{v_1}$ is a generalized cycle in $\Omega_R^1(B_{r,l})$, then $s(v_1) = t(u)$, $s(u) = t(v_0)$, $s(v_0) = t(v_1)$. Since $uv_0 \otimes \overline{v_1} \neq 0$ and $v_0 \otimes \overline{v_1u} \neq 0$, $v_0 \otimes \overline{v_1u}$ is a generalized cycle.

Similar reasoning shows the opposite implication. $\square$

Proposition 6.5 *For any algebra $B_{r,l}$, $r \geq 0$, $l = n(r+1) + l_0$, $1 \leq l_0 \leq r+1$, $n \geq 1$, the following conditions are satisfied:*

- (1) *For any generalized cycle $e_i \otimes \overline{c_i^s}$, $i = 0, 1, \dots, r$; $s = 1, 2, \dots, n$, we have $\llbracket e_i \otimes \overline{c_i^s} \rrbracket \neq 0$ in $DR_R^1(B_{r,l})$.*
- (2) *For any generalized cycle $u_i \otimes \overline{v_i}$, $i = 0, 1, \dots, r$ such that $u_i v_i = c_i^s$ for some $s = 1, 2, \dots, n$ and u_i, v_i are non-trivial paths in Q_r , we have $\llbracket u_i \otimes \overline{v_i} \rrbracket \neq 0$ in $DR_R^1(B_{r,l})$.*

Proof We know from Section 2 that for any derivation $\theta \in \text{Der}_R(B_{r,l})$ there is a linear transformation of contraction $i_\theta : DR_R^1(B_{r,l}) \rightarrow DR_R^0(B_{r,l})$ (see also [3,7]) such that $i_\theta(\llbracket u \otimes \overline{v} \rrbracket) = \llbracket u \cdot \theta(v) \rrbracket$.

For a proof of condition (1) notice that for the derivation $\theta_{(\alpha_0)}^0$, we have $i_{\theta_{(\alpha_0)}^0}(\llbracket e_i \otimes \overline{c_i^s} \rrbracket) = s \llbracket c_i^s \rrbracket$ by Lemma 4.4. Further, we infer by Lemma 5.1(2) that $\llbracket c_i^s \rrbracket \neq 0$ in $DR_R^0(B_{r,l})$. Therefore, $\llbracket e_i \otimes \overline{c_i^s} \rrbracket \neq 0$ in $DR_R^1(B_{r,l})$, which shows (1).

In order to prove condition (2), it is enough to consider an arrow α_{i_0} that appears in the path v_i . The contraction $i_{\theta_{(\alpha_{i_0})}^0}$ satisfies $i_{\theta_{(\alpha_{i_0})}^0}(\llbracket u_i \otimes \overline{v_i} \rrbracket) = p \llbracket u_i v_i \rrbracket = p \llbracket c_i^s \rrbracket$ for some $s = 1, 2, \dots, n$, where p is the multiplicity of appearance of the arrow α_{i_0} in the path v_i . Hence, again $\llbracket u_i \otimes \overline{v_i} \rrbracket \neq 0$ in $DR_R^1(B_{r,l})$, which shows (2). $\square$

Corollary 6.6 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq r + 2$, the K -linear space $DR_R^1(B_{r,l})$ is nonzero.*

Proof We know from Proposition 6.5 that $\llbracket e_i \otimes \overline{c_i} \rrbracket \neq 0$ in $DR_R^1(B_{r,l})$. $\square$

Lemma 6.7 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq r + 2$, we have $\llbracket e_i \otimes \overline{c_i^s} \rrbracket = \llbracket e_j \otimes \overline{c_j^s} \rrbracket$ in $DR_R^1(B_{r,l})$ for any $i, j \in \{0, 1, \dots, r\}$ and any $s = 1, 2, \dots, n$, where $l = n(r + 1) + l_0$, $1 \leq l_0 \leq r + 1$, $n \geq 1$.*

Proof We infer by Lemma 6.1 that $\llbracket u_0 \otimes \overline{u_1} \rrbracket + \llbracket u_1 \otimes \overline{u_0} \rrbracket = \llbracket e_j \otimes \overline{c_j^s} \rrbracket$ in $DR_R^1(B_{r,l})$ for $c_j^s = u_1 u_0$. In the same way, we have $\llbracket u_1 \otimes \overline{u_0} \rrbracket + \llbracket u_0 \otimes \overline{u_1} \rrbracket = \llbracket e_i \otimes \overline{c_i^s} \rrbracket$ for $c_i^s = u_0 u_1$. Therefore, $\llbracket e_j \otimes \overline{c_j^s} \rrbracket = \llbracket e_i \otimes \overline{c_i^s} \rrbracket$ for $i \neq j$. If $i = j$, then the required condition is obvious. $\square$

For any arrow α_i in the quiver Q_r , $i = 0, 1, \dots, r$, we denote by h_i the path in Q_r that satisfies $h_i \alpha_i = c_i$ for the oriented cycle c_i .

Proposition 6.8 *For any algebra $B_{r,r+2}$, $r \geq 0$, the system $\{\llbracket h_i \otimes \overline{\alpha_i} \rrbracket\}_{i=0,1,\dots,r}$ forms a basis of the K -linear space $DR_R^1(B_{r,r+2})$.*

Proof We deduce from Lemma 6.1 and Lemma 6.3 that the considered system generates the K -linear space $DR_R^1(B_{r,r+2})$.

In order to finish our proof, we have to check whether the system is linearly independent. Consider a linear combination $\sum_{i=0}^r k_i \llbracket h_i \otimes \overline{\alpha_i} \rrbracket = 0$, where $k_i \in K$, $i = 0, 1, \dots, r$. For any arrow α_i , $i = 0, 1, \dots, r$, we have the derivation $\theta_{(\alpha_i)}^0$ in $\text{Der}_R(B_{r,r+2})$ and the linear contraction $i_{\theta_{(\alpha_i)}^0} : DR_R^1(B_{r,r+2}) \rightarrow DR_R^0(B_{r,r+2})$. Notice that for a fixed arrow α_{i_0} we have $i_{\theta_{(\alpha_{i_0})}^0}(\sum_{i=0}^r k_i \llbracket h_i \otimes \overline{\alpha_i} \rrbracket) = \sum_{i=0}^r k_i \theta_{(\alpha_{i_0})}^0(\llbracket h_i \otimes \overline{\alpha_i} \rrbracket) = k_{i_0} \llbracket c_{i_0} \rrbracket$. Hence, $k_{i_0} = 0$, because $\llbracket c_{i_0} \rrbracket \neq 0$ in $DR_R^0(B_{r,r+2})$ by Lemma 5.2. Considering that α_{i_0} is arbitrary, we obtain that $k_0 = k_1 = \dots = k_r = 0$. Therefore, the considered system is linearly independent. $\square$

Corollary 6.9 *For any algebra $B_{r,r+2}$, $r \geq 0$, we have $\dim_K DR_R^1(B_{r,r+2}) = r + 1$.*

Proof It is an obvious consequence of Proposition 6.8. $\square$

7 Description of $DR_R^2(B_{r,l})$

The main aim of this section is to show that $DR_R^2(B_{r,r+2}) \neq 0$. Furthermore, we indicate some relations between elements of $DR_R^2(B_{r,r+2})$.

