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Twisted submanifolds of $\mathbb{R}^n$

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Abstract

We propose a general procedure to construct noncommutative deformations of an embedded submanifold M of $\mathbb{R}^n$ determined by a set of smooth equations $f^a(x) = 0$. We use the framework of Drinfel'd twist deformation of differential geometry of Aschieri et al. (Class Quantum Gravity 23:1883, 2006); the commutative pointwise product is replaced by a (generally noncommutative) $\star$-product determined by a Drinfel'd twist. The twists we employ are based on the Lie algebra Ξ_t of vector fields that are tangent to *all* the submanifolds that are level sets of the f^a (tangent infinitesimal diffeomorphisms); the twisted Cartan calculus is automatically equivariant under twisted Ξ_t. We can consistently project a connection from the twisted $\mathbb{R}^n$ to the twisted M if the twist is based on a suitable Lie subalgebra $\mathfrak{e} \subset \Xi_t$. If we endow $\mathbb{R}^n$ with a metric, then twisting and projecting to the normal and tangent vector fields commute, and we can project the Levi–Civita connection consistently to the twisted M, provided the twist is based on the Lie subalgebra $\mathfrak{k} \subset \mathfrak{e}$ of the Killing vector fields of the metric; a twisted Gauss theorem follows, in particular. Twisted algebraic manifolds can be characterized in terms of generators and $\star$-polynomial relations. We present in some detail twisted cylinders embedded in twisted Euclidean $\mathbb{R}^3$ and twisted hyperboloids embedded in twisted Minkowski $\mathbb{R}^3$ [these are twisted (anti-)de Sitter spaces dS_2, AdS_2].

Keywords Drinfel'd twist · Deformation quantization · Noncommutative geometry · Hopf algebras, their representations · Tangent, normal vector fields · First, second fundamental form

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1 Introduction

The notion of a submanifold N of a manifold M is a fundamental concept in (differential) geometry, playing a crucial role in various branches of mathematics and physics. A metric, connection, etc., on M uniquely induces (see, e.g., [43]) a metric, connection, etc., on N. In the last few decades, various deep physical and mathematical reasons have stimulated the generalization of differential geometry to so-called noncommutative geometry (NCG) [15,41,45,47,48]. In particular, NCG has been advocated as a suitable framework for formulating a fundamental (or at least an effective) theory of quantum spacetime allowing the quantization of gravity (see, e.g., [1,20,21]) and/or for unifying fundamental interactions (see, e.g., [12,13,16]). It is therefore natural and important to investigate whether and to what extent a notion of a submanifold is possible and fruitful in the NCG framework. Surprisingly, this question has received little systematic attention so far (rather isolated exceptions are the papers [40,50,58]). On several noncommutative (NC) spaces, one can make sense of special classes of NC submanifolds, but some features of the latter may depart from their commutative counterparts. For instance, from the noncommutative algebra "of functions on the quantum group $U_q(n)$", which is generated by the n^2 matrix elements of a $n \times n$ unitary matrix, one can obtain the one $\mathcal{A}$ on the quantum group $SU_q(n)$ by imposing that the so-called q-determinant (a suitable central element) be 1, as in the commutative ($q = 1$) limit, but the so-called quantum group bicovariant differential calculus on $\mathcal{A}$ (i.e., the corresponding $\mathcal{A}$-bimodule Ω of 1-forms) remains of dimension n^2 instead of $n^2 - 1$ [60]. The same phenomenon occurs, e.g., obtaining the $SO_q(n)$-covariant quantum Euclidean spheres S_q^{n-1} from the $SO_q(n)$-covariant quantum Euclidean spaces $\mathbb{R}_q^n$, by imposing that the [central and $SO_q(n)$-invariant] square distance r^2 from the origin be 1; said differently, the 1-form dr^2 cannot be imposed to vanish, and actually the graded commutator $\left[\frac{1}{q^2-1}r^{-2}dr^2,\,\cdot\,\right]$ acts as the exterior derivative [11,30,33,55].

In the present work, we wish to address the above question systematically within the framework of deformation quantization [6,7] (for a review see [56]) in the particular approach based on Drinfel'd twisting [22] of Hopf algebras. We restrict our attention to the noncommutative generalization of embedded submanifolds of $\mathbb{R}^n$, because by the Whitney embedding theorems [46] one can always embed a smooth manifold M in $\mathbb{R}^n$ with a sufficiently high dimension n. More precisely, we shall assume that $M \subset \mathbb{R}^n$ consists of points of $x \in \mathbb{R}^n$ fulfilling a set of equations

$$f^a(x) = 0, \qquad a = 1, 2, \ldots, k < n, \tag{1}$$

where $f \equiv (f^1, \ldots, f^k) : \mathbb{R}^n \to \mathbb{R}^k$ are smooth functions such that the Jacobian matrix $J = \partial f/\partial x$ is of rank k on all $\mathbb{R}^n$; or, more generally, that f is well defined and J is of rank k on an open subset $\mathcal{D}_f \subset \mathbb{R}^n$, and M consists of the points of $\mathcal{D}_f$ fulfilling (1). In all our examples here $\mathcal{E}_f := \mathbb{R}^n \backslash \mathcal{D}_f$ will be empty or of zero measure. By replacing in (1) $f^a(x) \mapsto f_c^a(x) := f^a(x) - c^a$, with $c \equiv (c^1, \ldots, c^k) \in f\left(\mathcal{D}_f\right)$, we obtain a whole k-parameter family of embedded manifolds M_c ($M_0 = M$) of dimension $n-k$ that are level sets of f. Embedded submanifolds $N \subset M$ can be obtained by adding more equations to (1).[1] The $*$-algebra $\mathcal{X}^M$ of smooth complex-valued functions on M can be expressed as the quotient of the $*$-algebra $\mathcal{X} = C^\infty(\mathcal{D}_f)$ of smooth functions on $\mathcal{D}_f$ over the ideal $\mathcal{C} \subset \mathcal{X}$ of smooth functions vanishing on M:

$$\mathcal{X}^M := \mathcal{X}/\mathcal{C} \equiv \left\{ [\alpha] := \alpha + \mathcal{C} \mid \alpha \in \mathcal{X} \right\}; \tag{2}$$

In the appendix, we prove that $\mathcal{C}$ is generated by the left-hand sides (lhs) of (1):

