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Temperature and entropy–area relation of quantum matter near spherically symmetric outer trapping horizons

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Abstract

We consider spherically symmetric spacetimes with an outer trapping horizon. Such spacetimes are generalizations of spherically symmetric black hole spacetimes where the central mass can vary with time, like in black hole collapse or black hole evaporation. While these spacetimes possess in general no timelike Killing vector field, they admit a Kodama vector field which in some ways provides a replacement. The Kodama vector field allows the definition of a surface gravity of the outer trapping horizon. Spherically symmetric spacelike cross sections of the outer trapping horizon define in- and outgoing lightlike congruences. We investigate a scaling limit of Hadamard 2-point functions of a quantum field on the spacetime onto the ingoing lightlike congruence. The scaling limit 2-point function has a universal form and a thermal spectrum with respect to the time parameter of the Kodama flow, where the inverse temperature $\beta = 2\pi/\kappa$ is related to the surface gravity κ of the horizon cross section in the same way as in the Hawking effect for an asymptotically static black hole. Similarly, the tunnelling probability that can be obtained in the scaling limit between in- and outgoing Fourier modes with respect to the time parameter of the Kodama flow shows a thermal distribution with the same inverse temperature, determined by the surface gravity. This can be seen as a local counterpart of the Hawking effect for a dynamical horizon in the scaling limit. Moreover, the scaling limit 2-point function allows it to define a scaling limit theory, a quantum field theory on the ingoing lightlike congruence emanating from a horizon cross section. The scaling limit 2-point

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function as well as the 2-point functions of coherent states of the scaling limit theory is correlation-free with respect to separation along the horizon cross section; therefore, their relative entropies behave proportional to the cross-sectional area. We thus obtain a proportionality of the relative entropy of coherent states of the scaling limit theory and the area of the horizon cross section with respect to which the scaling limit is defined. Thereby, we establish a local counterpart, and microscopic interpretation in the setting of quantum field theory on curved spacetimes, of the dynamical laws of outer trapping horizons, derived by Hayward and others in generalizing the laws of black hole dynamics originally shown for stationary black holes by Bardeen, Carter and Hawking.

Keywords Black hole temperature · Black hole entropy · Quantum fields in curved spacetime

Mathematics Subject Classification 83C57 · 83C47 · 81T05 · 81T20

1 Introduction

The famous four laws of black hole mechanics and their analogy with the laws of thermodynamics have been derived and developed in [3] assuming stationarity. The temperature thereby assigned to a black hole is related to the horizon's surface gravity and can physically be interpreted in terms of Hawking radiation [28] in the framework of quantum field theory in curved spacetime, see also [21,37,55,59]. Similarly, the area of the black hole horizon surface is analogous to an entropy. Discussion of black hole entropy and its physical nature has been given in a variety of contexts ([4,19,42,47,56, 62] and literature cited therein is just a small sample of references on the topic), but it has been difficult to find a simple, direct counterpart of the entropy–area relation for black holes in the setting of quantum field theory in curved spacetime (see, however, [33] and further discussion below).

Although Hawking radiation is derived neglecting backreaction, assuming that the spacetime geometry is stationary (or asymptotically stationary), the emission rate of Hawking radiation is usually associated with the rate of black hole mass loss due to evaporation, see e.g. [9,19,28]. However, black hole evaporation is a dynamical process and should be described locally. A local theory for the geometry of non-stationary black holes using concepts of dynamical horizons and trapped horizons has been developed, see e.g. [2,29]. In particular, in [29] it is shown that the first law holds as an energy balance along the trapped horizon. In contrast with the (asymptotically) stationary case, Hawking radiation and a relation between temperature and local geometrical quantities of dynamical or trapped horizons have so far not been derived for quantum fields in the background of *non-stationary (or dynamical)* black holes.

An essentially local derivation of the Hawking effect has been proposed by Parikh and Wilczeck [46]. In that approach, an estimate is given for the tunnelling probability of quantum particles across the horizon, showing that this probability has a thermal distribution. This idea has been generalized to the case of dynamical black holes in [16,17,27], hence furnishing a connection between the surface gravity and a thermal distribution of the tunnelling probability. These considerations didn't use quantum field theoretical methods as in the original derivation of the Hawking effect but relied on single-particle quantum mechanics in a WKB-type approximation. In order to overcome the limitations of such a quantum mechanical treatment, it has been shown in [43] that for a scalar quantum field on a stationary black hole spacetime—more generally, any spacetime with a bifurcate Killing horizon—a thermal distribution in the tunnelling probability is obtained in a certain scaling limit located on the horizon whenever the quantum field is in a Hadamard state. The associated temperature is the Hawking temperature and is independent of the chosen Hadamard state. A similar result can also be obtained in the case of self-interacting fields, see [11]. For some related results, focusing on the thermal nature of field theories restricted on null surfaces (horizons) and thus not focussing on the local aspect related to tunnelling processes, see [21,22,26,37,55,57].

In this paper, we aim at generalizing the result of [43] to the case of spherically symmetric, dynamical black holes. For generic spherically symmetric black holes, there is no Killing vector field which generates an horizon. Nevertheless, there are generalizations available that serve a similar purpose in the context of black hole thermodynamics. In particular, we shall use the concept of *outer trapping horizons* [2] and the *Kodama vector field* [39]. A spherically symmetric (non-stationary) black hole spacetime is a warped product of a 2-dimensional Lorentzian space and a 2-dimensional Euclidean sphere. The future-directed light rays in the two-dimensional Lorentzian space determine, at each spacetime point, two geodesic congruences of null type; one is called *outgoing* and the other *ingoing*. The *outer trapping horizon* $\mathcal{H}$ is the 3-dimensional hypersurface which divides the *inside* region where the expansion parameters $\theta_\pm$ of the ingoing($-$) and outgoing($+$) null geodesic congruences are both negative from the *outside* region where $\theta_+ > 0$ and $\theta_- < 0$. The outside region usually reaches out to spatial infinity. If the expansion parameter of the null geodesic congruence is positive (negative), the area of a congruence orthogonal spatial sphere grows (decreases) towards the future along the congruence. Hence, in the region where both $\theta_\pm$ are negative, all light rays tend to fall into the black hole while in the region where $\theta_+ > 0$ the outgoing lightrays tend to reach points which are far away (measured by the radius of the orthogonal spatial sphere) from the centre of the black hole. Thus, an outer trapping horizon $\mathcal{H}$ is the surface from which nothing can escape instantaneously. It is worth noting that in a dynamical spherically symmetric spacetime, $\mathcal{H}$ need not be lightlike but can have timelike or spacelike parts.

In [39], Kodama has shown that in the case of spherically symmetric spacetimes, it is possible to find a vector field[1] K^a which can be used as a replacement of the timelike Killing vector field of a stationary black hole (the full definition is given in Sect. 2.2). This *Kodama vector field* K^a is a conserved current, and also $W^{ab}K_b$ is a conserved current whenever W^{ab} is a symmetric tensor field that is invariant under the spherical symmetries of the spacetime. Furthermore, K^a is timelike outside of, spacelike inside of, and lightlike on an outer trapping horizon $\mathcal{H}$, respectively. On $\mathcal{H}$,

[1] We shall mostly employ the abstract index notation for vector and tensor fields as in [60].

one has

$$\frac{1}{2}K^a(\nabla_a K_b - \nabla_b K_a) = \kappa K_b \tag{1}$$

where the function κ is the *surface gravity* along $\mathcal{H}$.

Outer trapping horizons and the conservation of currents generated by the Kodama vector field have been used by Hayward [29] to derive a first thermodynamical law for dynamical black holes. In particular, it holds that

$$\mathcal{M}' = \frac{\kappa}{8\pi}\mathcal{A}' + w\mathcal{V}'$$

with the derivative $f' = z^a \nabla_a f$ where z^a is any (nowhere vanishing) vector field having zero angular components tangent to the outer trapping horizon. Furthermore, $\mathcal{M}$ is the Hawking mass of the black hole, $\mathcal{A} = 4\pi r^2$ is the area of the surface, $\mathcal{V} = \frac{4}{3}\pi r^3$ is the surface-enclosed volume, and κ is the surface gravity associated with the Kodama vector field. The term $w = -G_{UV} g^{UV}$ is related to the trace of the Einstein tensor taken with respect to the lightlike coordinates of the horizon, symbolized by indices U and V; see Sect. 2 for full details. As usual, $\mathcal{M}$ is interpreted as the black hole's internal energy, see (7), and $w\mathcal{V}'$ is the work done on the system. Interpreting $\kappa/(2\pi)$ as a temperature, $\mathcal{A}'/4$ represents the variation in entropy.

We will consider a (for simplicity, scalar) quantum field $\phi(x)$ propagating on a spherically symmetric spacetime with an outer trapping horizon and a Kodama vector field. Here, we follow common practice to write symbolically $\phi(x)$ where x is a spacetime point as if $\phi(x)$ was an operator-valued function, while actually it is an operator-valued distribution. We will take due care of this circumstance whenever required in the main body of the text. To further simplify matters, we assume that $\phi(x)$ is a quantized Klein–Gordon field fulfilling the field equation $(\nabla^a \nabla_a - M(x))\phi(x) = 0$ where ∇ is the covariant derivative of the spacetime metric g_{ab} and M is a smooth, real-valued function on spacetime. (This assumption could, in fact, be generalized.)

States (and in particular, quasifree states) of the quantized Klein–Gordon field on curved spacetimes admitting a physical interpretation consistent with the principles that apply for quantum field theory on Minkowski spacetime are *Hadamard states*. These states are defined as having a 2-point function of *Hadamard form*, meaning that

$$w^{(2)}(x_1, x_1) = \frac{1}{8\pi^2}\frac{\Delta^{1/2}(x_1, x_2)}{\sigma_\epsilon(x_1, x_2)} + W_\epsilon(x_1, x_2), \tag{2}$$

where Δ is the *van Vleck–Morette determinant* of the spacetime metric and $\sigma(x_1, x_2)$ is its Synge function, i.e. the squared geodesic distance divided by 2. Both quantities are determined by the spacetime metric; the subscript ϵ denotes a regularization that is used to properly define the quantity on the right-hand side as a distribution (after integration with test functions) in the limit $\epsilon \to 0$ (see [37] and Sect. 3 for further details). Similarly, in the limit $\epsilon \to 0$, $W_\epsilon(x_1, x_2)$ is a distribution which diverges at most logarithmically in σ for $\sigma \to 0$ and contains the state dependence as a smooth

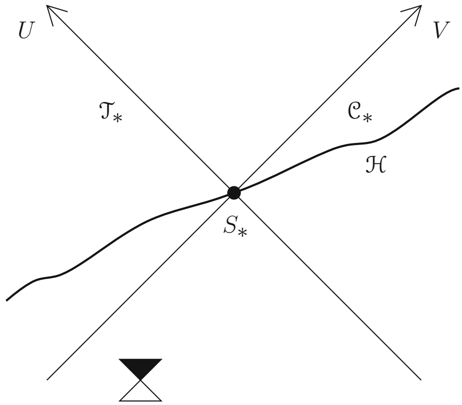

Fig. 1 The picture represents the U, V plane in adapted coordinates $(U, V, \vartheta, \varphi)$. The thick line corresponds to the outer trapping horizon $\mathcal{H}$; $S_* \subset \mathcal{H}$ is a sphere which is used to identify the null congruence $\mathcal{C}_*$ towards which we compute the scaling limit of the quantum states. The scaling limit state is then restricted onto the null congruence $\mathcal{T}_*$

contribution. For a discussion as to why Hadamard states are of particular significance, see e.g. [20,38,61] and references cited there.

We will show that close to an outer trapping horizon of a spherically symmetric spacetime, the universal leading short-distance singularity behaviour of any Hadamard state results, in a scaling limit, in an interpretation of the surface gravity κ as a temperature parameter, in close analogy to previous considerations for the case of quantum fields on stationary black holes [22,31,37,43,57]. Our approach follows the spirit of [43] very closely and thus makes contact with the tunnelling interpretation of Hawking radiation. For a spherically symmetric spacetime with outer trapping horizon $\mathcal{H}$, one introduces Eddington–Finkelstein coordinates v, r, ϑ, φ. Some point on $\mathcal{H}$ will be determined by certain coordinate values $(v_*, r_*, \vartheta_*, \varphi_*)$, and, by spherical symmetry, it determines the associated spatial spherical cross section $S_* = S(v_*, r_*)$ of $\mathcal{H}$. The outgoing null geodesic congruence emanating from S_* defines a null hypersurface denoted by $\mathcal{C}_*$, and similarly, the ingoing null geodesic congruence emanating from S_* defines a null hypersurface denoted by $\mathcal{T}_*$. As will be discussed in the main body of this paper, given S_*, there exists a natural choice of an affine parameter V along the geodesic generators of $\mathcal{C}_*$ and of an affine parameter U along the geodesic generators of $\mathcal{T}_*$ so that local coordinates $(U, V, \vartheta, \varphi)$ near S_* can be introduced, with the following properties:

(1) $U = 0$ and $V = 0$ exactly for the points on S_*,
(2) $U = 0$ exactly for the points on $\mathcal{C}_*$,
(3) $V = 0$ exactly for the points on $\mathcal{T}_*$,
(4) $ds^2 = -2A(U, V)dU\,dV + r^2(U, V)d\Omega^2$ is the metric line element where $d\Omega^2$ denotes
the line element of the two-dimensional Euclidean sphere, and $A = 1$ on $\mathcal{C}_* \cup \mathcal{T}_*$,
(5) $dU_a K^a(U, V = 0, \vartheta, \varphi) = -\kappa_* U + O(U^2)$ on $\mathcal{T}_*$ near $U = 0$, with $\kappa_* = \kappa|_{S_*}$.
We call $(U, V, \vartheta, \varphi)$ with the properties stated above *adapted coordinates* with respect to S_* (see Fig. 1 for an illustration.)

To analyse the short distance behaviour of the 2-point function of Hadamard states when both x_1 and x_2 are very close to $\mathcal{H}$, we proceed as follows. Once a sphere S_* (having radius r_*) of the outer trapping horizon is chosen and the null surface $\mathcal{C}_*$ of outgoing null geodesics is determined, we take a suitable scaling limit of the 2-point function towards $\mathcal{C}_*$. As we shall prove in Theorem 4.1, the 2-point function (distribution) Λ thus obtained is universal, and it can be tested with compactly supported smooth functions on $\mathcal{T}_*$. Using adapted coordinates, its regularized integral kernel has the form

$$\Lambda_\varepsilon(U, \boldsymbol{v}; U', \boldsymbol{v}') = -\frac{1}{\pi} \frac{r_*^2}{(U - U' + i\epsilon)^2} \delta(\boldsymbol{v}, \boldsymbol{v}'), \tag{3}$$

where $(U, \boldsymbol{v})$ denotes a point on $\mathcal{T}_*$, U is the null coordinate and $\boldsymbol{v} = (\vartheta, \varphi)$ denotes standard angular coordinates on the sphere S_*. Furthermore, $\varepsilon > 0$ is a regulator (to be taken to 0 after integrating against test functions) and $\delta(\boldsymbol{v}, \boldsymbol{v}')$ is the Dirac delta function supported on coinciding angles.[2] As we shall see in Sect. 5.2, the thermal properties are manifest when Λ is tested with respect to the flow Φ_τ ($\tau \in \mathbb{R}$) generated by K^a. Applied to Λ, the flow acts as $\Phi_\tau(U) = e^{\kappa_* \tau} U$ where κ_* is the surface gravity on S_*, and the Fourier frequencies, or energies, with respect to the flow parameter τ in the spectrum of Λ are distributed according to the spectral density

$$\rho(E) = \frac{E}{1 - e^{-\frac{2\pi}{\kappa_*} E}}\,. \tag{4}$$

The presence of a Bose factor with inverse temperature $2\pi/\kappa_*$ in the spectral density distribution makes the thermal interpretation manifest, analogously as in [43]. Making use of this fact, in following [43] we show that the tunnelling probability, or transition probability, between a one-particle state inside the outer trapping horizon $\mathcal{H}$, i.e. for $U > 0$, and another one particle state outside of $\mathcal{H}$, i.e. for $U < 0$, takes in the scaling limit the high-energy asymptotic form $\mathrm{e}^{-\beta E}$, when the one-particle states have a Fourier distribution peaked at E. This is the form of a transition probability for a thermal energy level occupation at inverse temperature $\beta = 2\pi/\kappa_*$.

The 2-point function Λ obtained in our scaling limit is very similar to the restriction of 2-point functions to Killing horizons considered in [22,26,37,55,57]. In these articles, the restrictions or scaling limits of 2-point functions to the analogues of $\mathcal{C}_*$ exhibit a thermal spectrum with respect to the Killing flow. In contrast, in the case of dynamical black holes the relevant part of the state is the transversal component of the 2-point function (the component supported on $\mathcal{T}_*$), showing thermal properties with respect to the Kodama flow in the scaling limit. The $\mathcal{C}_*$-part of the 2-point function depends on the details of the quantum matter entering the horizon, blurring an exact thermal spectrum. On the other hand, at least in the case of static black holes, the $\mathcal{T}_*$-part is related to the radiation emitted by the black hole and is the source of Hawking radiation, see e.g. [21].

The 2-point function Λ can be used to define a quantum field theory—the "scaling limit theory"—on the lightlike hypersurface $\mathcal{T}_*$; Λ also induces a quasifree state ω_Λ on the algebra of observables $\mathcal{W}$ of the scaling limit theory, where $\mathcal{W}$ is a CCR–Weyl algebra. This state turns out to be a KMS-state [25] at inverse temperature $\beta = 2\pi/\kappa_*$

[2] Formally, this means that $f(\boldsymbol{v}) = \int_{S^2} \delta(\boldsymbol{v}, \boldsymbol{v}') f(\boldsymbol{v}')\, d\Omega^2(\boldsymbol{v}')$ for any continuous function f on the unit sphere.

with respect to the Kodama flow. It is then possible to define and calculate the relative entropy $S(\omega_\Lambda|\omega_\varphi)$ in the sense of Araki [1] between ω_Λ and coherent states ω_φ on $\mathcal{W}$ analogously as in [32,41] where the function φ describes a coherent excitation of the scalar field over the state ω_Λ. We find that $S(\omega_\Lambda|\omega_\varphi)$ coincides with the classical energy of the coherent excitation, measured by an observer moving along the Kodama flow, multiplied with the inverse temperature $\beta = 2\pi/\kappa_*$ (see equation (74)). Furthermore, we observe that $S(\omega_\Lambda|\omega_\varphi)$ is proportional to r_*^2, the geometrical area of the outer trapping horizon's cross section S_* with respect to which the scaling limit theory is constructed. We argue that this is not accidental but a consequence of the fact that ω_Λ and ω_φ are correlation-free product states with respect to separation in the angular coordinate $\boldsymbol{\nu}$ of S_*, together with the additivity of the relative entropy for correlation-free product states. Thus, we arrive at $S(\omega_\Lambda|\omega_\varphi) \sim r_*^2$, analogous to the entropy–area relation suggested in the classic article of Bardeen, Carter and Hawking [3].

