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An infinite family of higher-order difference operators that commute with Ruijsenaars operators of type *A*

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Abstract
We introduce a new infinite family of higher-order difference operators that commute with the elliptic Ruijsenaars difference operators of type A. These operators are related to Ruijsenaars' operators through a formula of Wronski type.

Keywords Ruijsenaars operators · Wronski formula · Macdonald polynomials

Mathematics Subject Classification 81R12 · 33D67

1 Introduction

In Ruijsenaars's study of relativistic quantum integrable systems [11,12], he introduced a commuting family of linear difference operators in n variables, denoted by $D_1, \ldots, D_n$, involving sigma functions as coefficients. In this paper, we construct an explicit infinite family of difference operators $H_0, H_1, H_2, \ldots$ in the commutative algebra $\mathbb{C}[D_1, \ldots, D_n]$ which are related to D_r $(r = 1, \ldots, n)$ through a formula of Wronski type. This construction is applicable also to difference operators with trigonometric and rational coefficients. In order to deal with the elliptic, trigonometric and rational cases simultaneously, as in [5,6] we formulate our results in terms of an entire function $[z]$ satisfying the three-term relation (1.1).

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We fix a nonzero entire function $[z]$ in one complex variable $z \in \mathbb{C}$ satisfying the three-term relation of Hirota type

$$[z \pm \alpha][\beta \pm \gamma] + [z \pm \beta][\gamma \pm \alpha] + [z \pm \gamma][\alpha \pm \beta] = 0 \qquad (1.1)$$

for any $\alpha, \beta, \gamma \in \mathbb{C}$, where $[\alpha \pm \beta] = [\alpha + \beta][\alpha - \beta]$. We remark that a generic solution of this functional equation is given by

$$[z] = \text{const.}\, e^{cz^2} \sigma(z; \Omega) \qquad (c \in \mathbb{C}), \qquad (1.2)$$

where $\sigma(z; \Omega)$ denotes the Weierstrass sigma function associated with a period lattice $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. It is also satisfied by the functions

$$[z] = \text{const.}\, e^{cz^2} \sin(\pi z/\omega), \quad [z] = \text{const.}\, e^{cz^2} z, \qquad (1.3)$$

which are trigonometric and rational degenerations of the generic solution above. It is known that any solution of (1.1) belongs to one of these three categories. Throughout this paper, we denote by $D_r = D_r^x$ $(r = 1, \ldots, n)$ the Ruijsenaars operators in n variables $x = (x_1, \ldots, x_n)$ with parameters (δ, κ), associated with $[z]$. They are defined by

$$D_r^x = \sum_{I \subseteq \{1,\ldots,n\};\, |I|=r} \prod_{i \in I;\, j \notin I} \frac{[x_i - x_j + \kappa]}{[x_i - x_j]} \prod_{i \in I} T_{x_i}^{\delta}, \qquad (1.4)$$

where I runs over all subsets of indices of cardinality r and $T_{x_i}^{\delta}$ stands for the δ-shift operator $x_i \to x_i + \delta$ in x_i for each $i = 1, \ldots, n$. It is proved by Ruijsenaars [11] that these operators D_r commute with each other, namely

$$D_r D_s = D_s D_r \qquad \text{for all } r, s = 1, \ldots, n, \qquad (1.5)$$

on the basis of a certain functional identity for the sigma function. We denote by $\mathcal{R} = \mathbb{C}[D_1, \ldots, D_n]$ the commutative algebra generated by D_r $(r = 1, \ldots, n)$ and refer to it as the commutative algebra of *Ruijsenaars operators* (of type A_{n-1}). We also define $D_0 = 1$, and $D_r = 0$ for $r > n$. Note also that these operators D_r $(r = 1, \ldots, n)$ in the trigonometric case are the Macdonald q-difference operators expressed additively.

We define an infinite family of difference operators $H_l = H_l^x$ $(l = 0, 1, 2 \ldots)$ by

$$H_l^x = \sum_{\mu_1+\cdots+\mu_n=l} \prod_{1 \le i < j \le n} \frac{[x_i - x_j + (\mu_i - \mu_j)\delta]}{[x_i - x_j]} \prod_{i=1}^{n} \prod_{j=1}^{n} \prod_{k=0}^{\mu_i - 1} \frac{[x_i - x_j + \kappa + k\delta]}{[x_i - x_j + \delta + k\delta]} \prod_{i=1}^{n} T_{x_i}^{\mu_i \delta}. \qquad (1.6)$$

These operators are expressed briefly as

$$H_l^x = \sum_{\mu \in \mathbb{N}^n;\, |\mu|=l} \frac{\Delta(x + \mu\delta)}{\Delta(x)} \prod_{i,j=1}^{n} \frac{[x_i - x_j + \kappa]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i}} T_x^{\mu\delta}, \qquad (1.7)$$

in terms of the difference product $\Delta(x) = \prod_{1\le i<j\le n}[x_i - x_j]$, and the δ-shifted factorials

$$[z]_k = [z][z+\delta]\cdots[z+(k-1)\delta] \quad (k=0,1,2,\ldots) \tag{1.8}$$

for $[z]$. Also, for each multi-index $\mu = (\mu_1,\ldots,\mu_n) \in \mathbb{N}^n$, $\mathbb{N} = \{0,1,2,\ldots\}$, we define $|\mu| = \mu_1 + \cdots + \mu_n$ and $T_x^{\mu\delta} = T_{x_1}^{\mu_1\delta}\cdots T_{x_n}^{\mu_n\delta}$.

Theorem 1.1 *The linear difference operators $H_l = H_l^x$ $(l=0,1,2,\ldots)$ defined as above belong to the commutative algebra $\mathcal{R} = \mathbb{C}[D_1,\ldots,D_n]$ of Ruijsenaars operators. In particular, one has*

$$D_r H_s = H_s D_r, \qquad H_r H_s = H_s H_r \qquad (r,s=0,1,2,\ldots). \tag{1.9}$$