Theorem 1 $\mathcal{C} = \bigoplus_{a=1}^k \mathcal{X} f^a = \bigoplus_{a=1}^k f^a \mathcal{X}$, *i.e., for all* $h \in \mathcal{C}$ *there exist* $h^a \in \mathcal{X}$ *such that*

$$h(x) = \sum_{a=1}^{k} h^a(x) f^a(x) = \sum_{a=1}^{k} f^a(x) h^a(x). \tag{3}$$

Similarly, $\mathcal{X}^N$ can be obtained as the quotient of $\mathcal{X}^M$ over the ideal generated by further equations of the type (1), or equivalently as the quotient of $\mathcal{X}$ over the larger ideal generated by all such equations. Identifying vector fields with first-order differential operators, we denote as $\Xi := \{X = X^i \partial_i \mid X^i \in \mathcal{X}\}$ the Lie algebra of smooth vector fields X on $\mathcal{D}_f$; here and below we abbreviate $\partial_i \equiv \partial/\partial x^i$. Those vector fields $X \in \Xi$ such that $X(f^a)$ belong to $\mathcal{C}$ for all a, or equivalently such that $X(h)$ belongs to $\mathcal{C}$ if h does (i.e., vanishes when restricted to M) make up a Lie subalgebra $\Xi_{\mathcal{C}}$, which is also a $\mathcal{X}$-bimodule; those such that $X(h)$ belongs to $\mathcal{C}$ for all $h \in \mathcal{X}$ make up a smaller Lie subalgebra $\Xi_{\mathcal{C}\mathcal{C}}$, which is actually an ideal in $\Xi_{\mathcal{C}}$ and itself a $\mathcal{X}$-bimodule. It decomposes as $\Xi_{\mathcal{C}\mathcal{C}} = \bigoplus_{a=1}^k f^a \Xi$. The Lie algebra Ξ_M of vector fields tangent to M can be identified with that of derivations of $\mathcal{X}^M$, namely with the quotient

$$\Xi_M := \Xi_{\mathcal{C}}/\Xi_{\mathcal{C}\mathcal{C}} \equiv \left\{ [X] := X + \Xi_{\mathcal{C}\mathcal{C}} \mid X \in \Xi_{\mathcal{C}} \right\}. \tag{4}$$

If $f^a(x)$ are polynomial functions fulfilling suitable irreducibility conditions and we set $\mathcal{X} = \mathrm{Pol}^\bullet(\mathbb{R}^n)$, the $*$-algebra of complex-valued polynomial functions on $\mathbb{R}^n$ (instead of $\mathcal{X} = C^\infty(\mathcal{D}_f)$), then again the $*$-algebra $\mathcal{X}^M$ of complex-valued polynomial functions on M can be expressed as the quotient $\mathcal{X}^M = \mathcal{X}/\mathcal{C}$, where $\mathcal{C} \subset \mathcal{X}$ is the ideal of polynomial functions vanishing on M, $\Xi := \{X = X^i \partial_i \mid X^i \in \mathcal{X}\}$ is the Lie algebra of polynomial vector fields X on $\mathbb{R}^n$, etc. $\mathcal{C}$ can be decomposed again in the form (3), with $\mathcal{X} = \mathrm{Pol}^\bullet(\mathbb{R}^n)$ [38].

Often one is interested in noncommutative *deformations* of differential geometry on a manifold, i.e., in *families* of NCGs depending on a formal parameter ν and reducing to the original one if we formally set $\nu = 0$. Deformation quantization [6,7,44,56] provides a general framework to deform $\mathcal{X}$ into a noncommutative algebra $\mathcal{X}_\star$ over $\mathbb{C}[[\nu]]$ (the ring of formal power series in ν with coefficients in $\mathbb{C}$): as a

[1] Given a smooth manifold M, one could introduce such a $N \subset M$ also *patchwise*, i.e., as a level set of equations of the type (1) on each chart of an atlas of M fulfilling suitable matching conditions in intersecting charts.

module over $\mathbb{C}[[\nu]]$ $\mathcal{X}_\star$ coincides with $\mathcal{X}[[\nu]]$, but the commutative pointwise product $\alpha\beta$ of $\alpha, \beta \in \mathcal{X}$ ($\mathbb{C}[[\nu]]$-bilinearly extended to $\mathcal{X}[[\nu]]$) is deformed into a possibly noncommutative (but still associative) product,

$$\alpha\star\beta = \alpha\beta + \sum\nolimits_{l=1}^{\infty} \nu^l B_l(\alpha, \beta), \tag{5}$$

where B_l are suitable bidifferential operators of degree l at most. We wish to deform $\mathcal{X}^M$ into a noncommutative algebra $\mathcal{X}^M_\star$ in the form of a quotient:

$$\mathcal{X}^M_\star := \mathcal{X}_\star/\mathcal{C}_\star \equiv \left\{ [\alpha] := \alpha + \mathcal{C}_\star \mid \alpha \in \mathcal{X}_\star \right\}, \tag{6}$$

with $\mathcal{C}_\star$ a two-sided ideal of $\mathcal{X}_\star$. To make also $\mathcal{X}^M_\star = \mathcal{X}^M[[\nu]]$ hold as an equality of $\mathbb{C}[[\nu]]$-modules we require that $\mathcal{C}_\star = \mathcal{C}[[\nu]]$, i.e., that $c\star\alpha,\ \alpha\star c \in \mathcal{C}[[\nu]]$ for all $\alpha \in \mathcal{X}, c \in \mathcal{C}$, so that $(\alpha + c)\star(\alpha' + c') - \alpha\star\alpha' \in \mathcal{C}[[\nu]]$ for all $\alpha, \alpha' \in \mathcal{X}[[\nu]]$ and $c, c' \in \mathcal{C}[[\nu]]$; as a result, taking the quotient and deforming the product commute: $(\mathcal{X}/\mathcal{C})_\star = \mathcal{X}_\star/\mathcal{C}_\star$. This is fulfilled if[2]

$$\alpha\star f^a = \alpha f^a = f^a\star\alpha \Leftrightarrow B_l(\alpha, f^a) = 0 = B_l(f^a, \alpha) \qquad \forall l \in \mathbb{N} \tag{7}$$

for all $\alpha \in \mathcal{X}, a = 1, .., k$ [(7) implies that the f^a are central in $\mathcal{X}_\star$, again].

In [22] Drinfel'd has introduced a general deformation quantization procedure of universal enveloping algebras $U\mathfrak{g}$ (seen as Hopf algebras) of Lie groups G and of their module algebras, based on *twisting*; a *twist* is a suitable element (a 2-cocycle, see Sect. 2.1.1)

$$\mathcal{F} = \mathbf{1}\otimes\mathbf{1} + \sum_{l=1}^{\infty} \nu^l \sum_{I_l} \mathcal{F}_1^{I_l} \otimes \mathcal{F}_2^{I_l} \in U\mathfrak{g} \otimes U\mathfrak{g}[[\nu]] \tag{8}$$

(here $\otimes = \otimes_{\mathbb{C}[[\nu]]}$). It acts on the tensor product of any two $U\mathfrak{g}$-modules or module algebras, in particular algebras of functions on any G-manifold, including some symplectic[3] manifolds [4]. In [1] the authors consider the Lie algebra $\mathfrak{g} = \Xi_M$ of smooth vector fields on a generic smooth manifold M (this is the Lie algebra of the infinite-dimensional Lie group of diffeomorphisms of M) and the $U\Xi_M$-module algebra $\mathcal{X}^M = C^\infty(M)$; $\mathcal{F}_1^{I_l}, \mathcal{F}_2^{I_l}$ seen as differential operators acting on $\mathcal{X}^M$ have order l