The idea to relate the relative entropy of quantum field states on a spacetime containing a horizon to a form of black hole entropy goes back to Longo [40] (in the setting of quantum field theory on Minkowski spacetime, where the lightlike boundaries of a wedge region play the role of a horizon). These ideas have been extended in [36] to a relation between relative entropy of quantum field states and non-commutative geometrical quantities for area. In a series of articles by Schroer [53,54], it was mentioned that—in the setting of quantum field theory in Minkowski spacetime—quantum field theories restricted to lightlike hyperplanes typically show no correlations in the transversal spacelike directions of the hyperplane, which would result in an additive behaviour of entropy quantities for the restricted quantum fields and an area proportionality. In our present article, we see that the earlier ideas of Longo and of Schroer can indeed be combined to result, in our scaling limit, in a proportionality between the relative entropy of quantum field states and the horizon area of a black hole spacetime and, more generally, the cross-sectional area of an outer trapping horizon. In a recent work, Hollands and Ishibashi [33] consider linearized perturbations of the spacetime metric around Schwarzschild spacetime which are quantized similarly like a linear scalar field. Using preferred states for the characteristic data of the perturbations (which on the black hole horizon take a form as our scaling limit state), they define relative entropies in a similar way, and taking into account the backreaction of the coherent excitation on the background geometry, they show that the combined variation in the relative entropy and a cross-sectional area of the black-hole horizon along Schwarzschild time equals the future out- and ingoing flux of radiation. Related content, in the context of spherically symmetric dynamical black holes, appears also in [12].

This article is organized as follows. In Sect. 2, we discuss the geometric setup of spherically symmetric spacetimes in which a Kodama vector field and outer trapping horizons can be defined. Section 3 contains the specification of the quantum field theory on the spacetimes we consider, together with a discussion about Hadamard 2-point function. Section 4 begins by introducing a conformal transformation of the spacetimes considered which is useful for deriving the scaling limit of Hadamard 2-point functions, presented thereafter in Theorem 4.1. The behaviour of the scaling limit 2-point function Λ under the Kodama flow is also discussed. In Sect. 5, we show how the scaling limit theory is constructed from the scaling limit 2-point function

Λ, and derive and discuss several of its properties, like the thermal spectrum and thermal tunnelling probability with respect to the Kodama time. We also consider the coherent states in the scaling limit theory and their relative entropy which we find to be proportional to the area of the horizon cross section S_* with respect to which the scaling limit theory is defined. A conclusion is given in Sect. 6. Section 1 is a technical appendix containing the proof of Theorem 4.1.

2 Geometric setup

2.1 Spherically symmetric spacetime, Eddington–Finkelstein coordinates

We consider a spacetime (M, g_{ab}), where M is the 4-dimensional spacetime manifold and g_{ab} is the spacetime metric, with signature $(- \ + \ + +)$. It will be assumed that the spacetime is (spatially) spherically symmetric, i.e. its set of isometries contains the group $SO(3)$, and all the orbits of the $SO(3)$ action are spacelike. We also assume that the spacetime has an *outer trapping horizon* $\mathcal{H}$ and a *Kodama vector field*. Thus, we assume that M contains an open subset N, diffeomorphic to $\mathcal{L} \times S^2$ with an open, connected subset $\mathcal{L}$ of $\mathbb{R}^2$, on which advanced Eddington–Finkelstein coordinates $(v, r, \vartheta, \varphi)$ can be introduced, where (v, r) are coordinates on $\mathcal{L}$ and (ϑ, φ) are angular coordinates for the sphere. The spherical symmetry group then acts on the S^2 part of N. Using such coordinates, the metric g_{ab} on N assumes the line element

$$ds^2 = -e^{2\Psi(v,r)}C(v, r)dv^2 + 2e^{\Psi(v,r)}dvdr + r^2 d\Omega^2 \tag{5}$$

where $d\Omega^2$ is the normalized spherically symmetric Riemannian metric on S^2. With respect to angular coordinates (ϑ, φ), one has

$$d\Omega^2 = d\vartheta^2 + (\sin\vartheta)^2 d\varphi^2 \,. \tag{6}$$

The coordinate v takes values in a real interval and r in a positive real interval; the precise form of the intervals depends on the smooth coordinate functions $C \geq 0$ and Ψ. Furthermore, the function C can be written in terms of the *Hawking mass*

$$\mathcal{M}(v, r) = \frac{r}{2}\Big(1 - g^{ab}(v, r)\nabla_a r \nabla_b r\Big) \tag{7}$$

as

$$C(v, r) = 1 - \frac{2\mathcal{M}(v, r)}{r} \,. \tag{8}$$

As a side remark, we notice that if $\Psi(v, r) = 0$ and $\mathcal{M}(v, r) = \mathcal{M}(v)$, the metric g_{ab} reduces to the Vaidya metric, which is one of the simpler models of dynamical black holes (see [23] and references cited there).

Consider the following null vector fields,

$$\ell^a = 2\frac{\partial}{\partial v}^a + e^{\Psi} C \frac{\partial}{\partial r}^a, \qquad \underline{\ell}^a = -e^{-\Psi}\frac{\partial}{\partial r}^a. \tag{9}$$

Then, ℓ^a is a future pointing outgoing null vector field and $\underline{\ell}^a$ a future pointing ingoing null vector field. Furthermore, in terms of these vector fields we have that

$$g^{ab} = -\frac{1}{2}\left(\ell^a \underline{\ell}^b + \underline{\ell}^a \ell^b\right) + h^{ab} \tag{10}$$

where $h_{ab} = r^2 d\Omega^2$ and it also holds that

$$g_{ab}\ell^a \underline{\ell}^b = -2. \tag{11}$$

The expansion parameters of the congruences of outgoing and ingoing null geodesics tangent to ℓ^a and $\underline{\ell}^a$ are given by

$$\theta_+ = h^{ab}\nabla_a \ell_b = \frac{2e^{\Psi}C}{r}, \qquad \theta_- = h^{ab}\nabla_a \underline{\ell}_b = -\frac{2e^{-\Psi}}{r}. \tag{12}$$

On N, θ_- is always negative, while θ_+ has the same sign as C, and vanishes if $C = 0$. In the case of a black hole, we have that C is positive far from the centre; thus, far from $r = 0$, the transversal area of the congruence tangent to ℓ^a increases towards the future while the transversal area of the congruence tangent to $\underline{\ell}^a$ decreases.

2.2 Outer trapping horizons

The set of points where $C = 0$ is the union of trapped surfaces. The possibility arises that this set has several disjoint connected components. Thus, we define the *outer trapping horizon* $\mathcal{H}$ as the outermost connected component, in the following sense: We assume that there is a time function T on N so that all the r-coordinate values of $\mathcal{H}$ on hypersurfaces of constant T are larger than the respective r-coordinate values of the other connected components. If there is only one connected component then, writing $C = 0$ in terms of the Hawking mass,

$$\mathcal{H} = \left\{(v, r, \boldsymbol{v}) \in N : \quad \frac{2\mathcal{M}(v, r)}{r} = 1\right\}. \tag{13}$$

The hypersurface $\mathcal{H}$ is spacelike for black holes which are growing in a collapse, it is lightlike for stationary black holes, and it is timelike for black holes which evaporate.

In contrast with the case of Schwarzschild spacetime, there is in general no timelike or causal Killing vector field near $\mathcal{H}$ that could be used to define and test black hole thermodynamical quantities. Hayward [29] proposed to use the *Kodama vector field* as a replacement (see also [30] for a review). The Kodama vector field [39] can be defined in terms of ℓ, $\underline{\ell}$ and r as

$$K^a := \frac{1}{2}\left(\ell[r]\underline{\ell}^a - \underline{\ell}[r]\ell^a\right) = e^{-\Psi}\frac{\partial}{\partial v}^a\,, \tag{14}$$

with $\ell[r] = \ell^a \nabla_a r$ and similarly for $\underline{\ell}[r]$. The Kodama vector field is conserved and can be used to build other conserved quantities: It holds that

$$\nabla_a K^a = 0\,, \qquad \nabla_a(W^{ab}K_b) = 0, \tag{15}$$

for any symmetric tensor field W^{ab} that is invariant under the spherical symmetries of the spacetime.

Notice that K^a is timelike in the region where $\theta_+ > 0$ and is lightlike on $\mathcal{H}$. Furthermore, the *surface gravity* κ associated with the Kodama vector field is a function on $\mathcal{H}$ defined by

$$\frac{1}{2}K^a(\nabla_a K_b - \nabla_b K_a) = \kappa K_b \quad \text{on } \mathcal{H}\,. \tag{16}$$

With respect to the metric component function $C(v, r)$ of (5), one obtains

$$\kappa = \frac{1}{2}\frac{\partial C(v,r)}{\partial r} \quad \text{if } C(v,r) = 0, \tag{17}$$

and in terms of the Hawking mass it is

$$\kappa = \left(\frac{\mathcal{M}(v,r)}{r^2} - \frac{\partial_r \mathcal{M}(v,r)}{r}\right)\bigg|_{r=2\mathcal{M}(v,r)}. \tag{18}$$

This definition generalizes the concept of surface gravity known for stationary black hole horizons, or for bifurcate Killing horizons. In the case of a stationary black hole, it is known that the surface gravity is proportional to the Hawking temperature.

2.3 Lightlike congruences emanating from the outer trapping horizon and adapted null coordinates

We have already indicated in Introduction that there are lightlike congruences emanating from the outer trapping horizon $\mathcal{H}$. They are determined once one chooses any point $(v_*, r_*, \boldsymbol{v}_*)$ on $\mathcal{H}$. Any such point then determines its orbit $S_* = \{(v_*, r_*)\} \times S^2$ under the spherical symmetry group of the spacetime. Clearly, S_* is a subset of $\mathcal{H}$. The lightlike vector fields ℓ^a and $\underline{\ell}^a$ restricted to S_* are tangent to two lightlike congruences $\mathcal{C}_*$ ("outgoing") and $\mathcal{T}_*$ ("ingoing"), respectively. Owing to the spherical symmetry, these lightlike congruences are 2-dimensional lightlike hypersurfaces. It holds that $S_* = \mathcal{C}_* \cap \mathcal{T}_*$.

One can introduce local coordinates (U, V) covering an open neighbourhood of (v_*, r_*) in the $\mathcal{L}$-part of N. U and V are null (or lightlike) coordinates, so that the metric line element (5) takes the form

$$ds^2 = -2A(U,V)dU\,dV + r^2(U,V)d\Omega^2\,, \tag{19}$$

where the radial coordinate is now a function of U and V, $r = r(U,V)$. This can actually always be achieved for a spherically symmetric spacetime metric; in the case at hand, there is an integrating factor $\alpha = \alpha(v,r)$ so that the required coordinates can be defined on an open neighbourhood of S_* by

$$\alpha \cdot dU_a = \frac{1}{2}e^{2\Psi(v,r)}C(v,r)dv_a - e^{\Psi(v,r)}dr_a\,, \qquad dV_a = dv_a\,. \tag{20}$$

One can further re-define the coordinates U and V so that they have additional properties. First, we have the freedom to choose the U and V coordinates such that $U = 0$ and $V = 0$ exactly for the points in S_*. Furthermore, we can choose the U and V coordinates so that $U = 0$ exactly for the points on $\mathcal{C}_*$ and $V = 0$ exactly for the points on $\mathcal{T}_*$; this freedom of choice is related to the fact that we have $\ell_a = \beta(U) \cdot dU_a$ on $\mathcal{T}_*$ and $\underline{\ell}_a = \underline{\beta}(V) \cdot dV_a$ on $\mathcal{C}_*$, with smooth, nonzero functions $\beta(U)$ and $\underline{\beta}(V)$. Re-defining U and V again so that $\beta = 2$ and $\underline{\beta} = 1$, one obtains from (11) that

$$A(U,0) = 1 \quad \text{and} \quad A(0,V) = 1\,. \tag{21}$$

Thus, given S_*, we can choose coordinates $(U, V, \vartheta, \varphi) = (U, V, \boldsymbol{\nu})$ in an open neighbourhood of S_* with the properties (1) to (4) stated in introduction. In the next section, we shall see that also property (5) is satisfied.

2.4 Kodama flow near S_*

As discussed above, once a sphere S_* contained in the outer trapping horizon $\mathcal{H}$ is fixed, we can determine the cone $\mathcal{C}_*$ formed by outgoing radial null geodesics passing through S_*, and the transversal cone $\mathcal{T}_*$, formed by ingoing radial null geodesics passing through S_*. Later we shall analyse the scaling towards $\mathcal{C}_*$ of the 2-point function of any Hadamard state. The resulting distribution can be restricted to $\mathcal{T}_*$, and it will be tested with respect to an observer moving along the integral line of the Kodama field. Hence, we need to analyse the form of the action of K^a on $\mathcal{T}_*$ near S_*. We denote by Φ_τ ($\tau \in \mathbb{R}$) the flow generated by K^a. We recall that this means that whenever $p \in N$ and $\tau_0 \in \mathbb{R}$ so that $\Phi_\tau(p) \in N$ for all τ in an open interval around τ_0, it holds that $K^a\nabla_a f|_{\Phi_{\tau_0}(p)} = \frac{d}{d\tau}|_{\tau=\tau_0} f(\Phi_\tau(p))$ for all smooth, real-valued functions f on N.

We use adapted coordinates $(U, V, \boldsymbol{\nu})$ with respect to S_* as described in the previous section. Then, we write

$$\Phi_\tau(U,V,\boldsymbol{\nu}) = (\mathrm{u}_\tau(U,V,\boldsymbol{\nu}), \mathrm{v}_\tau(U,V,\boldsymbol{\nu}), \boldsymbol{\nu}) \tag{22}$$

i.e. $\mathrm{u}_\tau(U, V, \boldsymbol{\nu})$ is the U-coordinate of $\Phi_\tau(U, V, \boldsymbol{\nu})$ and $\mathrm{v}_\tau(U, V, \boldsymbol{\nu})$ is the V-coordinate. (Note that Φ_τ doesn't act on $\boldsymbol{\nu}$, and therefore, u_τ and v_τ actually do not depend on $\boldsymbol{\nu}$.)

Lemma 2.1 *With κ_* the value of κ on S_* (corresponding to $U = 0$ and $V = 0$), there is an open interval of U coordinate values around 0 so that, on using the notation $O(\tau, U^2) = O(\tau) \cdot O(U^2)$ with the usual meaning of the Landau symbol for a single argument,*

(a) $dU_a K^a(U, V = 0, \boldsymbol{\nu}) = -\kappa_* U + O(U^2)$,
(b) $\mathrm{u}_\tau(U, V = 0, \boldsymbol{\nu}) = \mathrm{e}^{-\kappa_*\tau} U + O(\tau, U^2)$

Proof On using that $\underline{\ell}^a = \frac{\partial}{\partial U}^a$ and the definition of K^a, one obtains that $dU_a K^a = -\frac{1}{2}\ell[r] = \partial_\tau \mathrm{u}_\tau$. On the other hand, $\frac{1}{2}\ell[r]$ vanishes on S_*, and it holds that

$$-\frac{1}{2}\underline{\ell}[\ell[r]]\Big|_{S_*} = \frac{1}{2}\frac{\partial C}{\partial r}\Big|_{S_*} = \kappa_* \,. \tag{23}$$

This yields $dU_a K^a(U, V = 0, \boldsymbol{\nu}) = -\kappa_* U + O(U^2)$, having used $\underline{\ell}^a = \frac{\partial}{\partial U}^a$ once more and the fact that S_* is the locus of $U = 0$ and $V = 0$. This proves (a).

Furthermore, we have $\partial_\tau \mathrm{u}_\tau(U, V = 0, \boldsymbol{\nu}) = dU_a K^a(U, V = 0, \boldsymbol{\nu}) = -\kappa_* U + O(U^2)$, yielding on integration

$$\mathrm{u}_\tau(U, V = 0, \boldsymbol{\nu}) = \mathrm{e}^{-\kappa_*\tau} U + O(\tau, U^2)$$

on fixing the constants of integration such that $\mathrm{u}_0(U, 0, \boldsymbol{\nu}) = (U, 0, \boldsymbol{\nu})$ to be consistent with $\Phi_{\tau=0}(p) = p$. This proves (b). □

This shows that the property (5) stated for adapted coordinates in Introduction is also fulfilled.

3 The quantized linear scalar field

The main point of our article is an investigation of quantized fields on a spherically symmetric spacetime (M, g_{ab}) with an outer trapping horizon $\mathcal{H}$ and Kodama vector field K^a. To this end, our investigation starts with the free quantized scalar field $\phi(x)$. We assume that the underlying spacetime (M, g_{ab}) is globally hyperbolic. Actually, global hyperbolicity of the spacetime at large distances from $\mathcal{H}$ is not required for our considerations; what we need is a spherically symmetric, globally hyperbolic open neighbourhood of the outer trapping horizon $\mathcal{H}$ contained in the open set $N \simeq \mathcal{L} \times S^2$ on which the Eddington–Finkelstein coordinates discussed before can be introduced. For notational convenience, we assume in the following that this spherically symmetric globally hyperbolic open neighbourhood just coincides with M.