We first remark that the family of difference operators H_l $(l=0,1,2,\ldots)$ originates from the kernel identities for Ruijsenaars operators ([6]) and the duality transformation formulas for multiple elliptic hypergeometric series ([5,10]). Let $G(z)$ be a nonzero meromorphic function on $\mathbb{C}$ such that $G(z+\delta) = [z]\,G(z)$, and define the kernel function $\Phi(x;y)$ of Cauchy type by

$$\Phi(x;y) = \prod_{i=1}^{n}\prod_{k=1}^{n}\frac{G(x_i+y_k)}{G(x_i+y_k+\kappa)} \tag{1.10}$$

for $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$. Then, from [5], Theorem 1.3, we have the kernel identity

$$D_r^x\,\Phi(x;y) = D_r^y\,\Phi(x;y) \qquad (r=0,1,\ldots,n) \tag{1.11}$$

for the Ruijsenaars operators D_r $(r=0,1,\ldots,r)$. On the other hand, the duality transformation formula for multiple elliptic hypergeometric series implies

$$\begin{aligned}
&\sum_{\mu\in\mathbb{N}^m;\,|\mu|=r}\frac{\Delta(x+\mu\delta)}{\Delta(x)}\prod_{i,j=1}^{n}\frac{[x_i-x_j+\kappa]_{\mu_i}}{[x_i-x_j+\delta]_{\mu_i}}\prod_{i,k=1}^{n}\frac{[x_i+y_k]_{\mu_i}}{[x_i+y_k+\kappa]_{\mu_i}}\\
&=\sum_{\nu\in\mathbb{N}^m;\,|\nu|=r}\frac{\Delta(y+\nu\delta)}{\Delta(y)}\prod_{k,l=1}^{n}\frac{[y_k-y_l+\kappa]_{\nu_k}}{[y_k-y_l+\delta]_{\nu_k}}\prod_{k,i=1}^{n}\frac{[y_k+x_i]_{\nu_k}}{[y_k+x_i+\kappa]_{\nu_k}} \quad (r=0,1,2,\ldots),
\end{aligned} \tag{1.12}$$

as the special case where $m=n$ and $a_i=b_i=\kappa$ $(i=1,\ldots,n)$ in the notation of [5], Theorem 2.2. This means that

$$H_r^x\,\Phi(x;y) = H_r^y\,\Phi(x;y) \quad (r=0,1,2,\ldots). \tag{1.13}$$

Namely, the kernel function for the Ruijsenaars operators D_r $(r=0,1,\ldots,n)$ simultaneously intertwines the operators H_r $(r=0,1,2\ldots)$. In view of this fact, it would be reasonable to expect that the operators H_r should already belong to the commutative algebra $\mathbb{C}[D_1,\ldots,D_n]$ of Ruijsenaars operators. Theorem 1.1 ensures that it is actually the case.

In this paper, we prove Theorem 1.1 as a consequence of the following recurrence formula *of Wronski type* for H_l $(l = 0, 1, 2, \ldots)$.

Theorem 1.2 *The difference operators H_l $(l = 0, 1, 2, \ldots)$ satisfy the following recurrence formula in relation to D_r $(r = 1, \ldots, n)$:*

$$\sum_{r+s=l} (-1)^r [r\kappa + s\delta]\, D_r\, H_s = 0 \qquad (l = 1, 2, \ldots). \tag{1.14}$$

Recall that the elementary symmetric functions

$$e_r = e_r(\xi) = \sum_{1\le i_1<\cdots<i_r\le m} \xi_{i_1}\cdots\xi_{i_r} \qquad (r = 0, 1, 2, \ldots) \tag{1.15}$$

in $\xi = (\xi_1, \ldots, \xi_n)$, and the complete homogeneous symmetric functions

$$h_l = h_l(\xi) = \sum_{\mu_1+\cdots+\mu_n=l} \xi_1^{\mu_1}\cdots\xi_n^{\mu_n} \qquad (l = 0, 1, 2, \ldots) \tag{1.16}$$

are related to each other through the *Wronski formula*[1]

$$\sum_{r+s=l} (-1)^r e_r\, h_s = 0 \qquad (l = 1, 2, \ldots). \tag{1.17}$$

See [7]. Theorem 1.2 can be thought of as an operator version of this Wronski formula for symmetric functions. A proof of Theorem 1.2 will be given in Sect. 2 by using a functional identity for $[z]$ (Lemma 2.3).

From the recurrence formulas

$$\begin{aligned} &[\delta]H_1 - [\kappa]D_1 = 0,\\ &[2\delta]H_2 - [\kappa+\delta]D_1H_1 + [2\kappa]D_2 = 0,\\ &[3\delta]H_3 - [\kappa+2\delta]D_1H_2 + [2\kappa+\delta]D_2H_1 - [3\kappa]D_3 = 0,\\ &\quad\ldots, \end{aligned} \tag{1.18}$$

we see inductively that H_l belongs to the commutative algebra $\mathcal{R} = \mathbb{C}[D_1, \ldots, D_n]$ of Ruijsenaars operators for all $l = 0, 1, 2, \ldots$. In fact, we have

$$\begin{aligned} H_1 &= \frac{[\kappa]}{[\delta]}D_1,\\ H_2 &= \frac{[\kappa][\kappa+\delta]}{[\delta][2\delta]}D_1^2 - \frac{[2\kappa]}{[2\delta]}D_2,\\ H_3 &= \frac{[\kappa][\kappa+\delta][\kappa+2\delta]}{[\delta][2\delta][3\delta]}D_1^3 - \frac{[2\kappa][\kappa+2\delta]}{[2\delta][3\delta]}D_2D_1 - \frac{[\kappa][2\kappa+\delta]}{[\delta][3\delta]}D_1D_2 + \frac{[3\kappa]}{[3\delta]}D_3,\\ &\ldots. \end{aligned} \tag{1.19}$$

[1] These relations (1.17) are attributed to Józef Maria Hoene-Wroński in [1] and [14] (see also [15]).

The relationship between the two families of difference operators D_r $(r = 01, 2, \dots)$ and H_l $(l = 0, 1, 2, \dots)$ is described in the following.

Proposition 1.3 *For each $l = 0, 1, 2, \dots$, the difference operator H_l is expressed in terms of D_r $(r = 0, 1, \dots)$ by the determinant formula*

$$H_l = \det\left(\frac{[(i-j+1)\kappa+(j-1)\delta]}{[i\delta]} D_{i-j+1}\right)_{i,j=1}^{l} \qquad (l = 0, 1, 2, \dots). \tag{1.20}$$

Conversely,

$$D_l = \det\left(\frac{[(i-j+1)\delta+(j-1)\kappa]}{[i\kappa]} H_{i-j+1}\right)_{i,j=1}^{l} \qquad (l = 0, 1, 2, \dots). \tag{1.21}$$

Proposition 1.4 *For each $l = 1, 2, \dots$, the difference operator H_l is expressed explicitly as*

$$H_l = \sum_{d=1}^{l} (-1)^{l-d} \sum_{r_1+\cdots+r_d=l;\, r_i\geq 1} \left(\prod_{i=1}^{d} \frac{[(r_1+\cdots+r_{i-1})\delta + r_i\kappa]}{[(r_1+\cdots+r_i)\delta]}\right) D_{r_1}\cdots D_{r_d} \tag{1.22}$$

in terms of D_r $(r = 0, 1, \dots)$.

We also summarize the kernel identities relevant to the difference operators D_r and H_r for the sake of reference.