Lemma 7.1 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq r + 2$, the following condition holds: for each oriented cycle c_i , $i = 0, 1, \dots, r$, if c_i^n, c_i^m are nonzero for some integers $n, m \geq 1$ then $\llbracket e_i \otimes \overline{c_i^n} \otimes \overline{c_i^m} \rrbracket = -\llbracket e_i \otimes \overline{c_i^m} \otimes \overline{c_i^n} \rrbracket$ in $DR_R^2(B_{r,l})$.*

In particular, $\llbracket e_i \otimes \overline{c_i^n} \otimes \overline{c_i^n} \rrbracket = 0$ for any $n \geq 1$.

Proof We have $[e_i \otimes \overline{c_i^n}, e_i \otimes \overline{c_i^m}] = (e_i \otimes \overline{c_i^n})(e_i \otimes \overline{c_i^m}) - (-1)^{1 \cdot 1}(e_i \otimes \overline{c_i^m})(e_i \otimes \overline{c_i^n}) = (e_i \otimes \overline{c_i^n})(e_i \otimes \overline{c_i^m}) + (e_i \otimes \overline{c_i^m})(e_i \otimes \overline{c_i^n}) = -c_i^n \otimes \overline{e_i} \otimes \overline{c_i^m} + e_i \otimes \overline{c_i^n} \otimes \overline{c_i^m} - c_i^m \otimes \overline{e_i} \otimes \overline{c_i^n} + e_i \otimes \overline{c_i^m} \otimes \overline{c_i^n} = e_i \otimes \overline{c_i^n} \otimes \overline{c_i^m} + e_i \otimes \overline{c_i^m} \otimes \overline{c_i^n}$. Hence, we obtain that $\llbracket e_i \otimes \overline{c_i^n} \otimes \overline{c_i^m} \rrbracket = -\llbracket e_i \otimes \overline{c_i^m} \otimes \overline{c_i^n} \rrbracket$ in $DR_R^2(B_{r,l})$ and $\llbracket e_i \otimes \overline{c_i^n} \otimes \overline{c_i^n} \rrbracket = 0$ for $n \geq 1$, because $\text{char}(K) = 0$. $\square$

Proposition 7.2 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq r + 2$, the following conditions are satisfied:*

- (1) *For any generalized cycle $e_i \otimes \overline{w_i} \otimes \overline{v_i}$ in $\Omega_R^2(B_{r,l})$, $i = 0, 1, \dots, r$, such that $w_i v_i = c_i$ and w_i, v_i are non-trivial, we have $\llbracket e_i \otimes \overline{w_i} \otimes \overline{v_i} \rrbracket \neq 0$ in $DR_R^2(B_{r,l})$.*
- (2) *For any generalized cycle $u_i \otimes \overline{w_i} \otimes \overline{v_i}$ in $\Omega_R^2(B_{r,l})$, $i = 0, 1, \dots, r$, such that $u_i w_i v_i = c_i$ in $B_{r,l}$ and u_i, w_i, v_i are non-trivial paths, we have $\llbracket u_i \otimes \overline{w_i} \otimes \overline{v_i} \rrbracket \neq 0$ in $DR_R^2(B_{r,l})$.*

Proof For any derivation $\theta \in \text{Der}_R(B_{r,l})$, there is a linear transformation of contraction $i_\theta : DR_R^2(B_{r,l}) \rightarrow DR_R^1(B_{r,l})$ (see [3, 7]) such that $i_\theta(\llbracket u \otimes \overline{w} \otimes \overline{v} \rrbracket) = \llbracket u\theta(w) \otimes \overline{v} \rrbracket - \llbracket u \otimes \overline{w\theta(v)} \rrbracket$.

For a proof of condition (1) notice that if we take into account the derivation $\theta_{(\alpha)}^0$ for an arrow α appearing in the path v_i , then $i_{\theta_{(\alpha)}^0}(\llbracket e_i \otimes \overline{w_i} \otimes \overline{v_i} \rrbracket) = \llbracket e_i \theta_{(\alpha)}^0(w_i) \otimes \overline{v_i} \rrbracket - \llbracket e_i \otimes \overline{w_i \theta_{(\alpha)}^0(v_i)} \rrbracket = -\llbracket e_i \otimes \overline{w_i \theta_{(\alpha)}^0(v_i)} \rrbracket \neq 0$ in $DR_R^1(B_{r,l})$ by Proposition 6.5. Therefore, $\llbracket e_i \otimes \overline{w_i} \otimes \overline{v_i} \rrbracket \neq 0$ in $DR_R^2(B_{r,l})$, which shows condition (1).

A similar reasoning shows condition (2), we omit the details. $\square$

Corollary 7.3 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq r + 2$, the K -linear space $DR_R^2(B_{r,l})$ is nonzero.*

Proof We deduce from Proposition 7.2 that $\llbracket e_i \otimes \overline{w_i} \otimes \overline{v_i} \rrbracket \neq 0$ in $DR_R^2(B_{r,l})$ for any generalized cycle $e_i \otimes \overline{w_i} \otimes \overline{v_i}$ in $\Omega_R^2(B_{r,l})$ with $w_i v_i = c_i$ in $B_{r,l}$. $\square$

Proposition 7.4 *For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, assume that $u \otimes \overline{v} \otimes \overline{\beta_n \cdots \beta_1}$ is a generalized cycle in $\Omega_R^2(B_{r,l})$ such that u, v are non-trivial paths, β_j is an arrow, $j = 1, \dots, n$, $n \geq 1$. Then, the element $\llbracket u \otimes \overline{v} \otimes \overline{\beta_n \cdots \beta_1} \rrbracket$ in $DR_R^2(B_{r,l})$ is a linear combination of the elements $\llbracket uv \otimes \overline{\beta_n \cdots \beta_2} \otimes \overline{\beta_1} \rrbracket, \llbracket u \otimes \overline{v\beta_n \cdots \beta_2} \otimes \overline{\beta_1} \rrbracket, \llbracket \beta_1 u \otimes \overline{v\beta_n \cdots \beta_3} \otimes \overline{\beta_2} \rrbracket, \dots, \llbracket \beta_{n-2} \cdots \beta_1 uv \otimes \overline{\beta_n} \otimes \overline{\beta_{n-1}} \rrbracket, \llbracket \beta_{n-2} \cdots \beta_1 u \otimes \overline{v\beta_n} \otimes \overline{\beta_{n-1}} \rrbracket$.*

Proof We prove the proposition by induction on n . If $n = 1$, then the required condition holds obviously.

