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Set-theoretic Yang–Baxter & reflection equations and quantum group symmetries

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Abstract
Connections between set-theoretic Yang–Baxter and reflection equations and quantum integrable systems are investigated. We show that set-theoretic R-matrices are expressed as twists of known solutions. We then focus on reflection and twisted algebras and we derive the associated defining algebra relations for R-matrices being Baxterized solutions of the A-type Hecke algebra $\mathcal{H}_N(q = 1)$. We show in the case of the reflection algebra that there exists a "boundary" finite sub-algebra for some special choice of "boundary" elements of the B-type Hecke algebra $\mathcal{B}_N(q = 1, Q)$. We also show the key proposition that the associated double row transfer matrix is essentially expressed in terms of the elements of the B-type Hecke algebra. This is one of the fundamental results of this investigation together with the proof of the duality between the boundary finite subalgebra and the B-type Hecke algebra. These are universal statements that largely generalize previous relevant findings and also allow the investigation of the symmetries of the double row transfer matrix.

Keywords Set-theoretic Yang-Baxter equation · Braces · Reflection equation · Hecke algebras · Open quantum spin chains

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1 Introduction

The Yang–Baxter equation and the R-matrix are central objects in the framework of quantum integrable systems. The Yang–Baxter equation was first introduced by Yang [73] when investigating many particle systems with δ-type interactions and later in the celebrated work of Baxter, who solved the anisotropic Heisenberg magnet (XYZ model) [2]. The solution of the model by Baxter was achieved by implementing the so-called Q-operator method, a sophisticated approach leading to sets of functional relations known as T-Q relations, that provide information on the spectrum of the model. A different approach on the resolution of the spectrum of 1D statistical models is the Quantum Inverse Scattering (QISM) method, an elegant algebraic technique [50], that led directly to the invention of quasitriangular Hopf algebras known as quantum groups, which then formally developed by Jimbo and Drinfeld independently [29,30,44,45].

Drinfeld [28] also suggested the idea of set-theoretic solutions to the Yang–Baxter equation, and since then a lot of research activity has been devoted to this issue (see for instance [32,40]). Set-theoretical solutions and Yang–Baxter maps have been investigated in the context of classical discrete integrable systems related also to the notion of Darboux-Bäcklund transformation [1,61,72]. Links between the set-theoretical Yang–Baxter equation and geometric crystals [3,33], or soliton cellular automatons [39,71] have been also revealed. Set-theoretical solutions of the Yang–Baxter equations have been investigated by employing the theory of braces and skew-braces. The theory of braces was established by W. Rump who developed a structure called a brace to describe all finite involutive set-theoretic solutions of the Yang–Baxter equation [63,64]. He showed that every brace provides a solution to the Yang–Baxter equation, and every non-degenerate, involutive set-theoretic solution of the Yang–Baxter equation can be obtained from a brace, a structure that generalizes nilpotent rings. Skew-braces were then developed in [38] to describe non-involutive solutions. Key links between set-theoretical solutions and quantum integrable systems and the associated quantum algebras were uncovered in [26].

Following the works of Cherednik [9] and Sklyanin [65], who introduced and studied the reflection equation, much attention has been focused on the issue of incorporating boundary conditions in to integrable models. The boundary effects, controlled by the refection equation, shed new light on the bulk theories themselves, and also paved the way to new mathematical concepts and physical applications. The set-theoretical reflection equation together with the first examples of solutions first appeared in [5], while a more systematic study and a classification inspired by maps appearing in integrable discrete systems presented in [4]. Other solutions were also considered and used within the context of cellular automata [54]. In [48,69] methods coming from the theory of braces were used to produce families of new solutions to the reflection equation, and in [13] skew braces were used to produce reflections.

The outline of the paper. In this study, we consider set-theoretic solutions of the Yang–Baxter and reflection equations coming from braces and we construct quantum spin chains with open boundary conditions through Sklyanin's double row transfer matrix [65]. We should mention that typical well-studied solutions of the Yang–Baxter equation are the Yangians, expressed as $R(\lambda) = \mathcal{P} + \lambda I$, where $\mathcal{P}$ is the flip map:

$u \otimes v \to v \otimes u$. Here we consider more general classes of solutions of the Yang–Baxter equation that are expressed as $R(\lambda) = \mathcal{P} + \lambda \mathcal{P}\check{r}$, where $\check{r}$ is a map that can be obtained for instance from a brace. Such solutions are of particular interest, given that in general they have no semi-classical analogue and as such they are distinctly different from the known quantum group solutions. Let us describe below in more detail what is achieved in each section:

- In Sect. 2, we present some basic background information. More precisely, in Sect. 2.1, we review some background on R-matrices associated with non-degenerate, involutive, set-theoretic solutions of the Yang–Baxter equation as well as set-theoretic solutions of the reflection equation and some information on braces. Then, in Sect. 2.2, we provide a review on recent results on the connections of brace solutions of the Yang–Baxter equation and the corresponding quantum algebras and integrable quantum spin chains [26].
- In Sect. 3 examples of set-theoretic R-matrices expressed as simple twists of known solutions via isomorhisms within the finite set $\{1, \dots, \mathcal{N}\}$ are presented. Based on these solutions we construct explicitly the associated "twisted" co-products by employing the finite set isomorphisms. We then move on to show that the generic brace solution of the Yang–Baxter equation can be obtained from the permutation operator via suitable Drinfeld twists [31]. Note that the properties of the brace structures are instrumental in deriving the form of the twist. Certain generalizations regarding the q-deformed case are also discussed.
- In Sect. 4, we focus on quadratic algebras, i.e., the reflection and twisted algebras [60,65].

  (1) In Sects. 4.1 and 4.2, we review some background information on reflection algebras and B-type Hecke algebras. More precisely, in Sect. 4.1, we recall the links between the refection algebras and B-type Hecke algebras and the Baxterization process, whereas in Sect. 4.2, we discuss set-theoretic representations of the B-type Hecke algebra by essentially reviewing some recent results on solutions of the set-theoretic reflection equation [69].
  (2) In Sect. 4.3, we derive the associated defining algebra relations for Baxterized solutions of the A-type Hecke algebra $\mathcal{H}_N(q = 1)$, and we show in the case of the reflection algebra that there exist a finite sub-algebra for some special choice of "boundary" elements of the B-type Hecke algebra, which also turns out to be a symmetry of the double row transfer matrix for special boundary conditions as will be shown in Sect. 5.2.

- In Sect. 5, we introduce open spin chains like systems and we focus on the study of the associated quantum group symmetries. We first review the construction of open quantum spin chains via the use of tensorial representations of the reflection algebras and the derivation of the double row transfer matrix. The findings of each subsection are described below.

  (1) In Sect. 5.1, we study the symmetries of the double row transfer matrix constructed from Baxterized solutions of the B-type Hecke algebra $\mathcal{B}_N(q = 1, Q)$. We first prove the key proposition of this study, i.e., we show that almost all the factors, but one, of the λ-series expansion of the open transfer matrix can be

expressed in terms of the elements of the B-type Hecke algebra. Interestingly, when choosing special boundary conditions, the full open transfer matrix can be exclusively expressed in terms of elements of the A-type Hecke algebra. Another fundamental result is that that all elements of the of the B type Hecke algebra $\mathcal{B}_N(1 = 1, Q)$ commute with a finite sub algebra of the reflection algebra. This then leads to another important proposition regarding the symmetry of the associate double row transfer matrix. These are universal results that largely extends earlier partial findings (see, e.g., [23,24,62]), and are of particulate physical and mathematical significance.

(2) In Sect. 5.2, more symmetries of open transfer matrices associated with certain classes of set-theoretic solutions of the Yang–Baxter equation coming from braces are also discussed. The derivation of these symmetries is primarily based on the properties of the brace structures. Some of these symmetries generalize recent findings on periodic transfer matrices [26], while others are new.

(3) In Sect. 5.3, symmetries of the double row transfer matrix constructed from the special class of Lyubashenko's solutions are identified confirming also some of the findings of Sect. 3.

2 Preliminaries

We present in this section some basic background information regarding set-theoretic solutions of the Yang–Baxter and reflection equations and braces as well as a brief review on the recent findings of [26] on the links between set-theoretic solutions of the Yang–Baxter equation from braces and quantum algebras.

2.1 The set-theoretic Yang–Baxter equation

Let $X = \{\mathrm{x}_1, \ldots, \mathrm{x}_{\mathcal{N}}\}$ be a set and $\check{r} : X \times X \to X \times X$. Denote

$$\check{r}(x, y) = \big(\sigma_x(y), \tau_y(x)\big).$$

We say that r is non-degenerate if σ_x and τ_y are bijective functions. Also, the solutions $(X, \check{r})$ is involutive: $\check{r}(\sigma_x(y), \tau_y(x)) = (x, y)$, $(\check{r}\check{r}(x, y) = (x, y))$. We focus on non-degenerate, involutive solutions of the set-theoretic braid equation:

$$(\check{r} \times id_X)(id_X \times \check{r})(\check{r} \times id_X) = (id_X \times \check{r})(\check{r} \times id_X)(id_X \times \check{r}).$$

Let V be the space of dimension equal to the cardinality of X, and with a slight abuse of notation, let $\check{r}$ also denote the R-matrix associated with the linearization of $\check{r}$ on $V = \mathbb{C}X$ (see [68] for more details), i.e., $\check{r}$ is the $\mathcal{N}^2 \times \mathcal{N}^2$ matrix:

$$\check{r} = \sum_{x,y,z,w \in X} \check{r}(x, z|y, w) e_{x,z} \otimes e_{y,w}, \tag{2.1}$$

where $e_{x,y}$ is the $\mathcal{N} \times \mathcal{N}$ matrix: $(e_{x,y})_{z,w} = \delta_{x,z}\delta_{y,w}$. Then for the $\check{r}$-matrix related to $(X, \check{r})$: $\check{r}(x, z|y, w) = \delta_{z,\sigma_x(y)}\delta_{w,\tau_y(x)}$. Notice that the matrix $\check{r} : V \otimes V \to V \otimes V$ satisfies the (constant) Braid equation:

$$(\check{r} \otimes I_V)(I_V \otimes \check{r})(\check{r} \otimes I_V) = (I_V \otimes \check{r})(\check{r} \otimes I_V)(I_V \otimes \check{r}).$$

Notice also that $\check{r}^2 = I_{V\otimes V}$ the identity matrix, because $\check{r}$ is involutive.

For set-theoretical solutions, it is thus convenient to use the matrix notation:

$$\check{r} = \sum_{x,y\in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)}. \tag{2.2}$$

Define also, $r = \mathcal{P}\check{r}$, where $\mathcal{P} = \sum_{x,y\in X} e_{x,y} \otimes e_{y,x}$ is the permutation operator, consequently $r = \sum_{x,y\in X} e_{y,\sigma_x(y)} \otimes e_{x,\tau_y(x)}$. The Yangian is a special case: $\check{r}(x, z|y, w) = \delta_{z,y}\delta_{w,x}$.

Let $(X, \check{r})$ be a non-degenerate set-theoretic solution to the Yang–Baxter equation. A map $k : X \to X$ is a reflection of $(X, \check{r})$ if it satisfies

$$\check{r}(k \times id_X)\check{r}(k \times id_X) = (k \times id_X)\check{r}(k \times id_X)\check{r}.$$

We say that k is a set-theoretic solution to the reflection equation. We also say that k is involutive if $k(k(x)) = x$.

Using the matrix notation introduced above then the reflection matrix K is an $\mathcal{N} \times \mathcal{N}$ matrix represented as:

$$\mathrm{k} = \sum_{x\in X} e_{x,k(x)} \tag{2.3}$$

and satisfies the constant reflection equation:

$$\check{r}(\mathrm{k} \otimes I_V)\check{r}(\mathrm{k} \otimes I_V) = (\mathrm{k} \otimes I_V)\check{r}(\mathrm{k} \otimes I_V)\check{r}. \tag{2.4}$$

Let us now recall the role of braces in the derivation of set-theoretic solutions of the Yang–Baxter equation. In [63,64], Rump showed that every solution $(X, \check{r})$ can be in a good way embedded in a brace.

Definition 2.1 (Proposition 4, [64]) A *left brace* is an abelian group $(A; +)$ together with a multiplication $\cdot$ such that the circle operation $a \circ b = a \cdot b + a + b$ makes A into a group, and $a \cdot (b + c) = a \cdot b + a \cdot c$.

In many papers, an equivalent definition is used [7] . The additive identity of a brace A will be denoted by 0 and the multiplicative identity by 1. In every brace $0 = 1$. The same notation will be used for skew braces (in every skew brace $0 = 1$).

Throughout this paper, we will use the following result, which is implicit in [63,64] and explicit in Theorem 4.4 of [7].

Theorem 2.2 (Rump's theorem, [7,63,64]). *It is known that for an involutive, non-degenerate solution of the braid equation there is always an underlying brace* $(B, \circ, +)$, *such that the maps* σ_x *and* τ_y *come from this brace, and* X *is a subset* *in this brace such that* $\check{r}(X, X) \subseteq (X, X)$ *and* $\check{r}(x, y) = (\sigma_x(y), \tau_y(x))$*, where* $\sigma_x(y) = x \circ y - x$*,* $\tau_y(x) = t \circ x - t$*, where* t *is the inverse of* $\sigma_x(y)$ *in the circle group* $(B, \circ)$*. Moreover, we can assume that every element from* B *belongs to the additive group* $(X, +)$ *generated by elements of* X*. In addition, every solution of this type is a non-degenerate, involutive set-theoretic solution of the braid equation.*

We will call the brace B an underlying brace of the solution $(X, \check{r})$, or a brace associated with the solution $(X, \check{r})$. We will also say that the solution $(X, \check{r})$ is associated with brace B. Notice that this is also related to the formula of set-theoretic solutions associated with the braided group (see [32] and [36]).

The following remark was also discovered by Rump.

Remark 2.3 Let $(N, +, \cdot)$ be an associative ring which is a nilpotent ring. For $a, b \in N$ define $a \circ b = a \cdot b + a + b$, then $(N, +, \circ)$ is a brace.

2.2 Yang–Baxter equation & quantum groups

In this subsection, we briefly review the main results reported in [26] on the various links between braces, representations of the A-type Hecke algebras, and quantum algebras.

Recall first the Yang–Baxter equation in the braid form ($\delta = \lambda_1 - \lambda_2$):

$$\check{R}_{12}(\delta)\ \check{R}_{23}(\lambda_1)\ \check{R}_{12}(\lambda_2) = \check{R}_{23}(\lambda_2)\ \check{R}_{12}(\lambda_1)\ \check{R}_{23}(\delta). \tag{2.5}$$

We focus here on brace solutions[1] of the YBE, given by (2.2) and the Baxterized solutions:

$$\check{R}(\lambda) = \lambda \check{r} + \mathbb{I}, \tag{2.6}$$

where $\mathbb{I} = I_X \otimes I_X$ and I_X is the identity matrix of dimension equal to the cardinality of the set X. Let also $R = \mathcal{P}\check{R}$, (recall the permutation operator $\mathcal{P} = \sum_{x,y\in X} e_{x,y} \otimes e_{y,x}$), then the following basic properties for R matrices coming from braces were shown in [26]:

Basic Properties. The brace R-matrix satisfies the following fundamental properties:

$$R_{12}(\lambda)\ R_{21}(-\lambda) = (-\lambda^2 + 1)\mathbb{I}, \qquad \textit{Unitarity} \tag{2.7}$$

$$R_{12}^{t_1}(\lambda)\ R_{12}^{t_2}(-\lambda - \mathcal{N}) = \lambda(-\lambda - \mathcal{N})\mathbb{I}, \qquad \textit{Crossing-unitarity}$$

$$R_{12}^{t_1 t_2}(\lambda) = R_{21}(\lambda), \tag{2.8}$$

where ${}^{t_{1,2}}$ *denotes transposition on the fist, second space, respectively, and recall* $\mathcal{N}$ *is the same as the cardinality of the set* X*.*

Let us also recall the connection of the brace representation with the A-type Hecke algebra.

[1] All, finite, non-degenerate, involutive, set-theoretic solutions of the YBE (2.2) are coming from braces (Theorem 2.2), therefore we will call such solutions *brace solutions*.

Definition 2.4 The A-type Hecke algebra $\mathcal{H}_N(q)$ is defined by the generators g_l, $l \in \{1,\ 2, \dots, N-1\}$ and the exchange relations:

$$g_l\ g_{l+1}\ g_l = g_{l+1}\ g_l\ g_{l+1}, \tag{2.9}$$
$$\Big[g_l,\ g_m\Big] = 0, \quad |l-m| > 1 \tag{2.10}$$
$$\big(g_l - q\big)\big(g_l + q^{-1}\big) = 0. \tag{2.11}$$

Remark 2.5 The brace solution $\check{r}$ (2.2) is a representation of the A-type Hecke algebra for $q = 1$. Indeed, $\check{r}$ satisfies the braid relation and $\check{r}^2 = 1$, which is shown by using the involution property. Also, the braid relation is satisfied by means of the brace properties (see also Theorem 2.2 and [26]).