Theorem 1.5 (1) *For two sets of variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, the kernel function $\Phi(x, y)$ of Cauchy type in (1.10) satisfies the following two types of kernel identities*:

$$(DD) \qquad D_r^x \Phi(x; y) = D_r^y \Phi(x; y) \qquad (r = 0, 1, \dots, n), \tag{1.23}$$

$$(HH) \qquad H_r^x \Phi(x; y) = H_r^y \Phi(x; y) \qquad (r = 0, 1, 2, \dots). \tag{1.24}$$

(2) *For two sets of variables $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$, let*

$$\Psi(x; y) = \prod_{i=1}^{m}\prod_{k=1}^{n}[x_i - y_k] \tag{1.25}$$

be the kernel function of dual Cauchy type. Under the balancing condition $m\kappa + n\delta = 0$, $\Psi(x; y)$ satisfies the kernel identity

$$(HD) \qquad H_r^x \Psi(x; y) = (-1)^r \widehat{D}_r^y \Psi(x; y) \qquad (r = 0, 1, 2, \dots), \tag{1.26}$$

where $\widehat{D}_r^y$ denotes the difference operator obtained from D_r^y by exchanging the parameters δ and κ.

Propositions 1.3 and 1.4 are consequences of the recurrence formula of Wronski type (see Sect. 3). After a complement on kernel identities for the Ruijsenaars operators (Sect. 4), we finally give some remarks on the trigonometric case in Sect. 5.

Notes: This paper is based on a collaboration of the authors which was completed as master's thesis [13] of the second author in Japanese. Also, an earlier version of the present paper, written around 2012, has been circulated among some researchers. For these reasons, some of the results in this paper are already cited in several studies [2,3,8,9] with reference to a private communication or to a paper in preparation.

2 Recurrence formula of Wronski type

In this section, we give a proof of Theorem 1.2. Our goal is to establish the recurrence formula of Wronski type between the two sequences of difference operators D_r ($r = 0, 1, 2, \ldots$) and H_s ($s = 0, 1, 2, \ldots$).

Theorem 2.1 *The difference operators* H_l ($l = 0, 1, 2 \ldots$) *defined by* (1.6) *satisfy the recurrence formulas*

$$\sum_{r+s=l} (-1)^r [r\kappa + s\delta]\, D_r H_s = 0 \qquad (l = 1, 2, \ldots). \tag{2.1}$$

Since $D_0 = 1$, by this theorem we see inductively that H_l belong to $\mathbb{C}[D_1, \ldots, D_n]$ for all $l = 0, 1, 2, \ldots$.

Theorem 2.2 *The difference operators* H_l ($l = 0, 1, 2 \ldots$) *belong to the commutative algebra* $\mathbb{C}[D_1, \ldots, D_n]$ *of Ruijsenaars operators. In particular, one has*

$$D_r H_s = H_s D_r, \qquad H_r H_s = H_s H_r \qquad (r, s = 0, 1, 2, \ldots). \tag{2.2}$$

Proof of Theorem 2.1 We express the difference operators D_r as

$$D_r = \sum_{|I|=r} A_I(x) T_x^{\epsilon_I \delta}, \qquad A_I(x) = \prod_{i\in I;\ j\notin I} \frac{[x_i - x_j + \kappa]}{[x_i - x_j]} = \frac{\Delta(x + \epsilon_I \kappa)}{\Delta(x)}, \tag{2.3}$$

where we define $\epsilon_I = \sum_{i\in I} \epsilon_i$ by using the unit vectors $\epsilon_1, \ldots, \epsilon_n$ of $\mathbb{N}^n$. Similarly, we express H_l as

$$H_l = \sum_{|\mu|=l} H_\mu(x) T_x^{\mu\delta}, \quad H_\mu(x) = \frac{\Delta(x + \mu\delta)}{\Delta(x)} \prod_{i,j=1}^{n} \frac{[x_j - x_i + \kappa]_{\mu_j}}{[x_j - x_i + \delta]_{\mu_j}}. \tag{2.4}$$

When $r+s=l$, we compute

$$\begin{aligned} D_r H_s &= \sum_{|I|=r} \sum_{|\mu|=s} A_I(x) H_\mu(x+\epsilon_I\delta) T_x^{(\epsilon_I+\mu)\delta} \\ &= \sum_{|\lambda|=l} \sum_{I\subseteq \operatorname{supp}(\lambda);\, |I|=r} A_I(x) H_{\lambda-\epsilon_I}(x+\epsilon_I\delta) T_x^{\lambda\delta}, \end{aligned} \tag{2.5}$$

where $\operatorname{supp}(\lambda)=\{i\in\{1,\dots,n\}\mid \lambda_i>0\}$. Hence, the recurrence formula (2.1) is equivalent to

$$\sum_{|\lambda|=l}\Bigg(\sum_{I\subseteq \operatorname{supp}(\lambda)} (-1)^{|I|}\big[\,|I|\kappa+(|\lambda|-|I|)\delta\,\big] A_I(x) H_{\lambda-\epsilon_I}(x+\epsilon_I\delta)\Bigg) T_x^{\lambda\delta}=0, \tag{2.6}$$

saying that

$$\sum_{I\subseteq \operatorname{supp}(\lambda)} (-1)^{|I|}\big[\,|I|\kappa+(|\lambda|-|I|)\delta\,\big] A_I(x) H_{\lambda-\epsilon_I}(x+\epsilon_I\delta)=0 \tag{2.7}$$

for any $\lambda\in\mathbb{N}^n$ with $|\lambda|>0$. We now make the expression $A_I(x)H_{\lambda-\epsilon_I}(x+\epsilon_I\delta)$ explicit. Setting $L=\operatorname{supp}(\lambda)$, we have

$$\begin{aligned} A_I(x)H_{\lambda-\epsilon_I}(x+\epsilon_I\delta) &= \frac{\Delta(x+\epsilon_I\kappa)}{\Delta(x)}\frac{\Delta(x+\lambda\delta)}{\Delta(x+\epsilon_I\delta)} \\ &\cdot \prod_{i\in I;\, j\in I}\frac{[x_j-x_i+\kappa]_{\lambda_j-1}}{[x_j-x_i+\delta]_{\lambda_j-1}} \prod_{i\in I;\, j\in L\setminus I}\frac{[x_j-x_i+\kappa-\delta]_{\lambda_j}}{[x_j-x_i]_{\lambda_j}} \\ &\cdot \prod_{i\notin I;\, j\in I}\frac{[x_j-x_i+\kappa+\delta]_{\lambda_j-1}}{[x_j-x_i+2\delta]_{\lambda_j-1}} \prod_{i\notin I;\, j\in L\setminus I}\frac{[x_j-x_i+\kappa]_{\lambda_j}}{[x_j-x_i+\delta]_{\lambda_j}}. \end{aligned} \tag{2.8}$$