[2] In fact, for all $c \equiv \sum_{a=1}^k f^a c^a \in \mathcal{C}$ ($c^a \in \mathcal{X}$) (7) implies $c = \sum_{a=1}^k f^a\star c^a$, so that for all $\alpha \in \mathcal{X}$, by the associativity of $\star$, $c\star\alpha = (\sum_{a=1}^k f^a\star c^a)\star\alpha = \sum_{a=1}^k f^a\star(c^a\star\alpha) \overset{(7)}{=} \sum_{a=1}^k f^a(c^a\star\alpha) \in \mathcal{C}[[\nu]]$; and similarly for $\alpha\star c$. Note that it is not sufficient to require that $\alpha\star f^a - \alpha f^a$, $f^a\star\alpha - f^a\alpha$, or equivalently $B_l(\alpha, f^a), B_l(f^a, \alpha)$, belong to $\mathcal{C}$ to obtain the same results. As a more general condition ensuring $c\star\alpha,\ \alpha\star c \in \mathcal{C}[[\nu]]$ one could require that for all $a = 1, .., k$ and $\alpha \in \mathcal{X}$ the product $f^a\alpha = \alpha f^a$ can be expressed as a combination of $\star$-products: $f^a\alpha = f^b\star A^a_b(\alpha) = A'^a_b(\alpha)\star f^b$.

[3] However, this quantization procedure does not apply to every Poisson manifold: there are several symplectic manifolds, e.g., the symplectic 2-sphere and the symplectic Riemann surfaces of genus $g > 1$, which do not admit a $\star$-product induced by a Drinfel'd twist (c.f. [8,17]). Nevertheless, if one is not taking into account the Poisson structure, every G-manifold (i.e., smooth manifolds G acts on) can be quantized via the above approach.

at most and no zero-order term. The deformed product reads

$$\alpha\star\beta := \alpha\beta + \sum_{l=1}^{\infty} \nu^l \sum_{I_l} \overline{\mathcal{F}}_1^{I_l}(\alpha)\ \overline{\mathcal{F}}_2^{I_l}(\beta)\,, \tag{9}$$

where $\overline{\mathcal{F}} = \mathbf{1}\otimes\mathbf{1} + \sum_{l=1}^{\infty} \nu^l \sum_{I_l} \overline{\mathcal{F}}_1^{I_l} \otimes \overline{\mathcal{F}}_2^{I_l}$ is the inverse of the twist. In the sequel, we will abbreviate $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$, $\overline{\mathcal{F}} = \overline{\mathcal{F}}_1 \otimes \overline{\mathcal{F}}_2$ (Sweedler notation with suppressed summation symbols). In the presence of several copies of $\mathcal{F}$, we write $\mathcal{F}_1 \otimes \mathcal{F}_2$ and $\mathcal{F}_1' \otimes \mathcal{F}_2'$, etc., in order to distinguish the summations. Actually, Ref. [1] twists not only $U\Xi_M$ into a new Hopf algebra $U\Xi_M^{\mathcal{F}}$ and $\mathcal{X}^M$ into a $U\Xi_M^{\mathcal{F}}$-module algebra $\mathcal{X}_\star^M$, but also the $U\Xi_M$-equivariant $\mathcal{X}^M$-bimodule of differential forms on M, their tensor powers, the Lie derivative, and the geometry on M (metric, connection, curvature, torsion, etc.)—if present—into deformed counterparts.

Here, as in [50], we shall take the algebraic characterization (2–4) as the starting point for defining submanifolds in NCG, but use a twist-deformed differential calculus on it.[4]

$$\Xi_t := \{X \in \Xi \ \mid\ X(f^1) = 0,\ \ldots,\ X(f^k) = 0\} \subset \Xi_{\mathcal{C}} \tag{10}$$

is the Lie subalgebra of vector fields tangent to *all* submanifolds M_c (because they fulfill $X(f_c^a) = 0$ for all $c \in f(\mathcal{D}_f)$) at all points; it is an $\mathcal{X}$-bimodule, as well. The key observation is that, applying this deformation procedure to $\mathcal{X} = C^\infty(\mathcal{D}_f)$ with a twist $\mathcal{F} \in U\Xi_t \otimes U\Xi_t[[\nu]]$, we satisfy (7) and therefore obtain a deformation $\mathcal{X}_\star$ of $\mathcal{X}$ such that $\mathcal{X}_\star^{M_c} = \mathcal{X}^{M_c}[[\nu]] = \mathcal{X}_\star/\mathcal{C}_\star^c$, for all $c \in f(\mathcal{D}_f)$; moreover, $\Xi_{M_c\star} = \Xi_{M_c}[[\nu]] = \Xi_{\mathcal{C}^c\star}/\Xi_{\mathcal{C}\mathcal{C}^c\star}$, see Sect. 3. In other words, we obtain a noncommutative deformation, in the sense of deformation quantization and in the form of quotients as in (2–4), of the k-parameter family of embedded manifolds $M_c \subset \mathbb{R}^n$. Actually, for every $X \in \Xi_{\mathcal{C}}$ there is an element in the equivalence class $[X]$ that belongs to Ξ_t, namely its tangent projection X_t (see Proposition 6). $\mathcal{X}_\star, \Xi_\star, \ldots$ are $U\Xi^{\mathcal{F}}$-equivariant, while $\mathcal{X}_\star^{M_c}, \Xi_{M\star}, \Xi_{t\star}, \ldots$ are $U\Xi_t^{\mathcal{F}}$-equivariant. If $\mathcal{F}$ is unitary or real, then $U\Xi^{\mathcal{F}}$ and $\mathcal{X}_\star$ admit $*$-structures (involutions), making them a Hopf $*$-algebra and a $U\Xi^{\mathcal{F}}$-module $*$-algebra, respectively; thereby $U\Xi_t^{\mathcal{F}}$ is a Hopf $*$-subalgebra and $\mathcal{X}_\star^{M_c}, \Xi_{t\star}, \ldots$ are a $U\Xi_t^{\mathcal{F}}$-module $*$-algebra and $U\Xi_t^{\mathcal{F}}$-equivariant Lie $*$-algebras, respectively. By the same procedure, one can obtain noncommutative deformations of differential geometry on submanifolds $N \subset M \subset \mathbb{R}^n$.

The plan of the paper will be as follows.

In Sect. 2, we present preliminary material, first on twisting (Sect. 2.1) and then on its application [1–3] to the differential geometry on a generic manifold (Sect. 2.2).

[4] The derivation-based approach to differential calculi of Dubois-Violette and Michor [23], which was used in [50], does not encompass several differential calculi (e.g., quantum group covariant ones), or requires algebra extensions to succeed (see, e.g., [11]). The approaches to the differential calculus à la Connes [15] and Woronowicz [60] (which include the one considered here) are more general: the bimodule of noncommutative differential 1-forms is the primary object whereby the whole calculus can be derived by imposing the Leibniz rule and nilpotency of the exterior derivative. As a result, the dual module consists of noncommutative vector fields which are no longer derivations.