The quantum algebra associated with braces

Given a solution of the Yang–Baxter equation, the quantum algebra is defined via the fundamental relation [34] (we have multiplied the familiar RTT relation with the permutation operator):

$$\check{R}_{12}(\lambda_1 - \lambda_2)\ L_1(\lambda_1)\ L_2(\lambda_2) = L_1(\lambda_2)\ L_2(\lambda_1)\ \check{R}_{12}(\lambda_1 - \lambda_2). \tag{2.12}$$

$\check{R}(\lambda) \in \mathrm{End}(\mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}}),\ L(\lambda) \in \mathrm{End}(\mathbb{C}^{\mathcal{N}}) \otimes \mathfrak{A}$, where $\mathfrak{A}$[2] is the quantum algebra defined by (2.12). We shall focus henceforth on solutions associated with braces only given by (2.6), (2.2). The defining relations of the corresponding quantum algebra were derived in [26]:

The quantum algebra associated with the brace R matrix (2.6), (2.2) *is defined by generators $L^{(m)}_{z,w}$, $z, w \in X$, and defining relations*

$$\begin{aligned} L^{(n)}_{z,w} L^{(m)}_{\hat{z},\hat{w}} - L^{(m)}_{z,w} L^{(n)}_{\hat{z},\hat{w}} &= L^{(m)}_{z,\sigma_w(\hat{w})} L^{(n+1)}_{\hat{z},\tau_{\hat{w}}(w)} - L^{(m+1)}_{z,\sigma_w(\hat{w})} L^{(n)}_{\hat{z},\tau_{\hat{w}}(w)} \\ &\quad - L^{(n+1)}_{\sigma_z(\hat{z}),w} L^{(m)}_{\tau_{\hat{z}}(z),\hat{w}} + L^{(n)}_{\sigma_z(\hat{z}),w} L^{(m+1)}_{\tau_{\hat{z}}(z),\hat{w}}. \end{aligned} \tag{2.13}$$

The proof is based on the fundamental relation (2.12) and the form of the brace R-matrix (for the detailed proof see [26]). Recall also that in the index notation we define $\check{R}_{12} = \check{R} \otimes \mathrm{id}_{\mathfrak{A}}$:

$$L_1(\lambda) = \sum_{z,w\in X} e_{z,w} \otimes I \otimes L_{z,w}(\lambda), \qquad L_2(\lambda) = \sum_{z,w\in X} I \otimes e_{z,w} \otimes L_{z,w}(\lambda). \tag{2.14}$$

The exchange relations among the various generators of the affine algebra are derived below via (2.12). Let us express L as a formal power series expansion $L(\lambda) = \sum_{n=0}^{\infty} \frac{L^{(n)}}{\lambda^n}$. Substituting expressions (2.6), and the λ^{-1} expansion in (2.12) we

[2] Notice that in L in addition to the indices 1 and 2 in (2.12) there is also an implicit "quantum index'n associated with $\mathfrak{A}$, which for now is omitted, i.e., one writes $L_{1n},\ L_{2n}$.

obtain the defining relations of the quantum algebra associated with a brace R-matrix (we focus on terms $\lambda_1^{-n}\lambda_2^{-m}$):

$$\begin{aligned}\check{r}_{12}L_1^{(n+1)}L_2^{(m)} - \check{r}_{12}L_1^{(n)}L_2^{(m+1)} + L_1^{(n)}L_2^{(m)} \\ = L_1^{(m)}L_2^{(n+1)}\check{r}_{12} - L_1^{(m+1)}L_2^{(n)}\check{r}_{12} + L_1^{(m)}L_2^{(n)}.\end{aligned} \tag{2.15}$$

The latter relations immediately lead to the quantum algebra relations (2.13), after recalling: $L_1^{(k)} = \sum_{x,y\in X} e_{x,y} \otimes I \otimes L_{x,y}^{(k)}$, $L_2^{(k)} = \sum_{x,y\in X} I \otimes e_{x,y} \otimes L_{x,y}^{(k)}$, and $\check{r}_{12} = \check{r} \otimes \mathrm{id}_{\mathfrak{A}}$, $L_{x,y}^{(k)}$ are the generators of the associated quantum algebra. The quantum algebra is also equipped with a co-product $\Delta : \mathfrak{A} \mapsto \mathfrak{A} \otimes \mathfrak{A}$ [29,30,34]. Indeed, we define

$$\mathrm{T}_{1;23}(\lambda) = L_{13}(\lambda)L_{12}(\lambda), \tag{2.16}$$

which satisfies (2.12) and is expressed as $\mathrm{T}_{1;23} = \sum_{x,y\in X} e_{x,y} \otimes \Delta(L_{x,y}(\lambda))$.

Remark 2.6 In the special case $\check{r} = \mathcal{P}$ the $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$ algebra is recovered:

$$\left[L_{i,j}^{(n+1)},\ L_{k,l}^{(m)}\right] - \left[L_{i,j}^{(n)},\ L_{k,l}^{(m+1)}\right] = L_{k,j}^{(m)}L_{i,l}^{(n)} - L_{k,j}^{(n)}L_{i,l}^{(m)}. \tag{2.17}$$

The next natural step is the classification of solutions of the fundamental relation (2.12), for the brace quantum algebra. A first step towards this goal will be to examine the fundamental object $L(\lambda) = L_0 + \frac{1}{\lambda}L_1$, and search for finite and infinite representations of the respective elements. The fusion procedure [51] can be also exploited to yield higher dimensional representations of the associated quantum algebra. The classification of L-operators will allow the identification of new classes of quantum integrable systems, such as analogues of Toda chains or deformed boson models. A first obvious example to consider is associated with Lyubashenko's solutions, which are discussed in what follows. This is a significant direction to pursue and will be systematically addressed elsewhere.

3 Set-theoretic solutions as Drinfeld twists

In this section, we first introduce some special cases of solutions of the braid equation that are immediately obtained from fundamental known solutions. We show in particular that a special class of solutions known as Lyubashenko's solutions [28] can be expressed as simple twists. Although the construction is simple, it has significant implications on the associated symmetries of the braid solutions. We then move on to show that the generic brace solution of the Yang–Baxter equation (2.2) can be obtained from the permutation operator via a suitable Drinfeld twist [31], and we identify the specific form of the twist. Moreover, inspired by the isotropic case, we provide a similar construction for the q-deformed analogue of Lyubashenko's solution.

Before we derive the Lyubashenko solution as a suitable twist, we first introduce a useful Lemma.

Lemma 3.1 *Let $\check{r}' : V \otimes V \to V \otimes V$ (V is a finite dimensional space) satisfy the braid relation and $(\check{r}')^2 = I^{\otimes 2}$. Let also $\mathbb{V} : V \to V$ be an invertible map, such that $(\mathbb{V} \otimes \mathbb{V})\check{r}' = \check{r}'(\mathbb{V} \otimes \mathbb{V})$. We define $\check{r} = (\mathbb{V} \otimes I)\check{r}'(\mathbb{V}^{-1} \otimes I) = (I \otimes \mathbb{V}^{-1})\check{r}'(I \otimes \mathbb{V})$, then:*

(1) $\check{r}^2 = I^{\otimes 2}$
(2) *$\check{r}$ satisfies the braid relation.*

Proof (1) $\check{r}^2 = (\mathbb{V} \otimes I)(\check{r}')^2(\mathbb{V}^{-1} \otimes I) = (\mathbb{V} \otimes I)I^{\otimes 2}(\mathbb{V}^{-1} \otimes I) = I^{\otimes 2}$.
(2) It is given that $\check{r}'$ satisfies the braid relation:

$$(\check{r}' \otimes I)(I \otimes \check{r}')(\check{r}' \otimes I) = (I \otimes \check{r}')(\check{r}' \otimes I)(I \otimes \check{r}'). \tag{3.1}$$

We express: $\check{r}' \otimes I = (\mathbb{V}^{-1} \otimes I \otimes I)(\check{r} \otimes I)(\mathbb{V} \otimes I \otimes I)$ and $I \otimes \check{r}' = (I \otimes I \otimes \mathbb{V})(I \otimes \check{r})(I \otimes I \otimes \mathbb{V}^{-1})$. We then conclude for the left hand side of (3.1):

$$LHS:\ (\mathbb{V}^{-1} \otimes I \otimes \mathbb{V})(\check{r} \otimes I)(I \otimes \check{r})(\check{r} \otimes I)(\mathbb{V} \otimes I \otimes \mathbb{V}^{-1}). \tag{3.2}$$

Similarly, for the RHS of (3.1) :

$$RHS:\ (\mathbb{V}^{-1} \otimes I \otimes \mathbb{V})(I \otimes \check{r})(\check{r} \otimes I)(I \otimes \check{r})(\mathbb{V} \otimes I \otimes \mathbb{V}^{-1}). \tag{3.3}$$

From (3.2), (3.3), we conclude that $\check{r}$ also satisfies the braid relation.

□

Proposition 3.2 *Let $\tau,\ \sigma : X \to X$, $X = \{1, \dots, \mathcal{N}\}$ be isomorphisms, such that $\sigma(\tau(x)) = \tau(\sigma(x)) = x$ and let $\mathbb{V} = \sum_{x \in X} e_{x,\tau(x)}$ and $\mathbb{V}^{-1} = \sum_{x \in X} e_{\tau(x),x}$. Then any solution of the type (Lyubashenko's solution)*

$$\check{r} = \sum_{x,y \in X} e_{x,\sigma(y)} \otimes e_{y,\tau(x)}, \tag{3.4}$$

can be obtained from the permutation operator $\mathcal{P} = \sum_{x,y \in X} e_{x,y} \otimes e_{y,x}$ as

$$\check{r} = (\mathbb{V} \otimes I)\mathcal{P}(\mathbb{V}^{-1} \otimes I) = (I \otimes \mathbb{V}^{-1})\mathcal{P}(I \otimes \mathbb{V}) \tag{3.5}$$

Proof The proof relies on the definitions of $\mathcal{P},\ \mathbb{V},\ \mathbb{V}^{-1}$ and the fundamental property $e_{x,y}e_{z,w} = \delta_{y,z}e_{x,w}$:

$$\begin{aligned}
&(\mathbb{V} \otimes I)\mathcal{P}(\mathbb{V}^{-1} \otimes I)\\
&= \left(\sum_{z \in X} e_{\sigma(z),z} \otimes I\right)\left(\sum_{x,y \in X} e_{x,y} \otimes e_{y,x}\right)\left(\sum_{w \in X} e_{w,\sigma(w)} \otimes I\right)\\
&= \sum_{x,y \in X} e_{\sigma(x),\sigma(y)} \otimes e_{y,x}.
\end{aligned}$$

The latter is indeed equal to (3.4) given that $\sigma(\tau(x)) = \tau(\sigma(x)) = x$, and due to Lemma 3.1, $\check{r}$ (3.4) satisfies the braid relation and $\check{r}^2 = I^{\otimes 2}$. □

Note that $r = \mathcal{P}\check{r}$, and consequently $R = \mathcal{P}\check{R}$ take a simple form for this class of solutions:

$$r = \mathbb{V}^{-1} \otimes \mathbb{V} \;\Rightarrow\; R(\lambda) = \lambda \mathbb{V}^{-1} \otimes \mathbb{V} + \mathcal{P}. \tag{3.6}$$

Examples:

1. $\sigma(y) = y + 1,\ \tau(x) = x - 1$, (see also [68]).
2. $\sigma(y) = \mathcal{N} + 1 - y,\ \tau(x) = \mathcal{N} + 1 - x$.

Note that in both examples above $x, y \in \{1, \dots, \mathcal{N}\}$ and σ, τ in example 1 are defined mod$\mathcal{N}$.

Before we present our findings on the symmetry of Lyubashenko's $\check{r}$-matrix, we first introduce a useful Lemma.

Lemma 3.3 *Let $\mathfrak{l}_{x,y}$ be the generators of the $\mathfrak{gl}_{\mathcal{N}}$ algebra satisfying:*

$$\Big[\mathfrak{l}_{x,y}, \mathfrak{l}_{z,w}\Big] = \delta_{y,z}\mathfrak{l}_{x,w} - \delta_{x,w}\mathfrak{l}_{z,y}. \tag{3.7}$$

The $\mathfrak{gl}_{\mathcal{N}}$ algebra is equipped with a coproduct $\Delta : \mathfrak{gl}_{\mathcal{N}} \to \mathfrak{gl}_{\mathcal{N}} \otimes \mathfrak{gl}_{\mathcal{N}}$ such that

$$\Delta(\mathfrak{l}_{x,y}) = \mathfrak{l}_{x,y} \otimes id + id \otimes \mathfrak{l}_{x,y}. \tag{3.8}$$

The N-coproduct is obtained by iteration $\Delta^{(N)} = (\Delta^{(N-1)} \otimes id)\Delta = (id \otimes \Delta^{(N-1)})\Delta$ and is given as $\Delta^{(N)}(\mathfrak{l}_{x,y}) = \sum_{n=1}^{N} id \otimes \ldots \otimes \underbrace{\mathfrak{l}_{x,y}}_{n^{th}\ position} \otimes \ldots \otimes id.$

Let also $\mathfrak{F}^{(N)} : \mathfrak{gl}_{\mathcal{N}}^{\otimes N} \to \mathfrak{gl}_{\mathcal{N}}^{\otimes N}$ be an invertible element such that $\mathfrak{F}^{(N)}\Delta^{(N)}(\mathfrak{l}_{x,y}) = \Delta_T^{(N)}(\mathfrak{l}_{x,y})\mathfrak{F}^{(N)}$, then $\Delta_T^{(N)}(\mathfrak{l}_{x,y})$ also satisfy the $\mathfrak{gl}_{\mathcal{N}}$ algebraic relations.

Proof The N-coproducts satisfy the $\mathfrak{gl}_{\mathcal{N}}$ relations (3.7), i.e., $\Big[\Delta^{(N)}(\mathfrak{l}_{x,y}), \Delta^{(N)}(\mathfrak{l}_{z,w})\Big] = \delta_{y,z}\Delta^{(N)}(\mathfrak{l}_{x,w}) - \delta_{x,w}\Delta^{(N)}(\mathfrak{l}_{z,y})$. By acting from the left with $\mathfrak{F}^{(N)}$ and with $(\mathfrak{F}^{N})^{-1}$ from the right in the latter commutator we immediately obtain $\Big[\Delta_T^{(N)}(\mathfrak{l}_{x,y}), \Delta_T^{(N)}(\mathfrak{l}_{z,w})\Big] = \delta_{y,z}\Delta_T^{(N)}(\mathfrak{l}_{x,w}) - \delta_{x,w}\Delta_T^{(N)}(\mathfrak{l}_{z,y})$. □

Corollary 3.4 *Let $\rho : \mathfrak{gl}_{\mathcal{N}} \to End(\mathbb{C}^{\mathcal{N}})$ be the fundamental representation of $\mathfrak{gl}_{\mathcal{N}}$, such that $\mathfrak{l}_{x,y} \mapsto e_{x,y}$, where recall $e_{x,y}$ are $\mathcal{N} \times \mathcal{N}$ matrices with elements $(e_{x,y})_{z,w} = \delta_{x,z}\delta_{y,w}$. The special solution $\check{r}$ (3.4) is $\mathfrak{gl}_{\mathcal{N}}$ symmetric, i.e.,*

$$\Big[\check{r},\ \Delta_i(e_{x,y})\Big] = 0, \quad x,\ y \in X, \tag{3.9}$$

where we define the "twisted" co-products ($i = 1, 2$):

$$\Delta_1(e_{x,y}) = e_{\sigma(x),\sigma(y)} \otimes I + I \otimes e_{x,y},$$

$$\Delta_2(e_{x,y}) = e_{x,y} \otimes I + I \otimes e_{\tau(x),\tau(y)}, \tag{3.10}$$

$(\Delta_1(e_{\tau(x),\tau(y)}) = \Delta_2(e_{x,y}))$.

Proof This can be shown using the form of the special class of solutions (3.4). The permutation operator is $\mathfrak{gl}_{\mathcal{N}}$ symmetric, i.e.,

$$\Big[\mathcal{P},\ \Delta(e_{x,y})\Big] = 0, \tag{3.11}$$

where the co-products $\Delta(e_{x,y})$ are defined in Lemma (3.3) ($\mathfrak{l}_{x,y} \mapsto e_{x,y}$).

Let $\mathbb{V} = \sum_{x\in X} e_{x,\tau(x)}$, then (3.9) immediately follows from (3.11) and (3.5) after multiplying (3.11) from the left and right with $\mathbb{V}\otimes I,\ \mathbb{V}^{-1}\otimes I$ or $I\otimes\mathbb{V}^{-1},\ I\otimes\mathbb{V}$ respectively. $\Delta_i(e_{x,y})$ are then defined as

$$\begin{aligned}\Delta_1(e_{x,y}) &= \mathbb{V}e_{x,y}\mathbb{V}^{-1}\otimes I + I\otimes e_{x,y},\\ \Delta_2(e_{x,y}) &= e_{x,y}\otimes I + I\otimes \mathbb{V}^{-1}e_{x,y}\mathbb{V}\end{aligned} \tag{3.12}$$

and explicitly given by (3.10). Indeed, $\mathbb{V}e_{x,y}\mathbb{V}^{-1} = e_{\sigma(x),\sigma(y)}$ and $\mathbb{V}^{-1}e_{x,y}\mathbb{V} = e_{\tau(x),\tau(y)}$.

According to Lemma 3.3 $\Delta_i(e_{x,y})$ also satisfy the $\mathfrak{gl}_{\mathcal{N}}$ algebra relations, thus $\check{r}$ (3.4) is $\mathfrak{gl}_{\mathcal{N}}$ symmetric. In this particular case, as is clear from the computation above, two invertible linear maps are involved, $\mathcal{F}_i^{(2)} : \mathrm{End}(\mathbb{C}^{\mathcal{N}}\otimes\mathbb{C}^{\mathcal{N}}) \to \mathrm{End}(\mathbb{C}^{\mathcal{N}}\otimes\mathbb{C}^{\mathcal{N}})$, $i\in\{1,2\}$ such that $\mathcal{F}_1^{(2)} := \mathbb{V}\otimes I$ and $\mathcal{F}_2^{(2)} := I\otimes\mathbb{V}^{-1}$ and $\mathcal{F}_i^{(2)}\Delta(e_{x,y}) = \Delta_i(e_{x,y})\mathcal{F}_i^{(2)}$. □

By iteration one derives the N co-products: $\Delta_1^{(N)} = (\Delta_1^{(N-1)}\otimes \mathrm{id})\Delta_1$ and $\Delta_2^{(N)} = (\mathrm{id}\otimes\Delta_2^{(N-1)})\Delta_2$, which explicitly read as

$$\Delta_1^{(N)}(e_{x,y}) = \sum_{n=1}^{N} I\otimes\ldots\otimes e_{\sigma^{N-n}(x),\sigma^{N-n}(y)}\otimes\ldots\otimes I, \tag{3.13}$$

$$\Delta_2^{(N)}(e_{x,y}) = \sum_{n=1}^{N} I\otimes\ldots\otimes e_{\tau^{n-1}(x),\tau^{n-1}(y)}\otimes\ldots\otimes I, \tag{3.14}$$

The above expressions can be written in a compact form as: $\Delta_i^{(N)}(e_{x,y}) = \mathcal{F}_i^{(N)}\Delta^{(N)}(e_{x,y})(\mathcal{F}_i^{(N)})^{-1}$, where $\Delta^{(N)}(e_{x,y}) = \sum_{n=1}^{N}\mathrm{id}\otimes\ldots\otimes\underbrace{e_{x,y}}_{n^{th}\text{ position}}\otimes\ldots\otimes\mathrm{id}$, and we define $\mathcal{F}_1^{(N)} := \mathbb{V}^{N-1}\otimes\mathbb{V}^{N-2}\otimes\ldots\otimes\mathbb{V}\otimes I$ and $\mathcal{F}_2^{(N)} := I\otimes\mathbb{V}^{-1}\otimes\mathbb{V}^{-2}\otimes\ldots\otimes\mathbb{V}^{-(N-1)}$ (see also relevant findings in [27]).

It was shown in [26] that the periodic Hamiltonian for systems built with R-matrices associated with the Hecke algebra $\mathcal{H}_N(q=1)$ is expressed exclusively in terms of the A-type Hecke algebra elements. In the special case where $\check{r} = \mathcal{P}$, i.e., the Yangian the periodic transfer matrix is $\mathfrak{gl}_{\mathcal{N}}$ symmetric. However, if we focus on the more general class of Lyubashenko's solutions of Proposition 3.2 and Corollary 3.4 we conclude that because of the existence of the term $\check{r}_{N1}$ (due to periodicity) [26], and also due to the form of the modified co-products (3.13), (3.14), the periodic Hamiltonian and in general the periodic transfer matrix is not $\mathfrak{gl}_{\mathcal{N}}$ symmetric anymore. However, we shall be able to show in section 5 that for a special choice of boundary conditions not only the corresponding Hamiltonian is $\mathfrak{gl}_{\mathcal{N}}$ symmetric, but also the double row transfer matrix. This means that the open spin chain enjoys more symmetry compared to the periodic one similarly to the q-deformed case [12,19,22,53,62]. It is therefore clear that from this point of view open spin chains are rather more natural objects to consider compared to the periodic ones. In [26] a systematic investigation of symmetries of the periodic transfer matrix for generic representations of the A-type Hecke algebra $\mathcal{H}_N(q=1)$ as well as for certain solutions of the Yang–Baxter equation coming from braces is presented.