Now, assume that for $n \leq m$ the required condition holds. Consider a generalized cycle in $\Omega_R^2(B_{r,l})$ of the form $u \otimes \bar{v} \otimes \overline{\beta_{m+1} \cdots \beta_1}$. Then, we have $[\beta_1, u \otimes \bar{v} \otimes \overline{\beta_{m+1} \cdots \beta_2}] = \beta_1 u \otimes \bar{v} \otimes \overline{\beta_{m+1} \cdots \beta_2} - (uv \otimes \overline{\beta_{m+1} \cdots \beta_2} \otimes \bar{\beta}_1 - u \otimes \overline{v \beta_{m+1} \cdots \beta_2} \otimes \bar{\beta}_1 + u \otimes \bar{v} \otimes \overline{\beta_{m+1} \cdots \beta_1})$. Hence, we have that $[[u \otimes \bar{v} \otimes \overline{\beta_{m+1} \cdots \beta_1}]] = [[\beta_1 u \otimes \bar{v} \otimes \overline{\beta_{m+1} \cdots \beta_2}]] - [[uv \otimes \overline{\beta_{m+1} \cdots \beta_2} \otimes \bar{\beta}_1]] + [[u \otimes \overline{v \beta_{m+1} \cdots \beta_2} \otimes \bar{\beta}_1]]$ in $DR_R^2(B_{r,l})$.

Applying the inductive assumption to $[[\beta_1 u \otimes \bar{v} \otimes \overline{\beta_{m+1} \cdots \beta_2}]]$, we obtain the required condition for $[[u \otimes \bar{v} \otimes \overline{\beta_{m+1} \cdots \beta_1}]]$. $\square$

Proposition 7.5 For any algebra $B_{r,l}$, $r \geq 0$, $l \geq 2$, assume that $u \otimes \bar{v} \otimes \bar{w}$ is a generalized cycle in $\Omega_R^2(B_{r,l})$ such that u, v, w are non-trivial paths in Q_r . Then, the following conditions are satisfied in $DR_R^2(B_{r,l})$:

- (1) $[[u \otimes \bar{v} \otimes \bar{w}]] - [[v \otimes \bar{w} \otimes \bar{u}]] = [[e_{t(u)} \otimes \bar{u} \bar{v} \otimes \bar{w}]]$.
- (2) $[[u \otimes \bar{v} \otimes \bar{w}]] - [[w \otimes \bar{u} \otimes \bar{v}]] = [[e_{t(u)} \otimes \bar{u} \bar{v} \otimes \bar{w}]] + [[e_{t(v)} \otimes \bar{v} \bar{w} \otimes \bar{u}]]$.
- (3) $[[e_{t(u)} \otimes \bar{u} \bar{v} \otimes \bar{w}]] + [[e_{t(v)} \otimes \bar{v} \bar{w} \otimes \bar{u}]] + [[e_{t(w)} \otimes \bar{w} \bar{u} \otimes \bar{v}]] = 0$.

Proof Consider the commutator $[e_{t(u)} \otimes \bar{u}, v \otimes \bar{w}] = (e_{t(u)} \otimes \bar{u})(v \otimes \bar{w}) - (-1)^1(v \otimes \bar{w})(e_{t(u)} \otimes \bar{u}) = -u \otimes \bar{v} \otimes \bar{w} + e_{t(u)} \otimes \bar{u} \bar{v} \otimes \bar{w} - v \otimes \bar{w} \otimes e_{t(u)} \otimes \bar{u} + v \otimes \bar{w} \otimes \bar{u} = -u \otimes \bar{v} \otimes \bar{w} + v \otimes \bar{w} \otimes \bar{u} + e_{t(u)} \otimes \bar{u} \bar{v} \otimes \bar{w}$. Therefore, we obtain the equality in condition (1).

In order to prove condition (2) one applies condition (1) two times, and for a proof of condition (3) one applies condition (1) three times. $\square$

8 R -relative differential 2-forms

We are going to show in this section how important are exact R -relative differential 2-forms.

For a fixed $[\omega] \in DR_R^2(B_{r,l})$, we have a K -linear transformation $i_{[\omega]} : \text{Der}_R(B_{r,l}) \rightarrow DR_R^1(B_{r,l})$ that is given by $\theta \mapsto i_\theta([\omega])$, where $i_\theta : DR_R^\bullet(B_{r,l}) \rightarrow DR_R^{\bullet-1}(B_{r,l})$ is a super-derivation of degree (-1) .

Lemma 8.1 For any algebra $B_{r,r+2}$, $r \geq 0$, assume that $\omega = \alpha_{i-1} \cdots \alpha_j \otimes \overline{\alpha_{j-1} \cdots \alpha_t} \otimes \overline{\alpha_{t-1} \cdots \alpha_i}$ is a generalized cycle in $\Omega_r^2(B_{r,r+2})$ and $\alpha_{i-1} \cdots \alpha_j$ is a non-trivial path. Then, the R -relative differential 2-form $[[\alpha_{i-1} \cdots \alpha_j \otimes \overline{\alpha_{j-1} \cdots \alpha_t} \otimes \overline{\alpha_{t-1} \cdots \alpha_i}]]$ is degenerate.

Proof In order to have that the form $[[\omega]]$ is non-degenerate, the transformation $i_{[\omega]} : \text{Der}_R(B_{r,r+2}) \rightarrow DR_R^1(B_{r,r+2})$ has to be an isomorphism. But for any derivation $\theta \in \text{Der}_R(B_{r,r+2})$, we have $i_{[\omega]}(\theta) = \alpha_{i-1} \cdots \alpha_j \theta(\alpha_{j-1} \cdots \alpha_t) \otimes \overline{\alpha_{t-1} \cdots \alpha_i} - \alpha_{i-1} \cdots \alpha_j \otimes \overline{\alpha_{j-1} \cdots \alpha_t} \theta(\alpha_{t-1} \cdots \alpha_i)$. Thus, for each arrow α_s with $s \in \{i-1, \dots, j\}$ we have $i_{[\omega]}(\theta_{(\alpha_s)}^0) = 0$. Therefore, $i_{[\omega]}$ is not an isomorphism. Consequently, the R -relative differential 2-form $[[\omega]]$ is degenerate. $\square$

Lemma 8.2 For any algebra $B_{r,r+2}$, $r \geq 0$, assume that $u'_0 \alpha u''_0 \otimes \overline{u_1} \otimes \overline{u_2}$ is a generalized cycle in $\Omega_R^2(B_{r,r+2})$ such that α is an arrow in Q_r and u'_0 is a non-trivial path. Then, $\llbracket u'_0 \alpha u''_0 \otimes \overline{u_1} \otimes \overline{u_2} \rrbracket = \llbracket \alpha u''_0 u_1 \otimes \overline{u_2} \otimes \overline{u'_0} \rrbracket - \llbracket \alpha u''_0 \otimes \overline{u_1} \overline{u_2} \otimes \overline{u'_0} \rrbracket + \llbracket \alpha u''_0 \otimes \overline{u_1} \otimes \overline{u_2} u'_0 \rrbracket$ in $DR_R^2(B_{r,r+2})$.