In Sect. 3, we deal with twist deformations of embedded manifolds $M \subset \mathbb{R}^n$ in the smooth context. In Sect. 3.1, we pave the way for these deformations recalling or deriving basic facts about differential geometry on a submanifold M of $\mathbb{R}^n$, i.e., how the Cartan calculus and any connection, metric, etc., on $\mathbb{R}^n$ induces corresponding data on M, how to concretely build bases of the bimodules Ξ_t, $\Xi_\perp$ of tangent and normal vectors fields (i.e., sections in the tangent and normal bundle), the corresponding projections pr_t, $\mathrm{pr}_\perp$, etc. In Sect. 3.2, we first show that the whole twisted Cartan calculus on $\mathcal{X}_\star$ is projected to the one on $\mathcal{X}_\star^M$, in the same way as for its undeformed counterpart, and that projection commutes with twisting, for all twists $\mathcal{F} \in U\Xi_t \otimes U\Xi_t[[\nu]]$. Then, we show that the same can be done for: i) a connection ∇, using a twist $\mathcal{F} \in U\mathfrak{e} \otimes U\mathfrak{e}[[\nu]]$, where $\mathfrak{e}$ is the corresponding *equivariance Lie algebra* (a Lie subalgebra of Ξ_t); ii) the metric, and the associated Levi–Civita connection, using a twist $\mathcal{F} \in U\mathfrak{k} \otimes U\mathfrak{k}[[\nu]]$, where $\mathfrak{k} \subseteq \mathfrak{e}$ is the Lie subalgebra of the corresponding *Killing vector fields*. Under the latter assumptions one can build a twisted version not only of the first, but also of the second fundamental form, and prove a twisted version of Gauss theorem. Twisted Ξ_t, $\Xi_\perp$, pr_t, $\mathrm{pr}_\perp$ stay essentially undeformed; we find suitable $\mathfrak{k}$-invariant bases for them. Here we limit ourselves to developing (pseudo)Riemannian geometry for our physical interests, but other geometric structures (say projective, affine, conformal,...) could be explored as well. To build concrete examples of twisted submanifolds, one can look for $M \subset \mathbb{R}^n$ such that Ξ_t contains a finite-dimensional Lie subalgebra $\mathfrak{g}$, because the simplest known Drinfel'd twists are based on such a $\mathfrak{g}$; a nontrivial $\mathfrak{g}$ surely exists if M is symmetric under some Lie group.

In particular, one can apply [38] this procedure to algebraic submanifolds $M \subset \mathbb{R}^n$, e.g., quadrics (i.e., level sets of a polynomial function $f(x) = 0$ of degree 2); for the latter there exists a Lie subalgebra $\mathfrak{g}$ (of dimension at least 2) of both Ξ_t and the Lie algebra $aff(n)$ of the affine group $Aff(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL(n)$ of $\mathbb{R}^n$. If we choose a twist $\mathcal{F} \in U\mathfrak{g} \otimes U\mathfrak{g}[[\nu]]$ all the structures can be formulated in terms of generators and $\star$-polynomial relations. More precisely, the algebra $\mathcal{X} = \mathrm{Pol}^\bullet(\mathbb{R}^n)$ of polynomial functions (with complex coefficients) in the set of Cartesian coordinates $x^1, \ldots, x^n$ is deformed so that every $\star$-polynomial of degree k in x equals an ordinary polynomial of the same degree in x, and vice versa. The same occurs with the $\mathcal{X}_\star$-bimodules and algebras $\Omega_\star^\bullet$ of differential forms, that of differential operators, etc. In [38] the authors discuss in detail deformations of all families of quadric surfaces embedded in $\mathbb{R}^3$ that are induced by twists of the abelian [54] or Jordanian [52,53] type. In Sect. 4 of the present work, as illustrations of the approach, we just present cocycle twist deformations of elliptic (in particular, circular) cylinders (first family) as well as hyperboloids and cone (second family) embedded in $\mathbb{R}^3$; they are induced by unitary abelian or Jordanian twists. Endowing $\mathbb{R}^3$ with the Euclidean (resp. Minkowski) metric gives the circular cylinders (resp. hyperboloids and cone) a Lie algebra $\mathfrak{k}$ of isometries of dimension at least 2; choosing a twist $\mathcal{F} \in U\mathfrak{k} \otimes U\mathfrak{k}[[\nu]]$ we thus find twisted (pseudo)Riemannian M_c (with the metric given by the twisted first fundamental form) that are symmetric under the Hopf algebra $U\mathfrak{k}^\mathcal{F}$ (the "quantum group of isometries"), and the twisted Levi–Civita connection on all M_c equals the projection of the twisted Levi–Civita connection on $\mathbb{R}^3$ (the exterior derivative), while the twisted curvature can be expressed in terms of the twisted second fundamental form through the twisted Gauss theorem. Actually, the metric, Levi–Civita connection, intrinsic and extrinsic curvatures of the circular cylinders and hyperboloids, as elements in the appropriate tensor spaces, remain undeformed; the twist enters only their action on twisted tensor products of vector fields. The twisted hyperboloids can be seen as twisted (anti-)de Sitter spaces dS_2, AdS_2.

In Sect. 5, we summarize our results, add further remarks, mention possible mathematical developments, physical applications, issues worth further investigations.

2 Preliminaries

2.1 Twisted algebraic structures

2.1.1 Twisting Hopf algebras $H := U\mathfrak{g}$

As known, the universal enveloping algebra (UEA) $H := U\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of any Lie group G is a Hopf algebra. First, we briefly recall what this means. Let

$$\begin{aligned} &\varepsilon(\mathbf{1}) = 1, \qquad &&\Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}, \qquad &&S(\mathbf{1}) = \mathbf{1}, \\ &\varepsilon(g) = 0, &&\Delta(g) = g\otimes\mathbf{1} + \mathbf{1}\otimes g, &&S(g) = -g, \qquad \text{if } g \in \mathfrak{g}; \end{aligned}$$

ε, Δ are extended to all of H as algebra maps, S as an antialgebra map:

$$\begin{aligned} &\varepsilon : H \to \mathbb{C}, &&\Delta : H \to H\otimes H, &&S : H \to H, \\ &\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \quad &&\Delta(ab) = \Delta(a)\Delta(b), \quad &&S(ab) = S(b)S(a). \end{aligned} \tag{11}$$

The extensions of ε, Δ, S are unambiguous, because $\varepsilon(g) = 0$, $\Delta\big([g, g']\big) = \big[\Delta(g), \Delta(g')\big]$, $S\big([g, g']\big) = \big[S(g'), S(g)\big]$ if $g, g' \in \mathfrak{g}$. The maps ε, Δ, S are the abstract operations by which one constructs the trivial representation, the tensor product of any two representations and the contragredient of any representation, respectively. $H = U\mathfrak{g}$ equipped with ε, Δ, S is a Hopf algebra; this means that a number of properties (see, e.g., [14,26,48]) are fulfilled, in particular $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ (coassociativity), $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$ (counitality), $\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = \mu \circ (\mathrm{id} \otimes S) \circ \Delta$ (antipode property) [$\mu : H \otimes H \to H$ denotes the product in H, $\mu(a \otimes b) = ab$, and $\eta : \mathbb{C} \to H$ is defined by $\eta(\alpha) = \alpha\mathbf{1}$]. H is cocommutative, i.e., $\tau \circ \Delta = \Delta$, where τ is the flip operator: $\tau(a \otimes b) = b \otimes a$.