With the following proposition we generalize the results on Lyubashenko's solutions. Specifically, we express the generic brace $\check{r}$-matrix (2.2) as a twist of the permutation operator. Drinfeld introduced [31] the"twisting" (or deformation) of a (quasi) triangular Hopf algebra that produces yet another (quasi) triangular (quasi) Hopf algebra (see also relevant [52,56]). Let us briefly recall the notion of a twist. Let $\check{R}$ be the quantum group invariant matrix, i.e., it *commutes* with the the respective quantum algebra [8,44,45]. We are focusing on the finite algebra $\mathfrak{g}$, specifically we are considering here the algebras $\mathfrak{gl}_{\mathcal{N}}$ or $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$, although via the evaluation homomorphism one obtains the corresponding affine algebras, i.e., the Yangian $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$ or the affine $\mathfrak{U}_q(\widehat{\mathfrak{gl}_{\mathcal{N}}})$ respectively [8,44,45]. Consider the fundamental representation $\pi : \mathfrak{g} \mapsto \mathrm{End}(\mathbb{C}^{\mathcal{N}})$, the co-products $\Delta : \mathfrak{g} \mapsto \mathfrak{g} \otimes \mathfrak{g}$ and the $\check{R}$-matrix satisfy linear intertwining relations: $(\pi \otimes \pi)\Delta(X)\ \check{R} = \check{R}\ (\pi \otimes \pi)\Delta(X)$ for $X \in \mathfrak{g}$. Let also $\mathcal{F} \in \mathrm{End}(\mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}})$, then the $\check{R}$ matrix can be "twisted" as $\mathcal{F}\check{R}\mathcal{F}^{-1}$, where $\mathcal{F}$ also satisfies a set of constraints dictated by the YBE. Given the linear intertwining relations and the twisted $\check{R}$-matrix, one derives the twisted co-products of the finite algebra as $\mathcal{F}\ (\pi \otimes \pi)\Delta(X)\ \mathcal{F}^{-1}$ (for a more detailed exposition on the notions of quasi-triangular Hopf algebras and Drinfeld twists the interested reader is referred for instance to [8]).

Proposition 3.5 *Let $\check{r} = \sum_{x,y\in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)}$ be the brace solution of the Yang–Baxter equation (see also (2.2) and footnote 1 in page 6). Let also V_k, $k \in \{1, \dots, \mathcal{N}^2\}$ be the eigenvectors of the permutation operator $\mathcal{P} = \sum_{x,y\in X} e_{x,y} \otimes e_{y,x}$, and $\hat{V}_k$, $k \in \{1, \dots, \mathcal{N}^2\}$ be the eigenvectors of the brace $\check{r}$ matrix. Then the $\check{r}$ matrix can be expressed as a Drinfeld twist, such that $\check{r} = \mathcal{F}\mathcal{P}\mathcal{F}^{-1}$, where the twist $\mathcal{F}$ is explicitly expressed as $\mathcal{F} = \sum_{k=1}^{\mathcal{N}^2} \hat{V}_k\ V_k^T$.*

Proof We divide our proof in three parts:

(1) First we diagonalize the permutation operator. Let $\hat{e}_j$ be the $\mathcal{N}$ dimensional column vectors with one at the j^{th} position and zero elsewhere, and then the (normalized) eigenvectors of the permutation operator are $(x,\ y \in X)$:

$$V_k = \frac{1}{\sqrt{2}}\big(\hat{e}_x \otimes \hat{e}_y + \hat{e}_y \otimes \hat{e}_x\big), \qquad k \in \Big\{1, \dots, \frac{\mathcal{N}^2+\mathcal{N}}{2}\Big\},$$
$$V_k = \frac{1}{\sqrt{2}}\big(\hat{e}_x \otimes \hat{e}_y - \hat{e}_y \otimes \hat{e}_x\big), \qquad k \in \Big\{\frac{\mathcal{N}^2+\mathcal{N}}{2}+1, \dots, \mathcal{N}^2\Big\}, \quad x \neq y.$$

The first $\frac{\mathcal{N}^2+\mathcal{N}}{2}$ eigenvectors have the same eigenvalue 1, while the rest $\frac{\mathcal{N}^2-\mathcal{N}}{2}$ eigenvectors have eigenvalue -1. Also it is easy to check that V_k form an ortho-normal basis for the $\mathcal{N}^2$ dimensional space. Indeed, $V_k^T V_l = \delta_{kl}$ and $\sum_{k=1}^{\mathcal{N}^2} V_k V_k^T = I_{\mathcal{N}^2}$ (^{T} denotes usual transposition).

(2) Second we diagonalize the brace $\check{r}$-matrix. First we observe that

$$\check{r}\ e_x \otimes e_y = e_{\sigma_x(y)} \otimes e_{\tau_y(x)}, \quad \check{r}\ e_{\sigma_x(y)} \otimes e_{\tau_y(x)} = e_x \otimes e_y.$$

Then we find that the eigenvectors of the $\check{r}$ matrix are

$$\hat{V}_k = \frac{1}{\sqrt{2}}\big(\hat{e}_x \otimes \hat{e}_y + \hat{e}_{\sigma_x(y)} \otimes \hat{e}_{\tau_y(x)}\big), \qquad k \in \Big\{1, \dots, \frac{\mathcal{N}^2+\mathcal{N}}{2}\Big\},$$
$$\hat{V}_k = \frac{1}{\sqrt{2}}\big(\hat{e}_x \otimes \hat{e}_y - \hat{e}_{\sigma_x(y)} \otimes \hat{e}_{\tau_y(x)}\big), \quad (x, y) \neq (\sigma_x(y), \tau_y(x)),$$
$$k \in \Big\{\frac{\mathcal{N}^2+\mathcal{N}}{2}+1, \dots, \mathcal{N}^2\Big\}.$$

As in the case of the permutation operator the $\check{r}$ matrix has the same eigenvalues 1 and -1 and the same multiplicities, $\frac{\mathcal{N}^2+\mathcal{N}}{2}$ and $\frac{\mathcal{N}^2-\mathcal{N}}{2}$, respectively. Hence, the two matrices are similar, i.e., there exists some $\mathcal{F} \in \mathrm{End}(\mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}})$ (not uniquely defined) such that $\check{r} = \mathcal{F}\mathcal{P}\mathcal{F}^{-1}$.

(3) Our task now is to derive the explicit form of $\mathcal{F}$. This is quite straightforward, and indeed, the eigenvalue problem for $\mathcal{P}$ (and $\check{r}$) reads as

$$\mathcal{P} V_k = \lambda_k V_k \ \Rightarrow\ \check{r} \hat{V}_k = \lambda_k \hat{V}_k$$

where, via $\check{r} = \mathcal{F}\mathcal{P}\mathcal{F}^{-1}$, we identify $\mathcal{F} V_k = \hat{V}_k$, which by using $\sum_{k=1}^{\mathcal{N}^2} V_k V_k^T = I$, leads to the explicit expression $\mathcal{F} = \sum_{k=1}^{\mathcal{N}^2} \hat{V}_k\ V_k^T$.

□

Note that if $\check{r} = \mathcal{F}\mathcal{P}\mathcal{F}^{-1}$ ($\mathcal{P}$ the permutation operator) then $r = \mathcal{P}\check{r} = \mathcal{F}^{(op)}\mathcal{F}^{-1}$, where $\mathcal{F}^{(op)} = \mathcal{P}\mathcal{F}\mathcal{P}$, and consequently the Baxterized solution (2.6) is given as $R(\lambda) = \mathcal{P}\check{R}(\lambda) = \lambda\mathcal{F}^{(op)}\mathcal{F}^{-1} + \mathcal{P}$.

Corollary 3.6 *The brace solution $\check{r}$ (2.2) is $\mathfrak{gl}_{\mathcal{N}}$ symmetric, i.e., $\big[\check{r},\ \Delta_T(e_{x,y})\big] = 0$, where the twisted co-products are given as $\Delta_T(e_{x,y}) = \mathcal{F}\Delta(e_{x,y})\mathcal{F}^{-1}$.*

Proof The proof is straightforward as in Corollary 3.4 using the fact that the permutation operator is $\mathfrak{gl}_{\mathcal{N}}$ symmetric. □

Notice that here we identified the Drinfeld twist as a similarity transformation between the permutation operator and the brace solution. The twisted n-co-product as well as the n form of $\mathcal{F}$ should be identified and the admissibility of the twist should be also examined. Also, issues on the co-associativity of the co-product need to be addressed. We already observe in the simple case of Lyubashenko's solutions that the co-associativity of the twisted co-products is not guaranteed. These are significant issues that are addressed in [27].

3.1 Parenthesis: the *q*-deformed case

We slightly deflect in this subsection from our main issue, which is the set-theoretic solutions of the Yang–Baxter equation, and briefly discuss the q-deformed case. Inspired by the special class of Lyubashenko's solutions, we generalize in what follows Proposition 3.2 and Corollary 3.4 in the case of the $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ invariant representation of the A-type Hecke algebra [44,45]:

$$\mathrm{g} = \sum_{x \neq y=1}^{\mathcal{N}} \Big(e_{x,y} \otimes e_{y,x} - q^{-sgn(x-y)} e_{x,x} \otimes e_{y,y}\Big) + q. \tag{3.15}$$

Note that strictly speaking this solution is not a set-theoretic solution of the braid equation. Nevertheless, isomorphisms within the set of integers $\{1, \ldots, \mathcal{N}\}$ can be still exploited to yield generalized solutions based on (3.15).

Proposition 3.7 *Let $\sigma,\ \tau : X \to X$ be isomorprhisms ($X = \{1, \ldots, \mathcal{N}\}$) such that $\sigma(\tau(x)) = \tau(\sigma(x)) = x$. The quantity*

$$\begin{aligned} \mathrm{G} &= \sum_{x \neq y=1}^{\mathcal{N}} \Big(e_{x,y} \otimes e_{\tau(y),\tau(x)} - q^{-sgn(x-y)} e_{x,x} \otimes e_{\tau(y),\tau(y)}\Big) + q \\ &= \sum_{x \neq y=1}^{\mathcal{N}} \Big(e_{\sigma(x),\sigma(y)} \otimes e_{y,x} - q^{-sgn(x-y)} e_{\sigma(x),\sigma(x)} \otimes e_{y,y}\Big) + q \end{aligned} \tag{3.16}$$

can be obtained from the $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ invariant braid solution (3.15), provided that $sgn(x-y) = sgn(\tau(x) - \tau(y)) = sgn(\sigma(x) - \sigma(y))$, and is also a representation of the A-type Hecke algebra.

Proof Let $\mathbb{V} = \sum_w e_{w,\tau(w)}$, and $\mathbb{V}^{-1} = \sum_z e_{\tau(z),z}$. We show by explicit computation that,

$$(\mathbb{V} \otimes \mathbb{V})\ \mathrm{g} = \mathrm{g}\ (\mathbb{V} \otimes \mathbb{V}) \tag{3.17}$$

provided that $sgn(\tau(x)-y) = sgn(\sigma(x)-y)$. We then define, bearing in mind (3.17):

$$\mathrm{G} = (\mathbb{V} \otimes I)\ \mathrm{g}\ (\mathbb{V}^{-1} \otimes I) = (I \otimes \mathbb{V}^{-1})\ \mathrm{g}\ (I \otimes \mathbb{V}), \tag{3.18}$$

which leads to (3.16).

Also, g is a given representation of the A-type Hecke algebra, i.e.,

$$(\mathrm{g} \otimes I)\ (I \otimes \mathrm{g})\ (\mathrm{g} \otimes I) = (I \otimes \mathrm{g})\ (\mathrm{g} \otimes I)\ (I \otimes \mathrm{g}), \tag{3.19}$$

$$\big(\mathrm{g} - q\big)\big(\mathrm{g} + q^{-1}\big) = 0. \tag{3.20}$$

By multiplying (3.19) with $\mathbb{V} \otimes I \otimes \mathbb{V}^{-1}$ from the left and $\mathbb{V}^{-1} \otimes I \otimes \mathbb{V}$ from the right, and also multiplying (3.20) with $\mathbb{V} \otimes I$ from the left and $\mathbb{V}^{-1} \otimes I$ from the right, and using the definition (3.18) we immediately conclude that G is also a representation of the A-type Hecke algebra (see also Lemma 3.1). □

It will be useful for what follows to recall the basic definitions regarding the $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ algebra [44,45]. Let

$$a_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j}, \quad i,\ j \in \{1, \ldots, \mathcal{N} - 1\} \tag{3.21}$$

be the Cartan matrix of the associated Lie algebra.

Definition 3.8 The quantum algebra $\mathfrak{U}_q(\mathfrak{sl}_{\mathcal{N}})$ has the Chevalley–Serre generators e_i, $f_i, q^{\pm \frac{h_i}{2}}, i \in \{1, \ldots, \mathcal{N} - 1\}$ obeying the defining relations:

$$\Big[q^{\pm\frac{h_i}{2}},\ q^{\pm\frac{h_j}{2}}\Big] = 0 \qquad q^{\frac{h_i}{2}}\ e_j = q^{\frac{1}{2}a_{ij}} e_j\ q^{\frac{h_i}{2}} \qquad q^{\frac{h_i}{2}}\ f_j = q^{-\frac{1}{2}a_{ij}}\ f_j\ q^{\frac{h_i}{2}},$$
$$\Big[e_i,\ f_j\Big] = \delta_{ij}\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \quad i, j \in \{1, \ldots, , \mathcal{N} - 1\} \tag{3.22}$$

and the q deformed Serre relations

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix}_q \chi_i^{1-a_{ij}-n}\ \chi_j\ \chi_i^n = 0, \quad \chi_i \in \{e_i,\ f_i\}, \quad i \neq j. \tag{3.23}$$

Remark 3.9 $q^{\pm h_i} = q^{\pm(\epsilon_i - \epsilon_{i+1})}$, where the elements $q^{\pm\epsilon_i}$ belong to $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$. Recall that $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ is derived by adding to $\mathfrak{U}_q(\mathfrak{sl}_{\mathcal{N}})$ the elements $q^{\pm\epsilon_i}$ $i \in \{1, \ldots, \mathcal{N}\}$ so that $q^{\sum_{i=1}^{\mathcal{N}} \epsilon_i}$ belongs to the center [44,45], and $\big[q^{\epsilon_i},\ q^{\epsilon_j}\big] = 0,\ q^{\epsilon_i}\ e_j = q^{\delta_{i,j} - \delta_{i,j+1}} e_j\ q^{\epsilon_i}$, $q^{\epsilon_i}\ f_j = q^{-(\delta_{i,j} - \delta_{i,j+1})} f_j\ q^{\epsilon_i}$.

$\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ is equipped with a co-product $\Delta : \mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}}) \to \mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}}) \otimes \mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ such that

$$\Delta(\xi) = q^{-\frac{h_i}{2}} \otimes \xi + \xi \otimes q^{\frac{h_i}{2}}, \quad \xi \in \Big\{e_i,\ f_i\Big\}, \qquad \Delta(q^{\pm\frac{\epsilon_i}{2}}) = q^{\pm\frac{\epsilon_i}{2}} \otimes q^{\pm\frac{\epsilon_i}{2}}. \tag{3.24}$$

The N-fold co-product may be derived by using the recursion relations

$$\Delta^{(N)} = (\mathrm{id} \otimes \Delta^{(N-1)})\Delta = (\Delta^{(N-1)} \otimes \mathrm{id})\Delta, \tag{3.25}$$

and as is customary, $\Delta^{(2)} = \Delta$ and $\Delta^{(1)} = \mathrm{id}$.

Let us now consider the fundamental representation of $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ [44,45], π : $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}}) \to \mathrm{End}(\mathbb{C}^{\mathcal{N}})$:

$$\pi(e_i) = e_{i,i+1}, \quad \pi(f_i) = e_{i+1,i}, \quad \pi(q^{\frac{\epsilon_i}{2}}) = q^{\frac{e_{i,i}}{2}}, \tag{3.26}$$

and let us also introduce some useful notation:

$$(\pi\otimes\pi)\Delta(e_j) = \Delta(e_{j,j+1}),\ (\pi\otimes\pi)\Delta(f_j) = \Delta(e_{j+1,j}),\ (\pi\otimes\pi)\Delta(q^{\epsilon_j}) = \Delta(q^{e_{j,j}}). \tag{3.27}$$

Corollary 3.10 *The element* G *defined in (3.16) is* $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ *symmetric, i.e.,*

$$\Big[\mathrm{G},\ \Delta_i(Y)\Big] = 0, \quad Y \in \Big\{e_{j,j+1},\ e_{j+1,j},\ q^{e_{j,j}}\Big\} \tag{3.28}$$

where we define the modified co-products ($i = 1, 2$):

$$\begin{aligned}
&\Delta_1(q^{e_{i,i}}) = q^{e_{\sigma(i),\sigma(i)}} \otimes q^{e_{i,i}}, \quad \Delta_2(q^{e_{i,i}}) = q^{e_{i,i}} \otimes q^{e_{\tau(i),\tau(i)}}\\
&\Delta_1(\xi) = \xi_\sigma \otimes q^{\frac{H_j}{2}} + q^{-\frac{H_{\sigma(j)}}{2}} \otimes \xi,\\
&\Delta_2(\xi) = \xi \otimes q^{\frac{H_{\tau(j)}}{2}} + q^{-\frac{H_j}{2}} \otimes \xi_\tau.
\end{aligned} \tag{3.29}$$

$H_j = \big(e_{j,j} - e_{j+1,j+1}\big)$, $H_{F(j)} = \big(e_{F(j),F(j)} - e_{F(j+1),F(j+1)}\big)$, *for* $\xi \in \Big\{e_{j,j+1},\ e_{j+1,j}\Big\}$, *we define respectively:* $\xi_F \in \Big\{e_{F(j),F(j+1)},\ e_{F(j+1),F(j)}\Big\}$.

Proof This can be shown in a straightforward manner from the properties of (3.16). Indeed, g (3.15) is $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ invariant [44–46] (recall the fundamental representation (3.26))

$$\Big[\mathrm{g},\ \Delta(Y)\Big] = 0, \tag{3.30}$$

where $Y \in \Big\{e_{j,j+1},\ e_{j+1,j},\ q^{e_{j,j}}\Big\}$ and the co-products of the algebra elements are given in (3.24) (see also (3.26), (3.27)). We consider two invertible linear maps: $\mathcal{F}_i^{(2)} : \mathrm{End}(\mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}}) \to \mathrm{End}(\mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}})$, $i \in \{1, 2\}$ such that $\mathcal{F}_1^{(2)} := \mathbb{V} \otimes I$ and $\mathcal{F}_2^{(2)} := I \otimes \mathbb{V}^{-1}$, where $\mathbb{V}$ is defined in Proposition 3.7, then from (3.30)

$$\mathcal{F}_i^{(2)}\Big[\mathrm{g},\ \Delta(Y)\Big](\mathcal{F}_i^{(2)})^{-1} = 0 \ \Rightarrow\ \Big[\mathrm{G},\ \Delta_i(Y)\Big] = 0, \tag{3.31}$$

where the modified co-products are defined as $\Delta_i(Y) = \mathcal{F}_i^{(2)}\Delta(Y)(\mathcal{F}_i^{(2)})^{-1}$, and more specifically:

$$\begin{aligned}
&\Delta_1(q^{e_{i,i}}) = \mathbb{V}q^{e_{i,i}}\mathbb{V}^{-1} \otimes q^{e_{i,i}}, \quad \Delta_2(q^{e_{i,i}}) = q^{e_{i,i}} \otimes \mathbb{V}^{-1}q^{e_{i,i}}\mathbb{V},\\
&\Delta_1(\xi) = \mathbb{V}\xi\mathbb{V}^{-1} \otimes q^{\frac{H_j}{2}} + \mathbb{V}q^{-\frac{H_j}{2}}\mathbb{V}^{-1} \otimes \xi,\\
&\Delta_2(\xi) = \xi \otimes \mathbb{V}^{-1}q^{\frac{H_j}{2}}\mathbb{V} + q^{-\frac{H_j}{2}} \otimes \mathbb{V}^{-1}\xi\mathbb{V}, \quad \xi \in \Big\{e_{j,j+1},\ e_{j+1,j}\Big\}
\end{aligned} \tag{3.32}$$

and explicitly given by (3.29).