Proof In order to prove the lemma, it is enough to consider the commutator $[u'_0, \alpha u''_0 \otimes \overline{u_1} \otimes \overline{u_2}]$. We omit the details. $\square$

Proposition 8.3 For any algebra $B_{r,r+2}$, $r \geq 0$, assume that $\omega_s = \alpha_{i_s-1} \cdots \alpha_{j_s} \otimes \overline{\alpha_{j_s-1} \cdots \alpha_{i_s}} \otimes \overline{\alpha_{t_s-1} \cdots \alpha_{i_s}}$, $s = 1, \dots, n$, are generalized cycles in $\Omega_R^2(B_{r,r+2})$. Then, the following conditions are satisfied in $DR_R^2(B_{r,r+2})$:

- (1) If there is an arrow α in Q_r such that it appears in every path $\alpha_{i_s-1} \cdots \alpha_{j_s}$ for $s = 1, \dots, n$, then an R -relative differential 2-form $\llbracket \omega \rrbracket = \sum_{s=1}^n k_s \llbracket \omega_s \rrbracket$ satisfies that $i_{\llbracket \omega \rrbracket}$ is not a monomorphism, where $k_1, \dots, k_n \in K$.
- (2) If an R -relative differential 2-form $\llbracket \omega \rrbracket = \sum_{s=1}^n k_s \llbracket \omega_s \rrbracket$, $k_1, \dots, k_n \in K$, satisfies that $i_{\llbracket \omega \rrbracket}$ is a monomorphism, then there is a presentation $\llbracket \omega \rrbracket = \sum_{a=1}^m k'_a \llbracket \omega'_a \rrbracket + \sum_{b=1}^d k''_b \llbracket u_b \rrbracket$ such that $\omega'_a = \alpha \omega''_a$ for some arrow α , $a = 1, 2, \dots, m$, and for each $b = 1, 2, \dots, d$, it holds $u_b = e_b \otimes \overline{v_{b,0}} \otimes \overline{v_{b,1}}$, where $e_b = e_{j(b)}$ with $j(b) \in \{0, 1, \dots, r\}$.

Proof For a proof of condition (1) notice that if there is an arrow α in Q_r that appears in each path $\alpha_{i_s-1} \cdots \alpha_{j_s}$, $s = 1, 2, \dots, n$, then $i_{\sum_{s=1}^n k_s \llbracket \omega_s \rrbracket}(\theta_{(\alpha)}^0) = 0$. Hence, $i_{\sum_{s=1}^n k_s \llbracket \omega_s \rrbracket}$ is not a monomorphism for any $k_1, \dots, k_n \in K$.

In order to prove condition (2) suppose that an R -relative differential 2-form $\llbracket \omega \rrbracket = \sum_{s=1}^n k_s \llbracket \omega_s \rrbracket$ satisfies that $i_{\llbracket \omega \rrbracket}$ is a monomorphism. Then, we know from (1) that there is not an arrow α that appears in every path $\alpha_{i_s-1} \cdots \alpha_{j_s}$, $s = 1, 2, \dots, n$. Applying Lemma 8.2, we may assume that $\omega_1, \dots, \omega_{s_1-1}$ have a form $\alpha_{i_1-1} \cdots \alpha_{j_1} \otimes \overline{\alpha_{j_1-1} \cdots \alpha_{i_1}} \otimes \overline{\alpha_{t_1-1} \cdots \alpha_{i_1}}, \alpha_{i_1-1} \cdots \alpha_{j_2} \otimes \overline{\alpha_{j_2-1} \cdots \alpha_{i_2}} \otimes \overline{\alpha_{t_2-1} \cdots \alpha_{i_2}}, \dots, \alpha_{i_1-1} \cdots \alpha_{j_{s_1-1}} \otimes \overline{\alpha_{j_{s_1-1}-1} \cdots \alpha_{i_{s_1-1}}} \otimes \overline{\alpha_{t_{s_1-1}-1} \cdots \alpha_{i_{s_1-1}}}$.

Suppose that $\omega_{s_1} = \alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{i_{s_1}}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}}$ and $\alpha_{i_{s_1}-1} \neq \alpha_{i_1-1}$. If $\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} = \alpha_{i_1-1} \cdots \alpha_{i_1} \alpha_{i_1-1} \cdots \alpha_{j_{s_1}}$, then we infer by Lemma 8.2 that we may change the form $\llbracket \omega_{s_1} \rrbracket$ onto a sum of three forms with α_{i_1-1} on the left hand side.

If the arrow α_{i_1-1} does not appear in the path $\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}}$, then $\omega_{s_1} = \alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{i_1-1} \cdots \alpha_{i_{s_1}}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}}$ or $\omega_{s_1} = \alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{t_{s_1}}} \otimes \overline{\alpha_{i_1-1} \cdots \alpha_{i_{s_1}}}$.

In the first case, we have $[\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{i_1-1}} \otimes \overline{\alpha_{i_1-1} \cdots \alpha_{t_{s_1}}}, \alpha_{i_1-1} \cdots \alpha_{t_{s_1}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} \otimes \overline{\alpha_{i_1-1} \cdots \alpha_{j_{s_1}}}] = -\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \alpha_{j_{s_1}-1} \cdots \alpha_{i_1} \otimes \overline{\alpha_{i_1-1} \cdots \alpha_{t_{s_1}}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} + \alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{i_1-1} \cdots \alpha_{t_{s_1}}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} - \alpha_{i_1-1} \cdots \alpha_{t_{s_1}} \alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}} \otimes \overline{\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{i_1-1}} + \alpha_{i_1-1} \cdots \alpha_{t_{s_1}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} \alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{i_1-1}}$. Therefore, we deduce that $\llbracket \alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{i_1-1}} \otimes \overline{\alpha_{i_1-1} \cdots \alpha_{t_{s_1}}} \rrbracket \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} = \llbracket \alpha_{i_1-1} \cdots \alpha_{t_{s_1}} \alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}} \otimes \overline{\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{i_1-1}} \rrbracket - \llbracket \alpha_{i_{s_1}-1} \cdots \alpha_{i_1} \otimes \overline{\alpha_{i_1-1} \cdots \alpha_{t_{s_1}}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} \rrbracket + \llbracket \alpha_{i_1-1} \cdots \alpha_{t_{s_1}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} \alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{i_1-1}} \rrbracket$. Further, using Proposition 7.5(1) we obtain that $\llbracket \alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \alpha_{j_{s_1}-1} \cdots \alpha_{i_1} \otimes \overline{\alpha_{i_1-1} \cdots \alpha_{t_{s_1}}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} \rrbracket =$