The quantized real free scalar field on (M, g_{ab}) is then defined in the standard manner which we will briefly sketch. For a fuller discussion, the reader may consult [34,38,61]. As (M, g_{ab}) is globally hyperbolic by assumption, there are uniquely determined advanced and retarded fundamental solutions $G^{\mathrm{adv/ret}}$ ("Green's operators") for the second-order hyperbolic Klein–Gordon operator $\nabla^a\nabla_a - M$ defined on smooth scalar test functions on M. Here, ∇_a denotes the covariant derivative of the spacetime metric g_{ab}, and $M \equiv M(x)$ is a smooth, real-valued function on M. Then, one can define the causal Green's function

$$\mathscr{G}(F, F') = \int_M \Big(F(x)(G^{\mathrm{adv}}F')(x) - F(x)(G^{\mathrm{ret}}F')(x) \Big)\, d\mathrm{vol}_g(x)\,, \qquad F, F' \in C_0^\infty(M, \mathbb{R})\,,$$

where $d\mathrm{vol}_g$ denotes the volume form of the spacetime metric g_{ab}. Hence, there is a $*$-algebra $\mathscr{A} = \mathscr{A}(M, g_{ab}, M)$ which is generated by a family of elements $\phi(F)$, $F \in C_0^\infty(M, \mathbb{R})$, and a unit element $\mathbf{1}$, subject to the relations:

$$(i)\ \ F \mapsto \phi(F) \ \text{ is } \mathbb{R}\text{-linear} \qquad (ii)\ \ \phi((\nabla^a\nabla_a + M)F) = 0$$

$$(iii)\ \ \phi(F)^* = \phi(F) \qquad (iv)\ \ [\phi(F), \phi(F')] = i\mathscr{G}(F, F')\cdot \mathbf{1}\,, \qquad F, F' \in C_0^\infty(M, \mathbb{R})\,.$$

Here, $[\phi(F), \phi(F')] = \phi(F)\phi(F') - \phi(F')\phi(F)$ denotes the algebraic commutator. The $\phi(F)$ are abstract field operators, at this level without a Hilbert space representation. One can symbolically write $\phi(x)$ to mean that $\phi(F) = \int_M \phi(x)F(x)\, d\mathrm{vol}_g(x)$ which can best be made rigorous when the $\phi(F)$ are given in some Hilbert space representation.

We recall that $w^{(2)}$ is a 2-point function for the Klein–Gordon field operators $\phi(F)$ if $w^{(2)} : C_0^\infty(M, \mathbb{R}) \times C_0^\infty(M, \mathbb{R}) \to \mathbb{C}$, $F, F' \mapsto w^{(2)}(F, F')$ is real-bilinear, extends to a distribution in $\mathcal{D}'(M \times M)$, and moreover, fulfils

$$w^{(2)}(F, F) \geq 0\,, \quad w^{(2)}(F', F) = \overline{w^{(2)}(F, F')}\,, \quad \mathrm{Im}\, w^{(2)}(F, F') = \frac{1}{2}\mathscr{G}(F, F')\,, \tag{24}$$

$$w^{(2)}((\nabla^a\nabla_a - M)F, F') = 0 = w^{(2)}(F, (\nabla^a\nabla_a - M)F') \qquad (F, F' \in C_0^\infty(M, \mathbb{R}))\,. \tag{25}$$

There is a one-to-one correspondence between states on $\mathscr{A}$ and Hilbert space representations of $\mathscr{A}$ which is given by the Gelfand–Naimark–Segal (GNS) representation. At this point, however, we don't make use of this, but we come back to a more operator-algebraic point of view in Sect. 5. Rather, we are interested in quasifree Hadamard states on $\mathscr{A}$; as these are completely determined by their 2-point function $w^{(2)}$, it is the behaviour of these 2-point functions near the outer trapping horizon that will be in the focus of our investigation.

At this point, it is useful, for later purpose, to look at the Hadamard form of the 2-point function in more detail, following mainly [37,50,51] (see also [52] for the relation of the Hadamard condition with equilibrium states).

Having chosen some S_*, in the adapted coordinates, we can define the time function $T(x) = T(U, V, \vartheta, \varphi) = (U + V)/2$. We can consider the hypersurface $\Sigma = \{x =$

$(U, V, \vartheta, \varphi) : T(x) = 0\}$. Then, there is an open neighbourhood $\mathcal{B}$ in M of S_* so that $\Sigma_{\mathcal{B}} = \Sigma \cap \mathcal{B}$ is a spacelike, acausal hypersurface containing S_*, and the open interior of the domain of dependence $D(\Sigma_{\mathcal{B}})$ is a globally hyperbolic open neighbourhood of S_* having $\Sigma_{\mathcal{B}}$ as a Cauchy surface. Then, an open neighbourhood $N_{\mathcal{B}}$ of $\Sigma_{\mathcal{B}}$ is called a *causal normal neighbourhood* if, given x and x' in $N_{\mathcal{B}}$, with $x' \in J^+(x)$, there is a convex normal neighbourhood (with respect to the metric g_{ab}) containing $J^-(x') \cap J^+(x)$. It has been shown in [37] that causal normal neighbourhoods always exist.

Then, a 2-point function $w^{(2)}$ is said to be *of Hadamard form near* S_* if, for all $F, F' \in N_{\mathcal{B}}$, it holds that

$$w^{(2)}(F, F') = \lim_{\varepsilon \to 0+} \int w_\epsilon(x, x') F(x) F'(x') \, d\mathrm{vol}_g(x) \, d\mathrm{vol}_g(x') \qquad (F, F' \in C_0^\infty(N_{\mathcal{B}}, \mathbb{R})) , \tag{26}$$

where (for any $k \in \mathbb{N}$)

$$w_\varepsilon(x, x') = \psi(x, x') \left(\frac{1}{8\pi^2} \frac{\Delta^{1/2}(x, x')}{\sigma_\varepsilon(x, x')} + \ln(\sigma_\varepsilon(x, x')) Y_k(x, x') \right) + Z_k(x, x') \tag{27}$$

with

$$\sigma_\varepsilon(x, x') = \sigma(x, x') - 2i\varepsilon t(x, x') + \varepsilon^2 \quad \text{and} \quad t(x, x') = T(x) - T(x'). \tag{28}$$

The important point here is the appearance of a smooth cutoff function ψ whose purpose is to make the terms in the brackets in (27) well defined and smooth. We explain this here briefly and refer for full details to the references [37,50,51] and also the review [38]; note that in the references, our cutoff function ψ is denoted by χ (however, we use χ as a different cut-off function in the proof of Theorem 4.1).[3] We denote by $\mathcal{X}$ the set of causally related pairs of points $(x, x') \in N_{\mathcal{B}} \times N_{\mathcal{B}}$, i.e. $(J^+(x) \cap J^-(x')) \cup (J^-(x) \cap J^+(x'))$ is non-empty. Then, there is an open neighbourhood $\mathcal{U}$ of $\mathcal{X}$ in $N_{\mathcal{B}} \times N_{\mathcal{B}}$ on which $\sigma(x, x')$ (the half of the squared geodesic distance) and $\Delta(x, x')$ (the van Vleck–Morette determinant) are well defined and C^∞ for $(x, x') \in \mathcal{U}$—however, it seems that a proof of existence of such a neighbourhood $\mathcal{U}$ has never previously been given in the literature. That issue is discussed in a recent paper by Moretti [44], where an argument showing that actually there is such a $\mathcal{U}$ is presented. Furthermore, there is a well-defined sequence of functions $Y_k(x, x')$ (determined by the Hadamard recursion relations) which can be chosen in $C^k(\mathcal{U}, \mathbb{R})$ for any $k \in \mathbb{N}$. The functions $Z_k(x, x')$ are correspondingly in $C^k(N_{\mathcal{B}} \times N_{\mathcal{B}}, \mathbb{R})$. There is then a further open neighbourhood of $\mathcal{X}$ in $N_{\mathcal{B}} \times N_{\mathcal{B}}$, denoted by $\mathcal{U}_*$, so that $\overline{\mathcal{U}_*} \subset \mathcal{U}$. Then, choose some $\psi \in C^\infty(N_{\mathcal{B}} \times N_{\mathcal{B}}, [0, 1])$ with $\psi(x, x') = 1$ on $\mathcal{U}_*$ and $\psi(x, x') = 0$ outside of $\mathcal{U}$. Consequently, the bracket term in (27) is well defined

[3] We thank an anonymous referee for pointing out that a previous version of our definition of the cut-off function was incomplete.

and C^k due to the presence of the cut-off function ψ. There is a freedom of choice for ψ; a different choice is compensated by a re-definition of the Z_k. Otherwise, different sequences Z_k correspond to different two-point functions.

4 Scaling limit of Hadamard 2-point functions near S_* and restriction to $\mathcal{T}_*$

4.1 Conformal transformation

In the adapted coordinates $(U, V, \vartheta, \varphi)$ discussed in Sect. 2.3, the line element of the spacetime (M, g_{ab}) under consideration assumes the form

$$ds^2 = -2A(U, V)dUdV + r^2(U, V)d\Omega^2 .$$

Our investigation of the scaling limit of the quantized linear scalar field near points of an outer trapping horizon $\mathcal{H}$ that we will consider below will be facilitated by using a conformally transformed metric. This applies in particular to the proof of Theorem 4.1. To simplify notation, we will write again $\boldsymbol{\nu}$ for (ϑ, φ) noting that the angular variables (ϑ, φ) really represent an element $\boldsymbol{\nu}$ of the unit sphere.

The conformal transformation is defined with respect to an arbitrarily chosen point (v_*, r_*) on $\mathcal{H}$ defining S_* and consequently $\mathcal{C}_*$ and $\mathcal{T}_*$, as explained in Introduction. Given (v_*, r_*) (or equivalently, the corresponding S_*), we introduce the conformally transformed metric $\tilde{g}_{ab}$ on M by

$$\tilde{g}_{ab} = \eta^2 g_{ab} \qquad \text{with the conformal factor} \qquad \eta^2(U, V) = \frac{r_*^2}{r^2(U, V)} ; \tag{29}$$

the associated line element is

$$d\tilde{s}^2 = -2\frac{A(U, V)r_*^2}{r^2(U, V)}dUdV + r_*^2 d\Omega^2 . \tag{30}$$

One feature of $\tilde{g}_{ab}$ is the splitting of the squared geodesic distance between points $(U, V, \boldsymbol{\nu})$ and $(U', V', \boldsymbol{\nu}')$ according to the Pythagorean theorem:

$$\tilde{\sigma}(U, V, \boldsymbol{\nu}; U', V', \boldsymbol{\nu}') = \tilde{\sigma}^{(\mathcal{L})}(U, V; U', V') + s(\boldsymbol{\nu}; \boldsymbol{\nu}') \tag{31}$$

where $\tilde{\sigma}^{(\mathcal{L})}(U, V; U', V')$ denotes the squared geodesic distance between the points (U, V) and (U', V') on the two-dimensional "Lorentzian" part of the spacetime with metric line element $-2\frac{A(U,V)r_*^2}{r^2(U,V)}dUdV$ and where $s(\boldsymbol{\nu}; \boldsymbol{\nu}')$ is the squared geodesic distance between points $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ on the two-dimensional sphere with radius r_*.

It is worth noting that on S_* where $r = r_*$, the conformal factor is equal to 1: $\eta|_{S_*} = 1$.

At the level of 2-point functions, the conformal transformation has the following effect. Suppose that

$$w^{(2)}(F, F') = \lim_{\varepsilon \to 0} \int w_\varepsilon(x, x') F(x) F'(x')\, d\mathrm{vol}_g(x)\, d\mathrm{vol}_g(x') \qquad (F, F' \in C_0^\infty(N_{\mathcal{B}}, \mathbb{R}))$$

is a 2-point function of Hadamard form, near S_*, for the quantized scalar field that we consider on (M, g_{ab}). Then, defining

$$\tilde{w}_\varepsilon(x, x') = \eta^{-1}(x) w_\varepsilon(x, x') \eta^{-1}(x')\,,$$

the distribution

$$\tilde{w}^{(2)}(F, F') = \lim_{\varepsilon \to 0} \int \tilde{w}_\epsilon(x, x') F(x) F'(x')\, d\mathrm{vol}_{\tilde{g}}(x)\, d\mathrm{vol}_{\tilde{g}}(x') \qquad (F, F' \in C_0^\infty(N_{\tilde{\mathcal{B}}}, \mathbb{R})),$$

is a 2-point function of Hadamard form near S_* on the conformally related spacetime $(M, \tilde{g}_{ab})$, with a suitably small neighbourhood $\tilde{\mathcal{B}}$ of S_*, and an associated causal normal neighbourhood $N_{\tilde{\mathcal{B}}}$ defined with respect to $\tilde{g}_{ab}$. This has been shown in [48].

The scaling limit which we will consider in the next section gives the same results on $w^{(2)}$ or on $\tilde{w}^{(2)}$ on account of $\eta|_{S_*} = 1$, but it is easier to study the scaling limit using $\tilde{w}^{(2)}$ because of (31). To this end, we put on record the following observations for later use.

The volume form $d\mathrm{vol}_g$ of the original metric and the volume form $d\mathrm{vol}_{\tilde{g}}$ of the conformally transformed metric are related according to $d\mathrm{vol}_{\tilde{g}}(x) = \eta^4(x) d\mathrm{vol}_g(x)$ and therefore one has

$$w^{(2)}(F, F') = \tilde{w}^{(2)}(\tilde{F}, \tilde{F}') \quad \text{with } \tilde{F} = \eta^{-3} F\,,\ \tilde{F}' = \eta^{-3} F \tag{32}$$

for all $F, F' \in C_0^\infty(M, \mathbb{R})$. The statement that $\tilde{w}^{(2)}$ is of Hadamard form near S_* on $(M, \tilde{g}_{ab})$ means that

$$\tilde{w}_\varepsilon(x, x') = \tilde{\psi}(x, x') \left(\frac{1}{8\pi^2} \frac{\tilde{\Delta}^{1/2}(x, x')}{\tilde{\sigma}_\varepsilon(x, x')} + \ln(\tilde{\sigma}_\varepsilon(x, x')) \tilde{Y}(x, x') \right) + \tilde{Z}(x, x') \quad (x, x' \in N_{\tilde{\mathcal{B}}}) \tag{33}$$

where $\tilde{\Delta}$ and $\tilde{\sigma}$ refer to $\tilde{g}_{ab}$, $\tilde{\psi}$ has properties analogous to ψ, and $\tilde{Y}$ and $\tilde{Z}$ (dropping the index k on Y and Z) can be chosen as C^k function for any $k \in \mathbb{N}$.

4.2 Scaling limit and restriction

We select a sphere S_* of radius r_* lying in the outer trapping horizon, and a patch of adapted coordinates $(U, V, \boldsymbol{\nu})$ relative to S_*. Moreover, we assume that $N_{\mathcal{B}}$ is a causal normal neighbourhood of a partial Cauchy surface $\Sigma_{\mathcal{B}}$ so that $S_* \subset \Sigma_{\mathcal{B}}$, as described in Sect. 3. Then, if $l_0 > 0$ is small enough, the open set

$$\mathcal{O} = \{(U, V, \boldsymbol{\nu}) : |U| < l_0\,, \;\; |V| < l_0\,, \;\; \boldsymbol{\nu} \in S^2\} \tag{34}$$

is a subset of $N_{\mathcal{B}}$ and an open neighbourhood of S_*. We assume that l_0 is chosen small enough so that $\mathcal{O}$ is also contained in a causal normal neighbourhood $N_{\tilde{\mathcal{B}}}$ of S_* with respect to the conformally transformed metric $\tilde{g}_{ab}$ described in Sect. 4.1.

Note that if $0 < \lambda \leq 1$ and $0 < \mu \leq 1$, then $(U, V, \boldsymbol{\nu}) \in \mathcal{O} \Rightarrow (\lambda U, \mu V, \boldsymbol{\nu}) \in \mathcal{O}$. Consequently, when defining the *scaling transformations*

$$(u_\lambda F)(U, V, \boldsymbol{\nu}) = \frac{1}{\lambda} F(U/\lambda, V, \boldsymbol{\nu})\,, \tag{35}$$

for $0 < \lambda \leq 1$, one can see that the u_λ map the space of test functions $C_0^\infty(\mathcal{O}, \mathbb{R})$ into itself, and

$$\text{supp}(u_\lambda F) = \{(\lambda U, V, \boldsymbol{\nu}) : (U, V, \boldsymbol{\nu}) \in \text{supp}(F)\} \quad (0 < \lambda \leq 1)\,. \tag{36}$$

We stress that the scaling transformations are defined with respect to the chosen S_* and the corresponding adapted coordinates.

We also define another type of transformations which serve, in a limit, to restricting distributions to $\mathcal{T}_*$ by effectively acting like a δ-distribution concentrated at $V = 0$. Let $\zeta \in C_0^\infty((-l_0, l_0), \mathbb{R})$ with $\zeta(V) \geq 0$ and $\int \zeta(V)\, dV = 1$. Then, we define, for any $F \in C_0^\infty(\mathcal{O}, \mathbb{R})$,

$$(v_\mu F)(U, V, \boldsymbol{\nu}) = \frac{1}{\mu} \zeta(V/\mu) F(U, V, \boldsymbol{\nu}) \qquad (0 < \mu \leq 1)\,. \tag{37}$$

Clearly, also every v_μ maps $C_0^\infty(\mathcal{O}, \mathbb{R})$ into itself.
Adopting this notation, we now present the result on scaling limits of Hadamard 2-point functions near S_* and subsequent restriction to $\mathcal{T}_*$.

Theorem 4.1 *Let $w^{(2)}$ be any 2-point function of Hadamard form for the scalar field ϕ on the spacetime (M, g_{ab}) as in Sect. 3.*
(I) *For all $f, f' \in C_0^\infty(\mathcal{O}, \mathbb{R})$, it holds that*

$$\lim_{\lambda \to 0} w^{(2)}(u_\lambda(2\partial_U f), u_\lambda(2\partial_{U'} f')) = L(f, f')\,, \qquad \textit{where}$$

$$L(f, f') = \lim_{\varepsilon \to 0+} -\frac{1}{r_*^2 \pi} \int \frac{f(U, V, \boldsymbol{\nu}) f'(U', V', \boldsymbol{\nu})}{(U - U' + i\varepsilon)^2} \times Q(V, V', \boldsymbol{\nu})\, dU\, dU' dV\, dV'\, d\Omega^2(\boldsymbol{\nu}) \tag{38}$$

with

$$Q(V, V', \boldsymbol{v}) = \tilde{\Delta}^{1/2}(0, V, \boldsymbol{v}, 0, V', \boldsymbol{v}) P(0, V, 0, V'), \tag{39}$$

$$P(U, V, U', V') = r_*^4 \eta^{-1}(U, V) A(U, V) \eta^{-1}(U', V') A(U', V'). \tag{40}$$

(II) *For all* $f, f' \in C_0^\infty(\mathcal{O}, \mathbb{R})$ *it holds that*

$$\lim_{\mu \to 0} \lim_{\lambda \to 0} w^{(2)}(v_\mu u_\lambda(2\partial_U f), v_\mu u_\lambda(2\partial_{U'} f')) = \lim_{\mu \to 0} L(v_\mu f, v_\mu f') = \Lambda(f, f') \tag{41}$$

where

$$\Lambda(f, f') = \lim_{\varepsilon \to 0+} -\frac{r_*^2}{\pi} \int \frac{f(U, 0, \boldsymbol{v}) f'(U', 0, \boldsymbol{v})}{(U - U' + i\varepsilon)^2} dU\, dU'\, d\Omega^2(\boldsymbol{v}) \tag{42}$$

The proof of this theorem is given in Appendix (Sect. 1).