The coproducts $\Delta(Y)$ satisfy the $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ relations, then via $\Delta_i(Y) = \mathcal{F}_i^{(2)}\Delta(Y)(\mathcal{F}_i^{(2)})^{-1}$, the modified coproducts $\Delta_i(Y)$ also satisfy the $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ exchange relations, thus G is $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ symmetric. □

Explicit expressions for the modified N co-products are then given as:

$$\Delta_1^{(N)}(q^{e_{j,j}}) = \bigotimes_{n=1}^{N} q^{e_{\sigma^{N-n}(j),\sigma^{N-n}(j)}}, \quad \Delta_2^{(N)}(q^{e_{j,j}}) = \bigotimes_{n=1}^{N} q^{e_{\tau^{n-1}(j),\tau^{n-1}(j)}}$$

$$\Delta_1^{(N)}(\xi) = \sum_{n=1}^{N} q^{-\frac{H_{\sigma^{N-1}(j)}}{2}} \otimes \ldots \otimes q^{-\frac{H_{\sigma^{N-n+1}(j)}}{2}} \otimes \xi_{\sigma^{N-n}} \otimes q^{\frac{H_{\sigma^{N-n-1}(j)}}{2}} \ldots \otimes q^{\frac{H_j}{2}},$$

$$\Delta_2^{(N)}(\xi) = \sum_{n=1}^{N} q^{-\frac{H_j}{2}} \otimes \ldots \otimes q^{-\frac{H_{\tau^{n-2}(j)}}{2}} \otimes \xi_{\tau^{n-1}} \otimes q^{\frac{H_{\tau^{n}(j)}}{2}} \ldots \otimes q^{\frac{H_{\tau^{N-1}(j)}}{2}}, \tag{3.33}$$

where $\xi_{F^n} \in \Big\{e_{F^n(j),F^n(j+1)},\ e_{F^n(j+1),F^n(j)}\Big\}$. The above expressions can be written in a compact form as: $\Delta_i^{(N)}(Y) = \mathcal{F}_i^{(N)}\Delta^{(N)}(Y)(\mathcal{F}_i^{(N)})^{-1}$, where recall $Y \in \Big\{e_{j,j+1},\ e_{j+1,j},\ q^{e_{j,j}}\Big\}$, $\Delta^{(N)}(Y)$ are the $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ N-coproducts, and we define $\mathcal{F}_1^{(N)} := \mathbb{V}^{N-1} \otimes \mathbb{V}^{N-2} \otimes \ldots \otimes \mathbb{V} \otimes I$ and $\mathcal{F}_2^{(N)} := I \otimes \mathbb{V}^{-1} \otimes \mathbb{V}^{-2} \otimes \ldots \otimes \mathbb{V}^{-(N-1)}$ (see also [27]).

Some general comments are in order here. We should note that set-theoretic solutions from braces have no semi-classical analogue [26], thus they are fundamentally different from the known Yangian solutions or the q-deformed solutions of the YBE associated with $\mathfrak{gl}_{\mathcal{N}}$ or $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ [8,44,45,59]. This is evident even in the simple case of Lyubashenko's solution (please see Proposition 3.2 and simple examples 1 and 2 in page 9), recall $r = \mathbb{V}^{-1} \otimes \mathbb{V} \Rightarrow R(\lambda) = \lambda \mathbb{V}^{-1} \otimes \mathbb{V} + \mathcal{P}$, where $\mathbb{V} = \sum_{x,\in X} e_{x,\tau(x)}$ (more generally due to Proposition 3.5, $R(\lambda) = \lambda \mathcal{F}^{(op)}\mathcal{F}^{-1} + \mathcal{P}$ and $\mathcal{F}^{(op)} = \mathcal{P}\mathcal{F}\mathcal{P}$). Such R-matrices can not be expressed as $1 + \hbar r^{(1)} + \cdots$ (up to an overall multiplicative function $f(\lambda)$), given the form of $\mathbb{V}$ (or $\mathcal{F}$ explicitly given in [27]), a fact that makes our construction distinct compared to the known examples of quantum algebras (quasi triangular Hopf algebras) as described for instance by Drinfeld in [29,30] (a detailed analysis on these issues is presented in [27]). In this spirit, it would be also very interesting to consider general twists, in analogy to Proposition 3.5, for the q-deformed case as well as the corresponding quantum groups and make possible connections with the theory of braces.

4 Co-ideals: reflection & twisted algebras

We introduce two, in principle distinct, quadratic algebras associated with the classification of boundary conditions in quantum integrable models. To define these quadratic algebras in addition to the R-matrix, we also need to introduce the K-matrix, which physically describes the interaction of particle-like excitations displayed by the quantum integrable system, with the boundary of the system. The K-matrix satisfies [9,60,65]:

$$R_{12}(\lambda_1-\lambda_2)\mathbb{K}_1(\lambda_1)\hat{R}_{12}(\lambda_1+\lambda_2)\mathbb{K}_2(\lambda_2) = \mathbb{K}_2(\lambda_2)\hat{R}_{21}(\lambda_1+\lambda_2)\mathbb{K}_1(\lambda_1)R_{21}(\lambda_1-\lambda_2), \tag{4.1}$$

where we define in general $A_{21} = \mathcal{P}_{12}A_{12}\mathcal{P}_{12}$. We make two distinct choices for $\hat{R}$, which lead to the two district quadratic algebras:

$$\hat{R}_{12}(\lambda) = R_{12}^{-1}(-\lambda) \qquad \text{Reflection algebra} \tag{4.2}$$

$$\hat{R}_{12}(\lambda) = R_{12}^{t_1}\left(-\lambda - \frac{\mathcal{N}}{2}\right) \qquad \text{Twisted algebra,} \tag{4.3}$$

notice $\frac{\mathcal{N}}{2}$ is the Coxeter number for $\mathfrak{gl}_{\mathcal{N}}$.

In the self-conjugate cases, e.g., in the case of $\mathfrak{sl}_2,\ \mathfrak{U}_q(\mathfrak{sl}_2)$ or $\mathfrak{so}_n,\ \mathfrak{sp}_n$ R-matrices $R(\lambda) \sim \mathcal{C}_1 R_{12}^{t_1}(-\lambda - c)\mathcal{C}_1$, for some matrix $\mathcal{C} : \mathcal{C}^2 = I$, i.e., the R-matrix is crossing symmetric, and the two algebras, twisted and refection, coincide. The constant c is associated with the Coxeter number of the corresponding algebra. It is worth noting that these algebras are linked to two distinct types of integrable boundary conditions, extensively studied in the context of $A^{(1)}_{\mathcal{N}-1}$ affine Toda field theories [10,11,14,15,25], and quantum spin chains [65] associated with $\mathfrak{gl}_{\mathcal{N}}$, $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$, and $\mathfrak{gl}(\mathcal{N}|\mathcal{M})$ algebras [16,17,19–24,57].

4.1 Boundary Yang–Baxter equation & *B*-type Hecke algebra

Let us first focus in the case where $\hat{R}_{12}(\lambda) = R_{12}^{-1}(-\lambda) \propto R_{21}(\lambda)$, i.e., we consider the boundary Yang–Baxter or reflection equation [9,65], expressed in the braid form

$$\check{R}_{12}(\lambda_1-\lambda_2)\mathbb{K}_1(\lambda_1)\check{R}_{12}(\lambda_1+\lambda_2)\mathbb{K}_1(\lambda_2) = \mathbb{K}_1(\lambda_2)\check{R}_{12}(\lambda_1+\lambda_2)\mathbb{K}_1(\lambda_1)\check{R}_{12}(\lambda_1-\lambda_2). \tag{4.4}$$

As in the case of the Yang–Baxter equation, where representations of the A-type Hecke algebra are associated with solutions of the Yang–Baxter equation [44,45], via the Baxterization process, representations of the B-type Hecke algebra provide solutions of the reflection equation [18,55].

Definition 4.1 The B-type Hecke algebra $\mathcal{B}_N(q, Q)$ is defined by the generators g_l, $l \in \{1,\ 2, \ldots, N-1\}$ and G_0 and the exchange relations (2.9)-(2.11) and

$$G_0\ g_1\ G_0\ g_1 = g_1\ G_0\ g_1\ G_0 \tag{4.5}$$

$$\Big[G_0,\ g_l\Big] = 0, \quad l > 1 \tag{4.6}$$

$$\big(G_0 - Q\big)\big(G_0 - Q^{-1}\big) = 0. \tag{4.7}$$

We focus here on the case where $q = 1$ and Q arbitrary, and consider the brace solutions (2.2) as representation of the Hecke elements g_l. We can solve the quadratic relation (4.5) together with (4.7) to provide representation of the G_0 element. Then via Baxterization we are able to identify suitable solutions of the reflection equation. It is obvious that the identity is a solution of the relations (4.5), (4.7), and hence of the reflection equation.

Remark 4.2 Let $\mathrm{b} = \sum_{x,z\in X} \mathrm{b}_{z,w} e_{z,w}$ be a representation of the G_0 element of the B-type Hecke algebra and $\check{r}$ is the set-theoretic solution given in (2.2). Representations of G_0 can be identified.

Indeed, let us solve the quadratic relation (4.5)

$$(\mathrm{b} \otimes I)\ \check{r}\ (\mathrm{b} \otimes I)\ \check{r} = \check{r}\ (\mathrm{b} \otimes I)\ \check{r}\ (\mathrm{b} \otimes I). \tag{4.8}$$

The LHS of the latter equation leads to

$$\sum \mathrm{b}_{z,x} \mathrm{b}_{\sigma_x(y),\hat{x}} e_{z,\sigma_{\hat{x}}(\hat{y})} \otimes e_{y,\tau_{\hat{y}}(\hat{x})}, \tag{4.9}$$

subject to: $\hat{y} = \tau_y(x)$, whereas the RHS gives:

$$\sum \mathrm{b}_{\sigma_x(y),\hat{x}} \mathrm{b}_{\sigma_{\hat{x}}(\hat{y}),\hat{w}} e_{x,\hat{w}} \otimes e_{y,\tau_{\hat{y}}(\hat{x})} \tag{4.10}$$

subject to: $\hat{y} = \tau_y(x)$. Comparison of the LHS and RHS provide conditions among $\mathrm{b}_{x,w}$. Moreover, b should satisfy condition (4.7) of the B-type Hecke algebra, which leads to

$$\sum_y \mathrm{b}_{z,y} \mathrm{b}_{y,w} = (Q - Q^{-1})\mathrm{b}_{z,w} + \delta_{z,w}. \tag{4.11}$$

Study of the fundamental relations above for any brace solution will lead to admissible representations for G_0.

Note that in the special case that $\mathrm{b}_{z,w} = \delta_{w,k(z)}$, where $k : X \to X$ satisfies $k(k(x)) = x$ ($Q = 1$), and some extra conditions that are discussed in the subsequent subsection, one recovers set-theoretic reflections (see also next subsection and [69] for a more detailed discussion). In general, the full classification of representations of the B-type Hecke algebra using the brace $\check{r}$-matrix (2.2) is an important problem itself, which however will be left for future investigations.

Remark 4.3 Let $\check{r} : V \otimes V \to V \otimes V$, $\mathrm{b} : V \to V$ provide representations of the B-type Hecke algebra, and assume that there exists some invertible $\mathbb{V} : V \to V$ (see also Lemma 3.1 and Proposition 3.2):

$$(\mathbb{V} \otimes \mathbb{V})\check{r} = \check{r}(\mathbb{V} \otimes \mathbb{V}). \tag{4.12}$$

We also define

$$\check{\rho} = (\mathbb{V} \otimes I)\ \check{r}\ (\mathbb{V}^{-1} \otimes I) = (I \otimes \mathbb{V}^{-1})\ \check{r}\ (I \otimes \mathbb{V}), \qquad \beta = \mathbb{V}\mathrm{b}\mathbb{V}^{-1} \tag{4.13}$$

It then follows that $\check{r}$, b as well as $\check{\rho}$, β provide presentations of the B-type Hecke algebra.

Remark 4.4 Let b be an $\mathcal{N} \otimes \mathcal{N}$ matrix and $\check{r}$ be an $\mathcal{N}^2 \otimes \mathcal{N}^2$ matrix. Let also b_1 (index notation) be a tensor realization of the G_0 element of the B-type Hecke algebra $\mathcal{B}_N(q = 1, Q)$ and $\check{r}_{ll+1}$ a tensor realization of the element g_l of $\mathcal{B}_N(q = 1, Q)$. Then solutions of the reflection equation (4.4) ($\check{R}(\lambda) = \lambda\check{r} + I^{\otimes 2}$) can be expressed as, up to an overall function of λ, (Baxterization):

$$\mathbb{K}(\lambda) = \lambda\big(\mathrm{b} - \frac{\kappa}{2}I\big) + \frac{\hat{c}}{2}I, \tag{4.14}$$

where $\hat{c}$ is an arbitrary constant, $\kappa = Q - Q^{-1}$ and I the $\mathcal{N} \times \mathcal{N}$ identity matrix.

This has been done in [18,55], but we briefly review the procedure here, in the special case $q = 1$. Indeed, recall $\check{R}$ is given by (2.6) and let $\mathbb{K}(\lambda) = \xi(\lambda)I + \zeta(\lambda)\mathrm{b}$ where the functions $\xi(\lambda),\ \zeta(\lambda)$ will be identified. We substitute the expressions for $\check{R}$ and $K(\lambda)$ in the reflection equation (4.4) and use repeatedly relations (4.5), (4.6), then after various terms cancellations the reflection equation (4.4) becomes:

$$2\lambda_1\xi_1\zeta_2 - 2\lambda_2\zeta_1\xi_2 + \kappa(\lambda_1 - \lambda_2)\zeta_1\zeta_2 = 0 \tag{4.15}$$

where we define: $\zeta_i = \zeta(\lambda_i),\ \xi_i = \xi(\lambda_i)$ and $\kappa = Q - Q^{-1}$. We divide (4.15) by $\zeta_1\zeta_2$ (provided that this is nonzero) and set $Q_i = \frac{\xi_i}{\zeta_i}$:

$$2\lambda_1 Q_1 - 2\lambda_2 Q_2 + \kappa(\lambda_1 - \lambda_2) = 0 \ \Rightarrow\ Q_i = \frac{\hat{c}}{2\lambda_i} - \frac{\kappa}{2}, \tag{4.16}$$

and the latter implies: $\frac{\xi(\lambda)}{\zeta(\lambda)} = \frac{\hat{c} - \lambda\kappa}{2\lambda}$ ($\hat{c}$ is an arbitrary constant).

The remark above 4.4 is of course valid at the abstract level, that is solutions of the spectral dependent braid and reflections equations can be expressed in terms of the generators $g_l,\ G_0$ of the B-type Hecke algebra $\mathcal{B}_N(q = 1, Q)$, i.e., $\check{R}_{ll+1}(\lambda) = \lambda g_l + id$ and $\mathbb{K}_1(\lambda) = \lambda\big(G_0 - \frac{\kappa}{2}id\big) + \frac{\hat{c}}{2}id$.

4.2 Set-theoretic representations of *B*-type Hecke algebras

In this section, we further investigate connections between the B-type Hecke algebra and the set-theoretic reflection equation, and give some specific examples of representations of Hecke algebras that correspond to set-theoretic reflections.

Lemma 4.5 *Let $(X, \check{r})$ be an involutive non-degenerate set-theoretic solution of the braid equation where $\check{r}(x, y) = (\sigma_x(y), \tau_y(x))$. Then $(X, \check{r}')$ is an involutive non-degenerate set-theoretic solution of the braid equation where $\check{r}'(x, y) = (\tau_x(y), \sigma_y(x))$.*

Let $k : X \to X$ be a function. Then the following are equivalent:

(1) *$k : X \to X$ is a solution to the set-theoretic reflection equation for the solution $(X, \check{r})$:*

$$\check{r}K_{[1]}\check{r}K_{[1]} = K_{[1]}\check{r}K_{[1]}\check{r}$$

where $K_{[1]}(x, y) = (k(x), y)$.

(1) $k : X \to X$ *is a solution to the following version of the reflection equation considered in [69] for the solution* $(X, \check{r}')$*:*

$$\check{r}' K_{[2]} \check{r}' K_{[2]} = K_{[2]} \check{r}' K_{[2]} \check{r}'$$

where $K_{[2]}(x, y) = (y, k(x))$.

Proof Observe that $\check{r}$ is non-degenerate, and hence maps σ_x, τ_y are bijections. Consequently, $\check{r}$ is non-degenerate. Let $P : X \times X \to X \times X$ be defined as usually as $P(x, y) = (y, x)$ for $x, y \in X$. Observe that $\check{r}' = P\check{r}P$, indeed $P\check{r}P(x, y) = P\check{r}(y, x) = P(\sigma_y(x), \tau_x(y)) = \check{r}'(x, y)$.

Notice that $\check{r}'$ is involutive: $\check{r}'\check{r}' = P\check{r}PP\check{r}P = P\check{r}^2P = P^2 = id_{X\times X}$. Observe that

$$\check{r}' K_{[2]} \check{r}' K_{[2]} = K_{[2]} \check{r}' K_{[2]} \check{r}'$$

is equivalent to

$$(P\check{r}'P)(PK_{[2]}P)(P\check{r}'P)PK_{[2]}P = (PK_{[2]}P)(P\check{r}'P)(PK_{[2]}P)(P\check{r}'P),$$

which immediately leads to

$$\check{r} K_{[1]} \check{r} K_{[1]} = K_{[1]} \check{r} K_{[1]} \check{r}.$$

It remains to check that $\check{r}'$ is also a solution to the braid equation. For this purpose let us introduce, in the index notation, P_{13}: $P_{13}(x, y, z) = (z, y, x)$, it then follows that $P_{13}(\check{r} \times id_X)P_{13} = id_X \times \check{r}'$ and $P_{13}(id_X \times \check{r})P_{13} = \check{r}' \times id_X$. This is easily shown, indeed $P_{13}(\check{r} \times id_X)P_{13}(x, y, z) = P_{13}(\check{r} \times id_X)(z, y, x) = P_{13}(\sigma_z(y), \tau_y(z), x) = (x, \tau_y(z), \sigma_z(y)) = (id_X \times \check{r}')(x, y, z)$. Similarly, we show that $P_{13}(id_X \times \check{r})P_{13} = \check{r}' \times id_X$. By acting on the braid equation for $\check{r}$ with P_{13} from the left and right it then immediately follows that $\check{r}'$ also satisfies the braid relation. □

Examples of functions k satisfying the reflection equation related to braces can be found in [13,48,69]. Recall that this set-theoretical version of the reflection equation together with the first examples of solutions first appeared in the work of Caudrelier and Zhang [5]

Notice that the element of the Hecke algebra can be used to construct c-number K-matrices satisfying equation (4.4), provided that $Q = 1$. Hence, by Lemma 4.5, constant K-matrices can be obtained from involutive set-theoretic solutions to the reflection equation. In particular, involutive τ-equivariant functions give c-number solutions of the parameter dependent equation (4.4), and every linear combination over $\mathbb{C}$ of such K-matrices is also a constant K-matrix, and hence gives a solution to equation (4.4) (by Theorem 5.6 [69] applied with interchanging σ and τ).