Noting that

$$\begin{aligned} \frac{\Delta(x+\epsilon_I\kappa)}{\Delta(x)}\frac{\Delta(x+\lambda\delta)}{\Delta(x+\epsilon_I\delta)} &= \frac{\Delta(x+\lambda\delta)}{\Delta(x)}\frac{\Delta(x+\epsilon_I\kappa)}{\Delta(x+\epsilon_I\delta)} \\ &= \frac{\Delta(x+\lambda\delta)}{\Delta(x)}\prod_{i\notin I;\, j\in I}\frac{[x_j-x_i+\kappa]}{[x_j-x_i+\delta]}, \end{aligned} \tag{2.9}$$

we can compute $A_I(x)H_{\lambda-\epsilon_I}(x+\epsilon_I\delta)$ as follows:

$$\begin{aligned} A_I(x)H_{\lambda-\epsilon_I}(x+\epsilon_I\delta) &= \frac{\Delta(x+\lambda\delta)}{\Delta(x)}\prod_{i\notin I;\, j\in L}\frac{[x_j-x_i+\kappa]_{\lambda_j}}{[x_j-x_i+\delta]_{\lambda_j}} \\ &\cdot \prod_{i\in I;\, j\in I}\frac{[x_j-x_i+\kappa]_{\lambda_j-1}}{[x_j-x_i+\delta]_{\lambda_j-1}} \prod_{i\in I;\, j\in L\setminus I}\frac{[x_j-x_i+\kappa-\delta]_{\lambda_j}}{[x_j-x_i]_{\lambda_j}} \end{aligned}$$

$$= \frac{\Delta(x+\lambda\delta)}{\Delta(x)} \prod_{i\in\{1,\ldots,n\};\, j\in L} \frac{[x_j - x_i + \kappa]_{\lambda_j}}{[x_j - x_i + \delta]_{\lambda_j}} \cdot \prod_{i\in I;\, j\in L\setminus I} \frac{[x_j - x_i + \kappa - \delta]}{[x_j - x_i]} \prod_{i\in I;\, j\in L} \frac{[x_j - x_i + \lambda_i\delta]}{[x_j - x_i + \kappa + (\lambda_j - 1)\delta]}. \tag{2.10}$$

Hence, (2.7) is equivalent to

$$\sum_{I\subseteq L} (-1)^{|I|} \big[\,|I|\kappa + (|\lambda| - |I|)\delta\,\big] \prod_{i\in I;\, j\in L\setminus I} \frac{[x_j - x_i + \kappa - \delta]}{[x_j - x_i]} \prod_{i\in I;\, j\in L} \frac{[x_j - x_i + \lambda_j\delta]}{[x_j - x_i + \kappa + (\lambda_j - 1)\delta]} = 0 \tag{2.11}$$

for any $L \neq \phi$ and $\lambda \in \mathbb{N}^n$ with $\mathrm{supp}(\lambda) = L$. Setting $\lambda = \epsilon_L + \nu$, we rewrite this in the form

$$\sum_{I\subseteq L} (-1)^{|I|} \big[\,|I|\kappa + (|\nu| + |L| - |I|)\delta\,\big] \prod_{i\in I;\, j\in L\setminus I} \frac{[x_j - x_i + \kappa - \delta]}{[x_j - x_i]} \prod_{i\in I;\, j\in L} \frac{[x_j - x_i + \delta + \nu_j\delta]}{[x_j - x_i + \kappa + \nu_j\delta]} = 0 \tag{2.12}$$

for any $\nu \in \mathbb{N}^n$ with $\mathrm{supp}(\nu) \subseteq L$. Since this formula contains only those variables x_i with $i \in L$, we have only to consider the case where $L = \{1, \ldots, n\}$ $(n \geq 1)$:

$$\sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|} \big[\,|I|\kappa + (|\nu| + n - |I|)\delta\,\big] \cdot \prod_{i\in I;\, j\notin I} \frac{[x_j - x_i + \kappa - \delta]}{[x_j - x_i]} \prod_{i\in I;\, j\in\{1,\ldots,n\}} \frac{[x_j - x_i + \delta + \nu_j\delta]}{[x_j - x_i + \kappa + \nu_j\delta]} = 0 \tag{2.13}$$

for any $\nu \in \mathbb{N}^n$. This identity follows from the following functional identity by the change of variables

$$z_i = x_i, \quad w_i = x_i + \delta + \nu_i\delta \quad (i = 1, \ldots, n); \quad a = \kappa - \delta. \tag{2.14}$$

□

Lemma 2.3 *Given two sets of variables* $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n)$ *and a parameter* a, *the following identity holds as a meromorphic function in* $(z_1, \ldots, z_n, w_1, \ldots, w_n)$ *for* $n \geq 1$:

$$\sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|} \frac{[|w| - |z| + |I|a]}{[\,|w| - |z|\,]} \prod_{i\in I;\, j\notin I} \frac{[z_j - z_i + a]}{[z_j - z_i]} \prod_{i\in I;\, k\in\{1,\ldots,n\}} \frac{[w_k - z_i]}{[w_k - z_i + a]} = 0, \tag{2.15}$$

where $|z| = \sum_{i=1}^{n} z_i$, $|w| = \sum_{k=1}^{n} w_k$.

Proof We give a proof of the functional identity (2.15) for the case where $[z] = \sigma(z; \Omega)$ is the Weierstrass sigma function associated with a period lattice Ω. By the classification of $[z]$, it is not difficult to derive (2.15) for any $[z]$ in this class from the case of $\sigma(z; \Omega)$, by the limiting procedures from $\sigma(z; \Omega)$ to $\sin(\pi z/\omega)$ and z, and by the invariance of (2.15) under the transformation $[z] \to e^{cz^2}[z]$.

Identity (2.15) for $[z] = \sigma(z; \Omega)$ can be proved by the induction on n. Since it holds trivially for $n = 1$, we assume $n \geq 2$. We regard the left-hand side of (2.15) as a meromorphic function of w_n and denote it by $F(w_n)$ assuming that the other variables are generic. Note first that $F(w_n)$ is an elliptic function possibly with simple poles at $w_n \equiv z_1 - a, \ldots, z_n - a$ and $w_n \equiv |z| - |w'|$ modulo the period lattice Ω, where $w' = (w_1, \ldots, w_{n-1})$. We first compute the residue of $F(w_n)$ at $w_n = z_n - a$. Nontrivial residues possibly arise from the terms corresponding to I containing n; we parametrize such I's as $I = J \cup \{n\}$ with $J \subseteq \{1, \ldots, n-1\}$. Then, we have