Remark (i) The more difficult step is proving Part (I) of the theorem, Part (II) then is merely a corollary. Actually, the statement follows easily when inserting the scaled test functions $u_\lambda(2\partial_U f)$ and $u_\lambda(2\partial_{U'} f')$ into the ε-regulated integral expression of the Hadamard form and exchanging the $\lambda \to 0$ and $\varepsilon \to 0$ limits. The more involved part of the proof consists in showing that this can be justified. We have opted to give a full, self-consistent proof in this article, despite some similarities of our proof with a related argument in [43] (that relied in parts also on results from [37]) which applies to the case of the quantized Klein–Gordon field on spacetimes with bifurcate Killing horizons.

(ii) As is familiar from the quantization of the massless free quantum field in 2-dimensional Minkowski spacetime, respectively, its chiral components on lightrays, the 2-point function is well defined for test functions which are first derivatives of compactly supported smooth functions. Without derivatives, an infrared divergence occurs, see e.g. [5], Sect. on the "Schwinger model". This is the reason why the test functions used for the scaling limit considerations are U-derivatives of compactly supported smooth functions.
(iii) One may choose U or V-coordinates so that the Van Vleck – Morette determinant is equal to 1; this simplifies the form of the function Q in the first part of Theorem 4.1; however, we need not make use of this possibility here.
(iv) The factor 2 in the definition of the scaling transformations u_λ has been introduced to match Λ with the convention for 2-point functions on lightlike hyperplanes used in the literature, see e.g. [15]. See also the remark towards the end of Sect. 5.5.

4.3 Kodama flow projected to $\mathcal{T}_*$ and its action in the scaling limit

Under the same assumptions as for the previous theorem, we can establish that the *projected action* T_τ *on* $\mathcal{T}_*$ of the flow of the Kodama vector field K^a acts like the dilation group in the scaling limit. To make this more precise, we define

$$\mathrm{T}_\tau(U,V,\boldsymbol{\nu}) = (\mathrm{u}_\tau(U,0,\boldsymbol{\nu}),V,\boldsymbol{\nu}) \quad (\tau \in \mathbb{R}) \tag{43}$$

for all $(U,V,\boldsymbol{\nu}) \in \mathcal{O}$, with the convention that the definition applies whenever $(\mathrm{u}_\tau(U,0,\boldsymbol{\nu}),V,\boldsymbol{\nu})$ is again in $\mathcal{O}$. Recall that (cf. Lemma 2.1) $\mathrm{u}_\tau(U,V=0,\boldsymbol{\nu}) = \mathrm{e}^{-\kappa_*\tau}U + O(\tau,U^2)$ so that the projected action of the Kodama flow on $\mathcal{T}_*$ takes the form

$$\mathrm{T}_\tau(U,V,\boldsymbol{\nu}) = (\mathrm{e}^{-\kappa_*\tau}U + O(\tau,U^2),V,\boldsymbol{\nu}) \tag{44}$$

and there is some $\tau_0 > 0$ and an open neighbourhood $\mathcal{O}_0$ of S_* with $\mathcal{O}_0 \subset \mathcal{O}$ so that $T_\tau(U,V,\boldsymbol{\nu}) \in \mathcal{O}$ for all $(U,V,\boldsymbol{\nu})$ in $\mathcal{O}_0$ and all $|\tau| < \tau_0$. We also define:

$$(T_\tau F)(U,V,\boldsymbol{\nu}) = F(\mathrm{T}_{-\tau}(U,V,\boldsymbol{\nu})) \quad (F \in C_0^\infty(\mathcal{O}_0,\mathbb{R}),\ |\tau| < \tau_0), \tag{45}$$

$$\mathrm{S}_\tau(U,\boldsymbol{\nu}) = (\mathrm{e}^{-\kappa_*\tau}U,\boldsymbol{\nu}) \quad ((U,\boldsymbol{\nu}) \in \mathcal{T}_*,\ \tau \in \mathbb{R}), \tag{46}$$

$$(S_\tau\varphi)(U,\boldsymbol{\nu}) = \varphi(\mathrm{S}_{-\tau}(U,\boldsymbol{\nu})) \quad (\varphi \in C_0^\infty(\mathcal{T}_*,\mathbb{R}),\ \tau \in \mathbb{R}) \tag{47}$$

Proposition 4.2 *For any 2-point function $w^{(2)}$ of ϕ that is of Hadamard form,*

$$\lim_{\mu\to 0+}\lim_{\lambda\to 0+} w^{(2)}(T_\tau v_\mu u_\lambda(\partial_U f), T_{\tau'}v_\mu u_\lambda(\partial_{U'} f')) = \Lambda(S_\tau f, S_{\tau'} f') \tag{48}$$

holds for all $f, f' \in C_0^\infty(\mathcal{O}_0,\mathbb{R})$ and $|\tau|, |\tau'| < \tau_0$.

Proof For any $F \in C_0^\infty(\mathcal{O}_0,\mathbb{R})$ and $|\tau| < \tau_0$, one obtains

$$\begin{aligned}(T_\tau u_\lambda F)(U,V,\boldsymbol{\nu}) &= (u_\lambda F)(\mathrm{T}_{-\tau}(U,V,\boldsymbol{\nu}))\\ &= (u_\lambda F)(\mathrm{e}^{\kappa_*\tau}U + O(\tau,U^2),V,\boldsymbol{\nu})\\ &= \frac{1}{\lambda}F(\lambda^{-1}(\mathrm{e}^{\kappa_*\tau}U + O(\tau,U^2)),V,\boldsymbol{\nu})\\ &= \frac{1}{\lambda}F(\mathrm{e}^{\kappa_*\tau}(U/\lambda) + O(\lambda)\cdot O(\tau,(U/\lambda)^2),V,\boldsymbol{\nu})\end{aligned} \tag{49}$$

for small enough $\lambda > 0$. One can now see that in the proof of Theorem 4.1, all estimates involving $F_\lambda(x) = (u_\lambda F)(U,V,\boldsymbol{\nu})$ (and similarly, the primed counterparts) are preserved when replacing $(u_\lambda F)(U,V,\boldsymbol{\nu})$ by $(T_\tau u_\lambda F)(U,V,\boldsymbol{\nu})$ (and similarly for the primed counterparts). Moreover, the limit considerations in the proof of Theorem 4.1 where $F(x) = F(U,V,\boldsymbol{\nu})$ appears (and the primed counterpart) render the analogous results upon replacing $F(U,V,\boldsymbol{\nu})$ by $F(\mathrm{e}^{\kappa_*\tau}U + O(\lambda)\cdot O(\tau,U^2),V,\boldsymbol{\nu})$ (analogously for the primed counterpart) as $\lambda \to 0$, except that $F(U,V,\boldsymbol{\nu})$ is in the limit to be replaced by $F(\mathrm{e}^{\kappa_*\tau}U,V,\boldsymbol{\nu})$ and $F'(U',V',\boldsymbol{\nu}')$ by $F'(\mathrm{e}^{\kappa_*\tau'}U',V',\boldsymbol{\nu}')$. That follows from the fact that $O(\lambda)\cdot O(\tau,U^2) \to 0$ as $\lambda \to 0$ uniformly as τ and U vary over compact sets. Observing this and carrying out the steps of the proof of Theorem 4.1 thus yield the claimed result. □

5 Thermal properties and entropy–area relation for the scaling limit theory on $\mathcal{T}_*$

5.1 The scaling limit theory on $\mathcal{T}_*$ (and its extension)

The 2-point function Λ defines a quantum field theory on $\mathcal{T}_*$ which naturally extends to a (chiral, conformal) quantum field theory on $\mathbb{R} \times S_* \simeq \mathbb{R} \times S^2$. We will refer to this as the "scaling limit theory" induced by the scaling limit 2-point function Λ.

To discuss this, fix again $S_* \subset \mathcal{H}$, which is a copy of the sphere S^2 with radius r_*. Then, one can introduce on the (real-linear) function space $\mathcal{D}_{S_*} = C_0^\infty(\mathbb{R} \times S_*, \mathbb{R})$ the symplectic form

$$\varsigma(\varphi, \varphi') = 2\text{Im}\,\Lambda(\varphi, \varphi') = r_*^2 \int (\partial_U \varphi(U, \boldsymbol{\nu})\varphi'(U, \boldsymbol{\nu}) - \varphi(U, \boldsymbol{\nu})\partial_U \varphi'(U, \boldsymbol{\nu}))\, dU\, d\Omega^2(\boldsymbol{\nu})\,. \tag{50}$$

Note the dependence of ς on r_*^2. Given this symplectic form, one can form the Weyl algebra $\mathcal{W}(\mathcal{D}_{S_*}, \varsigma)$ (the "exponentiated CCR algebra") over the symplectic space $(\mathcal{D}_{S_*}, \varsigma)$; by definition, it is a C^* algebra with unit element $\mathbf{1}$, generated by unitary elements $W(\varphi)$, $\varphi \in \mathcal{D}_{S_*}$, fulfilling the Weyl relations

$$W(0) = \mathbf{1}\,, \quad W(\varphi)^* = W(-\varphi)\,, \quad W(\varphi)W(\varphi') = \mathrm{e}^{-\frac{i}{2}\varsigma(\varphi,\varphi')} W(\varphi + \varphi')\,. \tag{51}$$

As is common in the operator algebraic approach to algebraic quantum field theory (cf. [24] and in the present context, see also [15,22,37,57]) one can introduce a family $\{\mathcal{W}(G)\}$ of C^* algebras indexed by open, relatively compact subsets G of $\mathbb{R} \times S_*$ by defining $\mathcal{W}(G)$ as the C^*-subalgebra generated by all $W(\varphi)$ with supp$(\varphi) \subset G$. Then, it is easy to see that $\{\mathcal{W}(G)\}$ fulfils the condition of *isotony*, meaning that $\mathcal{W}(G_1) \subset \mathcal{W}(G_2)$ if $G_1 \subset G_2$, and it fulfils also a condition of *locality*, which in the present case means that $\mathcal{W}(G_1)$ and $\mathcal{W}(G_2)$ commute elementwise if $G_1 \cap G_2 = \emptyset$. Furthermore, there are certain symmetries that act covariantly on the manifold $\mathbb{R} \times S_*$: The dilations $\mathrm{S}_\tau(U, \boldsymbol{\nu}) = (\mathrm{e}^{-\kappa_* \tau} U, \boldsymbol{\nu})$, the translations $\mathrm{L}_a(U, \boldsymbol{\nu}) = (U + a, \boldsymbol{\nu})$, and rotations $\mathrm{R}(U, \boldsymbol{\nu}) = (U, R\boldsymbol{\nu})$, where $\tau, a \in \mathbb{R}$ and $R \in SO(3)$. The actions of these symmetry operations can be lifted to $\mathcal{D}_{S_*}$ by setting $S_\tau \varphi = \varphi \circ \mathrm{S}_\tau^{-1}$, $L_a \varphi = \varphi \circ \mathrm{L}_a^{-1}$ and $R\varphi = \varphi \circ \mathrm{R}^{-1}$. Each of those is a symplectomorphism with respect to the symplectic form ς, i.e. one has $\varsigma(S_\tau \varphi, S_\tau \varphi') = \varsigma(\varphi, \varphi')$ for all $\varphi, \varphi' \in \mathcal{D}_{S_*}$, etc. This implies that these symplectomorphisms can be lifted to C^*-algebraic morphisms $\alpha_{(\tau,a,R)}$ of $\mathcal{W}(\mathcal{D}_{S_*}, \varsigma)$, given by

$$\alpha_{(\tau,a,R)} W(\varphi) = W(S_\tau L_a R\varphi)\,. \tag{52}$$

We also adopt the notation to write α_τ for $\alpha_{(\tau,0,1)}$ and α_a for $\alpha_{(0,a,1)}$ etc whenever no ambiguity can arise. It is plain that thereby, a representation of the group of symmetries generated by dilations, translations and rotations by automorphisms of $\mathcal{W}(\mathcal{D}_{S_*}, \varsigma)$ is established. It is also easily seen that these automorphisms act covariantly (or geometrically) on the family $\{\mathcal{W}(G)\}$ in the sense that

$$\alpha_\tau(\mathcal{W}(G)) = \mathcal{W}(\mathrm{S}_\tau G)\,, \quad \alpha_a(\mathcal{W}(G)) = \mathcal{W}(\mathrm{L}_a G)\,, \quad \alpha_R(\mathcal{W}(G)) = \mathcal{W}(\mathrm{R}G)\,. \tag{53}$$

We recall that a linear functional $\omega : \mathcal{W}(\mathcal{D}_{S_*}, \varsigma) \to \mathbb{C}$ is a *state* if it is positive, i.e. $\omega(A^*A) \geq 0$ for all $A \in \mathcal{W}(\mathcal{D}_{S_*}, \varsigma)$, and normalized, i.e. $\omega(\mathbf{1}) = 1$. Moreover, every state ω induces the associated GNS representation $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathcal{W}(\mathcal{D}_{S_*})$, characterized by the properties that π_ω is a unital $*$-representation of $\mathcal{W}(\mathcal{D}_{S_*}, \varsigma)$ by bounded linear operators on the Hilbert space $\mathscr{H}_\omega$, and Ω_ω is a unit vector in $\mathscr{H}_\omega$ so that $\pi_\omega(\mathcal{W}(\mathcal{D}_{S_*}, \varsigma))\Omega_\omega$ is dense in $\mathscr{H}_\omega$ and $\langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle = \omega(A)$ for all $A \in \mathcal{W}(\mathcal{D}_{S_*}, \varsigma)$ (on writing $\langle \xi, \psi \rangle$ for the scalar product of $\xi, \psi \in \mathscr{H}_\omega$). Thus, once given a state ω on $\mathcal{W}(\mathcal{D}_{S_*}, \varsigma)$, one can introduce the system $\{\mathcal{N}(G)\}$ of *local von Neumann algebras* in the GNS representation of ω given by

$$\mathcal{N}(G) = \overline{\pi_\omega(\mathcal{W}(G))} = \pi_\omega(\mathcal{W}(G))'' \tag{54}$$

where the overlining means taking the weak closure in $\mathcal{B}(\mathscr{H}_\omega)$ (the set of bounded linear operators on $\mathscr{H}_\omega$); the double prime denotes the double commutant: For $\mathcal{X} \subset \mathcal{B}(\mathscr{H}_\omega)$, $\mathcal{X}' = \{B \in \mathcal{B}(\mathscr{H}_\omega) : AB - BA = 0 \text{ for all } A \in \mathcal{X}\}$ is the *commutant* of $\mathcal{X}$, and $\mathcal{X}'' = (\mathcal{X}')'$. Whenever $\mathcal{X}$ contains the unit operator, it holds that $\overline{\mathcal{X}} = \mathcal{X}''$. For full details on these operator algebraic facts, see [6–8].
The 2-point function Λ induces a quasifree state ω_Λ on $\mathcal{W}(\mathcal{D}_{S_*}, \varsigma)$ defined by linear extension of the assignment $\omega_\Lambda(W(\varphi)) = \mathrm{e}^{-\Lambda(\varphi,\varphi)/2}$. We denote the associated local von Neumann algebras again by $\mathcal{N}(G)$ (unless a more detailed notation is required). Of particular interest are the von Neumann algebras $\mathcal{N}_R = \mathcal{N}((-\infty, 0) \times S_*)$ and $\mathcal{N}_L = \mathcal{N}((0, \infty) \times S_*)$.

Several important properties of ω_Λ have been established and are well known, from related contexts or from investigations of chiral conformal quantum field theory. We collect some of those properties here; proofs and further exposition can be found in [15,22,37,57]. For notational simplicity, the GNS representation of ω_Λ will be denoted by $(\mathscr{H}_\Lambda, \pi_\Lambda, \Omega_\Lambda)$.

(1) The state ω_Λ is invariant under the action α: $\omega_\Lambda \circ \alpha_{(\tau,a,R)} = \omega_\Lambda$. Consequently, there is a unitary action $U_{(\tau,a,R)}\pi_\Lambda(A)\Omega_\Lambda = \pi_\Lambda(\alpha_{(\tau,a,R)}A)\Omega_\Lambda$ ($A \in \mathcal{W}(\mathcal{D}_{S_*}, \varsigma)$) implementing the action of α in the GNS-representation of ω_Λ with $U_{(\tau,a,R)}\Omega_\Lambda = \Omega_\Lambda$.
(2) ω_Λ is a ground state for the translations α_a, i.e. there is a non-negative self-adjoint generator H in $\mathscr{H}_\Lambda$ so that $U_a = \mathrm{e}^{iHa}$. (We are here using the same convention as previously explained for α to write $U_a = U_{(0,a,1)}$, etc.)
(3) Ω_Λ is a cyclic and separating vector for the von Neumann algebras $\mathcal{N}_R$ and $\mathcal{N}_L$. Let Δ_R denote the modular operator with respect to $\mathcal{N}_R$ and Ω_Λ. Then, it holds that

$$\Delta_R^{i\tau} = U_{\beta\tau} \quad \text{with} \quad \beta = 2\pi/\kappa_* \quad (\tau \in \mathbb{R}) \tag{55}$$

(4) The previous relation (55) can equivalently be expressed as stating that the state ω_Λ restricted to the C^*-subalgebra $\mathcal{W}_R = \mathcal{W}((-\infty, 0) \times S_*)$ of $\mathcal{W}(\mathcal{D}_{S_*}, \varsigma)$ is a KMS-state for the action of the α_τ at inverse temperature $\beta = 2\pi/\kappa_*$. Analogously, ω_Λ restricted to $\mathcal{W}_L = \mathcal{W}((0, \infty) \times S_*)$ is a KMS state for the action of the α_τ at inverse temperature $\beta = -2\pi/\kappa_*$.

5.2 Thermal interpretation of the 2-point function Λ

We will now point out that the thermal properties expressed in (3) and (4) at the end of the previous subsection can be directly read off from the Fourier spectrum of Λ with respect to the Kodama time parameter, analogously as in [43].