As an application of Lemma 4.5 we obtain:

Proposition 4.6 *Let $(X, \check{r})$ be an involutive, non-degenerate solution of the braid equation. Let $\check{r} = \sum_{x,y\in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_x(y)}$, and let $g_n = I^{\otimes n-1} \otimes \check{r} \otimes I^{\otimes N-n-1}$. Let $\mathrm{b} = \sum_{x\in X} e_{x,k(x)}$ for some function $k : X \to X$ such that $k(k(x)) = x$ for all $x \in X$. Then $\mathrm{b}\otimes I$ is a representation of the G_0 element of the B-type Hecke algebra (together with $\check{r}$ used for representation of elements g_n) if and only if*

$$\tau_{\tau_y(x)}(k(\sigma_x(y))) = \tau_{\tau_y(k(x))}(k(\sigma_{k(x)}(y))).$$

Proof This follows immediately from Lemma 4.5 and Theorem 1.8 from [69], when we interchange σ with τ. □

Let $(X, \check{r})$ be an involutive, non-degenerate solution of the braid equation where we denote $\check{r}(x, y) = (\sigma_x(y), \tau_y(x))$, and let $k : X \to X$ be a function. We say that k is τ-equivariant if for every $x, y \in X$ we have

$$\tau_x(k(y)) = k(\tau_x(y)).$$

It was shown in [69] that every function $k : X \to X$ satisfying $k(\sigma_x(y)) = \sigma_x(k(y))$ satisfies the set-theoretic reflection equation. By interchanging σ with τ and applying Lemma 4.5, we get:

Corollary 4.7 *Let $(X, \check{r})$ be an involutive, non-degenerate solution of the braid equation. Let $\check{r} = \sum_{x,y\in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_x(y)}$, and let $g_n = I^{\otimes n-1} \otimes \check{r} \otimes I^{\otimes N-n-1}$. Let $\mathrm{b} = \sum_{x\in X} e_{x,k(x)}$ for some τ-equivariant function $k : X \to X$ such that $k(k(x)) = x$ for all $x \in X$. Then $\mathrm{b} \otimes I$ is a representation of the G_0 element of the B-type Hecke algebra (together with $\check{r}$ used for representation of elements g_n in this Hecke algebra).*

Examples of τ-equivariant functions can be defined by fixing $x, y \in X$ and defining for $k(r) = \tau_z(y)$ for $r = \tau_z(x)$ (provided that $\tau_v(x) = x$ implies $\tau_v(y) = y$ for every $v \in X$). In [48] Kyriakos Katsamaktsis used central elements to construct $\mathcal{G}(X, r)$ equivariant functions, his ideas also allow to define τ-equivariant functions in an analogous way as $k(x) = \tau_c(x)$, where c is central.

4.3 Reflection & twisted algebras

We shall discuss in more detail now the two distinct algebras associated with the quadratic equation (4.1). A solution of the quadratic equation (4.1) is of the form [60,65]

$$\mathbb{K}(\lambda|\theta_1) = L(\lambda - \theta_1)\,\big(K(\lambda) \otimes \mathrm{id}\big)\,\hat{L}(\lambda + \theta_1), \tag{4.17}$$

where $L(\lambda) \in \mathrm{End}(\mathbb{C}^{\mathcal{N}}) \otimes \mathfrak{A}$ satisfies the RTT relation (2.12) and $K(\lambda) \in \mathrm{End}(\mathbb{C}^{\mathcal{N}})$ is a c-number solution of the quadratic equation (4.1) (for some $R(\lambda) \in \mathrm{End}(\mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}})$, solution of the Yang–Baxter equation). We also define (in the index notation (see also Footnote 2, page 7))

$$\hat{L}_{1n}(\lambda) = L^{-1}_{1n}(-\lambda) \qquad \text{Reflection algebra}$$

$$\hat{L}_{1n}(\lambda) = L_{1n}^{t_1}\left(-\lambda - \frac{\mathcal{N}}{2}\right) \qquad \text{Twisted algebra.} \tag{4.18}$$

The quadratic algebra $\mathfrak{B}$ defined by (4.1) is a left co-ideal of the quantum algebra $\mathfrak{A}$ for a given R-matrix (see also, e.g., [14,15,22,65]), i.e., the algebra is endowed with a co-product $\Delta : \mathfrak{B} \to \mathfrak{B} \otimes \mathfrak{A}$ [65]. Indeed, we define (in the index notation)

$$\mathbb{T}_{0;12}(\lambda|\theta_1,\theta_2) = L_{02}(\lambda-\theta_2)\mathbb{K}_{01}(\lambda|\theta_1)\hat{L}_{02}(\lambda+\theta_2), \tag{4.19}$$

where $\mathbb{K}(\lambda|\theta_1)$ is given in (4.17) and in the index notation $\mathbb{K}_{01}(\lambda|\theta_1) = L_{01}(\lambda - \theta_1)K_0(\lambda)\hat{L}_{01}(\lambda+\theta_1)$. Let also $\mathbb{K}_{01}(\lambda|\theta_1) = \sum_{a,b=1}^{\mathcal{N}} e_{a,b} \otimes \mathbb{K}_{a,b}(\lambda|\theta_1) \otimes \mathrm{id}$, $L_{02} = \sum_{a,b=1}^{\mathcal{N}} e_{a,b} \otimes \mathrm{id} \otimes L_{a,b}(\lambda)$ and $\mathbb{T}_{0;12}(\lambda|\theta_1,\theta_2) = \sum_{a,b=1}^{\mathcal{N}} e_{a,b} \otimes \Delta(\mathbb{K}_{a,b}(\lambda|\theta_1,\theta_2))$, then via expression (4.19):

$$\Delta(\mathbb{K}_{a,b}(\lambda|\theta_1,\theta_2)) = \sum_{k,l} \mathbb{K}_{k,l}(\lambda|\theta_1) \otimes L_{a,k}(\lambda-\theta_2)\hat{L}_{l,b}(\lambda+\theta_2), \tag{4.20}$$

where the elements $\mathbb{K}_{k,l}(\lambda|\theta_1)$ can be also re-expressed in terms of the elements of the c-number matrix K and L when considering the realization (4.17).

In our analysis in the subsequent section, we shall be primarily focusing on tensor representations of $\mathbb{K}$ and on the special case: $L(\lambda) \to R(\lambda)$, $\hat{L}(\lambda) \to \hat{R}(\lambda)$ and for the rest of the present subsection and subsections 5.1–5.3 we shall be considering $R(\lambda) = \lambda\mathcal{P}\check{r} + \mathcal{P}$, where $\check{r}$ provides a representation of the A-type Hecke algebra $\mathcal{H}_N(q=1)$ and $\mathcal{P}$ is the permutation operator.

Before we move on with stating the next Proposition and Corollaries regarding the quadratic algebras defined by (4.1) we first introduce some useful notation associated with both the reflection and twisted algebras (4.1). We introduce $\check{r}^*$ and $\hat{\mathcal{P}}$:

$$\check{r}^*_{12} = \check{r}_{12}, \qquad \hat{\mathcal{P}}_{12} = I^{\otimes 2} \qquad \text{Reflection algebra} \tag{4.21}$$

$$\check{r}^*_{12} = r^{t_1}_{12}\mathcal{P}_{12}, \quad \hat{\mathcal{P}}_{12} = \left(\frac{\mathcal{N}}{2}r^{t_1}_{12} - \mathcal{P}^{t_1}_{12}\right)\mathcal{P}_{12} \qquad \text{Twisted algebra.} \tag{4.22}$$

Proposition 4.8 *Let $\check{R}(\lambda) = \lambda\check{r} + I^{\otimes 2}$, where $\check{r}$ provides a tensor realization of the Hecke algebra $\mathcal{H}_N(q=1)$, and let $\mathbb{K}(\lambda)$ satisfy the quadratic equation (4.1). Let also $\mathbb{K}(\lambda) = \sum_{n=0}^{\infty} \frac{\mathbb{K}^{(n)}}{\lambda^n}$ and $\mathbb{K}^{(n)} = \sum_{z,w\in X} e_{z,w} \otimes \mathbb{K}^{(n)}_{z,w}$, where $\mathbb{K}^{(n)}_{z,w}$ are the generators of the quadratic algebra defined by (4.1). The exchange relations among the quadratic algebra generators are encoded in:*

$$\begin{aligned}
&\check{r}_{12}\mathbb{K}_1^{(n+2)}\check{r}^*_{12}\mathbb{K}_1^{(m)} - \check{r}_{12}\mathbb{K}_1^{(n)}\check{r}^*_{12}\mathbb{K}_1^{(m+2)} + \check{r}_{12}\mathbb{K}_1^{(n+1)}\hat{\mathcal{P}}_{12}\mathbb{K}_1^{(m)}\\
&\quad -\check{r}_{12}\mathbb{K}_1^{(n)}\hat{\mathcal{P}}_{12}\mathbb{K}_1^{(m+1)} + \mathbb{K}_1^{(n+1)}\check{r}^*_{12}\mathbb{K}_1^{(m)} + \mathbb{K}_1^{(n)}\check{r}^*_{12}\mathbb{K}_1^{(m+1)} + \mathbb{K}_1^{(n)}\hat{\mathcal{P}}_{12}\mathbb{K}_1^{(m)}\\
&= \mathbb{K}_1^{(m)}\check{r}^*_{12}\mathbb{K}_1^{(n+2)}\check{r}_{12} - \mathbb{K}_1^{(m+2)}\check{r}^*_{12}\mathbb{K}_1^{(n)}\check{r}_{12} + \mathbb{K}_1^{(m)}\hat{\mathcal{P}}_{12}\mathbb{K}_1^{(n+1)}\check{r}_{12}\\
&\quad -\mathbb{K}_1^{(m+1)}\hat{\mathcal{P}}_{12}\mathbb{K}_1^{(n)}\check{r}_{12} + \mathbb{K}_1^{(m+1)}\check{r}^*_{12}\mathbb{K}_1^{(n)} + \mathbb{K}_1^{(m)}\check{r}^*_{12}\mathbb{K}_1^{(n+1)} + \mathbb{K}_1^{(m)}\hat{\mathcal{P}}_{12}\mathbb{K}_1^{(n)},
\end{aligned} \tag{4.23}$$

where $\check r^*$ and $\hat{\mathcal P}$ are defined in (4.21), (4.22).

Proof First we act from the left and right of (4.1) with the permutation operator $\mathcal{P}$, then (4.1) becomes

$$\check R_{12}(\lambda_1-\lambda_2)\mathbb{K}_1(\lambda_1)\check R^*_{12}(\lambda_1+\lambda_2)\mathbb{K}_1(\lambda_2) = \mathbb{K}_1(\lambda_2)\check R^*_{12}(\lambda_1+\lambda_2)\mathbb{K}_1(\lambda_1)\check R_{12}(\lambda_1-\lambda_2), \tag{4.24}$$

where $\check R(\lambda_1-\lambda_2) = (\lambda_1-\lambda_2)\check r + I^{\otimes 2}$ and $\check R^*(\lambda_1+\lambda_2) = (\lambda_1+\lambda_2)\check r^* + \hat{\mathcal P}$ ($\check r^*$, $\hat{\mathcal P}$ are defined in (4.21), (4.22)), and we recall that $\mathbb{K}(\lambda_i) = \sum_{n=0}^{\infty}\frac{\mathbb{K}^{(n)}}{\lambda_i^n}$ ($i \in \{1,\ 2\}$). We substitute the above expressions in (4.24), and we gather terms proportional to $\lambda_1^{-n}\lambda_2^{-m}$, $n, m \geq 0$ in the LHS and RHS of (4.24), which lead to (4.23). Recalling also that in general $A_{12} = A \otimes \mathrm{id}_{\mathfrak{A}}$, $\mathbb{K}_1^{(n)} = \sum_{z,w\in X} e_{z,w} \otimes I \otimes \mathbb{K}^{(n)}_{z,w}$, and substituting the latter expressions in (4.23) we obtain the exchange relations among the generators $\mathbb{K}^{(n)}_{z,w}$, which are particularly involved and are omitted here. □

It is useful for the following Corollaries to focus on terms proportional to $\lambda_1^2\lambda_2^{-m}$ and $\lambda_1\lambda_2^{-m}$ (or equivalently $\lambda_2^2\lambda_1^{-m}$ and $\lambda_2\lambda_1^{-m}$) in the $\lambda_{1,2}$ expansion of the quadratic algebra, and obtain

$$\check r_{12}\mathbb{K}_1^{(0)}\check r^*_{12}\mathbb{K}_1^{(m)} = \mathbb{K}_1^{(m)}\check r^*_{12}\mathbb{K}_1^{(0)}\check r_{12} \tag{4.25}$$

$$\begin{aligned}&\check r_{12}\mathbb{K}_1^{(1)}\check r^*_{12}\mathbb{K}_1^{(m)} + \mathbb{K}_1^{(0)}\check r^*_{12}\mathbb{K}_1^{(m)} + \check r_{12}\mathbb{K}_1^{(0)}\hat{\mathcal P}_{12}\mathbb{K}_1^{(m)}\\ &\quad = \mathbb{K}_1^{(m)}\check r^*_{12}\mathbb{K}_1^{(1)}\check r_{12} + \mathbb{K}_1^{(m)}\check r^*_{12}\mathbb{K}_1^{(0)} + \mathbb{K}_1^{(m)}\hat{\mathcal P}_{12}\mathbb{K}_1^{(1)}\check r_{12}.\end{aligned} \tag{4.26}$$

The two corollaries that follow concern the reflection algebra only, i.e., $\check r^* = \check r$, $\hat{\mathcal P} = I^{\otimes 2}$.

Corollary 4.9 *A finite non-abelian sub-algebra of the reflection algebra exists, realized by the elements of $\mathbb{K}^{(1)}$ when $\mathbb{K}^{(0)} \propto I$.*

Proof We focus on terms proportional $\lambda_1^2\lambda_2^{-m}$ and $\lambda_1\lambda_2^{-m}$ (4.25), (4.26) in the case of the reflection algebra:

$$\left[\check r_{12}\mathbb{K}_1^{(0)}\check r_{12},\ \mathbb{K}_1^{(m)}\right] = 0 \tag{4.27}$$

$$\begin{aligned}&\left[\check r_{12}\mathbb{K}_1^{(1)}\check r_{12},\ \mathbb{K}_1^{(m)}\right]\\ &\quad = \mathbb{K}_1^{(m)}\mathbb{K}_1^{(0)}\check r_{12} + \mathbb{K}_1^{(m)}\check r_{12}\mathbb{K}_1^{(0)} - \mathbb{K}_1^{(0)}\check r_{12}\mathbb{K}_1^{(m)} - \check r_{12}\mathbb{K}_1^{(0)}\mathbb{K}_1^{(m)}.\end{aligned} \tag{4.28}$$

Notice that due to (4.17) in the case of the reflection algebra $\mathbb{K}^{(0)} \propto I$ when the c-number matrix $K \propto I$. For $m = 1$ equation (4.28) provides the defining relations of a finite sub-algebra of the reflection algebra generated by $\mathbb{K}^{(1)}_{x,y}$. □

Corollary 4.10 *For the special class of Lyubashenkos's solutions $\check r$ of Proposition 3.2, a finite non-abelian sub-algebra of the reflection algebra exists, realized by the elements* *of* $\mathbb{K}^{(1)}$ *for any* $\mathbb{K}^{(0)}$. *When* $\mathbb{K}^{(0)} \propto I$ *the finite sub-algebra generated by the* $\mathbb{K}^{(1)}_{x,y}$ *is the* $\mathfrak{gl}_{\mathcal{N}}$ *algebra. Moreover, traces of* $\mathbb{K}^{(m)}$ *commute with the elements* $\mathbb{K}^{(1)}_{x,y}$,

$$\Big[\mathbb{K}^{(1)}_{x,y},\ tr_1(\mathbb{K}^{(m)}_1)\Big] = 0, \quad \forall x,\ y \in X. \tag{4.29}$$

Proof For the special class of solutions $\check{r}_{12} = \mathbb{V}_1\mathcal{P}_{12}\mathbb{V}_1^{-1}$ (3.4), equation (4.27) becomes $\Big[\tilde{\mathbb{K}}^{(0)}_2,\ \tilde{\mathbb{K}}^{(m)}_1\Big] = 0$, where we define $\tilde{\mathbb{K}}^{(m)} = \mathbb{V}^{-1}\mathbb{K}^{(m)}\mathbb{V}$ ($\mathbb{V} = \sum_{x\in X} e_{x,\tau(x)}$), which reads for the matrix elements as: $\mathbb{K}^{(m)}_{x,y} = \tilde{\mathbb{K}}^{(m)}_{\tau(x),\tau(y)}$. The latter commutator implies that $\mathbb{K}^{(0)}$ is a c-number matrix (i.e., the entries of $\mathbb{K}^{(0)}$ are c-numbers). Also, (4.28) becomes

$$\Big[\tilde{\mathbb{K}}^{(1)}_2,\ \tilde{\mathbb{K}}^{(m)}_1\Big] = \mathcal{P}_{12}\Big(\tilde{\mathbb{K}}^{(m)}_2\big(\tilde{\mathbb{K}}^{(0)}_1 + \tilde{\mathbb{K}}^{(0)}_2\big) - \big(\tilde{\mathbb{K}}^{(0)}_1 + \tilde{\mathbb{K}}^{(0)}_2\big)\tilde{\mathbb{K}}^{(m)}_1\Big). \tag{4.30}$$

Given that $\mathbb{K}^{(0)}$ is a c-number matrix, we conclude that expression (4.30) for $m = 1$ provides a closed algebra formed by the elements of $\mathbb{K}^{(1)}$. For $m = 1$ and for $\mathbb{K}^{(0)} \propto I$ (4.30) gives the $\mathfrak{gl}_{\mathcal{N}}$ exchange relations (up to an overall multiplicative factor, which can be absorbed by rescaling the generators). See also relevant results on tensor realizations of the sub-algebra in Corollary 5.17.

Taking the trace of (4.30) with respect to space 1 and using $\big[\tilde{\mathbb{K}}^{(0)}_2,\ \tilde{\mathbb{K}}^{(m)}_1\big] = 0$ we arrive at (4.29). □

5 Open quantum spin chains & associated symmetries

We consider in what follows spin-chain like representations, i.e., we are focusing on tensor representations of the quadratic algebra (4.1) (see also (4.17)): $L(\lambda) \to R(\lambda)$, $\hat{L}(\lambda) \to \hat{R}(\lambda)$ and $\mathbb{K}_{01}(\lambda|\theta_1) \to \mathcal{K}_{01}(\lambda|\theta_1) = R_{01}(\lambda - \theta_1)K_0(\lambda)\hat{R}_{01}(\lambda + \theta_1)$, where recall $K(\lambda)$ is a c-number solution of the quadratic equation (4.1), $R(\lambda)$ is a solution of the Yang–Baxter equation and $\hat{R}(\lambda)$ is defined in (4.2), (4.3).