$$
\begin{aligned}
&\mathrm{Res}_{w_n = z_n - a}(F(w_n) dw_n) \\
&= \frac{[a][|w'| - |z'|]}{[|w'| - |z'| - a]} \prod_{j \in \{1,\ldots,n-1\}} \frac{[z_j - z_n + a]}{[z_j - z_n]} \prod_{k \in \{1,\ldots,n-1\}} \frac{[w_k - z_n]}{[w_k - z_n + a]} \\
&\quad \cdot \sum_{J \subseteq \{1,\ldots,n-1\}} (-1)^{|J|} \frac{[|w'| - |z'| + |J|a]}{[|w'| - |z'|]} \\
&\qquad \cdot \prod_{i \in J;\, j \in \{1,\ldots,n-1\} \setminus J} \frac{[z_j - z_i + a]}{[z_j - z_i]} \prod_{i \in J;\, k \in \{1,\ldots,n-1\}} \frac{[w_k - z_i]}{[w_l - z_i + a]} \\
&= 0
\end{aligned}
\tag{2.16}
$$

by the induction hypothesis ($z' = (z_1, \ldots, z_{n-1})$). Since $F(w_n)$ is symmetric with respect to $(z_1, \ldots, z_n)$, we see that $w_n \equiv z_1 - a, \ldots, z_n - a$ are all removable poles of $F(w_n)$. Hence, $F(w_n)$ has at most one simple pole in each fundamental parallelogram, which is impossible unless $F(w_n)$ is a constant function since it is an elliptic function. We next look at the value of $F(w_n)$ at $w_n = z_n$. It is computed as

$$
\begin{aligned}
F(z_n) = &\sum_{I \subseteq \{1,\ldots,n-1\}} (-1)^{|I|} \frac{[|w'| - |z'| + |I|a]}{[|w'| - |z'|]} \\
&\cdot \prod_{i \in \{1,\ldots,n-1\} \setminus I;\, j \in I} \frac{[z_j - z_i + a]}{[z_j - z_i]} \prod_{i \in I;\, k \in \{1,\ldots,n-1\}} \frac{[w_k - z_i]}{[w_k - z_i + a]} \\
= &\ 0
\end{aligned}
\tag{2.17}
$$

again by the induction hypothesis. This implies that $F(w_n)$ is identically zero as a meromorphic function of w_n. $\square$

Remark 2.4 Lemma 2.3 can be proved in a different way if we make use of the argument of [5]. Recall that in (1.14) of Sect. 1, [5], the following identity is derived from the determinant formula of Frobenius:

$$\frac{E(T_z;u)D(z;w)}{D(z;w)} = \sum_{I\subseteq\{1,\ldots,n\}} u^{|I|}\frac{[\lambda+|z|+|w|+|I|\delta]}{[\lambda+|z|+|w|]}\prod_{i\in I;j\notin I}\frac{[z_i-z_j+\delta]}{[z_i-z_j]}\prod_{i\in I;j\in\{1,\ldots,n\}}\frac{[z_i+w_k]}{[z_i+w_k+\delta]}, \tag{2.18}$$

where $D(z;w)$ is the Frobenius determinant

$$D(z;w)=\det\left(\frac{[\lambda+z_i+w_j]}{[\lambda][z_i+w_j]}\right)_{i,j=1}^{n}=\frac{[\lambda+|z|+|w|]\Delta(z)\Delta(w)}{[\lambda]\prod_{i,j=1}^{n}[z_i+w_j]}, \tag{2.19}$$

and $E(T_z;u)=\prod_{i=1}^{n}(1+uT_{z_i}^{\delta})$. This means that

$$\frac{[\lambda]\prod_{i,j=1}^{n}[z_i+w_j]}{[\lambda+|z|+|w|]\Delta(z)\Delta(w)}\det\left(\frac{[\lambda+z_i+w_j]}{[\lambda][z_i+w_j]}+u\frac{[\lambda+z_i+w_j+\delta]}{[\lambda][z_i+w_j+\delta]}\right)_{i,j=1}^{n} = \sum_{I\subseteq\{1,\ldots,n\}} u^{|I|}\frac{[\lambda+|z|+|w|+|I|\delta]}{[\lambda+|z|+|w|]}\prod_{i\in I;j\notin I}\frac{[z_i-z_j+\delta]}{[z_i-z_j]}\prod_{i\in I;j\in\{1,\ldots,n\}}\frac{[z_i+w_k]}{[z_i+w_k+\delta]}. \tag{2.20}$$

By setting $u=-1$, we obtain

$$\frac{[\lambda]\prod_{i,j=1}^{n}[z_i+w_j]}{[\lambda+|z|+|w|]\Delta(z)\Delta(w)}\det\left(\frac{1}{[\lambda]}\left(\frac{[\lambda+z_i+w_j]}{[z_i+w_j]}-\frac{[\lambda+z_i+w_j+\delta]}{[z_i+w_j+\delta]}\right)\right)_{i,j=1}^{n} = \sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\frac{[\lambda+|z|+|w|+|I|\delta]}{[\lambda+|z|+|w|]}\prod_{i\in I;j\notin I}\frac{[z_i-z_j+\delta]}{[z_i-z_j]}\prod_{i\in I;j\in\{1,\ldots,n\}}\frac{[z_i+w_k]}{[z_i+w_k+\delta]}. \tag{2.21}$$

Note that in the limit $\lambda\to 0$, each entry of the matrix of the left-hand side has a finite limit

$$\lim_{\lambda\to 0}\frac{1}{[\lambda]}\left(\frac{[\lambda+z_i+w_j]}{[z_i+w_j]}-\frac{[\lambda+z_i+w_j+\delta]}{[z_i+w_j+\delta]}\right)=\frac{[z_i+w_j]'}{[z_i+w_j]}-\frac{[z_i+w_j+\delta]'}{[z_i+w_j+\delta]}. \tag{2.22}$$

Hence, the left-hand side converges to zero as $\lambda\to 0$. This implies that

$$\sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\frac{[|z|+|w|+|I|\delta]}{[|z|+|w|]}\prod_{i\in I;j\notin I}\frac{[z_i-z_j+\delta]}{[z_i-z_j]}\prod_{i\in I;j\in\{1,\ldots,n\}}\frac{[z_i+w_k]}{[z_i+w_k+\delta]}=0. \tag{2.23}$$

By replacing each z_i with $-z_i$, and δ with a, we obtain Lemma 2.3.