To this end, we recall the results presented in Sect. 2.4 and the action of the Kodama flow on the scaling limit state discussed in Sect. 4.3. In particular, points of $\mathcal{T}_*$ outside the Horizon $\mathcal{H}$ can be parametrized by $(u, \boldsymbol{v})$ where here the coordinate u is related to U by the following coordinate transformation

$$U = -\mathrm{e}^{-\kappa_* u}, \quad U < 0. \tag{56}$$

In particular, we have seen in Proposition 4.2 that the Kodama flow in the scaling limit, described by S_τ, acts as u-translation, $u \mapsto u + \tau$. Thus, if φ and φ' are both contained in $C_0^\infty((-\infty, 0) \times S_*, \mathbb{R})$, i.e. they are supported on $U < 0$, one obtains

$$\Lambda(\varphi, \varphi') = \lim_{\epsilon \to 0^+} -\frac{r_*^2 \kappa_*^2}{4\pi} \int \frac{\varphi(u, \boldsymbol{v})\varphi'(u', \boldsymbol{v})}{\sinh\left((u - u')\frac{\kappa_*}{2} + i\epsilon\right)^2} \, du \, du' \, d\Omega^2(\boldsymbol{v}) \, . \tag{57}$$

A similar relation holds if φ and φ' are both supported on $U > 0$, on using the coordinate transformation $U = \mathrm{e}^{\kappa_* u}$. The Fourier transform along $u - u'$ of that distribution can be directly computed, see e.g. Appendix of [14]; it yields

$$\Lambda(\varphi, \varphi') = 2r_*^2 \int \frac{\overline{\hat{\varphi}(E, \boldsymbol{v})}\hat{\varphi}'(E, \boldsymbol{v})}{1 - \mathrm{e}^{-\beta E}} E \, dE \, d\Omega^2(\boldsymbol{v}) \, , \quad \beta = 2\pi/\kappa_* \, , \tag{58}$$

if φ and φ' are both supported either on $U < 0$ or $U > 0$, where the Fourier transform with respect to u has been denoted by a hat

$$\hat{\varphi}(E, \boldsymbol{v}) = \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-iEu} \varphi(u, \boldsymbol{v}) \, du \, . \tag{59}$$

The appearance of the Bose thermal distribution factor $(1 - \mathrm{e}^{-\beta E})^{-1}$ for the Fourier "energies" in the integral expression (58) manifestly shows the thermal Fourier spectrum of the 2-point function for an observer moving along the Kodama flow, where the inverse temperature is given by $\beta = 2\pi/\kappa_*$.

5.3 Tunneling probability

Again proceeding as in [43], we now look at the Fourier transformed expression for the 2-point function Λ in the case that φ is supported on $U < 0$, i.e. outside of the outer trapping horizon, while φ' is supported on its inside, on $U > 0$. The result is (cf. [43], Sect. 3.3 b))

$$\Lambda(\varphi, \varphi') = r_*^2 \int \frac{\overline{\hat{\varphi}(E, \boldsymbol{\nu})}\hat{\varphi}'(E, \boldsymbol{\nu})}{\sinh(\frac{\beta}{2}E)} E \, dE \, d\Omega^2(\boldsymbol{\nu}) \,. \tag{60}$$

This formula is the basis for estimating the tunnelling probability or rather, transition probability between a one-particle state inside, and another one outside of the outer trapping horizon $\mathcal{H}$ in the scaling limit. To this end, we choose some $E_0 > 0$, and define, for small $a > 0$, $\hat{\eta}_a(E) = 1$ if $|E - E_0| < a$, and $\hat{\eta}_a(E) = 0$ otherwise (i.e. $\hat{\eta}_a$ is the characteristic function of an interval of width $2a$ around E_0). We furthermore choose any nonzero, real, integrable, bounded function b on S_* and define

$$\hat{h}_a(E, \boldsymbol{\nu}) = \frac{1}{\sqrt{2a}} \hat{\eta}_a(E) b(\boldsymbol{\nu}) \,, \qquad \hat{\varphi}_a(E, \boldsymbol{\nu}) = \frac{\hat{h}(E, \boldsymbol{\nu})}{||h_a||_{(\Lambda)}} \tag{61}$$

where

$$||h_a||^2_{(\Lambda)} = \Lambda(h_a, h_a) = 2r_*^2 \int \frac{|\hat{h}_a(E, \boldsymbol{\nu})|^2}{1 - \mathrm{e}^{-\beta E}} E \, dE \, d\Omega^2(\boldsymbol{\nu}) \,, \quad \beta = 2\pi/\kappa_* \,, \tag{62}$$

defines the one-particle norm on $\mathcal{D}_{S_*}$ which is evidently finite for any $\hat{h}_a$. Therefore, h_a (the inverse Fourier transform of $\hat{h}_a$) defines an element in $\overline{\mathcal{D}_{S_*}}^{(\Lambda)}$, the completion of $\mathcal{D}_{S_*}$ with respect to the norm $||\,.\,||_{(\Lambda)}$, supported on $U < 0$, the outside of $\mathcal{H}$. Consequently, φ_a, the inverse Fourier transform of $\hat{\varphi}_a$, is an element of $\overline{\mathcal{D}_{S_*}}^{(\Lambda)}$ which is supported on $U < 0$ and which is normalized, $||\varphi_a||_{(\Lambda)} = 1$. An a-parametrized family φ'_a of elements in $\overline{\mathcal{D}_{S_*}}^{(\Lambda)}$ with $||\varphi'_a||_{(\Lambda)} = 1$, but supported on $U > 0$, is defined in complete analogy. We note that for each a, there is a sequence $\varphi_a^{(n)} \in \mathcal{D}_{S_*}$ ($n \in \mathbb{N}$) supported on $U < 0$ with $||\varphi_a - \varphi_a^{(n)}||_{(\Lambda)} \underset{n\to\infty}{\longrightarrow} 0$. The same holds for primed counterparts of the functions involved, supported on $U > 0$.

It also follows from the properties of the GNS representations of $\mathcal{W}$ for quasifree states that, defining "one-particle vectors" $\psi[\varphi_a^{(n)}] = -i \ln(\pi_\Lambda(W(\varphi_a^{(n)})))\Omega_\Lambda$ in $\mathscr{H}_\Lambda$,

$$||\psi[\varphi_a^{(n)}] - \psi[\varphi_a^{(m)}]||_{\mathscr{H}_\Lambda} = ||\varphi_a^{(n)} - \varphi_a^{(m)}||_{(\Lambda)} \tag{63}$$

holds. Therefore, the one-particle vectors $\psi[\varphi_a^{(n)}]$ form a Cauchy sequence, converging for any fixed a to a unit vector, denoted by $\psi[\varphi_a]$, in $\mathscr{H}_\Lambda$.

We may now insert the expressions for h_a and the analogously defined h'_a into (58) and (60). Making use of the fact that $|\hat{h}_a|^2$, $|\hat{h}'_a|^2$ and $\overline{\hat{h}_a}\hat{h}'_a$ are delta-sequences with respect to E peaked at E_0 as $a \to 0$, one finds in the limit $a \to 0$ for the transition probability

$$\lim_{a\to 0} |\langle \psi[\varphi_a], \psi[\varphi'_a]\rangle|^2 = \lim_{a\to 0} \left|\Lambda(\varphi_a, \varphi'_a)\right|^2 = \frac{1}{4}\left(\frac{1-\mathrm{e}^{-\beta E_0}}{\sinh(\beta E_0/2)}\right)^2 = \mathrm{e}^{-\beta E_0}. \tag{64}$$

For large enough values of βE_0, this approaches the Boltzmann thermal distribution since, if e.g. $\beta E_0 \geq \ln(2)$, then

$$\left|\frac{\mathrm{e}^{-\beta E_0}}{1-\mathrm{e}^{-\beta E_0}} - \mathrm{e}^{-\beta E_0}\right| \leq 2\mathrm{e}^{-2\beta E_0} \tag{65}$$

showing that in the limit of large E_0, the transition probability becomes exponentially suppressed as is characteristic of a thermal occupation of energy levels at inverse temperature $\beta = 2\pi/\kappa_*$. This observation is in agreement with [16,17,27].

5.4 Coherent states of the scaling limit theory and their relative entropy

Given the state ω_Λ, its associated *coherent states* are of the form

$$\omega_\varphi(A) = \omega_\Lambda(W(\varphi)^* A W(\varphi)) \quad (A \in \mathcal{W}(\mathcal{D}_{S_*}, \varsigma)) . \tag{66}$$

This definition applies, in the first place, for all $\varphi \in \mathcal{D}_{S_*}$, but it is easy to see that it may be extended to all functions which lie in the completion $\overline{\mathcal{D}_{S_*}}^{(\Lambda)}$ of $\mathcal{D}_{S_*}$ with respect to the norm $||\varphi||_{(\Lambda)} = \Lambda(\varphi,\varphi)^{1/2}$. It is also easy to check that $\overline{\mathcal{D}_{S_*}}^{(\Lambda)}$ contains, e.g. $C_0^\infty(\mathbb{R},\mathbb{R}) \otimes L^2_{\mathbb{R}}(S_*, d\Omega^2)$ (algebraic tensor product without completion). There is a particular feature shared by all coherent states: They are *completely uncorrelated with respect to the spatial (i.e. spherical) degrees of freedom*. This means if there are finitely many subsets $G_j = I_j \times \Sigma_j$ $(j = 1, \ldots, n;\ n \geq 2)$ where the I_j are real open intervals (admitting the full real line) and the Σ_j are open subsets of $S_* \simeq S^2$ which are pairwise disjoint, $\Sigma_j \cap \Sigma_k = \emptyset$ if $j \neq k$, then

$$\omega_\varphi(A_1 A_2 \cdots A_n) = \omega_\varphi(A_1) \cdot \omega_\varphi(A_2) \cdots \omega_\varphi(A_n) \tag{67}$$

holds for all $A_j \in \mathcal{W}(G_j)$. This relation generalizes to the case that $A_j \in \mathcal{N}(G_j)$, on extending ω_Λ in the GNS representation to $\mathcal{B}(\mathscr{H}_\Lambda)$ as $\omega_\Lambda(B) = \langle \Omega_\Lambda, B\Omega_\Lambda\rangle$ $(B \in \mathcal{B}(\mathscr{H}_\Lambda))$. Note that ω_Λ itself is a coherent state (corresponding to $\varphi = 0$).

For coherent states, the *relative entropy* can be easily calculated. Without going into full details at his point, the relative entropy of a faithful, normal state on a von Neumann algebra with respect to another faithful, normal state was introduced by Araki [1] (see also [58]). It is a concept with an information theoretic background, see e.g. [18,45] for further discussion. If ω_φ is any coherent state on $\mathcal{W}(\mathcal{D}_{S_*}, \varsigma)$ as just described, then in the GNS representation $(\mathscr{H}_\Lambda, \pi_\Lambda, \Omega_\Lambda)$ it is induced by the unit vector $\Omega_\varphi = \pi_\Lambda(W(\varphi))\Omega_\Lambda$. If φ is compactly supported in $(-\infty, 0) \times S_*$ so that

$\pi_\Lambda(W(\varphi))$ is contained in $\mathcal{N}_R$, then it is not difficult to see that Ω_φ is a standard vector for $\mathcal{N}_R$, meaning that Ω_φ is cyclic and separating for $\mathcal{N}_R$. In this case, the definition of relative entropy in the sense of Araki applies for any pair of coherent states. In particular, the relative entropy of ω_φ with respect to ω_Λ on $\mathcal{N}_R$ is given as [32,41]

$$S(\omega_\Lambda|\omega_\varphi) = i \left.\frac{d}{dt}\right|_{t=0} \langle \Omega_\varphi, \Delta_R^{it}\Omega_\varphi\rangle \tag{68}$$

where Δ_R is, as above, the modular operator with respect to $\mathcal{N}_R$ and Ω_Λ.

To calculate $S(\omega_\Lambda|\omega_\varphi)$ in the case at hand (cf. again [32,41] for similar calculations), we use (55) to obtain

$$S(\omega_\Lambda|\omega_\varphi) = i \left.\frac{d}{dt}\right|_{t=0} \langle \Omega_\varphi, \Delta_R^{it}\Omega_\varphi\rangle = i \left.\frac{d}{dt}\right|_{t=0} \langle \Omega_\varphi, \Omega_{\varphi^t}\rangle \tag{69}$$

where

$$\varphi^t(U, \boldsymbol{\nu}) = (S_{2\pi t/\kappa_*}\varphi)(U, \boldsymbol{\nu}) = \varphi(\mathrm{e}^{2\pi t}U, \boldsymbol{\nu})\,. \tag{70}$$

Then, we observe

$$\begin{aligned}\langle \Omega_\varphi, \Omega_{\varphi^t}\rangle &= \omega_\Lambda(W(-\varphi)W(\varphi^t)) = \mathrm{e}^{\frac{i}{2}\varsigma(\varphi,\varphi^t)}\omega_\Lambda(W(\varphi^t - \varphi)) \\ &= \mathrm{e}^{\frac{i}{2}\varsigma(\varphi,\varphi^t)}\mathrm{e}^{-\Lambda(\varphi^t-\varphi,\varphi^t-\varphi)/2}\end{aligned} \tag{71}$$

Now we note that

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \Lambda(\varphi^t - \varphi, \varphi^t - \varphi) &= \left.\left(\Lambda(\frac{d}{dt}(\varphi^t - \varphi), \varphi^t - \varphi) + \Lambda(\varphi^t - \varphi, \frac{d}{dt}(\varphi^t - \varphi))\right)\right|_{t=0} \\ &= 0\end{aligned} \tag{72}$$

since $\varphi^t|_{t=0} = \varphi$. Hence, we find

$$S(\omega_\Lambda|\omega_\varphi) = \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} \varsigma(\varphi^t, \varphi) = -2\pi r_*^2 \int_{(-\infty,0)\times S^2} U(\partial_U\varphi)^2(U, \boldsymbol{\nu})\, dU\, d\Omega^2(\boldsymbol{\nu}). \tag{73}$$

In order to relate this entropy with the energy content of the coherent state measured by an observer moving along the Kodama flow, we rewrite the relative entropy formula with respect to the coordinate (56). We then obtain

$$S(\omega_\Lambda|\omega_\varphi) = \beta\mathcal{E}_\varphi \tag{74}$$

where

$$\mathcal{E}_\varphi = r_*^2 \int_{\mathbb{R}\times S^2} (\partial_u\varphi)^2(u, \boldsymbol{\nu})\, du\, d\Omega^2(\boldsymbol{\nu}) \tag{75}$$

is the energy content of the coherent state ω_φ measured by the Kodama observer and $\beta = \frac{2\pi}{\kappa}$ is the inverse temperature of the KMS state ω_Λ (cf. Sect. 6.4 in [37]).

5.5 Relative entropy is proportional to outer trapping horizon surface area

The previous equality (73) establishes a proportionality between the relative entropy of coherent states of the scaling limit theory and the cross section S_* of the outer trapping horizon, having the geometrical area $4\pi r_*^2$, with respect to which the scaling limit and the restriction to $\mathcal{T}_*$ of the quantized scalar field on the ambient spacetime are taken. This is justified if, for different such cross sections, say S_{*1} and S_{*2}, with respective radii r_{*1} and r_{*2}, the associated coherent states ω_Λ and ω_φ are identified. This is certainly very natural for the scaling limit state ω_Λ, but for ω_φ that may, at first sight, not appear compelling. Let us therefore provide further motivation why the proportionality between the relative entropy of coherent states and the surface area of the cross section of the outer trapping horizon at which the scaling limit theory is considered arises naturally. The key point lies in the fact that the coherent states in the scaling limit theory are completely correlation-free across spatial (i.e. spherical) separation as expressed in (67), together with the additivity of the relative entropy with respect to correlation-free states.

To discuss this in more detail, fix a horizon cross section S_* with radius r_* and consider the corresponding scaling limit Weyl-algebra $\mathcal{W}(\mathcal{D}_{S_*}, \varsigma)$ with the scaling limit state ω_Λ, its GNS representation $(\mathscr{H}_\Lambda, \pi_\Lambda, \Omega_\Lambda)$ and the von Neumann algebras $\mathcal{N}(G)$ for open subsets G of $\mathbb{R} \times S_*$ as introduced in Sect. 5.1. Specifically, for open subsets Σ of $S_* \simeq S^2$, we define the von Neumann algebras

$$\mathcal{N}_R(\Sigma) = \mathcal{N}((-\infty, 0) \times \Sigma). \tag{76}$$

We recall that $\mathcal{N}_R(\Sigma)$ is the von Neumann algebra contained in $\mathcal{B}(\mathscr{H}_\Lambda)$ generated by the $\pi_\Lambda(W(\varphi))$ where $\mathrm{supp}(\varphi) \subset (-\infty, 0)$. Hence, on account of (55), the $\mathcal{N}_R(\Sigma)$ are invariant under the adjoint action of the modular group Δ_R^{it} ($t \in \mathbb{R}$) with respect to $\mathcal{N}_R$ and Ω_Λ.

When we denote by $\omega_{\varphi,\Sigma}$ the state on $\mathcal{N}_R(\Sigma)$ given by

$$A \mapsto \omega_{\varphi,\Sigma}(A) = \langle \Omega_\varphi, A\Omega_\varphi \rangle \quad (A \in \mathcal{N}_R(\Sigma)), \tag{77}$$

i.e. the restriction of the coherent state ω_φ defined previously to $\mathcal{N}_R(\Sigma)$, and if likewise the restriction of ω_Λ to $\mathcal{N}_R(\Sigma)$ is denoted by $\omega_{\Lambda,\Sigma}$, then we find for the relative entropy in the same way as before,

$$S(\omega_{\Lambda,\Sigma}|\omega_{\varphi,\Sigma}) = -2\pi r_*^2 \int_{(-\infty,0)\times S^2} U(\partial_U\varphi)^2(U, \boldsymbol{\nu})\, dU\, d\Omega^2(\boldsymbol{\nu}) = S(\omega_\Lambda|\omega_\varphi). \tag{78}$$

Then, if Σ_1 and Σ_2 are any two *disjoint* open subsets of S^2, and if $\text{supp}(\varphi_j) \subset (-\infty, 0) \times \Sigma_j$ $(j = 1, 2)$, and setting $\varphi_{12} = \varphi_1 + \varphi_2$, one finds

$$\begin{aligned}
S(\omega_{\Lambda,\Sigma_1\cup\Sigma_2}|\omega_{\varphi_{12},\Sigma_1\cup\Sigma_2}) &= S(\omega_\Lambda|\omega_{\varphi_{12}}) \\
&= -2\pi r_*^2 \int_{(-\infty,0)\times S^2} U\,(\partial_U\varphi_{12})^2(U,\boldsymbol{\nu})\,dU\,d\Omega^2(\boldsymbol{\nu}) \\
&= -2\pi r_*^2 \int_{(-\infty,0)\times S^2} U\left[(\partial_U\varphi_1)^2(U,\boldsymbol{\nu}) + (\partial_U\varphi_2)^2(U,\boldsymbol{\nu})\right] dU\,d\Omega^2(\boldsymbol{\nu}) \\
&= S(\omega_\Lambda|\omega_{\varphi_1}) + S(\omega_\Lambda|\omega_{\varphi_2}) \\
&= S(\omega_{\Lambda,\Sigma_1}|\omega_{\varphi_1,\Sigma_1}) + S(\omega_{\Lambda,\Sigma_2}|\omega_{\varphi_2,\Sigma_2})
\end{aligned} \tag{79}$$

where we passed from the 3rd equality to the 4th since φ_1 and φ_2 are assumed to have disjoint $\boldsymbol{\nu}$-supports. This shows that the relative entropy of coherent states in any scaling limit is *additive with respect to angular separation*; actually, a corresponding additivity of the relative entropy across angular separation holds upon replacing the two open, disjoint subsets Σ_1 and Σ_2 of S^2 by finitely many $\Sigma_1, \ldots, \Sigma_N$, and similarly φ_1 and φ_2 by finitely many $\varphi_1, \ldots, \varphi_N$ with $\text{supp}(\varphi_j) \subset (-\infty) \times \Sigma_j$.