We introduce the open monodromy matrix $\mathcal{T}_{0,1,2\ldots N}(\lambda|\{\theta_i\}) \in \mathrm{End}((\mathbb{C}^{\mathcal{N}})^{\otimes(N+1)})$ [65], which provides a tensor representation of (4.1):

$$\mathcal{T}_{0;12\ldots N}(\lambda|\{\theta_i\}) = T_{0;12\ldots N}(\lambda|\{\theta_i\})\ K_0(\lambda)\ \hat{T}_{0;12\ldots N}(\lambda|\{\theta_i\}), \tag{5.1}$$

where $\{\theta_i\} := \{\theta_1, \ldots, \theta_N\}$ and the monodromy matrix
$T_{0;12\ldots N}(\lambda|\{\theta_i\}) \in \mathrm{End}((\mathbb{C}^{\mathcal{N}})^{\otimes(N+1)})$ is given by

$$T_{0;12\ldots N}(\lambda|\{\theta_i\}) = R_{0N}(\lambda - \theta_N)\cdots R_{02}(\lambda - \theta_2)R_{01}(\lambda - \theta_1) \tag{5.2}$$

and satisfies (2.12). Also, $\hat{T}_{0;12\ldots N}(\lambda|\{\theta_i\}) = T^{-1}_{0;12\ldots N}(-\lambda|\{\theta_i\})$ in the case of the reflection algebra and $\hat{T}_{0;12\ldots N}(\lambda|\{\theta_i\}) = T^{t_0}_{0;12\ldots N}(-\lambda - \frac{\mathcal{N}}{2}|\{\theta_i\})$ in the case of twisted algebra. We shall consider henceforth in expression (5.2) $\theta_i = 0,\ i \in \{1, \ldots N\}$. Such a choice is justified by the fact that we wish to construct local Hamiltonians, based on the fact that $R(0) \propto \mathcal{P}$ ($\mathcal{P}$ the permutation operator), as will be transparent in the next subsection. The fact that the monodromy matrix T satisfies the RTT relation and K is a c-number solution of the refection equation guarantee that the modified monodromy $\mathcal{T}$ also satisfies the reflection equation, The elements of the modified monodromy matrix are $\mathcal{T}_{x,y}(\lambda) = \Delta^{(N)}(\mathcal{K}_{x,y}(\lambda))$ (see also discussion in the first paragraph of subsection 4.3). We also define the open or double row transfer matrix [65] as

$$\mathfrak{t}(\lambda) = tr_0\big(\hat{K}_0 \mathcal{T}_0(\lambda)\big), \tag{5.3}$$

where $\hat{K}$ is a solution of a dual quadratic equation[3] (4.1). Note that for historical reasons the space indexed by 0 is usually called the *auxiliary space*, whereas the spaces indexed by $1, 2, \dots, N$ are called *quantum spaces*. Notice also that the quantum indices are suppressed in the definitions of T, $\hat{T}$ and $\mathcal{T}$ for brevity.

To prove integrability of the open spin chain, constructed from the brace R-matrix and the corresponding K-matrices we make use of the two important properties for the R-matrix, i.e., the unitarity and crossing-unitarity (2.7) and (2.8) respectively. Indeed, using the fact that $\mathcal{T}$ and $\hat{K}$ satisfy the quadratic and dual equations (4.1), and also R satisfies the fundamental properties (2.8), (2.8) it can be shown that (see [20,21,65] for detailed proofs on the commutativity of the open transfer matrices associated to both reflection and twisted algebras):

$$\Big[\mathfrak{t}(\lambda),\ \mathfrak{t}(\mu)\Big] = 0. \tag{5.4}$$

We focus henceforth on the reflection algebra only, and we investigate the symmetries associated with the open transfer matrix for generic boundary conditions. The main goal in the context of quantum integrable systems is the derivation of the eigenvalues and eigenstates of the transfer matrix. This is in general an intricate task and the typical methodology used is the Bethe ansatz formulation, or suitable generalizations [35,50]. In the algebraic Bethe ansatz scheme the symmetries of the transfer matrices and the existence of a reference state are essential components. When an obvious reference state is not available, which is the typical scenario when considering set-theoretic solutions, certain Bethe ansatz generalizations can be used. Specifically, the methodology implemented by Faddeev and Takhtajan in [35] to solve the XYZ model, based on the application of local gauge (Darboux) transformations at each site of the spin chain can be used. The separation of variables technique, introduced by Sklyanin [66], and recently further developed for open quantum spin chains [49], can also be employed, in particular when addressing the issue of Bethe ansatz completeness, but also as a further consistency check. Moreover, we plan to generalize the findings of [56] on the role of Drinfeld twists in the algebraic Bethe ansatz, for set-theoretic solutions. This will lead to new significant connections, for instance, with generalized Gaudin-type models.

[3] The dual quadratic equation is similar to (4.1), but $\lambda_i \to -\lambda_i - \frac{\mathcal{N}}{2}$ in the arguments of R, $\hat{R}$.

5.1 Symmetries of the open transfer matrix

We shall prove in what follows some fundamental Propositions that will provide significant information on the symmetries of the double row transfer matrix (5.3). Note that henceforth we consider $\hat{K} \propto I$ in (5.3).

Let us first prove a useful lemma for the brace $\check{r}$ matrix.

Lemma 5.1 *Let $(X, \check{r})$ be a finite, involutive, non-degenerate set-theoretic solution of the Yang–Baxter equation (i.e., a solution obtained from a finite brace). Let $\check{r}$ be the brace matrix $\check{r} = \sum_{x,y\in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)}$, then $tr_0(\check{r}_{n0}) = I$.*

Proof Let $(X, \check{r})$ be our underlying set-theoretic solution. Recall that $\check{r} = \sum_{x,y\in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)}$. Observe that

$$tr_0(\check{r}_{n0}) = \sum_{(x,y)\in W} e_{x,\sigma_x(y)},$$

where $(x, y) \in W$ if and only if $y = \tau_y(x)$. Notice that if $(x, y) \in W$ then $x = \tau_y^{-1}(y)$. Observe that $\tau_y^{-1}(y)$ is always in the set X (because our sets are finite so the inverse of map τ is some power of map τ), so for each y there exist x such that $(x, y) \in W$. This implies that that for each y in X there is exactly one x in X such that (x, y) is in W, we will denote this x as $x_{[y]}$. This implies that $tr_0(\check{r}_{n0}) = \sum_{y\in X} e_{x_{[y]},\sigma_{x_{[y]}}(y)}$. We notice that $\sigma_{x_{[y]}}(y) = x_{[y]}$, it follows from the fact that $(x_{[y]}, y)$ is in W. Consequently, $tr_0(\check{r}_{n0}) = \sum_{y\in X} e_{x_{[y]},x_{[y]}}$. We notice further that if (x, y) in W and (x, z) in W then $y = z$, so for each x there is exactly one y such that (x, y) is in W. Therefore,

$$tr_0(\check{r}_{n0}) = \sum_{z\in X} e_{z,z}$$

(where z equals elements $x_{[y]}$ for different y). Hence, that $tr_0(\check{r}_{n0}) = I$ where recall I is the identity matrix of dimension equal to the cardinality of X. $\square$

The following Proposition is quite general and holds for any $R(\lambda) = \lambda\mathcal{P}\check{r} + \mathcal{P}$, and $K(\lambda) = \lambda c(\mathrm{b} - \frac{\kappa}{2}I) + I$, ($c$ is an arbitrary constant and $\kappa = Q - Q^{-1}$, see also Remark 4.4). Also, $\check{r}$ and b provide a representation of the B-type Hecke algebra $\mathcal{B}_N(q = 1, Q)$, and $\mathcal{P}$ is the permutation operator. Recall we consider $\hat{K} = I$ in the definition of the open transfer matrix (5.3).

Proposition 5.2 *Let $R(\lambda) = \lambda\mathcal{P}\check{r} + \mathcal{P}$, and $K(\lambda) = \lambda c(\mathrm{b} - \frac{\kappa}{2}I) + I$, where $\check{r}$ and b provide representations of the the B-type Hecke algebra $\mathcal{B}_N(q = 1, Q)$ and $\mathcal{P}$ is the permutation operator (c is an arbitrary constant and $\kappa = Q - Q^{-1}$). Consider the λ-series expansion of the corresponding modified monodromy matrix (5.1) : $\mathcal{T}(\lambda) = \lambda^{2N+1}\sum_{k=0}^{2N+1}\frac{\mathcal{T}^{(k)}}{\lambda^k}$, and the series expansion of the double row transfer matrix $\mathfrak{t}(\lambda) = \lambda^{2N+1}\sum_{k=0}^{2N+1}\frac{\mathfrak{t}^{(k)}}{\lambda^k}$, where $\mathfrak{t}^{(k)} = tr_0(\mathcal{T}_0^{(k)})$. Then the commuting quantities, $\mathfrak{t}^{(k)}$ for $k = 1, \ldots, 2N + 1$, are expressed exclusively in terms of the elements $\check{r}_{nn+1}$, $n = 1, \ldots, N - 1$, and b_1, provided that $tr_0(\check{r}_{N0}) = I$.*

Proof Let $T(\lambda) = \lambda^N \sum_{k=0}^{N} \frac{T^{(k)}}{\lambda^k}$, $k \in \{0, 1, \ldots, N\}$. Let us also introduce some useful notation:

$$\mathfrak{T}^{(N-k-1)} = \sum_{[n_k,n_1]} \overleftarrow{\prod_{1\le j\le k}} \check{r}_{n_j n_j+1}, \qquad \hat{\mathfrak{T}}^{(N-k-1)} = \sum_{[n_k,n_1]} \overrightarrow{\prod_{1\le j\le k}} \check{r}_{n_j n_j+1},$$

where we define $[n_k, n_1] : 1 \le n_k < \cdots < n_1 \le N-1$, and the ordered products are given as $\prod^{\rightarrow}_{1\le j\le k} \check{r}_{n_j n_j+1} = \check{r}_{n_k n_k+1}\check{r}_{n_2 n_2+1}\ldots\check{r}_{n_1 n_1+1}$, $\prod^{\leftarrow}_{1\le j\le k} \check{r}_{n_j n_j+1} = \check{r}_{n_1 n_1+1}\check{r}_{n_2 n_2+1}\ldots\check{r}_{n_k n_k+1}$, $n_1 > n_2 > \ldots n_k$.

In the proof of Proposition 4.1 in [26], all the members of the expansion of the monodromy $T^{(k)}$, were computed using the notation introduced above and the definition of the monodromy, and were expressed as: $T_0^{(N-k)} = \Big(\mathfrak{T}^{(N-k-1)} + \check{r}_{N0}\mathfrak{T}^{(N-k)}\Big)\mathcal{P}_{01}\Pi$, and similarly: $\hat{T}_0^{(N-k)} = \hat{\Pi}\mathcal{P}_{01}\Big(\hat{\mathfrak{T}}^{(N-k-1)} + \hat{\mathfrak{T}}^{(N-k)}\check{r}_{N0}\Big)$, where $\Pi = \mathcal{P}_{12}\ldots\mathcal{P}_{N-1N}$ and $\hat{\Pi} = \mathcal{P}_{N-1N}\mathcal{P}_{N-2N-1}\ldots\mathcal{P}_{12}$.

Let us also express the c-number K-matrix (4.14) (derived up to an overall constant) as: $K(\lambda) = \lambda\hat{\mathrm{b}} + I$, where $\hat{\mathrm{b}} = c\big(\mathrm{b} - \frac{\kappa}{2}I\big)$ (see also (4.14)), and recall here $\hat{K} = I$. Also, in accordance to the expansion of the monodromy matrix in the previous section we express the modified monodromy as a formal series expansion: $\mathcal{T}(\lambda) = \lambda^{2N+1}\sum_k \frac{\mathcal{T}^{(k)}}{\lambda^k}$, then each term of the expansion is expressed as:

$$\mathcal{T}_0^{(2N-n+1)} = \sum_{k,l} T_0^{(N-k)}\hat{\mathrm{b}}_0\hat{T}_0^{(N-l)}|_{k+l=n-1} + \sum_{k,l} T_0^{(N-k)}\hat{T}_0^{(N-l)}|_{k+l=n}. \tag{5.5}$$

After taking the trace and using the fact the $tr_0(\check{r}_{N0}) = I$ we conclude for the first term of the expression (5.5) above:

$$\begin{aligned} tr_0\big(T_0^{(N-k)}\hat{\mathrm{b}}_0\hat{T}_0^{(N-l)}\big)_{k+l=n-1} = {}& \mathfrak{T}^{(N-k)}\hat{\mathrm{b}}_1\hat{\mathfrak{T}}^{(N-l-1)} + \mathfrak{T}^{(N-k-1)}\hat{\mathrm{b}}_1\hat{\mathfrak{T}}^{(N-l)} \\ & + \mathcal{N}\mathfrak{T}^{(N-k-1)}\hat{\mathrm{b}}_1\hat{\mathfrak{T}}^{(N-l-1)} \\ & + tr_0\big(\check{r}_{N0}\mathfrak{T}^{(N-k)}\hat{\mathrm{b}}_1\hat{\mathfrak{T}}^{(N-l)}\check{r}_{N0}\big). \end{aligned} \tag{5.6}$$

Analogous expression is derived for the second term in (5.5), given that $\hat{\mathrm{b}} \to I$ and $k + l = n$ in the expression above. The first three terms of (5.6) are clearly expressed only in terms of the elements of the B-type Hecke algebra $\check{r}_{nn+1}$, b_1 (recall $\hat{\mathrm{b}} = c\big(\mathrm{b} - \frac{\kappa}{2}I\big)$). Let us focus on the last term: $tr_0\big(\check{r}_{N0}\mathfrak{T}^{(N-k)}\hat{\mathrm{b}}_1\hat{\mathfrak{T}}^{(N-l)}\check{r}_{N0}\big) = tr_0\Big(\check{r}_{N0}\big(\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}\big)\check{r}_{N0}\Big)$, where we define

$$\mathcal{A} = \sum_{[n_{k-1},n_1]} \overleftarrow{\prod_{1\le j\le k-1}} \check{r}_{n_j n_j+1}\hat{\mathrm{b}}_1 \sum_{[m_{l-1},m_1]} \overrightarrow{\prod_{1\le j'\le l-1}} \check{r}_{m_{j'} m_{j'}+1}$$

$$\mathcal{B} = \sum_{[n_{k-1},n_1)} \overleftarrow{\prod_{1\le j\le k-1}} \check{r}_{n_j n_j+1}\hat{\mathrm{b}}_1 \sum_{[m_{l-1},m_1]} \overrightarrow{\prod_{1\le j'\le l-1}} \check{r}_{m_{j'} m_{j'}+1}$$

$$\mathcal{C} = \sum_{[n_{k-1},n_1]} \overleftarrow{\prod_{1\le j\le k-1}} \check{r}_{n_j n_j+1} \hat{\mathrm{b}}_1 \sum_{[m_{l-1},m_1)} \overrightarrow{\prod_{1\le j'\le l-1}} \check{r}_{m_{j'} m_{j'}+1}$$

$$\mathcal{D} = \sum_{[n_{k-1},n_1)} \overleftarrow{\prod_{1\le j\le k-1}} \check{r}_{n_j n_j+1} \hat{\mathrm{b}}_1 \sum_{[m_{l-1},m_1)} \overrightarrow{\prod_{1\le j'\le l-1}} \check{r}_{m_{j'} m_{j'}+1}, \tag{5.7}$$

and $[n_{k-1}, n_j) : 1 \le n_{k-1}, < \cdots < n_j < N-1$, and $[n_{k-1}, n_j] : 1 \le n_{k-1}, < \cdots < n_j = N-1$. The last three terms above ($\mathcal{B},\ \mathcal{C},\ \mathcal{D}$) lead to the following expressions, after using the braid relation, involution and the fact that $tr_0(\check{r}_{N0}) = I$:

$$tr_0\Big(\check{r}_{N0}\mathcal{B}\check{r}_{N0}\Big) = \sum_{[n_{k-1},n_1)} \overleftarrow{\prod_{1\le j\le k-1}} \check{r}_{n_j n_j+1} \hat{\mathrm{b}}_1 \sum_{[m_{l-1},m_2)} \overrightarrow{\prod_{2\le j'\le l-1}} \check{r}_{m_{j'} m_{j'}+1}$$

$$tr_0\Big(\check{r}_{N0}\mathcal{C}\check{r}_{N0}\Big) = \sum_{[n_{k-1},n_2)} \overleftarrow{\prod_{2\le j\le k-1}} \check{r}_{n_j n_j+1} \hat{\mathrm{b}}_1 \sum_{[m_{l-1},m_1)} \overrightarrow{\prod_{1\le j'\le l-1}} \check{r}_{m_{j'} m_{j'}+1}$$

$$tr_0\Big(\check{r}_{N0}\mathcal{D}\check{r}_{N0}\Big) = \mathcal{N} \sum_{[n_{k-1},n_1)} \overleftarrow{\prod_{1\le j\le k-1}} \check{r}_{n_j n_j+1} \hat{\mathrm{b}}_1 \sum_{[m_{l-1},m_1)} \overrightarrow{\prod_{1\le j'\le l-1}} \check{r}_{m_{j'} m_{j'}+1}.$$

The terms above clearly they depend only on $\check{r}_{nn+1},\ \mathrm{b}_1$. Let us now focus on the more complicated first term of (5.7), and consider:

$$tr_0\Big(\check{r}_{N0}\mathcal{A}\check{r}_{N0}\Big) = \sum_{[n_{k-1},n_1)} \overleftarrow{\prod_{1\le j\le k-1}} \check{r}_{n_j n_j+1} \hat{\mathrm{b}}_1 \sum_{[m_{l-1},m_1)} \overrightarrow{\prod_{1\le j'\le l-1}} \check{r}_{m_{j'} m_{j'}+1}$$

$$= \sum_{[n_{k-1},n_1)} \overleftarrow{\prod_{k'+1\le j\le k-1}} \check{r}_{n_j n_j+1} \hat{\mathrm{b}}_1\ tr_0\Big(\check{r}_{N0} \overleftarrow{\prod_{1\le j\le k'}} \check{r}_{n_j n_j+1}\Big|_{c_j=0,c_{k'}>0}$$

$$\times \sum_{[m_{l-1},m_1)} \overrightarrow{\prod_{1\le j'\le l'}} \check{r}_{m_{j'} m_{j'}+1}\check{r}_{N0}\Big) \overrightarrow{\prod_{l'+1\le j'\le l-1}} \check{r}_{m_{j'} m_{j'}+1}\Big|_{c_{j'}=0,c_{l'}>0}.$$

We distinguish the following cases:

(1) $l' = k'$, then

$$tr_0\Big(\check{r}_{N0} \overleftarrow{\prod_{1\le j\le k'}} \check{r}_{n_j n_j+1} \overrightarrow{\prod_{1\le j'\le k'}} \check{r}_{m_{j'} m_{j'}+1}\check{r}_{N0}\Big) = \mathcal{N} I^{\otimes k'}.$$

(2) $|l' - k'| = 1$, then

$$tr_0\Big(\check{r}_{N0} \overleftarrow{\prod_{1\le j\le k'}} \check{r}_{n_j n_j+1} \overrightarrow{\prod_{1\le j'\le k'}} \check{r}_{m_{j'} m_{j'}+1}\ \check{r}_{N0}\Big) = I^{\otimes m}.$$

where $m = max(k', l')$

(3) $k' - l' = m + 1$, then

$$tr_0\Big(\check r_{N0} \overleftarrow{\prod_{1\le j\le k'}} \check r_{n_j n_j+1} \overrightarrow{\prod_{1\le j'\le k'}} \check r_{m_{j'} m_{j'}+1}\, \check r_{N0}\Big) = \overleftarrow{\prod_{l'+2\le j\le l'+m+1}} \check r_{n_j n_j+1}\Big|_{c_j=0}$$

(4) $l' - k' = m + 1$, then

$$tr_0\Big(\check r_{N0} \overleftarrow{\prod_{1\le j\le k'}} \check r_{n_j n_j+1} \overrightarrow{\prod_{1\le j'\le k'}} \check r_{m_{j'} m_{j'}+1}\, \check r_{N0}\Big) = \overrightarrow{\prod_{k'+2\le j\le k'+m+1}} \check r_{n_j n_j+1}\Big|_{c_j=0},$$

where we define $c_j = n_j - n_{j+1} - 1$.