3 Explicit relations between the two commuting families

Setting

$$D_r^{(l)} = \frac{[r\kappa + (l-r)\delta]}{[l\delta]} D_r \qquad (0 \le r \le l), \tag{3.1}$$

we rewrite the recurrence formula of Theorem 2.1 as

$$(-1)^l H_l + (-1)^{l-1} D_1^{(l)} H_{l-1} + \cdots - D_{l-1}^{(l)} H_1 = -D_l^{(l)} \qquad (l = 1, 2, \ldots). \tag{3.2}$$

In the matrix form, this means that

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ D_1^{(2)} & 1 & 0 & \ldots & \vdots \\ D_2^{(3)} & D_1^{(3)} & 1 & & \\ \vdots & \vdots & & \ddots & 0 \\ D_{l-1}^{(l)} & D_{l-2}^{(l)} & \ldots & D_1^{(l)} & 1 \end{bmatrix} \begin{bmatrix} -H_1 \\ H_2 \\ -H_3 \\ \vdots \\ (-1)^l H_l \end{bmatrix} = - \begin{bmatrix} D_1^{(1)} \\ D_2^{(2)} \\ D_3^{(3)} \\ \vdots \\ D_l^{(l)} \end{bmatrix}. \tag{3.3}$$

Hence, by Cramer's formula we obtain

$$H_l = \det \begin{bmatrix} D_1^{(1)} & 1 & 0 & \ldots & 0 \\ D_2^{(2)} & D_1^{(2)} & 1 & & \vdots \\ \vdots & D_2^{(3)} & D_1^{(3)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ D_l^{(1)} & D_{l-1}^{(l)} & \ldots & \ldots & D_1^{(l)} \end{bmatrix} = \det \left(D_{i-j+1}^{(i)} \right)_{i,j=1}^{l} \qquad (l = 1, 2, \ldots). \tag{3.4}$$

Namely, we have

$$H_l = \det \left(\frac{[(i-j+1)\kappa + (j-1)\delta]}{[i\delta]} D_{i-j+1} \right)_{i,j=1}^{l} \qquad (l = 1, 2. \ldots). \tag{3.5}$$

By symmetry, we also have

$$D_l = \det \left(\frac{[(i-j+1)\delta + (j-1)\kappa]}{[i\kappa]} H_{i-j+1} \right)_{i,j=1}^{l} \qquad (l = 1, 2. \ldots). \tag{3.6}$$

The recurrence formula (2.1) can also be written as

$$H_l = D_1^{(l)} H_{l-1} - \cdots + (-1)^{l-2} D_{l-1}^{(l)} H_1 + (-1)^{l-1} D_l^{(l)} \qquad (l = 1, 2, \ldots). \tag{3.7}$$

Applying this formula repeatedly, we obtain

$$\begin{aligned} H_l &= \sum_{d=1}^{l} (-1)^{l-d} \sum_{0=l_0<l_1<\cdots<l_d=l} D^{(l_d)}_{l_d-l_{d-1}} D^{(l_{d-1})}_{l_{d-1}-l_{d-2}} \cdots D^{(l_1)}_{l_1-l_0} \\ &= \sum_{d=1}^{l} (-1)^{l-d} \sum_{r_1+\cdots+r_d=l;\, r_i>0} \prod_{i=1}^{d} D^{(r_1+\cdots+r_i)}_{r_i} \\ &= \sum_{d=1}^{l} (-1)^{l-d} \sum_{r_1+\cdots+r_d=l;\, r_i>0} \left(\prod_{i=1}^{d} \frac{[(r_1+\cdots+r_{i-1})\delta + r_i\kappa]}{[(r_1+\cdots+r_i)\delta]} \right) D_{r_1}\cdots D_{r_d}. \end{aligned} \tag{3.8}$$

4 Kernel identities

We recall from [5] the duality transformation formula for multiple elliptic hypergeometric series (of type A): Under the balancing condition $a_1+\cdots+a_m = b_1+\cdots+b_n$,

$$\begin{aligned} &\sum_{\mu\in\mathbb{N}^m;\,|\mu|=r} \frac{\Delta(x+\mu\delta)}{\Delta(x)} \prod_{i,j=1}^{m} \frac{[x_i-x_j+a_j]_{\mu_i}}{[x_i-x_j+\delta]_{\mu_i}} \prod_{i=1}^{m}\prod_{k=1}^{n} \frac{[x_i+y_k-b_k]_{\mu_i}}{[x_i+y_k]_{\mu_i}} \\ &= \sum_{\nu\in\mathbb{N}^n;\,|\nu|=r} \frac{\Delta(y+\nu\delta)}{\Delta(y)} \prod_{k,l=1}^{n} \frac{[y_k-y_l+b_l]_{\nu_k}}{[y_k-y_l+\delta]_{\nu_k}} \prod_{k=1}^{n}\prod_{i=1}^{m} \frac{[y_k+x_i-a_i]_{\nu_k}}{[y_k+x_i]_{\nu_k}} \qquad (r=0,1,2,\ldots), \end{aligned} \tag{4.1}$$

where $x=(x_1,\ldots,x_m)$ and $y=(y_1,\ldots,y_n)$. As we already remarked, when $m=n$ and $a_i=b_i=\kappa$ $(i=1,\ldots,n)$, this implies

$$H^x_r\Phi(x;y) = H^y_r\Phi(x;y) \qquad (r=0,1,2,\ldots). \tag{4.2}$$

Let $a_1=\cdots=a_m=\kappa$, $b_1=\cdots=b_n=-\delta$. Then, this transformation formula implies that under the condition $m\kappa+n\delta=0$,

$$\begin{aligned} &\sum_{\mu\in\mathbb{N}^m;\,|\mu|=r} \frac{\Delta(x+\mu\delta)}{\Delta(x)} \prod_{i,j=1}^{m} \frac{[x_i-x_j+\kappa]_{\mu_i}}{[x_i-x_j+\delta]_{\mu_i}} \prod_{i=1}^{m}\prod_{k=1}^{n} \frac{[x_i+y_k+\mu_i\delta]}{[x_i+y_k]} \\ &= (-1)^r \sum_{K\subseteq\{1,\ldots,n\};\ |K|=r} \prod_{k\in K,\, l\notin K} \frac{[y_k-y_l-\delta]}{[y_k-y_l]} \prod_{k\in K}\prod_{i=1}^{m} \frac{[y_k+x_i-k_i]}{[y_k+x_i]} \qquad (r=0,1,2,\ldots). \end{aligned} \tag{4.3}$$

By replacing y_k by $-y_k$ for $k=1,\ldots,n$, we obtain

$$\sum_{\mu\in\mathbb{N}^m;\,|\mu|=r} \frac{\Delta(x+\mu\delta)}{\Delta(x)} \prod_{i,j=1}^{m} \frac{[x_i-x_j+\kappa]_{\mu_i}}{[x_i-x_j+\delta]_{\mu_i}} \prod_{i=1}^{m}\prod_{k=1}^{n} \frac{[x_i-y_k+\mu_i\delta]}{[x_i-y_k]}$$

$$= (-1)^r \sum_{K\subseteq\{1,\ldots,n\};\ |K|=r} \prod_{k\in K,\, l\notin K} \frac{[y_k - y_l + \delta]}{[y_k - y_l]} \prod_{k\in K}\prod_{i=1}^{m} \frac{[x_i - y_k - \kappa]}{[x_i - y_k]} \qquad (r = 0, 1, 2, \ldots). \tag{4.4}$$

This means that the dual Cauchy kernel

$$\Psi(x; y) = \prod_{i=1}^{m}\prod_{k=1}^{n}[x_i - y_k] \tag{4.5}$$

satisfies

$$H_r^{(x;\delta,\kappa)}\Psi(x; y) = (-1)^r D_r^{(y;\kappa,\delta)}\Psi(x; y) \qquad (r = 0, 1, 2, \ldots) \tag{4.6}$$

under the condition $m\kappa + n\delta = 0$.