In fact, this can be seen to be, more generally, a consequence of the fact that the coherent states in the scaling limit are correlation-free across angular separation and the additivity of the relative entropy of correlation-free states. One can show that there is a joint unitary equivalence $\omega_{\varphi_{12},\Sigma_1\cup\Sigma_2} \simeq \omega_{\varphi_1,\Sigma_1} \otimes \omega_{\varphi_2,\Sigma_2}$ and $\omega_{\Lambda,\Sigma_1\cup\Sigma_2} \simeq \omega_{\Lambda,\Sigma_1} \otimes \omega_{\Lambda,\Sigma_2}$, where the *correlation-free product state* $\omega_{\varphi_1,\Sigma_1} \otimes \omega_{\varphi_2,\Sigma_2}$ is the state defined on $\mathcal{N}_R(\Sigma_1) \otimes \mathcal{N}_R(\Sigma_2)$ by linear extension of

$$A_1 \otimes A_2 \mapsto \omega_{\varphi_1,\Sigma_1} \otimes \omega_{\varphi_2,\Sigma_2}(A_1 \otimes A_2) = \omega_{\varphi_1,\Sigma_1}(A_1) \cdot \omega_{\varphi_2,\Sigma_2}(A_2)\,. \tag{80}$$

For (faithful, normal) correlation-free product states, the equation

$$S(\omega_{\Lambda,\Sigma_1} \otimes \omega_{\Lambda,\Sigma_2}|\omega_{\varphi_1,\Sigma_1} \otimes \omega_{\varphi_2,\Sigma_2}) = S(\omega_{\Lambda,\Sigma_1}|\omega_{\varphi_1,\Sigma_1}) + S(\omega_{\Lambda,\Sigma_2}|\omega_{\varphi_1,\Sigma_2}) \tag{81}$$

holds (cf. [45], eq. (5.22)), whereupon one may conclude that

$$S(\omega_{\Lambda,\Sigma_1\cup\Sigma_2}|\omega_{\varphi_{12},\Sigma_1\cup\Sigma_2}) = S(\omega_{\Lambda,\Sigma_1}|\omega_{\varphi_1,\Sigma_1}) + S(\omega_{\Lambda,\Sigma_2}|\omega_{\varphi_1,\Sigma_2}) \tag{82}$$

obtains.

Therefore, the scaling of the relative entropy of coherent states proportional to the geometric area of the horizon cross section arises naturally. This is seen particularly clearly when considering coherent states corresponding to elements $\varphi \in \overline{\mathcal{D}_{S_*}}^{(\Lambda)}$ which are of the form $\varphi = h \odot \chi_\Sigma$ where

$$(h \odot \chi_\Sigma)(U, \boldsymbol{\nu}) = h(U) \cdot \chi_\Sigma(\boldsymbol{\nu}) \quad (U \in (-\infty, 0)\,,\ \boldsymbol{\nu} \in S^2) \tag{83}$$

with $h \in C_0^\infty((-\infty, 0), \mathbb{R})$ and χ_Σ the characteristic function of an open, or more generally, measurable subset Σ of S^2. In this case,

$$\begin{aligned} S(\omega_\Lambda|\omega_{h\odot\chi_\Sigma}) &= -2\pi \int_{-\infty}^{0} U(\partial_U h)^2(U)\, dU \cdot r_*^2 \int_{\Sigma\subset S^2} d\Omega^2(\boldsymbol{v}) \\ &= -2\pi \int_{-\infty}^{0} U(\partial_U h)^2(U)\, dU \cdot \mathcal{A}(\Sigma_{r_*} \subset S_*) \end{aligned} \tag{84}$$

where $\mathcal{A}(\Sigma_{r_*} \subset S_*)$ is the geometrical surface of Σ viewed as subset of the horizon cross section S_* which is a copy of the 2-dimensional sphere with radius r_*, i.e. the surface of Σ as subset of S^2, scaled by r_*^2.

In the light of these observations, it is entirely natural to identify if $\Sigma = S^2$, the coherent states $\omega_{h\odot 1}$ for different horizon cross sections S_* with different radii r_*, which renders a proportionality of the relative entropies with the horizon cross-sectional area $\mathcal{A}(S_*)$,

$$S(\omega_\Lambda|\omega_{h\odot 1}) = -2\pi \int_{-\infty}^{0} U(\partial_U h)^2(U)\, dU \cdot \mathcal{A}(S_*) \tag{85}$$

for the coherent states of the said type, when considering the scaling limit theory taken at S_*, arising from any Hadamard state of the quantum field theory on the underlying spherically symmetric spacetime with an outer trapping horizon.

Remark Without the factor 2 in the definition of the scaling transformations u_λ, one would obtain that the relative entropy $S(\omega_\Lambda|\omega_{h\odot 1})$ equals one quarter of the horizon cross-sectional area times $-2\pi \int_{-\infty}^{0} U(\partial_U h)^2(U)\, dU$, where the latter is the relative entropy of the coherent state induced by h of the free chiral conformal quantum field theory defined on the real line with the vacuum two-point function

$$\Lambda_{(1)}(h, h') = \lim_{\varepsilon\to 0+} -\frac{1}{\pi} \int \frac{h(U)h'(U')}{(U - U' + i\varepsilon)^2} dU\, dU' \quad (h, h' \in C_0^\infty(\mathbb{R}, \mathbb{R})) \tag{86}$$

This is in close analogy to the classical derivation where black hole entropy is equated to one quarter of the cross-sectional horizon area. Yet, it should be borne in mind that it refers not to the entropy of the outer trapping horizon itself but to quantities of a quantum field theory arising in the scaling limit towards a spherical cross section of the outer trapping horizon. Therefore, the value of the relative entropy depends on the states chosen and also on the field content of the initially considered quantum field theory. Nevertheless, regardless of such choices, there is a characteristic scaling of that relative entropy proportional to (one quarter of) the cross-sectional area of the outer trapping horizon with respect to which the scaling limit is considered.

6 Conclusion

In this paper, we have investigated the scaling limits of Hadamard 2-point functions on the lightlike submanifold $\mathcal{T}_*$ of a spherically symmetric outer trapping horizon generated by lightlike geodesics traversing the outer trapping horizon. The scaling limit 2-point function Λ was found to have a universal form, independent of which Hadamard 2-point function of the quantum field theory on the underlying spherically symmetric spacetime is initially chosen. The projected Kodama flow acts in the scaling limit like a dilation, and the scaling limit 2-point function Λ shows a thermal spectrum with respect to the projected Kodama flow at inverse temperature $\beta = 2\pi/\kappa_*$ where κ_* is the surface gravity of the horizon cross section S_* where the lightlike generators of $\mathcal{T}_*$ traverse the outer trapping horizon. Consequently, one can derive a tunnelling probability in the scaling limit for Fourier modes peaked at Fourier energy E_0 with respect to the Kodama time behaving like $\mathrm{e}^{-\beta E_0}$ for large E_0, analogous to a thermal distribution of energy modes. These results are in agreement with earlier, related results for stationary black horizons or bifurcate Killing horizons, in particular [43] (see also [16,17,37]), and also with the first law of non-stationary black hole dynamics discussed by Hayward [29], $\mathcal{M}' = \frac{\kappa}{8\pi}\mathcal{A}' + w\mathcal{V}'$ mentioned in Introduction.

Furthermore, the scaling limit 2-point function Λ defines a quantum field theory on each $\mathcal{T}_*$, the scaling limit theory, determined by the horizon cross section S_*. The thermal Fourier spectrum with respect to the Kodama time in the scaling limit is equivalent to the KMS property of the scaling limit state ω_Λ induced by Λ when restricted to observables localized on the part of $\mathcal{T}_*$ lying either inside or outside of the outer trapping horizon. Furthermore, the state ω_Λ as well as all the coherent states ω_φ in the scaling limit theory are correlation-free product states with respect to separation in the angular coordinate $\boldsymbol{\nu}$ of S_*, and we have seen that this leads naturally to a proportionality of the relative entropy $S(\omega_\Lambda|\omega_\varphi)$ with $4\pi r_*^2$, the area of the cross section S_* defining the scaling limit theory. Again, this is in keeping with the classical theory of black hole thermodynamics [3,4,29]. We emphasize that this is a consequence of our scaling limit analysis and seems to be the first such result in the setting of quantum field theory in curved spacetime (apart from the related arguments of [33]).

We should remark that our scaling limit consideration is akin to an adiabatic limit in the sense that effectively, in the scaling limit all processes or dynamical changes at finite time scales are scaled away. In this sense, our concepts of inverse temperature and of relative entropy in the scaling limit theory are not dynamical, and that is a considerable limitation of our approach. The entropy concept, in our the scaling limit, bears some similarity to that in Bekenstein's early article [4] on the subject: When an object (e.g. a table, a chair or a tankard) traverses the horizon, then the information about the object is lost outside of the horizon. In [4], the example of beams of light entering a black hole horizon is used. Our scaling limit theory can be seen as a bunch of elementary theories for such beams of light, namely, a free chiral conformal field theory, one for each point on S_*. As the area of S_* is increased, for example, it accommodates for more such ingoing light beams as measured by the area, and correspondingly, a larger amount of information along "elementary light beams" passing the horizon through S_* can be absorbed, which corresponds to the scaling of entropy—a measure for the lost amount of information—proportional to the area of S_*. See also the article [10] which can be regarded as a quantum version of Bekenstein's attribution of entropy to beams of light.

We think that similar results can also be obtained for other types of horizons, like cosmological horizons [12] or isolated horizons. A greater challenge is to attempt to obtain a more dynamical concept of temperature and entropy for dynamical black hole horizons in the setting of quantum field theory in curved spacetimes and semiclassical gravity, in the spirit of the approach of [33] which takes dynamical metric perturbations around a static Schwarzschild black hole horizon into account (see e.g. the recent work [13]). It would also be of interest to see if the notions of temperature and entropy in the context of our semiclassical approach to the temperature and entropy of black hole horizons can be linked to more "holographic" entropy concepts [35].

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Appendix

In Appendix, we present the proof of Theorem 4.1. The proof will be facilitated by the following auxiliary result.

Lemma 7.1 *Let* $(\varrho_\alpha)_{0<\alpha_k\leq 1}$ $(k = 1, \dots, n)$ *be a family of measurable functions* $\varrho_\alpha : \mathbb{R}^m \to \mathbb{C}$*, indexed by* $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$*, so that for any compact subset* C *of* $\mathbb{R}^m$ *there is some* $0 < a_0 \leq 1$ *with*

$$\sup_{0<\alpha_k\leq a_0} \sup_{y\in C} |\varrho_\alpha(y)| \leq \frac{1}{2} \tag{87}$$

Furthermore, let $(j_\alpha)_{0<|\lambda|\leq 1}$ *be a family of continuous functions* $j_\alpha : \mathbb{R}^m \to \mathbb{C}$*, with the properties:*

(i) $\text{supp}(j_\alpha) \subset J^m$ *with some fixed compact real interval* J*;*
(ii) $|j_\alpha(y)| \leq b$ *for all* α *and all* $y \in J^m$ *with a fixed finite constant* $b > 0$*.*

Then, the following statements hold.

(A) *There are some* $0 < a_0 \leq 1$ *and a finite positive constant* κ *so that, if* $0 < \delta \leq \frac{1}{2}$*, it holds that (writing* $y = (y_1, \underline{y})$ *and* $d^m y = dy_1\, d^{m-1}\underline{y}$*)*

$$\sup_{0<\alpha_k\leq a_0} \int_{\mathbb{R}^{m-1}} \int_0^\delta \left| \ln |\varrho_\alpha(y) + y_1^2| \right| \, |j_\alpha(y)|\, dy_1 d^{m-1}\underline{y} \;\leq\; \kappa\delta^{1/3}\,. \tag{88}$$

(B) *There are some* $0 < a_0 \leq 1$ *and a finite constant* κ *so that if* $\delta \leq \frac{1}{2}$*, it holds that (writing* $y = (y_0, y_1, \underline{y})$ *and* $d^m y = dy_0 dy_1 d^{m-2}\underline{y}$*))*

$$\sup_{0<\alpha_k\leq a_0} \int_{\mathbb{R}^{m-2}} \int_{|y_0^2 - y_1^2|<\delta^2} \left| \ln |\varrho_\alpha(y) + y_0^2 - y_1^2| \right| \, |j_\alpha(y)|\, dy_0\, dy_1 d^{m-2}\underline{y} \;\leq\; \kappa\delta^{1/3}\,. \tag{89}$$

Proof of Lemma 7.1 Part **(A)**
Making use of the integration coordinate substitution $z = y_1^2$, thus $dz = 2y_1 dy_1$,

$$\begin{aligned} &\int_{\mathbb{R}^{m-1}} \int_0^\delta \left| \ln |\varrho_\alpha(y) + y_1^2|\, j_\alpha(y) \right| \, d^m y \\ &\quad = \int_{\mathbb{R}^{m-1}} \int_0^{\delta^2} \left| \ln |\varrho_\alpha(\sqrt{z}, \underline{y}) + z| \frac{j_\alpha(\sqrt{z}, \underline{y})}{2\sqrt{z}} \right| \, dz\, d^{m-1}\underline{y}\,. \end{aligned} \tag{90}$$

Hölder's integral inequality with $p = 3, q = 3/2$ (so that $1/p + 1/q = 1$) with respect to the z-integration yields

$$\begin{aligned} &\int_{\mathbb{R}^{m-1}} \int_0^{\delta^2} \left| \ln |\varrho_\alpha(\sqrt{z}, \underline{y}) + z| \frac{j_\alpha(\sqrt{z}, \underline{y})}{2\sqrt{z}} \right| \, dz\, d^{m-1}\underline{y} \\ &\quad \leq \frac{1}{2} \int_{\mathbb{R}^{m-1}} \left[\int_0^{\delta^2} |\, \ln |\varrho_\alpha(\sqrt{z}, \underline{y}) + z|\, |^3\, dz \right]^{1/3} \\ &\qquad \times \left[\int_0^{\delta^2} z^{-3/4} |j_\alpha(\sqrt{z}, \underline{y})|^{3/2}\, dz \right]^{2/3} d^{m-1}\underline{y} \end{aligned} \tag{91}$$

Choosing $0 < a_0 \leq 1$ so that

$$\sup_{0<\alpha_k\leq a_0} \sup_{y\in J^m} |\varrho_\alpha(y)| \leq \frac{1}{2}\,, \tag{92}$$

then for $0 < \alpha_k \leq a_0$, the last integral can be estimated by

$$\sup_{0<|\varrho|\leq 1/2} \frac{1}{2} \left[\int_0^{\delta^2} |\, \ln |\varrho + z|\, |^3\, dz \right]^{1/3} \left[\int_0^{\delta^2} z^{-3/4} b^{3/2}\, dz \right]^{2/3} |J|^{m-1} \tag{93}$$

where $|J|$ is the length of the interval J.

We observe that since $|\varrho| \leq 1/2$ and $0 < z \leq \delta^2$ with $\delta \leq 1/2$, we obtain $|\mathrm{Re}(\varrho)+z|^2+|\mathrm{Im}(\varrho)|^2 = |\varrho+z|^2 < 1$. Consequently, under the integral, $|\ \ln|\varrho+z|\ | \leq |\ \ln|\mathrm{Re}(\varrho)+z|\ |$. This results in

$$\sup_{0<|\varrho|\leq 1/2}\left[\int_0^{\delta^2} |\ \ln|\varrho+z|\ |^3\, dz\right]^{1/3} \leq \sup_{0<|\varrho|\leq 1/2}\left[\int_0^{\delta^2} |\ \ln|\mathrm{Re}(\varrho)+z|\ |^3\, dz\right]^{1/3}$$
$$= \sup_{0<|r|\leq 1/2}\left[\int_{-r}^{\delta^2-r} |\ \ln|z|\ |^3\, dz\right]^{1/3} \leq K \tag{94}$$

with a finite, positive real constant K. On the other hand, we obtain

$$\left[\int_0^{\delta^2} z^{-3/4}\, dz\right]^{2/3} = 4^{2/3}\delta^{1/3}. \tag{95}$$

Putting all the previous steps together, we find

$$\sup_{0<\alpha_k\leq a_0} \int_{\mathbb{R}^{m-1}}\int_0^{\delta} \left|\ln|\varrho_\alpha(y)+y_1^2|\right|\ |j_\alpha(y)|\, dy_1 d^{m-1}\underline{y} \leq \frac{4^{2/3}}{2} b|J|^{m-1}K\delta^{1/3}. \tag{96}$$

This proves the statement of Part **(A)**, with $\kappa = \frac{4^{2/3}}{2} b|J|^{m-1}K$.