$$\begin{aligned} & \llbracket \alpha_{i_1-1} \cdots \alpha_{t_{s_1}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} \otimes \overline{\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \alpha_{j_{s_1}-1} \cdots \alpha_{i_1}} \rrbracket + \llbracket e_{i_{s_1}} \otimes \\ & \overline{\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \alpha_{j_{s_1}-1} \cdots \alpha_{i_1} \alpha_{i_1}-1 \cdots \alpha_{t_{s_1}}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} \rrbracket. \text{ Hence, we obtain that} \\ & \frac{\llbracket \alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{i_1}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} \rrbracket}{\otimes \overline{\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{i_1}}} + \frac{\llbracket \alpha_{i_1-1} \cdots \alpha_{t_{s_1}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} \otimes \overline{\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}}} \rrbracket}{\otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{i_1}}} \\ & - \frac{\llbracket \alpha_{i_1-1} \cdots \alpha_{t_{s_1}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} \otimes \overline{\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \alpha_{j_{s_1}-1} \cdots \alpha_{i_1}} \rrbracket}{\otimes \overline{\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \alpha_{j_{s_1}-1} \cdots \alpha_{i_1} \alpha_{i_1}-1 \cdots \alpha_{t_{s_1}}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}}}. \text{ Replacing } \llbracket \omega_{s_1} \rrbracket \text{ by the} \\ & \text{above sum, we obtain } \llbracket \omega \rrbracket = \sum_{a_1=1}^{m_1} k_{a_1}^{(1)} \llbracket \omega_{a_1}^{(1)} \rrbracket - k_{s_1} \llbracket e_{i_{s_1}} \otimes \\ & \overline{\alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \alpha_{j_{s_1}-1} \cdots \alpha_{i_1} \alpha_{i_1}-1 \cdots \alpha_{t_{s_1}}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_{s_1}}} \rrbracket, \text{ where } \omega_{a_1}^{(1)} = \omega_{a_1} \text{ for} \\ & a_1 < s_1, \text{ the next summands } \omega_{s_1}^{(1)}, \omega_{s_1+1}^{(1)}, \omega_{s_1+2}^{(1)} \text{ are the first three summands in the} \\ & \text{above presentation of } \omega_{s_1}. \text{ Further, } \omega_{a_1}^{(1)} = \omega_{a_1-2} \text{ for } a_1 \geq s_1 + 3. \text{ Moreover,} \end{aligned}$$

$$k_{a_1}^{(1)} = \begin{cases} k_{a_1} & \text{if } a_1 \leq s_1 \\ k_{s_1} & \text{if } a_1 = s_1 + 1 \\ -k_{s_1} & \text{if } a_1 = s_1 + 2 \\ k_{a_1-2} & \text{if } a_1 \geq s_1 + 3. \end{cases}$$

If $\omega_{s_1} = \alpha_{i_{s_1}-1} \cdots \alpha_{j_{s_1}} \otimes \overline{\alpha_{j_{s_1}-1} \cdots \alpha_{t_{s_1}}} \otimes \overline{\alpha_{t_{s_1}-1} \cdots \alpha_{i_1-1} \alpha_{i_1} \cdots \alpha_{i_{s_1}}}$, then we replace ω_{s_1} applying Proposition 7.5(1),(2) and again, we obtain a new presentation $\llbracket \omega \rrbracket = \sum_{a_2=1}^{m_2} k_{a_2}^{(2)} \llbracket \omega_{a_2}^{(2)} \rrbracket + \sum_{b_2=1}^{d_2} k_{b_2}^{(2)} \llbracket u_{b_2} \rrbracket$.

Continuing the above procedure for $\llbracket \omega_{s_1+1} \rrbracket$ and the following forms we obtain the required presentation of $\llbracket \omega \rrbracket$. $\square$

The above proposition indicates an importance of R -relative differential 2-forms that are exact.

Example 8.4 Consider the algebra $B_{1,3}$. Then, we deduce from Corollary 4.6 that $\dim_K \text{Der}_R(B_{1,3}) = 2$. Moreover, $DR_R^1(B_{1,3})$ has nonzero elements $\llbracket e_0 \otimes \overline{c_0} \rrbracket$, $\llbracket \alpha_1 \otimes \overline{\alpha_0} \rrbracket$ in view of Proposition 6.5. Using Lemma 6.7 and Lemma 6.1, it is easy to see that the above two elements form a basis of $DR_R^1(B_{1,3})$. Hence, $\dim_K DR_R^1(B_{1,3}) = 2$. Therefore, it can happen that there is a symplectic manifold $(B_{1,3}, \llbracket \omega \rrbracket)$. But $B_{1,3}$ is a preprojective algebra for the hereditary algebra $K(0 \xrightarrow{\alpha_0} 1)$. Hence, it is known that such manifolds exist in the differential calculus $(DR_R^\bullet(B_{1,3}), d)$.

Example 8.5 Consider the algebra $B_{2,3}$. Then, one can check that $\dim_K \text{Der}_R(B_{2,3}) = 3$. Moreover, we can check straightforward that the elements $\llbracket \alpha_2 \alpha_1 \otimes \overline{\alpha_0} \rrbracket$, $\llbracket \alpha_0 \alpha_2 \otimes \overline{\alpha_1} \rrbracket$, $\llbracket \alpha_1 \alpha_0 \otimes \overline{\alpha_2} \rrbracket$ span the K -linear space $DR_R^1(B_{2,3})$. But these elements are linearly dependent. Thus, $\dim_K DR_R^1(B_{2,3}) < 3$. Consequently, there is no symplectic manifold $(B_{2,3}, \llbracket \omega \rrbracket)$ in the differential calculus $(DR_R^\bullet(B_{2,3}), d)$.

Lemma 8.6 For any algebra $B_{r,r+2}$, $r \geq 0$, and for any generalized cycle $e_i \otimes \overline{u_1} \otimes \overline{u_2}$ in $\Omega_R^2(B_{r,r+2})$, we have the following equality $\llbracket e_i \otimes \overline{u_1} \otimes \overline{u_2} \rrbracket = -\llbracket e_j \otimes \overline{u_2} \otimes \overline{u_1} \rrbracket$ in $DR_R^2(B_{r,r+2})$, where $i = t(u_1)$, $j = t(u_2)$.