5 The trigonometric cases

In this section, we consider the trigonometric cases where $[x] = e(x/2) - e(-x/2)$ in the notation $e(u) = \exp(2\pi\sqrt{-1}u)$ of the exponential function. Instead of the parameter $\delta, \kappa \in \mathbb{C}$, we use the multiplicative parameters $q = e(\delta)$ and $t = e(\kappa)$ assuming that $\mathrm{Im}(\delta) > 0$ so that $|q| < 1$. Note that when $z = e(x)$, we have $[x] = z^{\frac{1}{2}} - z^{-\frac{1}{2}} = -z^{-\frac{1}{2}}(1 - z)$, and hence,

$$[x]_k = (-1)^k q^{-\frac{1}{2}\binom{k}{2}} z^{-\frac{1}{2}}(z; q)_k \qquad (k = 0, 1, 2, \ldots), \tag{5.1}$$

in the standard notation $(z; q)_k = (1-z)(1-qz)\cdots(1-q^{k-1}z)$ of q-shifted factorials.

We denote by $z = (z_1, \ldots, z_n)$ the multiplicative variables defined by $z_i = e(x_i)$ $(i = 1, \ldots, n)$ corresponding to $x = (x_1, \ldots, x_n)$. For these z variables, we denote by T_{q,z_i} the q-shift operator with respect to z_i $(i = 1, \ldots, n)$ and set $T_{q,z}^{\mu} = T_{q,z_1}^{\mu_1}\cdots T_{q,z_n}^{\mu_n}$ for each multi-index $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$. In this multiplicative notation, it is convenient to introduce the q-difference operators $\mathcal{D}_r^z$ and $\mathcal{H}_l^z$ normalized so that

$$D_r^x = t^{-\frac{1}{2}r(n-r)}\mathcal{D}_r^z \quad (r = 0, 1, \ldots, n), \quad H_l^x = q^{-\frac{1}{2}l}t^{-\frac{1}{2}ln}\mathcal{H}_l^z \quad (l = 0, 1, 2, \ldots). \tag{5.2}$$

These q-difference operators are given explicitly by

$$\mathcal{D}_r^z = t^{\binom{r}{2}} \sum_{I\subseteq\{1,\ldots,n\};\ |I|=r} \prod_{i\in I;\ j\notin I} \frac{tz_i - z_j}{z_i - z_j} T_{q,z}^{\epsilon_I} \tag{5.3}$$

and

$$\mathcal{H}_l^z = \sum_{\mu\in\mathbb{N}^n;\ |\mu|=l} \prod_{1\le i<j\le n} \frac{q^{\mu_i}z_i - q^{\mu_j}z_j}{z_i - z_j} \prod_{1\le i,j\le n} \frac{(tz_i/z_j; q)_{\mu_i}}{(qz_i/z_j; q)_{\mu_i}} T_{q,x}^{\mu}. \tag{5.4}$$

The recurrence relation (2.1) of Wronski type is then rewritten as follows:

$$\sum_{r+s=l}(-1)^r(1-t^rq^s)\mathcal{D}_r^z\,\mathcal{H}_s^z=0\quad(l=1,2,\ldots).\tag{5.5}$$

It is known by [7] that the commuting family of q-difference operators $\mathcal{D}_r^z$ ($r=0,1,\ldots,n$) act on the ring $\mathbb{C}[z]^{\mathfrak{S}_n}=\mathbb{C}[z_1,\ldots,z_n]^{\mathfrak{S}_n}$ of symmetric polynomials in $z=(z_1,\ldots,z_n)$ and that they are simultaneously diagonalized by the (monic) Macdonald polynomials $P_\lambda(z)=P_\lambda(z|q,t)$ indexed by partitions $\lambda=(\lambda_1,\ldots,\lambda_n)$ with $l(\lambda)\le n$:

$$\mathcal{D}_r^zP_\lambda(z)=P_\lambda(z)\,e_r(t^\delta q^\lambda)\quad(r=0,1,2,\ldots,n),\tag{5.6}$$

where $e_r(\xi)$ stands for the elementary symmetric polynomial of degree r for each $r=0,1,\ldots,n$, and $t^\delta q^\lambda=(t^{n-1}q^{\lambda_1},t^{n-2}q^{\lambda_2},\ldots,q^{\lambda_n})$. In terms of the generating function

$$\mathcal{D}^z(u)=\sum_{r=0}^n(-u)^r\mathcal{D}_r^z=\sum_{I\subseteq\{1,\ldots,n\}}t^{\binom{|I|}{2}}\prod_{i\in I;\,j\notin I}\frac{tz_i-z_j}{z_i-z_j}\,T_{q,z}^{\epsilon_I},\tag{5.7}$$

formula (5.6) is equivalent to

$$\mathcal{D}^z(z)P_\lambda(z)=P_\lambda(z)\prod_{i=1}^n(1-ut^{n-i}q^{\lambda_i}).\tag{5.8}$$

Since $\mathcal{H}_l^z\in\mathbb{C}[\mathcal{D}_1^z,\ldots,\mathcal{D}_n^z]$, the q-difference operators $\mathcal{H}_l^z$ ($l=0,1,2\ldots$) satisfy

$$\mathcal{H}_l^z\,P_\lambda(z)=P_\lambda(z)\,g_l(t^\delta q^\lambda)\qquad(l=0,1,2,\ldots)\tag{5.9}$$

for some symmetric polynomials $g_l(\xi)\in\mathbb{C}[\xi]^{\mathfrak{S}_n}$. By the Wronski-type formula (5.5), these polynomials are determined by the recurrence relation

$$\sum_{r+s=0}(-1)^r(1-t^rq^s)e_r(\xi)\,g_s(\xi)=0\quad(l=1,2,\ldots).\tag{5.10}$$