It is thus clear that the factor $tr_0\big(\check r_{N0}\mathcal{A}\check r_{N0}\big)$ is also expressed in terms of the elements $\check r_{nn+1}$ and b_1. Indeed, then all the factors $\mathfrak{t}^{(k)}$, $k \in \{1, \ldots, 2N+1\}$ are expressed in terms of $\check r_{nn+1},\ \mathrm{b}_1$. However, the term
$\mathfrak{t}^{(0)} = tr_0\big(\check r_{N0}\check r_{N-1N}\ldots\check r_{12}\hat{\mathrm{b}}_1\check r_{12}\ldots\check r_{N-1N}\check r_{N0}\big)$ can not be expressed in the general case in terms of $\check r_{nn+1},\ \mathrm{b}_1$. Notice that in the special case where $\mathrm{b} = I$ we obtain $\mathfrak{t}^{(0)} \propto I^{\otimes N}$

The local Hamiltonian of the system for instance is given by the following explicit expression

$$\mathfrak{t}^{(2N)} = tr_0(\mathcal{T}_0^{(2N)}) = 2\sum_{n=1}^{N-1} \check r_{nn+1} + \hat{\mathrm{b}}_1 + 2tr_0(\check r_{N0}). \tag{5.8}$$

□

We prove below a useful Lemma:

Lemma 5.3 *The elements $\mathfrak{T}^{(i)}$ and $\hat{\mathfrak{T}}^{(i)}$, $i \in \{0,\ 1\}$, introduced on Proposition 5.2, satisfy the following relations with the A-type Hecke algebra $\mathcal{H}_N(q = 1)$ elements $\check r_{nn+1}$:*

$$\begin{aligned} \mathfrak{T}^{(i)}\check r_{nn+1} &= \check r_{n-1n}\mathfrak{T}^{(i)}, \qquad n \in \{2, \ldots N-1\} \\ \hat{\mathfrak{T}}^{(i)}\check r_{nn+1} &= \check r_{n+1n+2}\hat{\mathfrak{T}}^{(i)}, \qquad n \in \{1, \ldots N-2\} \end{aligned}$$

Proof The proof is straightforward for $\mathfrak{T}^{(0)},\ \hat{\mathfrak{T}}^{(0)}$ due to the form of $\mathfrak{T}^{(0)},\ \hat{\mathfrak{T}}^{(0)}$ and the use of the braid relation.

For $\mathfrak{T}^{(1)},\ \hat{\mathfrak{T}}^{(1)}$ the proof is a bit more involved. Let us focus on $\mathfrak{T}^{(1)}$ acting on $\check r_{nn+1}$, which can be explicitly expressed as

$$\mathfrak{T}^{(1)}\check r_{nn+1} = \Big(\mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D}\Big)\check r_{nn+1} \tag{5.9}$$

where we define: $\mathrm{A} = \sum_m \check r_{N-1N}\ldots\check r_{m+1m+2}\check r_{m-1m}\ldots\check r_{12}$ for $m \ge n+2$ or $m \le n-2$, $\mathrm{B} = \check r_{N-1N}\ldots\check r_{nn+1}\check r_{n-2n-1}\ldots\check r_{12}$, $\mathrm{C} = \check r_{N-1N}\ldots\check r_{n+2n+1}\check r_{nn+1}\ldots\check r_{12}$ and $\mathrm{D} = \check r_{N-1N}\ldots\check r_{n+12n+2}\check r_{n-1n}\ldots\check r_{12}$.

Using the braid relations and the fact that $\check{r}^2 = I^{\otimes 2}$, we show that: $\mathrm{A}\check{r}_{nn+1} = \check{r}_{n-1n}\mathrm{A}$, $\mathrm{B}\check{r}_{nn+1} = \check{r}_{n-1n}\mathrm{D}$, $\mathrm{C}\check{r}_{nn+1} = \check{r}_{n-1n}\mathrm{C}$ and $\mathrm{D}\check{r}_{nn+1} = \check{r}_{n-1n}\mathrm{B}$, which immediately lead to $\mathfrak{T}^{(1)}\check{r}_{nn+1} = \check{r}_{n-1n}\mathfrak{T}^{(1)}$, $n \in \{2, \dots N-1\}$.

The proof for $\hat{\mathfrak{T}}^{(1)}$is in exact analogy, so we omit the details here for brevity. □

For the rest of the section we focus on representations of the the B-type Hecke algebra $\mathcal{B}_N(q = 1, Q = 1)$.

Proposition 5.4 *Let $R(\lambda) = \lambda\mathcal{P}\check{r} + \mathcal{P}$, and $K(\lambda) = \lambda c\mathrm{b} + I$ (c is an arbitrary constant), where $\check{r}$ and b provide a representation of the B-type Hecke algebra $\mathcal{B}_N(q = 1, Q = 1)$, and $\mathcal{P}$ is the permutation operator. The elements of $\mathcal{T}^{(i)}$, $i \in \{0, 1\}$, introduced on Proposition 5.2, commute with the B-type Hecke algebra $\mathcal{B}_N(q = 1, Q = 1)$ generators:*

$$\left[\mathcal{T}^{(i)}_{x,y},\ \check{r}_{nn+1}\right] = \left[\mathcal{T}^{(i)}_{x,y},\ \mathrm{b}_1\right] = 0, \quad n \in \{1, \dots, N-1\}, \quad x,\ y \in X. \tag{5.10}$$

Proof We first write down explicitly the elements $\mathcal{T}^{(0)}$ and $\mathcal{T}^{(1)}$. Recall that $\mathcal{T}^{(0)} = \check{r}_{N0}\mathfrak{T}^{(0)}\hat{\mathfrak{b}}_1\hat{\mathfrak{T}}^{(0)}\check{r}_{N0}$, ($\hat{\mathfrak{b}} = c\mathrm{b}$) and from the proof of Proposition 5.2:
$\mathcal{T}^{(1)} = \mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}+I^{\otimes(N+1)}$, where $\mathfrak{a} = \check{r}_{N0}\mathfrak{T}^{(1)}\hat{\mathfrak{b}}_1\hat{\mathfrak{T}}^{(0)}\check{r}_{N0}$, $\mathfrak{b} = \check{r}_{N0}\mathfrak{T}^{(0)}\hat{\mathfrak{b}}_1\hat{\mathfrak{T}}^{(1)}\check{r}_{N0}$
$\mathfrak{c} = \mathfrak{T}^{(0)}\hat{\mathfrak{b}}_1\hat{\mathfrak{T}}^{(0)}\check{r}_{N0}$, $\mathfrak{d} = \check{r}_{N0}\mathfrak{T}^{(0)}\hat{\mathfrak{b}}_1\hat{\mathfrak{T}}^{(0)}$.

Using Lemma 5.3 and the expressions just above, we conclude: $\left[\mathcal{T}^{(i)},\ \check{r}_{nn+1}\right] = 0$, $n \in \{1, \dots N-2\}, i \in \{0, 1\}$. Moreover, using the quadratic relation of the B-type algebra $\check{r}_{12}\mathrm{b}_1\check{r}_{12}\mathrm{b}_1 = \mathrm{b}_1\check{r}_{12}\mathrm{b}_1\check{r}_{12}$ and the form of $\mathcal{T}^{(0)}$ we show that $\left[\mathcal{T}^{(0)},\ \mathrm{b}_1\right] = 0$, while use of the braid relation and the form of $\mathcal{T}^{(0)}$ lead to $\left[\mathcal{T}^{(0)},\ \check{r}_{N-1N}\right] = 0$.

It now remains to show that $\left[\mathcal{T}^{(1)},\ \check{r}_{N-1N}\right] = \left[\mathcal{T}^{(1)},\ \mathrm{b}_1\right] = 0$, the proof of the latter is more involved. Indeed, let us first focus on $\left[\mathcal{T}^{(1)},\ \mathrm{b}_1\right]$, it is convenient in this case to express the first two terms of $\mathcal{T}^{(1)}$ as $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$ and $\mathfrak{b} = \mathfrak{b}_1 + \mathfrak{b}_2$, where
$\mathfrak{a}_1 = \check{r}_{N0}\sum_{n=2}^{N-1}\left(\check{r}_{N-1N}\cdots\check{r}_{n+1n+2}\check{r}_{n-1n}\cdots\check{r}_{23}\right)\check{r}_{12}\hat{\mathfrak{b}}_1\hat{\mathfrak{T}}^{(0)}\check{r}_{N0}$,
$\mathfrak{a}_2 = \check{r}_{N0}\check{r}_{N-1N}\cdots\check{r}_{23}\hat{\mathfrak{b}}_1\mathfrak{T}^{(0)}\check{r}_{N0}$,
$\mathfrak{b}_1 = \check{r}_{N0}\mathfrak{T}^{(0)}\hat{\mathfrak{b}}_1\check{r}_{12}\sum_{n=2}^{N-1}\left(\check{r}_{23}\cdots\check{r}_{n-1n}\check{r}_{n+1n+2}\cdots\check{r}_{N-1N}\right)\check{r}_{N0}$,
$\mathfrak{b}_2 = \check{r}_{N0}\mathfrak{T}^{(0)}\hat{\mathfrak{b}}_1\check{r}_{23}\cdots\check{r}_{N-1N}\check{r}_{N0}$.
Using the quadratic relation $\check{r}_{12}\mathrm{b}_1\check{r}_{12}\hat{\mathrm{b}}_1 = \mathrm{b}_1\check{r}_{12}\mathrm{b}_1\check{r}_{12}$, and the fact that $\mathrm{b}^2 = I$ we show that: $\mathfrak{a}_1\mathrm{b}_1 = \mathrm{b}_1\mathfrak{a}_1$, $\mathfrak{b}_1\mathrm{b}_1 = \mathrm{b}_1\mathfrak{b}_1$, $\mathfrak{a}_2\mathrm{b}_1 = \mathrm{b}_1\mathfrak{b}_2$ and $\mathfrak{b}_2\mathrm{b}_1 = \mathrm{b}_1\mathfrak{a}_2$, $\mathfrak{c}\mathrm{b}_1 = \mathrm{b}_1\mathfrak{c}$, $\mathfrak{d}\mathrm{b}_1 = \mathrm{b}_1\mathfrak{d}$, which lead to $\left[\mathcal{T}^{(1)},\ \mathrm{b}_1\right] = 0$.

We lastly focus on $\left[\mathcal{T}^{(1)},\ \check{r}_{N-1N}\right]$, it is convenient in this case as well to express the first two terms of $\mathcal{T}^{(1)}$ as $\mathfrak{a} = \hat{\mathfrak{a}}_1 + \hat{\mathfrak{a}}_2$ and $\mathfrak{b} = \hat{\mathfrak{b}}_1 + \hat{\mathfrak{b}}_2$, where we define
$\hat{\mathfrak{a}}_1 = \check{r}_{N0}\check{r}_{N-1N}\sum_{n=1}^{N-2}\left(\check{r}_{N-2N-1}\cdots\check{r}_{n+1n+2}\check{r}_{n-1n}\cdots\check{r}_{12}\right)\hat{\mathfrak{b}}_1\hat{\mathfrak{T}}^{(0)}\check{r}_{N0}$,
$\hat{\mathfrak{a}}_2 = \check{r}_{N0}\check{r}_{N-2N-1}\cdots\check{r}_{12}\hat{\mathfrak{b}}_1\mathfrak{T}^{(0)}\check{r}_{N0}$,
$\hat{\mathfrak{b}}_1 = \check{r}_{N0}\mathfrak{T}^{(0)}\hat{\mathfrak{b}}_1\sum_{n=1}^{N-2}\left(\check{r}_{12}\cdots\check{r}_{n-1n}\check{r}_{n+1n+2}\cdots\check{r}_{N-2N-1}\right)\check{r}_{N-1N}\check{r}_{N0}$,
$\hat{\mathfrak{b}}_2 = \check{r}_{N0}\mathfrak{T}^{(0)}\hat{\mathfrak{b}}_1\check{r}_{12}\cdots\check{r}_{N-2N-1}\check{r}_{N0}$.
Using the braid relation and the fact that $\check{r}^2 = I^{\otimes 2}$ we show that: $\hat{\mathfrak{a}}_1\check{r}_{N-1N} = \check{r}_{N-1N}\hat{\mathfrak{a}}_1$, $\hat{\mathfrak{b}}_1\check{r}_{N-1N} = \check{r}_{N-1N}\hat{\mathfrak{b}}_1$, $\hat{\mathfrak{a}}_2\check{r}_{N-1N} = \check{r}_{N-1N}\mathfrak{c}$ and $\hat{\mathfrak{b}}_2\check{r}_{N-1N} = \check{r}_{N-1N}\mathfrak{d}$, $\mathfrak{c}\check{r}_{N-1N} = \check{r}_{N-1N}\hat{\mathfrak{a}}_2$, $\mathfrak{d}\check{r}_{N-1N} = \check{r}_{N-1N}\hat{\mathfrak{b}}_2$, which lead to $\left[\mathcal{T}^{(1)},\ \check{r}_{N-1N}\right] = 0$.
And this concludes our proof. □

Corollary 5.5 *Let $R(\lambda) = \lambda\mathcal{P}\check{r} + \mathcal{P}$, and $K(\lambda) = \lambda c \mathrm{b} + I$ (c is an arbitrary constant), where $\check{r}$ and* b *provide a representation of the B-type Hecke algebra $\mathcal{B}_N(q = 1, Q = 1)$, and $\mathcal{P}$ is the permutation operator. Let also $\mathrm{t}^{(k)}$, $k \in \{1, \ldots, 2N + 1\}$ be the mutually commuting charges as defined in Proposition 5.2, and $tr_0(\check{r}_{N0}) \propto I^{\otimes 2}$, then*

$$\left[\mathrm{t}^{(k)},\ \mathcal{T}^{(i)}_{x,y}\right] = 0, \qquad i \in \{0,\ 1\}. \tag{5.11}$$

Proof The proof is straightforward, based on Propositions 5.2 and 5.4. □

Corollary 5.6 *Let $R(\lambda) = \lambda\mathcal{P}\check{r} + \mathcal{P}$, and $K(\lambda) = \lambda c \mathrm{b} + I$ (c is an arbitrary constant), where $\check{r}$ and* b *provide a representation of the B-type Hecke algebra $\mathcal{B}_N(q = 1, Q = 1)$, and $\mathcal{P}$ is the permutation operator. Let also $\mathrm{t}^{(k)}$, $k \in \{0, \ldots, 2N + 1\}$ be the mutually commuting charges as defined in Proposition 5.2, and $tr_0(\check{r}_{N0}) \propto I^{\otimes 2}$. In the special case* $\mathrm{b} = I$*:*

$$\left[\mathrm{t}^{(k)},\ \mathcal{T}^{(1)}_{x,y}\right] = 0 \Rightarrow \left[\mathrm{t}(\lambda),\ \mathcal{T}^{(1)}_{x,y}\right] = 0. \tag{5.12}$$

Proof The proof follows directly from Propositions 5.2 and 5.4, and the fact that for $\mathrm{b} = I$, $\mathcal{T}^{(0)} \propto I^{\otimes(N+1)}$ and $\mathrm{t}^{(0)} \propto I^{\otimes N}$. □

Remark 5.7 The twisted co-products for the finite algebra generated by the element of $\mathcal{T}^{(1)}$, in the special case $\mathrm{b} = I$ can be expressed as follows, after recalling the notation introduced in the proof of Proposition 5.2:

$$\mathcal{T}^{(1)} = 2c \sum_{n=1}^{N} \left(r_{0N} r_{0N-1} \ldots r_{0n+1} \check{r}_{n0} \hat{r}_{0n+1} \ldots \hat{r}_{0N-1} \hat{r}_{0N}\right), \tag{5.13}$$

where $r = \mathcal{P}\check{r}$, $\hat{r} = \check{r}\mathcal{P}$ and $\mathcal{P}$ the permutation operator. After using expression (5.13), the brace relation and recalling that $r = \mathcal{P}\check{r},\ \hat{r} = \check{r}\mathcal{P}$, we have

$$\mathcal{T}^{(1)} = 2c \sum_{n=1}^{N} \left(\check{r}_{nn+1} \check{r}_{n+1n+2} \ldots \check{r}_{N-1N} \check{r}_{N0} \check{r}_{N-1N} \ldots \check{r}_{n+1n+2} \check{r}_{nn+1}\right). \tag{5.14}$$

Note that explicit expressions of the above co-products for (5.14) can be computed for the brace solution. We shall derive in the next subsection the co-products associated with Lyubashenko's solutions recovering the twisted co-products of Corollary 3.4.

An interesting direction to pursue is the derivation of analogous results in the case of the twisted algebras extending the findings of [12] on the duality between twisted Yangians and Brauer algebras [59] to include set-theoretic solutions. We aim at examining whether the corresponding transfer matrix can be expressed in terms of the elements of the Brauer algebra, and also check if the elements of the Brauer algebra commute with a finite sub-algebra of the twisted algebra. These findings will have significant implications on the symmetries of open transfer matrices providing valuable information on their spectrum.

5.2 More examples of symmetries

In this subsection, we present examples of symmetries of the double row transfer matrix partly inspired by the symmetries in [26], but also some new ones. Let $(X, \check{r})$ be a set-theoretic solution, as usually we denote $\check{r}(x, y) = (\sigma_x(y), \tau_y(x))$. In all the examples below, we assume that the solution $(X, \check{r})$ is involutive, non-degenerate and finite. Also, we always assume that $\hat{K} = I$ in (5.3).

The following class of symmetries is similar to those of Proposition 4.6 in [26].