Proof Consider the commutator $[e_i \otimes \overline{u_1}, e_j \otimes \overline{u_2}] = (e_i \otimes \overline{u_1})(e_j \otimes \overline{u_2}) - (-1)^1(e_j \otimes \overline{u_2})(e_i \otimes \overline{u_1}) = u_1 \otimes \overline{e_j} \otimes \overline{u_2} - e_i \otimes \overline{u_1} e_j \otimes \overline{u_2} + u_2 \otimes \overline{e_i} \otimes \overline{u_1} - e_j \otimes \overline{u_2} \overline{e_i} \otimes \overline{u_1} = -e_i \otimes \overline{u_1} \otimes \overline{u_2} - e_j \otimes \overline{u_2} \otimes \overline{u_1}$. Then, the required condition follows. $\square$

9 Exact R -relative differential 2-forms

Lemma 9.1 For an algebra $B_{r,r+2}$, $r \geq 0$, let $[\omega] = \sum_{i=0}^r k_i [e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]$, $0 \neq k_i \in K$, $i = 0, 1, \dots, r$, and $0 \neq \theta = \sum_{j=0}^r k'_j \theta_{(\alpha_j)}^0 \in \text{Der}_R(B_{r,r+2})$, $k'_j \in K$, $j = 0, 1, \dots, r$. If $i_{[\omega]}(\theta) = 0$ in $DR_R^1(B_{r,r+2})$ then $\theta = 0$.

Proof First notice that we have $i_{[\omega]}(\theta) = \sum_{i=0}^r \sum_{j=0}^r k_i k'_j ([\theta_{(\alpha_j)}^0(h_i) \otimes \bar{\alpha}_i] - [e_i \otimes \overline{h_i \theta_{(\alpha_j)}^0(\alpha_i)}]) = \sum_{i \neq j} k_i k'_j [h_i \otimes \bar{\alpha}_i] - \sum_{i=j} k_i k'_j [e_i \otimes \bar{c}_i] = \sum_{i \neq j} k_i k'_j [h_i \otimes \bar{\alpha}_i] - \sum_{i=0}^r k_i k'_i ([h_0 \otimes \bar{\alpha}_0] + [h_1 \otimes \bar{\alpha}_1] + \dots + [h_r \otimes \bar{\alpha}_r])$, because we infer by Lemma 6.1 and Lemma 6.3 that $[e_i \otimes \bar{c}_i] = [h_0 \otimes \bar{\alpha}_0] + [h_1 \otimes \bar{\alpha}_1] + \dots + [h_r \otimes \bar{\alpha}_r]$.

If $i_{[\omega]}(\theta) = 0$, then for each $i = 0, 1, \dots, r$ we have the equality $0 = k_i \sum_{j \neq i} (k'_j - k'_i) - k_i k'_i$. Since all $k_i \neq 0$, $i = 0, 1, \dots, r$, so we obtain the following system of equations:

$$\begin{aligned} 0 &= \sum_{j \neq 0} (k'_j - k'_0) - k'_0 = \sum_{j \neq 0} k'_j - (r+1)k'_0 \\ 0 &= \sum_{j \neq 1} (k'_j - k'_1) - k'_1 = \sum_{j \neq 1} k'_j - (r+1)k'_1 \\ &\vdots \\ 0 &= \sum_{j \neq r} (k'_j - k'_r) - k'_r = \sum_{j \neq r} k'_j - (r+1)k'_r. \end{aligned}$$

The matrix of this system of equations has the following form

$$M = \begin{bmatrix} -(r+1) & 1 & 1 & \dots & 1 \\ 1 & -(r+1) & 1 & \dots & 1 \\ 1 & 1 & -(r+1) & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 1 & -(r+1) \end{bmatrix}.$$

Since $\text{char}(K) = 0$, $\det(M) \neq 0$ and $k'_0 = k'_1 = \dots = k'_r = 0$. Thus, $\theta = 0$. $\square$

Lemma 9.2 For an algebra $B_{r,r+2}$, $r \geq 0$, let $[\omega] = \sum_{i=0}^r k_i [e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]$, $k_i \in K$, $i = 0, 1, \dots, r$. If there are $i_0 \neq i_1$ in $\{0, 1, \dots, r\}$ such that $k_{i_0} = 0 = k_{i_1}$, then $i_{[\omega]}$ is not a monomorphism.

Proof Let $[\omega] = \sum_{i=0}^r k_i [e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]$ and suppose that $k_{i_0} = 0 = k_{i_1}$ for some $i_0 \neq i_1$. Then, $i_{[\omega]}(\theta_{(\alpha_{i_0})}^0) = \sum_{i \neq i_0} k_i [h_i \otimes \bar{\alpha}_i] - k_{i_0} [e_{i_0} \otimes \bar{c}_{i_0}] = \sum_{i \neq i_0, i_1} k_i [h_i \otimes \bar{\alpha}_i] + k_{i_1} [h_{i_1} \otimes \bar{\alpha}_{i_1}] - k_{i_0} [e_{i_0} \otimes \bar{c}_{i_0}] = \sum_{i \neq i_0, i_1} k_i [h_i \otimes \bar{\alpha}_i]$. Further, $i_{[\omega]}(\theta_{(\alpha_{i_1})}^0) = \sum_{i \neq i_1} k_i [h_i \otimes \bar{\alpha}_i] - k_{i_1} [e_{i_1} \otimes \bar{c}_{i_1}] = \sum_{i \neq i_0, i_1} k_i [h_i \otimes \bar{\alpha}_i] + k_{i_0} [h_{i_0} \otimes \bar{\alpha}_{i_0}] - k_{i_1} [e_{i_1} \otimes \bar{c}_{i_1}] = \sum_{i \neq i_0, i_1} k_i [h_i \otimes \bar{\alpha}_i]$. Thus, $i_{[\omega]}(\theta_{(\alpha_{i_0})}^0) = i_{[\omega]}(\theta_{(\alpha_{i_1})}^0)$. Hence, $i_{[\omega]}$ is not a monomorphism. $\square$

Lemma 9.3 For an algebra $B_{r,r+2}$, $r \geq 0$, the following condition holds: if $[\omega] = \sum_{i=0}^r k_i [e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]$, $k_i \in K$, $i = 0, 1, \dots, r$, and there is $i_0 \in \{0, 1, \dots, r\}$ such that $k_{i_0} = 0$ and every $k_i \neq 0$ for $i \neq i_0$ then $i_{[\omega]}$ is an epimorphism.

Proof Without loss of generality, we may assume that $i_0 = 0$. Then, $k_0 = 0$ and $k_i \neq 0$ for $i \neq 0$. Let $\theta = \sum_{j=0}^r k'_j \theta_{(\alpha_j)}^0$, $k'_j \in K$, $j = 0, 1, \dots, r$. Fix $p \in \{0, 1, \dots, r\}$. Let $[[h_p \otimes \bar{\alpha}_p]] = i_{[[\omega]]}(\theta) = \sum_{i \neq j, i \neq 0} k_i k'_j [[h_i \otimes \bar{\alpha}_i]] - \sum_{i=1}^r k_i k'_i [[e_i \otimes \bar{c}_i]] = \sum_{i \neq j, i \neq 0} k_i k'_j [[h_i \otimes \bar{\alpha}_i]] - \sum_{i=1}^r k_i k'_i ([[h_0 \otimes \bar{\alpha}_0]] + [[h_1 \otimes \bar{\alpha}_1]] + \dots + [[h_r \otimes \bar{\alpha}_r]])$.