If G is a real form of a Lie group, then there exists also a $*$-structure on $H = U\mathfrak{g}$, i.e., an involution $* : H \to H$ such that for all $a, b \in H$ and $\alpha, \beta \in \mathbb{C}$

$$\mathbf{1}^* = \mathbf{1}, \qquad (\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*, \qquad (ab)^* = b^* a^*, \tag{12}$$

$$\varepsilon(a^*) = [\varepsilon(a)]^* \qquad \Delta(a^*) = [\Delta(a)]^{*\otimes *}, \qquad S\left\{\left[S(a^*)\right]^*\right\} = a. \tag{13}$$

H equipped with $*, \varepsilon, \Delta, S$ is a Hopf $*$-algebra.

Secondly, we recall how to deform a Hopf algebra using a *twist* [22] (see also [14,57]). Let $\hat{H} = H[[\nu]]$. Given a twist, i.e., an element $\mathcal{F} = \mathbf{1}\otimes\mathbf{1} + \mathcal{O}(\nu) \in$

$(H\otimes H)[[\nu]]$ fulfilling

$$(\epsilon\otimes id)\mathcal{F}=(id\otimes\epsilon)\mathcal{F}=\mathbf{1}, \tag{14}$$
$$(\mathcal{F}\otimes\mathbf{1})[(\Delta\otimes id)(\mathcal{F})]=(\mathbf{1}\otimes\mathcal{F})[(id\otimes\Delta)(\mathcal{F})]=:\mathcal{F}^3, \tag{15}$$

we shall call $H_s\subseteq H$ the smallest Hopf subalgebra such that $\mathcal{F}\in(H_s\otimes H_s)[[\nu]]$, and

$$\beta:=\mathcal{F}_1 S(\mathcal{F}_2)\in H_s\Rightarrow\beta^{-1}=S\left(\overline{\mathcal{F}}_1\right)\overline{\mathcal{F}}_2. \tag{16}$$

Extending the product, Δ, ε, S linearly to the formal power series in ν and setting

$$\Delta_{\mathcal{F}}(a):=\mathcal{F}\Delta(a)\overline{\mathcal{F}},\qquad S_{\mathcal{F}}(a):=\beta\, S(a)\beta^{-1},\qquad \mathcal{R}:=\mathcal{F}_{21}\overline{\mathcal{F}}, \tag{17}$$

one finds that the analogs of conditions (11), as well as analogs of the coassociativity, counitality and antipode property are satisfied and therefore $H^{\mathcal{F}}=(\hat{H},\Delta_{\mathcal{F}},\varepsilon,S_{\mathcal{F}})$ is a Hopf algebra deformation of (H,Δ,ε,S). While the latter was cocommutative, $H^{\mathcal{F}}$ is triangular noncocommutative (or quasi-cocommutative), i.e., $\tau\circ\Delta_{\mathcal{F}}(a)=\mathcal{R}\Delta_{\mathcal{F}}(a)\overline{\mathcal{R}}$, where $\overline{\mathcal{R}}=\mathcal{R}_{21}$ is the inverse of the so-called *universal R-matrix* or *triangular structure* $\mathcal{R}$. Correspondingly, $\Delta_{\mathcal{F}}$, $S_{\mathcal{F}}$ replace Δ, S in the tensor product of any two representations and the contragredient of any representation, respectively. Drinfel'd has shown [22] that any triangular deformation of the Hopf algebra H can be obtained in this way (up to isomorphisms).

To obtain a new Hopf $*$-algebra, we need some further assumption on the twist. Without loss of generality, ν can be assumed real. If $\mathcal{F}$ is *real* (i.e., $\mathcal{F}^{*\otimes *}=(S\otimes S)[\mathcal{F}_{21}]$), then $\beta^*=\beta$ and $\mathcal{R}^{*\otimes *}=(\beta\otimes\beta)^{-1}\overline{\mathcal{R}}(\beta\otimes\beta)=(\beta\otimes\beta)\overline{\mathcal{R}}(\beta\otimes\beta)^{-1}$. Introducing the new $*$-structure

$$g^{*_{\mathcal{F}}}:=\beta g^*\beta^{-1} \tag{18}$$

makes $(H^{\mathcal{F}},*_{\mathcal{F}})$ into a triangular Hopf $*$-algebra, i.e., also (12, 13) and $\mathcal{R}^{*_{\mathcal{F}}\otimes *_{\mathcal{F}}}=\overline{\mathcal{R}}$ are satisfied. $*_{\mathcal{F}}$ can be transformed back to $*$ by the Hopf algebra isomorphism (31), which transforms the product of $\hat{H}$ into the $\star$-product induced by $\mathcal{F}$. Another possibility is that $\mathcal{F}$ is *unitary* (i.e., $\mathcal{F}^{*\otimes *}=\overline{\mathcal{F}}$). Then, $\beta^*=S(\beta^{-1})$, $\mathcal{R}^{*\otimes *}=\overline{\mathcal{R}}$, and $(H^{\mathcal{F}},*)$ itself is a triangular Hopf $*$-algebra.

Equations (15), (17) imply the generalized intertwining relation $\Delta_{\mathcal{F}}^{(n)}(a)=\mathcal{F}^n\Delta^{(n)}(a)(\mathcal{F}^n)^{-1}$ for the iterated coproduct. By definition

$$\Delta_{\mathcal{F}}^{(n)}:\hat{H}\to\hat{H}^{\otimes n},\qquad \Delta^{(n)}:H[[\nu]]\to(H)^{\otimes n}[[\nu]],\qquad \mathcal{F}^n\in(H_s)^{\otimes n}[[\nu]]$$

reduce to $\Delta_{\mathcal{F}}$, Δ, $\mathcal{F}$ for $n=2$, whereas for $n>2$ they can be defined recursively as

$$\begin{aligned}&\Delta_{\mathcal{F}}^{(n+1)}=(id^{\otimes^{n-1}}\otimes\Delta_{\mathcal{F}})\circ\Delta_{\mathcal{F}}^{(n)},\qquad \Delta^{(n+1)}=(id^{\otimes(n-1)}\otimes\Delta)\circ\Delta^{(n)},\\&\mathcal{F}^{n+1}=(\mathbf{1}^{\otimes(n-1)}\otimes\mathcal{F})[(id^{\otimes(n-1)}\otimes\Delta)\mathcal{F}^n].\end{aligned} \tag{19}$$