Lemma 5.8 *Let $(X, \check{r})$ be a set-theoretic solution of the braid equation and let $f : X \to X$ be an isomorphism of solutions, so $f(\sigma_x(y)) = \sigma_{f(x)}(f(y))$ and $f(\tau_x(y)) = \tau_{f(x)}(f(y))$. Denote $M = \sum_{x\in X} e_{x,f(x)}$, and let $\mathfrak{t}(\lambda)$ be the double row transfer matrix for $R(\lambda) = \mathcal{P} + \lambda\mathcal{P}\check{r}$ and $K(\lambda) = \lambda c \mathrm{b} + I$, where c is an arbitrary constant and $\mathrm{b} = \sum_{x\in X} e_{x,k(x)}$. Then, given that $f(k(x)) = k(f(x))$:*

$$\Big[M^{\otimes N},\ \mathfrak{t}(\lambda)\Big] = 0.$$

Proof Notice that $M \otimes M$ commutes with $r = \mathcal{P}\check{r}$ and $\hat{r} = \check{r}\mathcal{P}$, which leads to $M^{\otimes(N+1)}T(\lambda) = T(\lambda)M^{\otimes(N+1)}$ and $M^{\otimes(N+1)}\hat{T}(\lambda) = \hat{T}(\lambda)M^{\otimes(N+1)}$, also due to $f(k(x)) = k(f(x))$ we have that $\mathrm{b}M = M\mathrm{b}$. These commutation relations then lead to $\big[\mathcal{T}(\lambda),\ M^{\otimes(N+1)}\big] = 0$, and from the latter we obtain, following the proof of Proposition 4.6 in [26], $M^{\otimes N}\mathcal{T}_{f(x),f(x)} = \mathcal{T}_{x,x}M^{\otimes N}$, which directly leads to $\big[\mathfrak{t}(\lambda),\ M^{\otimes N}\big] = 0$. $\square$

The following Lemma also follows from Proposition 4.9 in [26].

Lemma 5.9 *Let $(X, \check{r})$ be a finite, non degenerate involutive set-theoretic solution of the braid equation. Let $x_1, \dots, x_\alpha \in X$ for some $\alpha \in \{1, \dots, \mathcal{N}\}$ and assume that $\check{r}(x_i, y) = (y, x_i)$, $\forall y \in X$. Then, $\forall i, j \in \{1, 2, \dots, \alpha\}$*

$$\Big[\Delta^{(N)}(e_{x_i,x_j}),\ \mathfrak{t}(\lambda)\Big] = 0,$$

where $\Delta^{(N)}(e_{x_i,x_j}) = \sum_{n=1}^{N} I \otimes \dots \otimes \underbrace{e_{x_i,x_j}}_{n^{th}\ position} \otimes \dots \otimes I$, $\mathfrak{t}(\lambda)$ is the double row transfer matrix for $R(\lambda) = \mathcal{P} + \lambda\mathcal{P}\check{r}$ and $K(\lambda)$ such that $\big[K(\lambda),\ e_{x_i,x_j}\big] = 0$.

Proof The co-product $\Delta(e_{x_i,x_j})$ commutes with both $r = \mathcal{P}\check{r}$ and $\hat{r} = \check{r}\mathcal{P}$, then as in the proof of Proposition 4.9 in [26] it can be shown that $\big[\Delta^{(N+1)}(e_{x_i,x_j}),\ T(\lambda)\big] = \big[\Delta^{(N+1)}(e_{x_i,x_j}),\ \hat{T}(\lambda)\big] = 0$, recall also that $\big[K(\lambda),\ e_{x_i,x_j}\big] = 0$. The three commutation relations then immediately lead to $\big[\Delta^{(N+1)}(e_{x_i,x_j}),\ \mathcal{T}(\lambda)\big] = 0$. Then following the proof of Proposition 4.9 in [26], we focus on the diagonal entries of the latter commutator: $\big[\Delta^{(N)}(e_{x_i,x_j}),\ \mathcal{T}_{x_i,x_i}(\lambda)\big] = -\mathcal{T}_{x_j,x_i}(\lambda) + \delta_{ij}\mathcal{T}_{x_j,x_i}(\lambda)$, $\big[\Delta^{(N)}(e_{x_i,x_j}),\ \mathcal{T}_{x_j,x_j}(\lambda)\big] = \mathcal{T}_{x_j,x_i}(\lambda) - \delta_{ij}\mathcal{T}_{x_j,x_i}(\lambda)$ and $\big[\Delta^{(N)}(e_{x_i,x_j}),\ \mathcal{T}_{z,z}(\lambda)\big] = 0,\ z \neq x_i,\ x_j$, and we conclude that $\big[\Delta^{(N)}(e_{x_i,x_j}),\ \mathfrak{t}(\lambda)\big] = 0$. $\square$

The following Lemma is similar to Proposition 4.11 in [26], but here for the double row transfer matrix, we obtain a stronger result:

Lemma 5.10 *Let $(X, \check{r})$ be a finite, non degenerate involutive set-theoretic solution of the braid equation. Let $x_1, \dots, x_\alpha \in X$ for some $\alpha \in \{1, \dots, \mathcal{N}\}$ and assume that $\check{r}(x_i, x_i) = (x_i, x_i)$. Then, $\forall i, j \in \{1, 2, \dots, \alpha\}$*

$$\left[e_{x_i, x_j}^{\otimes N},\ \mathfrak{t}(\lambda)\right] = 0$$

where $\mathfrak{t}(\lambda)$ is the double row transfer matrix for $R(\lambda) = \mathcal{P} + \lambda \mathcal{P}\check{r}$ and $K(\lambda) \propto I$.

Proof Similarly as in the proof of Proposition 4.11 from [26] it can be shown that $e_{x_i, x_j}^{\otimes N}$ commutes with $\check{r}_{nn+1}, \forall n \in \{1, \dots, N-1\}$. The result now immediately follows from Proposition 5.2 and from the fact that for $\mathrm{b} = I$, we have $\mathcal{T}^{(0)} = I^{\otimes (N+1)}$ and $\mathfrak{t}^{(0)} = I^{\otimes N}$. □

We also present the following new examples of symmetries, different to the ones derived in [26]. Let us first introduce some invariant subsets of a set-theoretic solution. Let $(X, \check{r})$ be an involutive, non-degenerate set-theoretic solution.

Definition 5.11 Let $(X, \check{r})$ be a finite set-theoretic solution of the braid equation and let $Y \subseteq X$. Denote $\check{r}(x, y) = (\sigma_x(y), \tau_y(x))$. We say that Y is a σ-equivariant set if whenever $x, y \in Y$ then $\sigma_x(y)$ and $\tau_y(x) \in Y$.

Proposition 5.12 *Let $(X, \check{r})$ be an involutive non-degenerate solution of the braid equation. Let $Y, Z \subseteq X$ be σ-equivariant sets. Define $M_{Y,Z} = \sum_{i \in Y, j \in Z} e_{i,j}$, then*

$$\left[M_{Y,Z}^{\otimes N},\ \mathfrak{t}(\lambda)\right] = 0$$

where $\mathfrak{t}(\lambda)$ is the double row transfer matrix for $K(\lambda) \propto I$ and $R(\lambda) = \mathcal{P} + \lambda \mathcal{P}\check{r}$.

Proof By Proposition 5.2 it suffices to show that $M_{Y,Z}$ commutes with $\check{r}_{nn+1}$, $\forall n \in \{1, \dots, N-1\}$.

Observe first that

$$M \otimes M = \sum_{i,j \in Y, k,l \in Z} e_{i,k} \otimes e_{j,l}.$$

Also, $\check{r}(M \otimes M) = M \otimes M$ and $(M \otimes M)\check{r} = M \otimes M$,

$$\begin{aligned}(M \otimes M)\check{r} &= \sum_{i,j \in Y, k,l \in Z} e_{i,k} \otimes e_{j,l} \sum_{x,y \in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)} \\ &= \sum_{i,j \in Y, k,l \in Z} e_{i,\sigma_k(l)} \otimes e_{j,\tau_l(k)} = M \otimes M,\end{aligned}$$

because mapping $\check{r} : Y \otimes Y \to Y \otimes Y$ with $(k, l) \to (\sigma_k(l), \tau_l(k))$ is bijective (as explained in the end of the proof).

To show that $\check{r}(M \otimes M) = M \otimes M$ observe that, because $\check{r}$ is involutive it follows that $\check{r} = \sum_{x,y \in X} e_{\sigma_x(y),x} \otimes e_{\tau_y(x),y}$. Therefore,

$$\check{r}(M \otimes M) = \sum_{x,y \in X} e_{\sigma_x(y),x} \otimes e_{\tau_y(x),y} \sum_{i,j \in Y,k,l \in Z} e_{\sigma_i(j),k} \otimes e_{\tau_j(i),l} = M \otimes M,$$

because $\check{r} : Z \otimes Z \to Z \otimes Z$ is a bijective function.

Therefore $M^{\otimes N}$ commutes with $\check{r}_{nn+1}$, $\forall n \in \{1, \ldots, N-1\}$. The results now follows from Proposition 5.2.

To show that $\check{r}$ is a bijective function on $Y \times Y$ observe that $\check{r}$ has the zero kernel on $X \otimes X$, so is injective on $Y \otimes Y$. Notice that $\check{r}(Y \otimes Y) \subseteq Y \otimes Y$ since Y is σ-equivariant set. Because $\check{r} : Y \otimes Y \to Y \otimes Y$ is injective then $\check{r}(Y \otimes Y)$ has the same cardinality as $Y \otimes Y$, hence $\check{r} : Y \otimes Y \to Y \otimes Y$ is surjective and hence bijective. □

Remark 5.13 We we choose σ-equivariant subsets of X which have pairwise empty intersections we get similar algebra of symmetries as in the previous Lemma.

Definition 5.14 Let $z \in X$. By the *orbit* of z we will mean the smallest set $Y \subseteq X$ such that $z \in Y$ and $\sigma_x(y) \in Y$ and $\tau_x(y) \in Y$, for all $y \in Y, x \in X$.

We have also the following symmetries:

Lemma 5.15 *Let $(X, \check{r})$ be an involutive, non degenerate solution of the braid equation and let $Q_1, \ldots, Q_t$ be orbits of X.*
Define $W_{p_1,\ldots,p_t,q_1,\ldots,q_t} = \{i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_n$: exactly p_i elements among $i_1, i_2, \ldots, i_n$ belong to the orbit Q_i and exactly q_i elements among $j_1, j_2, \ldots, j_n$ belong to the orbit Q_i for every $i \leq t\}$.

Fix non-negative integers $p_1, \ldots, p_t, q_1, \ldots, q_t$, and define

$$A_{p_1,\ldots,p_t,q_1,\ldots,q_t} = \sum_{i_1,i_2,\ldots,i_n,j_1,j_2,\ldots,j_n \in W_{p_1,\ldots,p_t,q_1,\ldots,q_t}} e_{i_1,j_1} \otimes e_{i_2,j_2} \otimes \cdots \otimes e_{i_n,j_n}.$$

Then $A_{p_1,\ldots,p_t,q_1,\ldots,q_t}$ commutes with $\check{r}_{n,n+1}$ and so it commutes with the double row transfer matrix when $K(\lambda) \propto I$.

Proof It follows from the fact that if $(X, \check{r})$ is an involutive, non-degenerate set-theoretic solution of the Braid equation then $r(Q_i, Q_j) \subseteq (Q_i, Q_j)$ and it is a bijective map, for every $i, j \leq t$. □

5.3 Symmetries associated with Lyubashenko's solution.

We focus in this subsection on the symmetries of the open transfer matrix constructed from the Lyubashenko solution of Proposition 3.2.

Corollary 5.16 *Let $\mathfrak{t}(\lambda)$ be the double row transfer matrix for $R(\lambda) = \lambda\mathcal{P}\check{r} + \mathcal{P}$, where $\check{r}$ is Lyubashenko's solution of Proposition 3.2, and $K(\lambda) = \lambda c\mathrm{b} + I$, c is an arbitrary* *constant and* b *satisfies:* $\mathrm{b}^2 = I$ *and* $\mathrm{b}_1\check{r}_{12}\mathrm{b}_1\check{r}_{12} = \check{r}_{12}\mathrm{b}_1\check{r}_{12}\mathrm{b}_1$. *Then,*

$$\Big[\mathfrak{t}(\lambda),\ \mathcal{T}^{(1)}_{x,y}\Big] = 0, \quad x, y \in X. \tag{5.15}$$

Proof Recall from the notation introduced in the proof of Proposition 5.2 that $\mathcal{T}^{(0)} = r_{0N}\cdots r_{01}c\mathrm{b}_0\hat{r}_{01}\cdots\hat{r}_{0N}$, $\hat{r} = \mathcal{P}r\mathcal{P}$. Then using the fact that in the special case of Lyubashenko's solutions, $r_{0n} = \mathbb{V}_0^{-1}\mathbb{V}_n$, we can explicitly write $\mathcal{T}^{(0)} = \mathbb{V}_0^{-N}c\mathrm{b}_0\mathbb{V}_0^N$, which is a c-number matrix and $\mathfrak{t}^{(0)} = tr_0(c\mathrm{b}_0)$, which immediately leads to $\big[\mathfrak{t}^{(0)},\ \mathcal{T}^{(1)}_{x,y}\big] = 0$, and via Propositions 5.2 and 5.4 we arrive at (5.15). □

Corollary 5.17 *Let* $\check{r}$ *be Lyubashenko's solution of Proposition 3.2,* $R(\lambda) = \lambda\mathcal{P}\check{r} + \mathcal{P}$, *and* $K(\lambda) \propto I$. *Then the elements* $\mathcal{T}^{(1)}_{x,y}$ *of (5.13) are twisted co-products of* $\mathfrak{gl}_{\mathcal{N}}$, *and hence the corresponding double row transfer matrix* $\mathfrak{t}(\lambda)$ *is* $\mathfrak{gl}_{\mathcal{N}}$ *symmetric.*

Proof Recall that in the special case where $\mathrm{b} = I$ the quantity $\mathcal{T}^{(1)}$ is given in (5.13). In the case of the special solutions of Proposition 3.2 recall that $\check{r}_{n0} = \mathbb{V}_0^{-1}\mathcal{P}_{0n}\mathbb{V}_0$, then expression (5.13) simplifies to

$$\mathcal{T}^{(1)} = 2c\sum_{n=1}^{N}\big(\mathbb{V}_n^{(N-n+1)}\mathcal{P}_{0n}\mathbb{V}_n^{-(N-n+1)}\big).$$

Recall also from Proposition 3.2 that $\mathbb{V} = \sum_{x\in X} e_{\sigma(x),x}$ and $\mathcal{P} = \sum_{x,y\in X} e_{x,y}\otimes e_{y,x}$, then $\mathcal{T}^{(1)}$ can be explicitly expressed as (we set for simplicity $2c = 1$)

$$\mathcal{T}^{(1)} = \sum_{x,y\in X} e_{x,y} \otimes \Big(\sum_{n=1}^{N} I\otimes\ldots\otimes \underbrace{e_{\sigma^{N-n+1}(y),\sigma^{N-n+1}(x)}}_{n^{th}\, position}\otimes\ldots\otimes I\Big).$$

The latter expression immediately provides the elements $\mathcal{T}^{(1)}_{x,y} = \Delta_1^{(N)}(e_{\sigma(y),\sigma(x)})$, where the twisted N co-product $\Delta_1^{(N)}$ of $\mathfrak{gl}_{\mathcal{N}}$ is defined in Corollary 3.4, expression (3.13). Then due to Corollary 5.6 we deduce that the corresponding double row transfer matrix $\mathfrak{t}(\lambda)$ is $\mathfrak{gl}_{\mathcal{N}}$ symmetric And with this we conclude our proof (compare also with the results in Corollary 4.10 for $\mathbb{K}^{(0)} = I$). □

Corollary 5.18 *Let* $\check{r}$ *be Lyubashenko's solution of Proposition 3.2, and* $M = \sum_{y\in X}\alpha_y M_y$, *where* $M_y = \sum_{x\in X} e_{x,\sigma^y(x)}$ *and* $\alpha_y \in \mathbb{C}$. *Let also* $\mathfrak{t}(\lambda)$ *be the double row transfer matrix for* $R(\lambda) = \mathcal{P} + \lambda\mathcal{P}\check{r}$ *and* $K = \lambda c\mathrm{b} + I$, *where* c *is an arbitrary constant and* $\mathrm{b} = \sum_{x\in X} e_{x,k(x)}$, *then*

$$\Big[M_y^{\otimes N},\ \mathfrak{t}(\lambda)\Big] = \Big[M^{\otimes N},\ \mathfrak{t}(\lambda)\Big] = 0, \tag{5.16}$$

provided that $\sigma^y(k(x)) = k(\sigma^y(x))$.

Moreover, let $\xi \in \mathbb{C}$ *and* $A = \sum_{x \in X} \xi^x e_{x,x}$*, then*

$$\left[A^{\otimes N},\ \mathfrak{t}(\lambda)\right] = 0, \tag{5.17}$$

provided that $\xi^x = \xi^{k(x)}$ *and* $\xi^{x+y} = \xi^{\sigma(y)+\tau(x)}$.

Proof Observe that

$$\check{r}\big(e_{z,w} \otimes e_{\hat{z},\hat{w}}\big) = \big(e_{\sigma(\hat{z}),\sigma(\hat{w})} \otimes e_{\tau(z),\tau(w)}\big)\check{r}, \tag{5.18}$$

then $\check{r}(M_y \otimes M_{\hat{y}}) = (M_{\hat{y}} \otimes M_y)\check{r}$, hence $\check{r}$ as well as $r = \mathcal{P}\check{r}$ and $\hat{r} = \check{r}\mathcal{P}$ commute with $M_y \otimes M_y$ and $M \otimes M$. Moreover, $M_y,\ M$ commute with $K(\lambda)$ due to $\sigma^y(k(x)) = k(\sigma^y(x))$. $T,\ \hat{T}$ and K commute with $M_y^{\otimes(N+1)}$ and $M^{\otimes(N+1)}$, and consequently so does the double row transfer matrix $\mathcal{T}(\lambda)$. From $\left[M_y^{\otimes(N+1)},\ \mathcal{T}(\lambda)\right] = 0$ we obtain $e_{x,\sigma^y(x)} \otimes M_y^{\otimes N}\mathcal{T}_{\sigma^y(x),\sigma^y(x)} = e_{x,\sigma^y(x)} \otimes \mathcal{T}_{x,x}M_y^{\otimes N}$, then $\sum_{x\in X} M_y^{\otimes N}\mathcal{T}_{\sigma^y(x),\sigma^y(x)} = \sum_{x\in X} \mathcal{T}_{x,x}M_y^{\otimes N}$, similarly for M, which lead to (5.16).

Notice that in the special case $\mathfrak{b} = I$ (5.16) follows also immediately from the fact that $M_y^{\otimes N}$ and $M^{\otimes N}$ commute with $\check{r}_{nn+1}$, $\forall n \in \{1, \ldots, N-1\}$ and Proposition 5.2.

Similarly, via (5.18) and the fact that $\xi^{x+y} = \xi^{\sigma(y)+\tau(x)}$ we show that $\left[A \otimes A,\ \check{r}\right] = 0$, and hence $\left[T(\lambda),\ A^{\otimes(N+1)}\right] = \left[\hat{T}(\lambda),\ A^{\otimes(N+1)}\right] = 0$. Moreover, due to $\xi^x = \xi^{k(x)}$ we show that $\left[K(\lambda),\ A\right] = 0$, and consequently $\left[\mathcal{T}(\lambda),\ A^{\otimes(N+1)}\right] = 0$. By taking the trace over the auxiliary space in the latter commutator, we arrive at (5.17). □

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