We deduce from the above equality that $[[h_p \otimes \bar{\alpha}_p]] = k_1 \sum_{j \neq 1} k'_j [[h_1 \otimes \bar{\alpha}_1]] - k_1 k'_1 \sum_{a=0}^r [[h_a \otimes \bar{\alpha}_a]] + k_2 \sum_{j \neq 2} k'_j [[h_2 \otimes \bar{\alpha}_2]] - k_2 k'_2 \sum_{a=0}^r [[h_a \otimes \bar{\alpha}_a]] + \dots + k_r \sum_{j \neq r} k'_j [[h_r \otimes \bar{\alpha}_r]] - k_r k'_r \sum_{a=0}^r [[h_a \otimes \bar{\alpha}_a]]$. Further, we have $[[h_p \otimes \bar{\alpha}_p]] = (k_1 \sum_{j \neq 1} k'_j - \sum_{i=1}^r k_i k'_i) [[h_1 \otimes \bar{\alpha}_1]] + (k_2 \sum_{j \neq 2} k'_j - \sum_{i=1}^r k_i k'_i) [[h_2 \otimes \bar{\alpha}_2]] + \dots + (k_r \sum_{j \neq r} k'_j - \sum_{i=1}^r k_i k'_i) [[h_r \otimes \bar{\alpha}_r]] - \sum_{i=1}^r k_i k'_i [[h_0 \otimes \bar{\alpha}_0]]$.

Since the system $\{[[h_i \otimes \bar{\alpha}_i]]\}_{i=0,1,\dots,r}$ is a basis of the K -linear space $DR_R^1(B_{r,r+2})$ by Proposition 6.8, for any $p = 0, 1, \dots, r$ we obtain the following system of equations with variables $k'_0, k'_1, \dots, k'_r$:

$$\begin{aligned} 0 &= -\sum_{i=1}^r k_i k'_i \\ 0 &= k_1 \sum_{j \neq 1} k'_j - \sum_{i=1}^r k_i k'_i \\ &\vdots \\ 0 &= k_{p-1} \sum_{j \neq p-1} k'_j - \sum_{i=1}^r k_i k'_i \\ 1 &= k_p \sum_{j \neq p} k'_j - \sum_{i=1}^r k_i k'_i \\ 0 &= k_{p+1} \sum_{j \neq p+1} k'_j - \sum_{i=1}^r k_i k'_i \\ &\vdots \\ 0 &= k_r \sum_{j \neq r} k'_j - \sum_{i=1}^r k_i k'_i \end{aligned}$$

that gives us the following system:

$$\begin{aligned} 0 &= \sum_{i=1}^r k_i k'_i \\ 0 &= k_1 \sum_{j \neq 1} k'_j \\ &\vdots \\ 0 &= k_{p-1} \sum_{j \neq p-1} k'_j \\ 1 &= k_p \sum_{j \neq p} k'_j \\ 0 &= k_{p+1} \sum_{j \neq p+1} k'_j \\ &\vdots \\ 0 &= k_r \sum_{j \neq r} k'_j \end{aligned}$$

for $p = 1, 2, \dots, r$ and the above system for $p = 0$ has the form:

$$\begin{aligned} -1 &= \sum_{i=1}^r k_i k'_i \\ -1 &= k_1 \sum_{j \neq 1} k'_j \\ -1 &= k_2 \sum_{j \neq 2} k'_j \\ &\vdots \\ -1 &= k_r \sum_{j \neq r} k'_j. \end{aligned}$$

The matrix M of these two systems has the form

$$M = \begin{bmatrix} 0 & k_1 & k_2 & k_3 & \cdots & k_{r-2} & k_{r-1} & k_r \\ k_1 & 0 & k_1 & k_1 & \cdots & k_1 & k_1 & k_1 \\ k_2 & k_2 & 0 & k_2 & \cdots & k_2 & k_2 & k_2 \\ \vdots & \vdots & & & & \vdots & & \vdots \\ k_p & k_p & & \cdots & k_p & 0 & \cdots & k_p \\ \vdots & \vdots & & & & & & \vdots \\ k_r & k_r & & \cdots & & & k_r & 0 \end{bmatrix}.$$

Since all $k_1, k_2, \dots, k_r \neq 0$, $\det(M) \neq 0$ and each of these systems has exactly one solution.

Consequently, each basis member $[[h_p \otimes \bar{\alpha}_p]] \in \text{im}(i_{[\omega]})$. Therefore, $i_{[\omega]}$ is an epimorphism. $\square$

10 Proofs of the main results

Proof of Theorem 1.1. In order to prove condition (1) notice that we infer by Lemma 9.1 and Lemma 9.3 that there are exact R -relative differential 2-forms $[\omega] \in DR_R^2(B_{r,r+2})$ such that $i_{[\omega]} : \text{Der}_R(B_{r,r+2}) \rightarrow DR_R^1(B_{r,r+2})$ is a monomorphism or an epimorphism. Hence, they are isomorphisms, because $\dim_K \text{Der}_R(B_{r,r+2}) = \dim_R DR_R^1(B_{r,r+2})$ by Corollary 4.6 and Corollary 6.9. Consequently, the forms $[\omega]$ are non-degenerate. Their forms show that they are exact, and so they are closed as well. Therefore, there are exact symplectic manifolds $(B_{r,r+2}, [\omega])$.

For a proof of condition (2) notice that a 2-form $[\omega]$ is exact provided that it is of the form $\sum_{i=0}^r k_i [e_i \otimes \bar{u}_i \otimes \bar{v}_i]$, because we infer by Proposition 6.8 that $\text{im}(d_1)$ is spanned by $\{[e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]\}_{i=0,1,\dots,r}$. Furthermore, we deduce from Lemmas 9.1, 9.2, 9.3 that if $(B_{r,r+2}, [\omega])$ is an exact symplectic manifold then there exists at most one index $i_0 \in \{0, 1, \dots, r\}$ such that $k_{i_0} = 0$.

The opposite implication is a consequence of Lemmas 9.1, 9.2, 9.3. $\square$

We say that two symplectic manifolds $(B_{r_1,l_1}, [\omega_1])$, $(B_{r_2,l_2}, [\omega_2])$ are *symplectomorphic* if there is a K -algebra isomorphism $f : B_{r_1,l_1} \rightarrow B_{r_2,l_2}$ such that $f^*([\omega_1]) = [\omega_2]$. If this is the case, then we call the map f a *symplectomorphism*.