The result for $\Delta_{\mathcal{F}}^{(n)}, \mathcal{F}^n$ is the same if in definitions (19) we iterate the coproduct on a different sequence of tensor factors [coassociativity of $\Delta_{\mathcal{F}}$; this follows from the coassociativity of Δ and the cocycle condition (15)]; for instance, for $n=3$ this amounts to (15) and $\Delta_{\mathcal{F}}^{(3)} = (\Delta_{\mathcal{F}} \otimes id) \circ \Delta_{\mathcal{F}}$. For any $a \in H[[h]] = \hat{H}$, we shall use the Sweedler notations (summations are understood)

$$\Delta^{(n)}(a) = a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(n)}, \qquad \Delta_{\mathcal{F}}^{(n)}(a) = a_{\widehat{(1)}} \otimes a_{\widehat{(2)}} \otimes \cdots \otimes a_{\widehat{(n)}}.$$

We consider the following examples of twists:

i.) Let $n \in \mathbb{N}$, $P := \sum_{i=1}^{n} e_i \otimes e_{n+i} \in \mathfrak{g} \otimes \mathfrak{g}$, with pairwise commuting elements $e_1, \ldots, e_{2n} \in \mathfrak{g}$. Then,

$$\mathcal{F} = \exp(i\nu P) \in (U\mathfrak{g} \otimes U\mathfrak{g})[[\nu]]$$

is a Drinfel'd twist on $U\mathfrak{g}$ ([54]). We refer to it as an abelian (or *Reshetikhin*) twist on $U\mathfrak{g}$. It is unitary if $P^{*\otimes *} = P$; this is, e.g., the case if the e_i are anti-Hermitian or Hermitian. It is immediate to check that the twist with P replaced by $P' = \frac{1}{2}\sum_{i=1}^{n}(e_i \otimes e_{n+i} - e_{n+i} \otimes e_i)$ is both unitary and real, leads to the same $\mathcal{R}$ and makes $\beta = \mathbf{1}$, whence $S_{\mathcal{F}} = S$, and the $*$-structure remains undeformed also for H-$*$-modules and module algebras, see (22).

ii.) Let $H, E \in \mathfrak{g}$ be elements of a Lie algebra such that $[H, E] = 2E$. Then,

$$\mathcal{F} = \exp\left[\frac{1}{2} H \otimes \log(\mathbf{1} + i\nu E)\right] \in (U\mathfrak{g} \otimes U\mathfrak{g})[[\nu]]$$

defines a Drinfel'd twist, called Jordanian twist [52,53]. If H and E are anti-Hermitian, $\mathcal{F}$ is unitary. More sophisticated twists can be obtained using this as a prototype [9,10,51].

There are numerous other examples of Drinfel'd twists. We refer to [25] for an explicit (recursive) construction of twists on UEA via a Fedosov method and a classification (of equivalence classes) of twists in terms of the Chevalley–Eilenberg cohomology of the Lie algebra.

2.1.2 Twisting *H*-modules and *H*-module algebras

We recall that, given a Hopf ($*$-)algebra H over $\mathbb{C}$, a left H-module $(\mathcal{M}, \triangleright)$ is a vector space $\mathcal{M}$ over $\mathbb{C}$ equipped with a left action, i.e., a $\mathbb{C}$-bilinear map $(g, a) \in H \times \mathcal{M} \mapsto g \triangleright a \in \mathcal{M}$ such that $(20)_1$ and $1 \triangleright a = a$ hold. An element $a \in \mathcal{M}$ of a left H-module is said to be H-invariant if $g \triangleright a = \epsilon(g)a$ for all $g \in H$. Equipped also with an antilinear involution $*$ fulfilling $(20)_2$, $(\mathcal{M}, \triangleright, *)$ is a left H-$*$-module. A left H-module ($*$-)algebra is a ($*$-)algebra $\mathcal{A}$ over $\mathbb{C}$ equipped with a left H-($*$-)module structure, such that $(20)_3$ and $g \triangleright 1 = \epsilon(g)1$ hold:

$$(gg')\triangleright a = g \triangleright (g' \triangleright a), \qquad (g \triangleright a)^* = [S(g)]^* \triangleright a^*, \qquad g \triangleright (ab) = \big(g_{(1)} \triangleright a\big)\big(g_{(2)} \triangleright b\big). \quad (20)$$

If $g \in \mathfrak{g}$, formula (20)$_3$ becomes the Leibniz rule. An $\mathcal{A}$-bimodule $\mathcal{M}$ for a left H-module algebra $\mathcal{A}$ is said to be an H-equivariant $\mathcal{A}$-bimodule if $\mathcal{M}$ is a left H-module such that

$$g \triangleright (a \cdot s \cdot b) = (g_{(1)} \triangleright a) \cdot (g_{(2)} \triangleright s) \cdot (g_{(3)} \triangleright b) \tag{21}$$

for all $g \in H$, $a, b \in \mathcal{A}$ and $s \in \mathcal{M}$, where we denoted the $\mathcal{A}$-module actions on $\mathcal{M}$ by $\cdot$. If in addition, $\mathcal{A}$ is a left H-module $*$-algebra and there is a $*$-involution on $\mathcal{M}$, we call $\mathcal{M}$ an H-equivariant $\mathcal{A}$-$*$-bimodule if $(a \cdot s \cdot b)^* = b^* \cdot s^* \cdot a^*$. We remark that any $\mathcal{A}$-($*$-)subbimodule of an H-equivariant $\mathcal{A}$-($*$-)bimodule is an H-equivariant $\mathcal{A}$-($*$-)bimodule if it is closed under the Hopf algebra action. A map $\phi\colon \mathcal{M} \to \mathcal{M}'$ between left H-modules is said to be H-equivariant if it commutes with the Hopf algebra actions, i.e., $g \triangleright \phi(s) = \phi(g \triangleright s)$ for all $g \in H$ and $s \in \mathcal{M}$. Extending the action $\triangleright$ $\mathbb{C}[[\nu]]$-bilinearly one can make any H-module $(\mathcal{M}, \triangleright)$ into an $\hat{H}$-module $(\mathcal{M}[[\nu]], \triangleright)$. If $\mathcal{F}$ is a real (resp. unitary) twist on H, the undeformed $*$-involution (resp. the $*$-involution $*_{\mathcal{F}}$) structures $H^{\mathcal{F}}$ as a triangular Hopf $*$-algebra. If $\mathcal{F}$ is real (resp. unitary) and $(\mathcal{M}, \triangleright, *)$ is an H-$*$-module, then (see, e.g., [1,34]) $(\mathcal{M}[[\nu]], \triangleright, *_\star)$ is an $H^{\mathcal{F}}$-$*$-module, where

$$a^{*_\star} := a^* \qquad (\text{resp. } a^{*_\star} := S(\beta) \triangleright a^*). \tag{22}$$

Given an H-module ($*$-)algebra $\mathcal{A}$ and choosing $\mathcal{M} = \mathcal{A}$, the twist gives also a systematic way to make $\mathcal{A}[[\nu]]$ into an $H^{\mathcal{F}}$-module ($*$-)algebra $\mathcal{A}_\star$, by endowing it with a new product,

$$a \star a' := \left(\overline{\mathcal{F}}_1 \triangleright a\right) \left(\overline{\mathcal{F}}_2 \triangleright a'\right), \tag{23}$$

the so-called $\star$-product. In fact, $\star$ is associative by (15), fulfills $(a \star a')^{*_\star} = a'^{*_\star} \star a^{*_\star}$ and

$$g \triangleright (a \star a') = \left(g_{\widehat{(1)}} \triangleright a\right) \star \left(g_{\widehat{(2)}} \triangleright a'\right). \tag{24}$$