Part **(B)**

First we note that the set $|y_0^2 - y_1^2| < \delta^2$ in the y_0-y_1-plane can be split into the four parts

$$H_1(\delta) = \left\{|y_1| \leq y_0 < \sqrt{y_1^2+\delta^2}\right\}, \quad H_2(\delta) = \left\{-\sqrt{y_1^2+\delta^2} < y_0 \leq -|y_1|\right\} \tag{97}$$

$$H_3(\delta) = \left\{|y_0| \leq y_1 < \sqrt{y_0^2+\delta^2}\right\}, \quad H_4(\delta) = \left\{-\sqrt{y_0^2+\delta^2} < y_1 \leq -|y_0|\right\} \tag{98}$$

The sets overlap only at their boundaries, $y_0 \pm y_1 = 0$. Therefore,

$$\int_{\mathbb{R}^{m-2}}\int_{|y_0^2-y_1^2|<\delta^2} \left|\ln|\varrho_\alpha(y)+y_0^2-y_1^2|\right|\ |j_\alpha(y)|\, dy_0\, dy_1\, d^{m-2}\underline{y}$$
$$= \sum_{\ell=1}^{4}\int_{\mathbb{R}^{m-2}}\int_{H_\ell(\delta)} \left|\ln|\varrho_\alpha(y)+y_0^2-y_1^2|\right|\ |j_\alpha(y)|\, dy_0\, dy_1\, d^{m-2}\underline{y} \tag{99}$$

The integrals involving the $H_\ell(\delta)$ all have a very similar structure, and thus, it suffices to show that, e.g.

$$\sup_{0<\alpha_k\le a_0}\int_{\mathbb{R}^{m-2}}\int_{H_1(\delta)}\left|\ln|\varrho_\alpha(y)+y_0^2-y_1^2|\right|\,|j_\alpha(y)|\,dy_0\,dy_1\,d^{m-2}\underline{y}\ \le\ \kappa_1\delta^{1/3} \tag{100}$$

since similar estimates for the other $H_\ell(\delta)$ can be deduced by analogous arguments. Carrying out a substitution $z = y_0^2$ followed by a Hölder-type integral inequality similarly as in the proof of Part **(A)**, we find, on making a_0 small enough so that (92) holds, for all $0 < \alpha_k \le a_0$,

$$\begin{aligned}
&\int_{\mathbb{R}^{m-2}}\int_{H_1(\delta)}\left|\ln|\varrho_\alpha(y)+y_0^2-y_1^2|\right|\,|j_\alpha(y)|\,dy_0\,dy_1\,d^{m-2}\underline{y}\\
&=\int_{\mathbb{R}^{m-2}}\int_J\int_{|y_1|}^{\sqrt{y_1^2+\delta^2}}\left|\ln|\varrho_\alpha(y)+y_0^2-y_1^2|\right|\,|j_\alpha(y)|\,dy_0\,dy_1\,d^{m-2}\underline{y}\\
&\le\frac{1}{2}\int_{\mathbb{R}^{m-2}}\int_J\left[\int_{y_1^2}^{y_1^2+\delta^2}\left|\ln|\varrho_\alpha(\sqrt{z},y_1,\underline{y})+z-y_1^2|\right|^3dz\right]^{1/3} \qquad (101)\\
&\quad\times\left[\int_{y_1^2}^{y_1^2+\delta^2}\frac{|j_\alpha(\sqrt{z},y_1,\underline{y})|^{3/2}}{z^{3/4}}\,dz\right]^{2/3}dy_1\,d^{m-2}\underline{y}\\
&\le\frac{1}{2}\sup_{|\varrho|<1/2}\left[\int_0^{\delta^2}|\ln|\varrho+z||^3\,dz\right]^{1/3}b|J|^{m-1}\sup_{y_1\in J}\left[\int_{y_1^2}^{y_1^2+\delta^2}z^{-3/4}\,dz\right]^{2/3}
\end{aligned} \tag{102}$$

where an obvious substitution of z by $z - y_1^2$ has been carried out in the integral involving the logarithm. It is easy to check that

$$\sup_{y_1\in J}\left[\int_{y_1^2}^{y_1^2+\delta^2}z^{-3/4}\,dz\right]^{2/3}\le\left[\int_0^{\delta^2}z^{-3/4}\,dz\right]^{2/3} \tag{103}$$

and therefore, we see that the integral expression in (102) can be estimated by

$$\frac{4^{2/3}}{2}b|J|^{m-1}K\delta^{1/3} \tag{104}$$

just as in the proof of Part **(A)**. This concludes the proof of Part **(B)** □

Proof of Theorem 4.1 It will be convenient to introduce the following abbreviations, referring to adapted coordinates $(U, V, \boldsymbol{\nu})$ near the chosen S_*:

$$x = (U, V, \boldsymbol{\nu}), \quad x' = (U', V', \boldsymbol{\nu}'), \quad x_\lambda = (\lambda U, V, \boldsymbol{\nu}), \quad x'_\lambda = (\lambda U', V', \boldsymbol{\nu}'), \tag{105}$$

$$dX = dX(x) = dU\, dV\, d\Omega^2(\boldsymbol{\nu}) = dU\, dV\, \sin(\vartheta) d\vartheta\, d\varphi \tag{106}$$

using $\boldsymbol{\nu} = (\vartheta, \varphi)$ in spherical angular coordinates as before; dX' is defined analogously. Other abbreviations that we will use are

$$\tilde{F}_\lambda(U, V, \boldsymbol{\nu}) = \eta^{-3}(U, V) F_\lambda(U, V, \boldsymbol{\nu}), \qquad F_\lambda(U, V, \boldsymbol{\nu}) = (u_\lambda 2\partial_U f)(U, V, \boldsymbol{\nu}) \tag{107}$$

and analogously for symbols endowed with primes.

Recalling (32), we have

$$\begin{aligned} w^{(2)}(F_\lambda, F'_\lambda) &= \tilde{w}^{(2)}(\tilde{F}_\lambda, \tilde{F}'_\lambda) \\ &= \lim_{\varepsilon \to 0+} \int \tilde{w}_\varepsilon(x, x') \tilde{F}_\lambda(x) \tilde{F}'_\lambda(x')\, d\mathrm{vol}_{\tilde{g}}(x)\, d\mathrm{vol}_{\tilde{g}}(x') \\ &= \lim_{\varepsilon \to 0+} \int \tilde{w}_\varepsilon(x, x') F_\lambda(x) F'_\lambda(x') P(U, V, U', V')\, dX\, dX' \end{aligned} \tag{108}$$

having made use of $d\mathrm{vol}_{\tilde{g}}(x) = \eta(U, V)^2 A(U, V) r_*^2\, dX$ in the adapted coordinates for S_* which, as we recall, is a copy of a 2-sphere with radius r_*. We introduce on a smooth partition of unity on $S^2 \times S^2$, consisting of two functions χ and $\chi^\perp$, as follows: Choose some $0 < \delta^2 < \pi^2/64$ and choose a non-negative C^∞ function χ, bounded by 1, on $S^2 \times S^2$ so that $\chi(\boldsymbol{\nu}, \boldsymbol{\nu}') = 1$ if $s(\boldsymbol{\nu}, \boldsymbol{\nu}')/r_*^2 \le \delta^2/2$, and $\chi(\boldsymbol{\nu}, \boldsymbol{\nu}') = 0$ if $s(\boldsymbol{\nu}, \boldsymbol{\nu}')/r_*^2 \ge \delta^2$. We then write $\chi^\perp(\boldsymbol{\nu}, \boldsymbol{\nu}') = 1 - \chi(\boldsymbol{\nu}, \boldsymbol{\nu}')$. Note that $\chi = \chi_\delta$ and $\chi^\perp = \chi_\delta^\perp$ depend on the choice of δ. With this notation, we can write

$$\tilde{w}_\varepsilon(x, x') = \tilde{w}_\varepsilon(x, x') \chi(\boldsymbol{\nu}, \boldsymbol{\nu}') + \tilde{w}_\varepsilon(x, x') \chi^\perp(\boldsymbol{\nu}, \boldsymbol{\nu}'). \tag{109}$$

In a further step, we observe that, on a change of the U and U' integration coordinates,

$$\begin{aligned} &\int \tilde{w}_\varepsilon(x, x') \chi^\perp(x, x') F_\lambda(x) F'_\lambda(x') P(U, V, U', V') r_*^2\, dX\, dX' \\ &= \int \left[\tilde{\psi}(x_\lambda, x'_\lambda) \left(\frac{1}{8\pi^2} \frac{\tilde{\Delta}^{1/2}(x_\lambda, x'_\lambda)}{\tilde{\sigma}_\varepsilon(x_\lambda, x'_\lambda)} + \ln(\tilde{\sigma}_\varepsilon(x_\lambda, x'_\lambda)) \tilde{Y}(x_\lambda, x'_\lambda) \right) + \tilde{Z}(x_\lambda, x'_\lambda) \right] \\ &\qquad \times P(\lambda U, V, \lambda U', V') \chi^\perp(\boldsymbol{\nu}, \boldsymbol{\nu}') F(x) F(x')\, dX\, dX'. \end{aligned} \tag{110}$$

In view of the particular form of the half of the squared geodesic distance (31), we have

$$\tilde{\sigma}_\varepsilon(x_\lambda, x'_\lambda) = \tilde{\sigma}^{(\mathcal{L})}(\lambda U, V, \lambda U', V') + s(\boldsymbol{\nu}, \boldsymbol{\nu}') + 2i\varepsilon t(\lambda U, V, \lambda U', V') + \varepsilon^2 \tag{111}$$

Since in the integral on the right-hand side of (110), $s(\boldsymbol{\nu}, \boldsymbol{\nu}') \geq \delta > 0$ owing to the presence of $\chi^{\perp}$, the integrand functions remain uniformly bounded in the limits as $\varepsilon \to 0$ and $\lambda \to 0$, and they converge almost everywhere to an integrable function. Therefore, one obtains

$$\begin{aligned} &\lim_{\lambda\to 0+}\lim_{\varepsilon\to 0+}\int \tilde{w}_{\varepsilon}(x,x')\chi^{\perp}(x,x')F_{\lambda}(x)F'_{\lambda}(x')P(U,V,U',V')\,dX\,dX' \\ &\quad = \int h(V,V',\boldsymbol{\nu},\boldsymbol{\nu}')\partial_U f(U,V,\boldsymbol{\nu})\partial_{U'} f'(U',V',\boldsymbol{\nu}')\,dX\,dX' = 0 \end{aligned} \tag{112}$$

for some bounded L^1 function h; the integral on the right-hand side vanishes since, after the limit $\lambda \to 0$, h is independent of U and U', and f and f' have compact support (in particular, compact support with respect to U, respectively, U'). Note that this holds no matter how small $\delta > 0$ has been chosen.

Next we notice that $s(\boldsymbol{\nu}, \boldsymbol{\nu}')$ is invariant under rotations $R \in SO(3)$, $s(R\boldsymbol{\nu}, R\boldsymbol{\nu}') = s(\boldsymbol{\nu}, \boldsymbol{\nu}')$; similarly, the surface-integration form $d\Omega^2$ is invariant under the rotations, $d\Omega^2(R\boldsymbol{\nu}) = d\Omega^2(\boldsymbol{\nu})$. Therefore, given $\boldsymbol{\nu}'$, we can regard it as obtained from a standard "north pole point" $\mathring{\boldsymbol{\nu}}'$ by a suitable rotation $R_{\boldsymbol{\nu}'} \in SO(3)$ so that $\boldsymbol{\nu}' = R_{\boldsymbol{\nu}'}\mathring{\boldsymbol{\nu}}'$, hence $s(\boldsymbol{\nu}, \boldsymbol{\nu}') = s(\boldsymbol{\nu}, R_{\boldsymbol{\nu}'}\mathring{\boldsymbol{\nu}}') = s(R_{\boldsymbol{\nu}}^{-1}\boldsymbol{\nu}, \mathring{\boldsymbol{\nu}}')$. The relation between $\boldsymbol{\nu}'$ and $R_{\boldsymbol{\nu}'}$ is bijective and smooth as long as $\boldsymbol{\nu}'$ is bounded away by a finite distance from the antipode point $-\mathring{\boldsymbol{\nu}}'$ (on identifying $\mathring{\boldsymbol{\nu}}' = (0, 0, 1) \in \mathbb{R}^3$). In the following integrals we will consider this is always the case owing to the presence of the function $\chi(\boldsymbol{\nu}, \boldsymbol{\nu}')$. Introducing the abbreviations

$$\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} = \tilde{\sigma}^{(\mathcal{L})}(\lambda U, V, \lambda U', V')\,, \quad t_{[\lambda]} = t(\lambda U, V, \lambda U', V')\,, \tag{113}$$

we thus have, for any λ-parametrized family $q_\lambda(x, x')$ of bounded, compactly supported C^1 functions, writing $R_{\boldsymbol{\nu}'}x = (U, V, R_{\boldsymbol{\nu}'}\boldsymbol{\nu})$ for $x = (U, V, \boldsymbol{\nu})$

$$\begin{aligned} \int \frac{q_\lambda(x,x')}{\tilde{\sigma}_\varepsilon(x_\lambda, x'_\lambda)}\chi(\boldsymbol{\nu},\boldsymbol{\nu}')\,dX\,dX' &= \int \frac{q_\lambda(x,x')}{\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + s(R^{-1}_{\boldsymbol{\nu}'}\boldsymbol{\nu}, \mathring{\boldsymbol{\nu}}') + 2i\varepsilon t_{[\lambda]} + \varepsilon^2}\chi(\boldsymbol{\nu},\boldsymbol{\nu}')\,dX\,dX' \\ &= \int \frac{q_\lambda(R_{\boldsymbol{\nu}'}x,x')}{\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + s(\boldsymbol{\nu}, \mathring{\boldsymbol{\nu}}') + 2i\varepsilon t_{[\lambda]} + \varepsilon^2}\chi(R_{\boldsymbol{\nu}'}\boldsymbol{\nu},\boldsymbol{\nu}')\,dX\,dX'\,. \end{aligned} \tag{114}$$

Using the standard spherical angular coordinates $(\vartheta, \varphi) = \boldsymbol{\nu}$, with $\vartheta = 0$ corresponding to the "north pole point" $= \mathring{\boldsymbol{\nu}}'$, the half of the squared geodesic distance on the sphere with radius r_* takes the simple form

$$s(\vartheta, \varphi, \mathring{\boldsymbol{\nu}}') = \frac{r_*^2\vartheta^2}{2}\,, \tag{115}$$

and we thus obtain, on writing $\xi_\lambda(x, x') = \chi(R_{\boldsymbol{\nu}'}\boldsymbol{\nu}, \boldsymbol{\nu}')q_\lambda(R_{\boldsymbol{\nu}'}x, x')$,

$$\int \frac{q_\lambda(R_{\boldsymbol{\nu}'}x, x')}{\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + s(\boldsymbol{\nu}, \mathring{\boldsymbol{\nu}}') + 2i\varepsilon t_{[\lambda]} + \varepsilon^2}\chi(R_{\boldsymbol{\nu}'}\boldsymbol{\nu}, \boldsymbol{\nu}')\, dX\, dX'$$
$$= \int\int_{\vartheta=0}^{\delta} \frac{\xi_\lambda(x, x')}{\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + r_*^2\vartheta^2/2 + 2i\varepsilon t_{[\lambda]} + \varepsilon^2}\sin(\vartheta)d\vartheta\, d\varphi\, dU\, dV\ dX' \tag{116}$$

since in the polar coordinates chosen, $r_*^2\vartheta^2/2$ is the half of the squared geodesic distance between $\mathring{\boldsymbol{\nu}}'$ and $\boldsymbol{\nu}$ on the sphere with radius r_*, and $\xi_\lambda(x, x') = 0$ if $\vartheta > \delta$ by the properties of χ. Now carrying out a partial integration with respect to ϑ and observing

$$\frac{1}{\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + r_*^2\vartheta^2/2 + 2i\varepsilon t_{[\lambda]} + \varepsilon^2} = \frac{1}{r_*^2\vartheta}\partial_\vartheta \ln(\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + r_*^2\vartheta^2/2 + 2i\varepsilon t_{[\lambda]} + \varepsilon^2)\,, \tag{117}$$

we are led to

$$\int\int_{\vartheta=0}^{\delta} \frac{\xi_\lambda(x, x')}{\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + r_*^2\vartheta^2/2 - 2i\varepsilon t_{[\lambda]} + \varepsilon^2}\sin(\vartheta)d\vartheta\, d\varphi\, dU\, dV\, dX'$$
$$= \int\left[\ln(\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + r_*^2\vartheta^2/2 + 2i\varepsilon t_{[\lambda]} + \varepsilon^2)\xi_\lambda(x, x')\frac{\operatorname{sinc}(\vartheta)}{r_*^2}\right]_{\vartheta=0}^{\delta} d\varphi\, dU\, dV\, dX' \tag{118}$$
$$-\int\int_{\vartheta=0}^{\delta} \ln(\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + r_*^2\vartheta^2/2 + 2i\varepsilon t_{[\lambda]} + \varepsilon^2)\,\partial_\vartheta\left(\xi_\lambda(x, x')\frac{\operatorname{sinc}(\vartheta)}{r_*^2}\right) d\vartheta\, d\varphi\, dU\, dV\, dX'\,. \tag{119}$$

We note that

$$\ln(\tilde{\sigma}_\varepsilon(x, x')) = \ln|\tilde{\sigma}^{(\mathcal{L})}(U, V, U', V') + s(\boldsymbol{\nu}, \boldsymbol{\nu}') + 2i\varepsilon t(x, x') + \varepsilon^2|$$
$$+ i\arg(\tilde{\sigma}^{(\mathcal{L})}(U, V, U', V') + s(\boldsymbol{\nu}, \boldsymbol{\nu}') + 2i\varepsilon t(x, x') + \varepsilon^2) \tag{120}$$

so that Lemma 7.1 applies to the expression in (119), with $\alpha = (\lambda, \varepsilon) \in \mathbb{R}^2$, on noting that the argument function part stays uniformly bounded in (λ, ε), resulting in a contribution in (119) which is $O(\delta)$ as $\delta \to 0$. Therefore, supposing that λ and ε have been chosen sufficiently small, and likewise that $\delta > 0$ is sufficiently small, we can conclude that

$$\sup_{\lambda,\varepsilon}\left|\int\int_{\vartheta=0}^{\delta} \ln(\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + r_*^2\vartheta^2/2 - 2i\varepsilon t_{[\lambda]} + \varepsilon^2)\right.$$
$$\left.\times\ \partial_\vartheta\left(\xi_\lambda(x, x')\frac{\operatorname{sinc}(\vartheta)}{r_*^2}\right) d\vartheta\, d\varphi\, dU\, dV\, dX'\right| \le \kappa_1\delta^{1/3} \tag{121}$$

with a suitable positive constant κ_1.

In a similar manner we find, provided that λ, ε and δ are sufficiently close to 0, on account of Lemma 7.1

$$\sup_{\lambda,\varepsilon}\left|\int \ln(\tilde{\sigma}_\varepsilon(x_\lambda, x'_\lambda))\tilde{Y}(x_\lambda, x'_\lambda)\chi(\boldsymbol{\nu}, \boldsymbol{\nu}')F(x)F(x')P(\lambda U, V, \lambda U', V')\, dX\, dX'\right|$$
$$\leq \sup_{\lambda,\varepsilon}\left|\int\int_{\vartheta=0}^{\delta} \ln(\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + r_*^2\vartheta^2/2 + 2i\varepsilon t_{[\lambda]} + \varepsilon^2)k_\lambda(x, x')\, d\vartheta\, d\varphi\, dU\, dV\, dX'\right|$$
$$\leq \kappa_2\delta^{1/3} \tag{122}$$

with a family of smooth functions $k_\lambda(x, x')$ of x and x' which is uniformly bounded and uniformly compactly supported in λ; $\kappa_2 > 0$ is a suitable constant.