In view of

$$E(\xi;u)=\sum_{r=0}^n(-u)^re_r(\xi)=\prod_{i=1}^n(1-u\xi_i),\tag{5.11}$$

let us introduce the generating function $G(\xi;u)=\sum_{l=0}^\infty u^l\,g_l(\xi)$. Then, the recurrence formula above is equivalent to the functional equation

$$E(\xi;u)\,G(\xi;u)=E(\xi;tu)\,G(\xi;qu),\tag{5.12}$$

namely

$$G(\xi;u)=\frac{E(\xi;tu)}{E(\xi;u)}G(\xi;qu)=\left(\prod_{i=1}^{n}\frac{1-tu\xi_i}{1-u\xi_i}\right)G(\xi;qu). \tag{5.13}$$

Hence, we have

$$G(\xi;u)=\sum_{l=0}^{\infty}u^l g_l(\xi)=\prod_{i=1}^{n}\frac{(tu\xi_i;q)_\infty}{(u\xi_i;q)_\infty}, \tag{5.14}$$

where $(u;q)_\infty=\prod_{i=0}^{\infty}(1-q^iu)$. This means that

$$g_l(\xi)=\sum_{\nu_1+\cdots+\nu_n=l}\frac{(t;q)_{\nu_1}\cdots(t;q)_{\nu_n}}{(q;q)_{\nu_1}\cdots(q;q)_{\nu_n}}\,\xi_1^{\nu_1}\cdots\xi_n^{\nu_n}=\frac{(t;q)_l}{(q;q)_l}P_{(l)}(\xi)\quad(l=0,1,2,\ldots). \tag{5.15}$$

We introduce the generation function

$$\mathcal{H}^z(u)=\sum_{l=0}^{\infty}u^l\,\mathcal{H}^z_l=\sum_{\mu\in\mathbb{N}^n}u^{|\mu|}\prod_{1\le i<j\le n}\frac{q^{\mu_i}z_i-q^{\mu_j}z_j}{z_i-z_j}\prod_{1\le i,j\le n}\frac{(tz_i/z_j;q)_{\mu_i}}{(qz_i/z_j;q)_{\mu_i}}\,T^{\mu}_{q,x} \tag{5.16}$$

for our q-difference operators $\mathcal{H}^z_l$ $(l=0,1,2,\ldots)$. Then, the argument above implies that

$$\mathcal{H}^z(u)\,P_\lambda(x)=P_\lambda(x)\prod_{i=1}^{n}\frac{(ut^{n-i+1}q^{\lambda_i};q)_\infty}{(ut^{n-i}q^{\lambda_i};q)_\infty} \tag{5.17}$$

for any partition $\lambda=(\lambda_1,\ldots,\lambda_n)$ with $l(\lambda)\le n$. Note also that the recurrence formula of Wronski type is equivalent to

$$\mathcal{D}^z(u)\mathcal{H}^z(u)=\mathcal{D}^z(tu)\mathcal{H}^z(qu). \tag{5.18}$$

Finally, we give comments on the kernel identities for the trigonometric case. Consider two sets of variables $z=(z_1,\ldots,z_m)$ and $w=(w_1,\ldots.w_n)$, assuming that $m\ge n$. The Cauchy-type kernel for this case is given by

$$\Pi(z;w)=\prod_{i=1}^{m}\prod_{k=1}^{n}\frac{(tz_iw_k;q)_\infty}{(z_iw_k;q)_\infty}. \tag{5.19}$$

Then, we have the kernel identities

$$(DD)\qquad \mathcal{D}^z(u)\Pi(z;w)=(u;t)_{m-n}\mathcal{D}^w(t^{m-n}u)\Pi(z;w), \tag{5.20}$$

$$(HH) \qquad \mathcal{H}^z(u)\Pi(z;w) = \frac{(t^{m-n}u;q)_\infty}{(u;q)_\infty}\mathcal{H}^w(t^{m-n}u)\Pi(z;w). \tag{5.21}$$

By the kernel function of dual Cauchy type

$$\Psi(z;w) = \prod_{i=1}^{m}\prod_{k=1}^{n}(z_i - w_k), \tag{5.22}$$

the two families of q-difference operators are exchanged as follows:

$$(HD) \qquad (u;q)_\infty \mathcal{H}^z(u)\Psi(z;w) = (t^m q^n u;q)_\infty \widehat{\mathcal{D}}^w(u)\Psi(z;w), \tag{5.23}$$

where $\widehat{\mathcal{D}}^w(u) = \mathcal{D}^{(w|t,q)}(u)$ denotes the q-difference operator obtained from $\mathcal{D}^w(u) = \mathcal{D}^{(w|q,t)}(u)$ by exchanging q and t.

The three kernel identities (DD), (HH) and (HD) are equivalent to certain special cases of Kajihara's Euler transformation formula [4]: For two sets of variables $z = (z_1,\ldots,z_m)$, $w = (w_1,\ldots,w_n)$ and parameters $a = (a_1,\ldots,a_m)$, $b = (b_1,\ldots,b_n)$,

$$\begin{aligned}
&\frac{(u/\alpha;q)_\infty}{(u;q)_\infty}\sum_{\mu\in\mathbb{N}^m}(u/\alpha)^{|\mu|}\prod_{1\le i<j\le m}\frac{q^{\mu_i}z_i - q^{\mu_j}z_j}{z_i - z_j}\prod_{i,j=1}^{m}\frac{(a_j z_i/z_j;q)_{\mu_i}}{(qz_i/z_j;q)_{\mu_i}}\\
&\quad\prod_{i=1}^{m}\prod_{l=1}^{n}\frac{(z_i w_l/b_l;q)_{\mu_i}}{(z_i w_l;q)_{\mu_i}}\\
&= \frac{(u/\beta;q)_\infty}{(u;q)_\infty}\sum_{\nu\in\mathbb{N}^n}(u/\beta)^{|\nu|}\prod_{1\le k<l\le n}\frac{q^{\nu_k}w_k - q^{\nu_l}w_l}{w_k - w_l}\prod_{k,l=1}^{n}\frac{(b_l w_k/w_l;q)_{\nu_k}}{(qw_k/w_l;q)_{\nu_k}}\\
&\quad\prod_{k=1}^{n}\prod_{j=1}^{m}\frac{(w_k z_j/a_j;q)_{\nu_k}}{(w_k z_j;q)_{\nu_k}},
\end{aligned} \tag{5.24}$$

where $\alpha = a_1\cdots a_m$ and $\beta = b_1\cdots b_n$. In fact, one can verify directly that these three kernel identities are equivalent to the following special cases of (5.24), respectively:

$$\begin{aligned}
&(DD): a_j = q^{-1}\ (j=1,\ldots,m),\ b_l = q^{-1}\ (l=1,\ldots,n),\\
&(HH): a_j = t\qquad (j=1,\ldots,m),\ b_l = t\qquad (l=1,\ldots,n),\\
&(HD): a_j = t\qquad (j=1,\ldots,m),\ b_l = q^{-1}\ (l=1,\ldots,n).
\end{aligned} \tag{5.25}$$

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