If $aa' = \pm a'a$, i.e., a, a' (anti)commute, then

$$a' \star a \;=\; \pm\, (\mathcal{R}_2 \triangleright a) \star (\mathcal{R}_1 \triangleright a'). \tag{25}$$

Consequently, twists leading to the same $\mathcal{R}$ [e.g., the abelian twists $\exp(i\nu P)$, $\exp(i\nu P')$ of the previous section] lead to the same commutation relations in $\mathcal{A}_\star$. More generally, for any H-equivariant $\mathcal{A}$-($*$-)bimodule $\mathcal{M}$ of a left H-module ($*$-)algebra $\mathcal{A}$, the twisted module actions

$$a \star s = (\overline{\mathcal{F}}_1 \triangleright a) \cdot (\overline{\mathcal{F}}_2 \triangleright s) \quad \text{and} \quad s \star a = (\overline{\mathcal{F}}_1 \triangleright s) \cdot (\overline{\mathcal{F}}_2 \triangleright a), \tag{26}$$

where $a \in \mathcal{A}$ and $s \in \mathcal{M}$, structure $\mathcal{M}$ as an $H^{\mathcal{F}}$-equivariant $\mathcal{A}_\star$-($*$-)bimodule $\mathcal{M}_\star$ (with $*$-involution (22) on $\mathcal{M}_\star$). We refer to [5,34] for proofs of the previous statements.

Given two H-modules $(\mathcal{M}, \triangleright)$, $(\mathcal{N}, \triangleright)$, the tensor product $(\mathcal{M}\otimes\mathcal{N}, \triangleright)$ is an H-module if we define $g \triangleright (a\otimes b) := \big(g_{(1)} \triangleright a\big)\otimes\big(g_{(2)} \triangleright b\big)$. As above, this is extended to an $H^{\mathcal{F}}$-($*$-)module $(\mathcal{M} \otimes \mathcal{N}[[\nu]], \triangleright)$. Introducing the "$\star$-tensor product" [1]

$$(a\otimes_\star b) := \overline{\mathcal{F}}(\triangleright\otimes\triangleright)(a\otimes b) \equiv \overline{\mathcal{F}}_1 \triangleright a \otimes \overline{\mathcal{F}}_2 \triangleright b \tag{27}$$

(an invertible endomorphism, i.e., a change of basis, of $\mathcal{M}\otimes\mathcal{N}[[\nu]]$), we find

$$g \triangleright (a\otimes_\star b) \;=\; g_{\widehat{(1)}} \triangleright a \;\otimes_\star\; g_{\widehat{(2)}} \triangleright b. \tag{28}$$

Given two H-module ($*$-)algebras $\mathcal{A}$, $\mathcal{B}$, this applies in particular to $\mathcal{M} = \mathcal{A}, \mathcal{N} = \mathcal{B}$. The tensor ($*$-)algebra $\mathcal{A}\otimes\mathcal{B}$ [whose product is defined by $(a\otimes b)(a'\otimes b') = (aa'\otimes bb')$] is an H-module ($*$-)algebra under the action $\triangleright$. By introducing the $\star$-product (23) $\mathcal{A}\otimes\mathcal{B}$ is deformed into an $H^{\mathcal{F}}$-module ($*$-)algebra $(\mathcal{A}\otimes\mathcal{B})_\star$, with product and $*$-structure related to those of $\mathcal{A}_\star$, $\mathcal{B}_\star$ by

$$(a\otimes_\star b)\star(a'\otimes_\star b') = a\star(\mathcal{R}_2 \triangleright a') \otimes_\star (\mathcal{R}_1 \triangleright b)\star b', \tag{29}$$

$$\begin{aligned} (a\otimes_\star b)^* &= \mathcal{R}_2 \triangleright a^* \;\otimes_\star \mathcal{R}_1 \triangleright b^* \quad \text{if } \mathcal{F} \text{ is real,}\\ (a\otimes_\star b)^{*_\star} &= \mathcal{R}_2 \triangleright a^{*_\star} \;\otimes_\star \mathcal{R}_1 \triangleright b^{*_\star} \text{ if } \mathcal{F} \text{ is unitary,} \end{aligned} \tag{30}$$

where $\mathcal{R}_1 \otimes \mathcal{R}_2$ (again a summation is understood) is the decomposition of $\mathcal{R}$ in $H^{\mathcal{F}} \otimes H^{\mathcal{F}}$. From (29) we recognize that $(\mathcal{A}\otimes\mathcal{B})_\star$ is isomorphic to the *braided tensor product (algebra)* [14,48] of $\mathcal{A}_\star$ with $\mathcal{B}_\star$; here the braiding is involutive and therefore spurious, as $\overline{\mathcal{R}} = \mathcal{R}_{21}$. So $(\mathcal{A}\otimes\mathcal{B})_\star$ encodes both the usual $\star$-product within $\mathcal{A}$, $\mathcal{B}$ and the $\star$-tensor product (or braided tensor product) between the two. (On the contrary, setting $(a\otimes b) := \mathcal{F}_1 \triangleright a \otimes_\star \mathcal{F}_2 \triangleright b$ 'unbraids' the braided tensor product, cf. [31].) By (15) the $\star$-tensor product is associative, and the previous results hold also for iterated $\star$-tensor products.

The algebra $(H[[\nu]], \star)$ itself is an $H^{\mathcal{F}}$-module algebra, and one can build a triangular Hopf algebra $H_\star = (H[[\nu]], \star, \eta, \Delta_\star, \epsilon, S_\star, \mathcal{R}_\star)$ isomorphic to $H^{\mathcal{F}} = (H[[\nu]], \mu, \eta, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}}, \mathcal{R})$, with isomorphism $D : H_\star \to H^{\mathcal{F}}$ and inverse given by [1,42] (cf. also [27,28,34])

$$D(\xi) := (\overline{\mathcal{F}}_1 \triangleright \xi)\overline{\mathcal{F}}_2 = \mathcal{F}_1\, \xi\, S(\mathcal{F}_2)\; \beta^{-1}, \qquad D^{-1}(\phi) = \overline{\mathcal{F}}_1\, \phi\, \beta\, S\big(\overline{\mathcal{F}}_2\big). \tag{31}$$

Namely, $D(\xi\star\xi') = D(\xi)D(\xi')$, and $\Delta_\star, S_\star, \mathcal{R}_\star$ are related to $\Delta_{\mathcal{F}}, S_{\mathcal{F}}, \mathcal{R}$ by the relations

$$\Delta_\star = (D^{-1} \otimes D^{-1}) \circ \Delta_{\mathcal{F}} \circ D, \qquad S_\star = D^{-1} \circ S_{\mathcal{F}} \circ D, \qquad \mathcal{R}_\star = (D^{-1} \otimes D^{-1})(\mathcal{R}). \tag{32}$$

One can think of D also as a change of generators within $H[[\nu]]$.