Furthermore, since $\tilde{Z}(x, x')$ is C^∞, we see that

$$\sup_{\lambda,\varepsilon}\left|\int \tilde{Z}(x, x')\chi(\boldsymbol{\nu}, \boldsymbol{\nu}')F_\lambda(x)F_\lambda(x')P(U, V, U', V')r_*^2\, dX\, dX'\right| = O(\delta) \tag{123}$$

if λ and δ are small enough.

Summarizing our findings up to this point, we see that, on choosing

$$q_\lambda(x, x') = \tilde{\psi}(x_\lambda, x'_\lambda)\frac{\tilde{\Delta}^{1/2}(x_\lambda, x'_\lambda)}{8\pi^2}F(x)F(x')P(\lambda U, V, \lambda U', V') \tag{124}$$

in (114), we obtain

$$\lim_{\lambda\to 0+} w^{(2)}(F_\lambda, F'_\lambda)$$
$$= \lim_{\lambda\to 0+}\lim_{\varepsilon\to 0+}\int\left[\ln(\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + r_*^2\vartheta^2/2 + 2i\varepsilon t_{[\lambda]} + \varepsilon^2)\xi_\lambda(x, x')\frac{\mathrm{sinc}(\vartheta)}{r_*^2}\right]_{\vartheta=0}^{\delta} d\varphi\, dU\, dV\, dX'$$
$$+ O(\delta^{1/3}) \tag{125}$$

for any sufficiently small $\delta > 0$. However, the integral expression is independent of δ: Recalling that $\xi_\lambda(x, x') = 0$ if $\vartheta > \delta$, the evaluation of the integral expression at $\vartheta = \delta$ vanishes, and the resulting expression, as we will see, is independent of χ_δ which is contained in the definition of $\xi_\lambda(x, x')$. Therefore, since δ may be chosen arbitrarily small, we now obtain

$$\lim_{\lambda\to 0+} w^{(2)}(F_\lambda, F'_\lambda)$$
$$= \lim_{\lambda\to 0+}\lim_{\varepsilon\to 0+}\int - \ln(\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + r_*^2\vartheta^2/2 + 2i\varepsilon t_{[\lambda]} + \varepsilon^2)\xi_\lambda(x, x')\frac{\mathrm{sinc}(\vartheta)}{r_*^2}\bigg|_{\vartheta=0} d\varphi\, dU\, dV\, dX'. \tag{126}$$

Evaluating the integral expression at $\vartheta = 0$, observing $\xi_\lambda(x, x') = \chi(R_{\boldsymbol{\nu}'}\boldsymbol{\nu}, \boldsymbol{\nu}')\, q_\lambda(R_{\boldsymbol{\nu}'}x, x')$, results in

$$\lim_{\lambda\to 0+} w^{(2)}(F_\lambda, F'_\lambda)$$
$$= \lim_{\lambda\to 0+}\lim_{\varepsilon\to 0+}\int -\ln(\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} + 2i\varepsilon t_{[\lambda]} + \varepsilon^2) q_\lambda(U, V, \boldsymbol{\nu}', U'V', \boldsymbol{\nu}')\frac{2\pi}{r_*^2}\, dU\, dV\, dU'\, dV'\, d\Omega^2(\boldsymbol{\nu}') \tag{127}$$

To see this, note first that, in the coordinates chosen, $x|_{\vartheta=0} = (U, V, \mathring{\boldsymbol{\nu}}')$, implying $s(R_{\boldsymbol{\nu}'}\boldsymbol{\nu}, \boldsymbol{\nu}') = s(\boldsymbol{\nu}, \mathring{\boldsymbol{\nu}}') = s(\mathring{\boldsymbol{\nu}}', \mathring{\boldsymbol{\nu}}') = 0$ and therefore, $\chi(R_{\boldsymbol{\nu}'}\boldsymbol{\nu}, \boldsymbol{\nu}') = 1$. On the other hand, $\boldsymbol{\nu} = \mathring{\boldsymbol{\nu}}'$ also means that there is no φ-dependence in the integrand, and the integral with respect to φ can be carried out, just contributing a factor 2π. Moreover, it implies that $q_\lambda(R_{\boldsymbol{\nu}'}x, x') = q_\lambda(U, V, \boldsymbol{\nu}', U', V', \boldsymbol{\nu}')$ in the last integral.
We are thus left with having to evaluate the limits of the right-hand side in (127). We note that

$$\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]}\Big|_{\lambda=0} = \tilde{\sigma}^{(\mathcal{L})}(\lambda U, V, \lambda U', V')\Big|_{\lambda=0} = 0 \tag{128}$$

and therefore, the Taylor expansion of $\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]}$ in λ at $\lambda = 0$ up to second-order yields

$$\tilde{\sigma}^{(\mathcal{L})}_{[\lambda]} = \lambda(U - U')(\tilde{V} - \tilde{V}') + R_\lambda(U, V, U', V') \tag{129}$$

with $R_\lambda = O(\lambda^2)$ uniformly on compact sets in U, U' and V, V' while $\tilde{V}$ (and $\tilde{V}'$) is a geodesic parameter of the lightlike curves $V \mapsto (0, V, \boldsymbol{\nu})$ with respect to the conformally transformed metric $\tilde{g}_{ab}$ chosen such that $dU_a = \tilde{g}_{ab}(\partial/\partial\tilde{V})^b$. This can be seen from eqns. (3.3) and (3.4) in [49]; note that V is an affine parameter for the said lightlike curves with respect to g_{ab} but not necessarily with respect to the conformally transformed metric $\tilde{g}_{ab}$. Using the form of the "Lorentzian" part of the conformally transformed metric

$$-2\eta^2(U, V)A(U, V)dUdV\,, \tag{130}$$

it is not difficult to check that $\tilde{V} = \tilde{V}(V)$ has the property

$$\tilde{V} - \tilde{V}' = \int_V^{V'} \eta^2(0, V_1)\, dV_1 \quad \text{implying} \quad \tilde{V} - \tilde{V}' = (V - V')\gamma(V, V') \tag{131}$$

with a positive, jointly continuous function $\gamma(V, V')$ where $\gamma(V, V')$ and $1/\gamma(V, V')$ are bounded when V and V' range over compact sets. Consequently, we have that

$$\begin{aligned}
&\lim_{\lambda\to 0+} w^{(2)}(F_\lambda, F'_\lambda)\\
&\quad = \lim_{\lambda\to 0+}\lim_{\varepsilon\to 0+}\int -\ln(\lambda(U-U')(V-V')\gamma(V,V') + R_\lambda + 2i\varepsilon t_{[\lambda]} + \varepsilon^2)\\
&\qquad\qquad \times\, q_\lambda(U,V,\boldsymbol{\nu}',U'V',\boldsymbol{\nu}')\frac{2\pi}{r_*^2}\,dU\,dV\,dU'\,dV'\,d\Omega^2(\boldsymbol{\nu}')\\
&\quad = \lim_{\lambda\to 0+}\lim_{\varepsilon\to 0+}\int -[\ln(\lambda\gamma) + \ln((U-U')(V-V') + \varrho_\lambda + \lambda^{-1}\gamma^{-1}[2i\varepsilon t_{[\lambda]} + \varepsilon^2])]\\
&\qquad\qquad \times\, q_\lambda(U,V,\boldsymbol{\nu}',U'V',\boldsymbol{\nu}')\frac{2\pi}{r_*^2}\,dU\,dV\,dU'\,dV'\,d\Omega^2(\boldsymbol{\nu}')
\end{aligned} \tag{132}$$

with $\varrho_\lambda = \lambda^{-1}\gamma^{-1}R_\lambda$, and we have abbreviated $\gamma = \gamma(V,V')$.

Let us first consider the ε-independent part in (132). We split $\ln(\lambda\gamma) = \ln(\lambda) + \ln(\gamma)$. Then, the limits can be carried out to yield[4]

$$\lim_{\lambda\to 0+}\int -\ln(\gamma(V,V'))q_\lambda(U,V,\boldsymbol{\nu}',U',V',\boldsymbol{\nu}')\frac{2\pi}{r_*^2}\,dU\,dV\,dX' = 0 \tag{133}$$

because the only U and U' dependence in $q_0(U,V,U',V')$ comes from $\partial_U f(U)$ and $\partial_{U'} f'(U')$ and therefore, since f and f' are compactly supported, we find that the resulting integral vanishes. On the other hand, we also have $q_\lambda(U,V,\boldsymbol{\nu}',U',V',\boldsymbol{\nu}') = q_0(U,V,\boldsymbol{\nu}',U',V',\boldsymbol{\nu}') + q_\lambda^{(1)}(U,V,U',V';\boldsymbol{\nu}')$ where $q_\lambda^{(1)}(U,V,U',V';\boldsymbol{\nu}') = O(\lambda)$ uniformly on compact sets in U, U' and V, V'. Then,

$$\lim_{\lambda\to 0+}\int \ln(\lambda)(q_0(U,V,\boldsymbol{\nu}',U',V',\boldsymbol{\nu}') + q_\lambda^{(1)}(U,V,U',V';\boldsymbol{\nu}'))\frac{2\pi}{r_*^2}\,dU\,dV\,dX' = 0 \tag{134}$$

since, as in the previous argument, $q_0(U,V,\boldsymbol{\nu}',U',V',\boldsymbol{\nu}')$ depends on U and U' only through $\partial_U f$ and $\partial_{U'} f'$, and $\ln(\lambda)q_\lambda^{(1)}(U,V,U',V';\boldsymbol{\nu}') \to 0$ as $\lambda \to 0$ uniformly on compact sets.

Therefore, setting $\beta_{\lambda,\varepsilon}(U,V,U',V') = \varrho_\lambda(U,V,U',V') + \gamma^{-1}[2i\varepsilon t_{[\lambda]} + \lambda\varepsilon^2]$, we obtain

$$\begin{aligned}
\lim_{\lambda\to 0+} w^{(2)}(F_\lambda, F'_\lambda) &= \lim_{\lambda\to 0+}\lim_{\varepsilon\to 0+}\int -\ln((U-U')(V-V') + \beta_{\lambda,\varepsilon}(U,V,U',V'))\\
&\qquad \times\, q_\lambda(U,V,\boldsymbol{\nu}',U',V',\boldsymbol{\nu}')\frac{2\pi}{r_*^2}\,dU\,dV\,dX'
\end{aligned} \tag{135}$$

where we have made use of the fact that, owing to the limit $\varepsilon \to 0+$ being taken prior to $\lambda \to 0+$, one may redefine ε as ε/λ without changing the result of the limits.

[4] We reinsert the abbreviation $dX' = dU'\,dV'\,d\Omega^2(\boldsymbol{\nu}')$.

Then, we choose $\delta > 0$ and split the integration over the U, U' and V, V' coordinates into the domains

$$D_>(\delta) = |(U - U')(V - V')| \geq \delta^2 , \qquad D_<(\delta) = |(U - U')(V - V')| < \delta^2 . \tag{136}$$

Furthermore, we change from the (U, V) coordinates to (T, Y) coordinates where

$$T = \frac{1}{2}(U + V) , \qquad Y = \frac{1}{2}(U - V) \tag{137}$$

and using a coordinate substitution $\bar{T} = T - T'$, $\bar{Y} = Y - Y'$, one arrives at

$$\begin{aligned} &\int_{D_<(\delta)} \ln((U - U')(V - V') + \beta_{\lambda,\varepsilon}(U, V, U', V'))q_\lambda(U, V, \boldsymbol{v}', U', V', \boldsymbol{v}')\frac{2\pi}{r_*^2}\, dU\, dV\, dX' \\ &= \int \int_{4|\bar{T}^2 - \bar{Y}^2| < \delta^2} \ln(\bar{T}^2 - \bar{Y}^2 + \bar{\beta}_{\lambda,\varepsilon}(\bar{T}, \bar{Y}, T', Y'))\bar{q}_\lambda(\bar{T}, \bar{Y}, T', Y'; \boldsymbol{v}')\frac{2\pi}{r_*^2}\, d\bar{T}\, d\bar{Y}\, dT'\, dY'\, d\Omega^2(\boldsymbol{v}') \end{aligned} \tag{138}$$

where $\bar{\beta}_{\lambda,\varepsilon}$ and $\bar{q}_\lambda$ are the $\bar{T}, \bar{Y}, T', Y'$-coordinate versions of $\beta_{\lambda,\varepsilon}$ and q_λ. We see that the right-hand side is in a form to which Part **(B)** of Lemma 7.1 applies (with $\alpha = (\lambda, \varepsilon)$), and arguing analogously as with (119) before, we conclude that, once ε, λ and δ have been chosen sufficiently small, it holds that

$$\begin{aligned} &\sup_{\lambda,\varepsilon} \left| \int_{D_<(\delta)} \ln((U - U')(V - V') + \beta_{\lambda,\varepsilon}(U, V, U', V'))q_\lambda(U, V, \boldsymbol{v}', U', V', \boldsymbol{v}')\frac{2\pi}{r_*^2}\, dU\, dV\, dX' \right| \\ &\quad = O(\delta^{1/3}) \end{aligned} \tag{139}$$

allowing to conclude

$$\begin{aligned} \lim_{\lambda \to 0+} w^{(2)}(F_\lambda, F'_\lambda) = \lim_{\delta \to 0} \lim_{\lambda \to 0+} \lim_{\varepsilon \to 0+} \int_{D_>(\delta)} &\ln((U - U')(V - V') + \beta_{\lambda,\varepsilon}(U, V, U', V')) \\ &\times q_\lambda(U, V, \boldsymbol{v}', U', V', \boldsymbol{v}')\frac{2\pi}{r_*^2}\, dU\, dV\, dX' \end{aligned} \tag{140}$$

The limits can now be carried out performing the limit $\varepsilon \to 0$ followed by $\lambda \to 0$ prior to integration, to yield

$$\begin{aligned} &\lim_{\lambda \to 0+} w^{(2)}(F_\lambda, F'_\lambda) \\ &= \int [\ln(|(U - U')|) - i\pi\theta(U - U')]q_0(U, V, \boldsymbol{v}', U', V', \boldsymbol{v}') \\ &\frac{2\pi}{r_*^2}\, dU\, dV\, dX' \end{aligned} \tag{141}$$

where θ denotes the Heaviside function. To see this, we note that

$$\lim_{\lambda\to 0+}\lim_{\varepsilon\to 0+}\ln((U-U')(V-V')+\beta_{\lambda,\varepsilon}(U,V,U',V'))q_\lambda(U,V,\boldsymbol{v}',U',V',\boldsymbol{v}')$$
$$=[\ln(|(U-U')(V-V')|)+i\pi\theta((U'-U)(V-V'))\text{sign}(\gamma^{-1}(V-V'))]q_0(U,V,\boldsymbol{v}',U',V',\boldsymbol{v}') \tag{142}$$

Using $\ln(|(U-U')(V-V')|)=\ln|U-U'|+\ln|V-V'|$, one can see again that the $\ln|V-V'|$ term gives a vanishing contribution on integration with respect to U and U' because $q_0(U,V,\boldsymbol{v}',U',V',\boldsymbol{v}')$ depends on U and U' only through $\partial_U f$ and $\partial_{U'}f'$. Moreover, since $\gamma^{-1}=\gamma^{-1}(V,V')>0$, we have $\theta((U'-U)(V-V'))\text{sign}(\gamma^{-1}(V-V'))=\theta((U'-U)\text{sign}(V-V'))\text{sign}(V-V')$; and since

$$\int[\theta(U-U')+\theta(U'-U)]\partial_U f(U,V,\boldsymbol{v}')\partial_{U'}f'(U',V',\boldsymbol{v}')dU\,dU'=0 \tag{143}$$

we can conclude that

$$\int i\pi\theta((U'-U)(V-V'))\text{sign}(\gamma^{-1}(V-V'))q_0(U,V,\boldsymbol{v}',U',V',\boldsymbol{v}')\,dU\,dV\,dX'$$
$$=-\int i\pi\theta(U-U')q_0(U,V,\boldsymbol{v}',U',V',\boldsymbol{v}')\,dU\,dV\,dX' \tag{144}$$

showing that (141) holds.

It is well known—or can easily be derived by arguments analogous to those given in the proof up to now, using partial integrations with respect to U and U'—that

$$\int[\ln(|(U-U')|)-i\pi\theta(U-U')]q_0(U,V,\boldsymbol{v}',U',V',\boldsymbol{v}')\frac{2\pi}{r_*^2}\,dU\,dV\,dX'$$
$$=\lim_{\varepsilon\to 0+}\int\frac{q_0(U,V,\boldsymbol{v}',U',V',\boldsymbol{v}')}{(U-U'+i\varepsilon)^2}\frac{2\pi}{r_*^2}\,dU\,dV\,dX'\,. \tag{145}$$

Finally, we observe that $\tilde{\psi}(0,V,\boldsymbol{v}',0,V',\boldsymbol{v}')=1$ because the points $(0,V,\boldsymbol{v}')$ and $(0,V',\boldsymbol{v}')$ are causally related and lie, by assumption, in a causal normal neighbourhood. Therefore, we have shown that

$$\lim_{\lambda\to 0+}w^{(2)}(F_\lambda,F'_\lambda)=\lim_{\varepsilon\to 0+}\int\frac{q_0(U,V,\boldsymbol{v}',U',V',\boldsymbol{v}')}{(U-U'+i\varepsilon)^2}\frac{2\pi}{r_*^2}\,dU\,dV\,dX'$$
$$=-\frac{1}{r_*^2\pi}\int\frac{f(U,V,\boldsymbol{v})f'(U',V',\boldsymbol{v})}{(U-U'-i\varepsilon)^2}Q(V,V',\boldsymbol{v})$$
$$\times\,dU\,dU'dV\,dV'\,d\Omega^2(\boldsymbol{v}) \tag{146}$$

as claimed in the statement (I) of the theorem.

The second statement is easily proved since the $\mu\to 0$ limit can be taken directly as the distribution L only involves integrations against continuous, bounded functions with respect to the V and V' coordinates. In fact, the limits $\varepsilon\to 0$ in the definition of

L (resp., Λ) can be interchanged with the limit $\mu \to 0$ used to pass from L to Λ. The result claimed in (II) then follows on observing that the square root of the van Vleck–Morette determinant equals the unit at coinciding points, so $\tilde{\Delta}^{1/2}(0, 0, \boldsymbol{\nu}, 0, 0, \boldsymbol{\nu}) = 1$, and $\eta^{-1}(0, 0) = 1$ as well as $A(0, 0) = 1$.

This completes the proof of the theorem. □

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