Proof of Theorem 1.2. Let $(B_{r,r+2}, [\omega])$ be an exact symplectic manifold. Then, we infer by Theorem 1.1 that $[\omega] = \sum_{i=0}^r k_i [e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i] \neq 0$ and there exists at most one index $i_0 \in \{0, 1, \dots, r\}$ such that $k_{i_0} = 0$. Now, consider a K -algebra automorphism $f : B_{r,r+2} \rightarrow B_{r,r+2}$ such that $f(e_i) = e_{i+s}$, $f(\alpha_i) = \alpha_{i+s}$ for $s = r+1-i_0$. Then, $f(\alpha_{i_0}) = \alpha_0$ and $f(e_{i_0}) = e_0$. Consequently, $f(h_i) = h_{i+s}$, $i \neq i_0$, and $f(h_{i_0}) = h_0$. Therefore, the induced map $f^* : DR_R^2(B_{r,r+2}) \rightarrow DR_R^2(B_{r,r+2})$ is the K -linear transformation such that $f^*([e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]) = [f(e_i) \otimes f(h_i) \otimes f(\alpha_i)] = [e_{i+s} \otimes \bar{h}_{i+s} \otimes \bar{\alpha}_{i+s}]$. Hence, $f^*([\omega]) = f^*(\sum_{i=0}^r k_i [e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]) = \sum_{i=0}^r k_i f^*([e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]) = \sum_{i=0}^r k_i [e_{i+s} \otimes \bar{h}_{i+s} \otimes \bar{\alpha}_{i+s}]$. Notice that in this sum coefficient $k_{i_0} = 0$ appears at $[e_0 \otimes \bar{h}_0 \otimes \bar{\alpha}_0]$. Then, the required condition is satisfied.

If all coefficients $k_i \neq 0, i = 0, 1, \dots, r$, then consider a K -algebra automorphism $f : B_{r,r+2} \rightarrow B_{r,r+2}$ such that $f(e_i) = e_i, i = 0, 1, \dots, r$, and $f(\alpha_i) = \alpha_i$ for $i = 1, 2, \dots, r$. Moreover, we put $f(\alpha_0) = \frac{1}{k_0}\alpha_0$. It is easy to see that this is a well-defined K -algebra automorphism of $B_{r,r+2}$. Further, we have $f^*([e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]) = [e_i \otimes f(\bar{h}_i) \otimes f(\bar{\alpha}_i)] = \frac{1}{k_0}[e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]$. Therefore, $f^*([\omega]) = \sum_{i=0}^r k_i f^*([e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]) = \sum_{i=0}^r \frac{k_i}{k_0}[e_i \otimes \bar{h}_i \otimes \bar{\alpha}_i]$ and $\frac{k_0}{k_0} = 1$ is the zero coefficient of the form. Consequently, the required condition holds. $\square$

11 Final results and comments

Lemma 11.1 *For any algebra $B_{r,r+1}, r \geq 1$, the following conditions are satisfied:*

- (1) *The system $\{[h_i \otimes \bar{\alpha}_i]\}_{i=0,1,\dots,r}$ spans the K -linear space $DR_R^1(B_{r,r+1})$.*
- (2) *The system $\{[h_i \otimes \bar{\alpha}_i]\}_{i=0,1,\dots,r}$ is linearly dependent.*
- (3) $\dim_K DR_R^1(B_{r,r+1}) \leq r$.

Proof Condition (1) is a straightforward consequence of Lemma 6.3 and Lemma 6.1.

For a proof of condition (2) notice that $[h_i \otimes \bar{\alpha}_i] + [\alpha_i \otimes \bar{h}_i] = [e_i \otimes \bar{c}_i]$, because $[\alpha_i, e_i \otimes \bar{h}_i] = \alpha_i \otimes \bar{h}_i + h_i \otimes \bar{\alpha}_i - e_i \otimes \bar{c}_i$. But $l = r + 1$, so $c_i = 0$ in $B_{r,r+1}$, and so $[h_i \otimes \bar{\alpha}_i] + [\alpha_i \otimes \bar{h}_i] = 0$. Further, applying Lemma 6.3 we obtain that $[\alpha_i \otimes \bar{h}_i] = [h_{i+1} \otimes \bar{\alpha}_{i+1}] + [h_{i+2} \otimes \bar{\alpha}_{i+2}] + \dots + [h_r \otimes \bar{\alpha}_r] + [h_0 \otimes \bar{\alpha}_0] + \dots + [h_{i-1} \otimes \bar{\alpha}_{i-1}]$. Therefore, $[h_i \otimes \bar{\alpha}_i] + [\alpha_i \otimes \bar{h}_i] = [h_i \otimes \bar{\alpha}_i] + [h_{i+1} \otimes \bar{\alpha}_{i+1}] + \dots + [h_r \otimes \bar{\alpha}_r] + [h_0 \otimes \bar{\alpha}_0] + \dots + [h_{i-1} \otimes \bar{\alpha}_{i-1}] = 0$. Consequently, the system $\{[h_i \otimes \bar{\alpha}_i]\}_{i=0,1,\dots,r}$ is linearly dependent.

In order to prove (3) notice that this system spans $DR_R^1(B_{r,r+1})$ by (1) and is linearly dependent by (2). Then, $\dim_K DR_R^1(B_{r,r+1}) \leq r$. $\square$

Theorem 11.2 *For any algebra $B_{r,r+1}, r \geq 0$, and the Karoubi-de Rham differential calculus $(DR_R^\bullet(B_{r,r+1}), d)$ there is no symplectic manifold $(B_{r,r+1}, [\omega])$.*

Proof Suppose to the contrary that for some algebra $B_{r,r+1}$ there is a symplectic manifold $(B_{r,r+1}, [\omega])$. Then, $[\omega] \in DR_R^2(B_{r,r+1})$ is an R -relative differential 2-form that is closed and non-degenerate. Hence, $i_{[\omega]} : \text{Der}_R(B_{r,r+1}) \rightarrow DR_R^1(B_{r,r+1})$ is an isomorphism. But we know from Lemma 11.1 that $\dim_K DR_R^1(B_{r,r+1}) \leq r$ and $\dim_K \text{Der}_R(B_{r,r+1}) = r + 1$ by Corollary 4.6. Consequently, $i_{[\omega]}$ cannot be an isomorphism, which shows that there is no symplectic manifold $(B_{r,r+1}, [\omega])$. $\square$

Remark 11.3 Theorem 11.2 is proved without computing the Karoubi-de Rham complex. It is not easy to check whether this complex is non-trivial for algebras $B_{r,r+1}$. Vanishing of the elements $[e_i \otimes \bar{c}_i]$ in $DR_R^1(B_{r,r+1})$ is a reason that we cannot repeat the arguments in proving the non-triviality of the complex $(DR_R^\bullet(B_{r,r+1}), d)$.

Remark 11.4 In many examples of algebras, non-existence of symplectic manifolds is a consequence of trivialisation of their Karoubi-de Rham complex. This is the case for hereditary algebras or canonical algebras (see [11]).

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