If $\mathcal{F}$ is real, then $(H[[\nu]], \star, *)$ is a left $H^{\mathcal{F}}$-module $*$-algebra, and $D : (H_\star, *) \to (H^{\mathcal{F}}, *_{\mathcal{F}})$ is an isomorphism of triangular Hopf $*$-algebras (cf. [48] Proposition 2.3.7, [1]).

If $\mathcal{F}$ is unitary, then $(H[[\nu]], \star, *_\star)$ is a left $H^{\mathcal{F}}$-module $*$-algebra, and $D : (H_\star, *_\star) \to (H^{\mathcal{F}}, *)$ is an isomorphism of triangular Hopf $*$-algebras, see Proposition 18 in "Appendix".

2.2 Twisted differential geometry

Reference [1] applies the previous machinery to $H = U\Xi$, where Ξ is the Lie algebra of the Lie group of diffeomorphisms of M, and $\mathcal{A}$ is the algebra $\mathcal{X} = C^\infty(M)$ of smooth functions on M, or more generally an $\mathcal{X}$-bimodule of tensor fields on M. Tensor fields of rank (p, r) $(p, r \in \mathbb{N}_0)$ on M can be described as elements in the tensor product

$$\mathcal{T}^{p,r} := \underbrace{\Omega \otimes \ldots \otimes \Omega}_{p\text{times}} \otimes \underbrace{\Xi \otimes \ldots \otimes \Xi}_{r\text{times}} \tag{33}$$

of the $\mathcal{X}$-bimodules $\Omega \equiv \Omega^1$, Ξ of differential 1-forms and vector fields on M, respectively. Here and below $\otimes$ stands for $\otimes_{\mathcal{X}}$ (rather than $\otimes_{\mathbb{C}}$), namely $T \otimes fT' = Tf \otimes T'$ for all $f \in \mathcal{X}$. We set $\mathcal{T}^{0,0} := \mathcal{X}$. The tensor product is associative; to avoid the need of reorderings we multiply $T \in \mathcal{T}^{p,r}$ by 1-form tensor factors only from the left if $r > 0$, by vector field tensor factors only from the right if $p > 0$. The tensor product between a function $f \in \mathcal{X} \equiv \mathcal{T}^{0,0}$ and another tensor field is as usual not explicitly written. All $\mathcal{T}^{p,r}$ are $\mathcal{X}$-bimodules, e.g., $f(T \otimes T') = (fT) \otimes T'$, $(T \otimes T')f = T \otimes (T'f)$. $\mathcal{T} := \bigoplus_{p,r\in\mathbb{N}_0} \mathcal{T}^{p,r}$ (endowed with the product $\otimes$) is a $U\Xi$-module algebra: the action $\triangleright : U\Xi \otimes \mathcal{T} \to \mathcal{T}$ is uniquely determined by $\mathbf{1}_H \triangleright T = T$ and

$$X \triangleright T = \mathcal{L}_X(T), \qquad X \in \Xi, \tag{34}$$

where $\mathcal{L}$ is the Lie derivative. It fulfills the Leibniz rule $g \triangleright (T \otimes T') = g_{(1)} \triangleright T \otimes g_{(2)} \triangleright T'$.

By setting $\mathcal{A} = \mathcal{T}$ we can apply the results of Sect. 2.1, in particular define a deformed tensor algebra $\mathcal{T}_\star$ with associative $\star$-tensor product defined by Eq. (27). $\mathcal{T}_\star$ is a $U\Xi^{\mathcal{F}}$-module algebra. All $\mathcal{T}_\star^{h,r}$ are $\mathcal{X}_\star$-bimodules. In $\mathcal{T}_\star$ we have in particular

$$T \otimes_\star h\star T' = T\star h \otimes_\star T' \ , \qquad h\star(T \otimes_\star T') = (h\star T) \otimes_\star T' \ . \tag{35}$$

The first formula shows that $\otimes_\star$ is actually $\otimes_{\mathcal{X}_\star}$, the tensor product over $\mathcal{X}_\star$. While the usual product of a tensor field T with a function h from the left and from the right coincide,[5] in general this no longer occurs with the $\star$-product.

In a chart U with coordinates x^μ any vector field X can be expressed in the ∂_μ basis as $X = X^\mu \partial_\mu$. It can be also uniquely expressed as $X = X^\mu_\star \star \partial_\mu$, where $X^\mu_\star$ are functions defined on U. The same occurs if $\{\partial_\mu\}$ is replaced by a more general (not necessarily holonomic or ν-independent) frame $\{e_a\}$: $X = X^a_\star \star e_a$. Similarly, every

[5] If $T \in \Xi$, then the product $T \cdot h$ of T with h from the right is the vector field that on a function g gives $(T \cdot h)(g) = T(g)h$. In Sect. 3.1.3, we shall denote it by $T \triangleleft h$, so as to distinguish it from the operator $Th = T(h) + hT$.

1-form ω can be uniquely written as $\omega = \omega_\mu dx^\mu = \omega^\star_\mu \star dx^\mu$, with $\omega_\mu, \omega^\star_\mu$ functions defined on U, and where $\{dx^\mu\}$ is the usual dual frame of the vector field frame $\{\partial_\mu\}$. More generally, in U every tensor field $T^{p,q} \in \mathcal{T}^{p,q}$ can be uniquely written using functions $T_\star{}^{\lambda_1 \ldots \lambda_q}_{\mu_1 \ldots \mu_p}$ defined on U as

$$T^{p,q} = T_\star{}^{\lambda_1 \ldots \lambda_q}_{\mu_1 \ldots \mu_p} \star dx^{\mu_1} \otimes_\star \ldots \otimes_\star dx^{\mu_p} \otimes_\star \partial_{\lambda_1} \otimes_\star \ldots \otimes_\star \partial_{\lambda_q}. \tag{36}$$

Let us twist the algebra $\mathcal{A} = \Omega^\bullet = \oplus_p \Omega^p$ of differential forms. We denote by $\Omega^\bullet_\star := (\Omega^\bullet, \wedge_\star)$ the $\mathbb{C}[[\nu]]$-module of forms equipped with the $\star$-deformed wedge product

$$\omega \wedge_\star \omega' := (\overline{\mathcal{F}}_1 \triangleright \omega) \wedge (\overline{\mathcal{F}}_2 \triangleright \omega') = \omega \otimes_\star \omega' - \mathcal{R}_2 \triangleright \omega' \otimes_\star \mathcal{R}_1 \triangleright \omega\,. \tag{37}$$

This can be seen as the tensor subspace of totally $\star$-antisymmetric (contravariant) tensor fields. The degree of the top form stays undeformed. Below we drop the symbols $\wedge, \wedge_\star$.

The usual exterior derivative $d : \Omega^\bullet \to \Omega^{\bullet+1}$ satisfies the graded Leibniz rule $d(\alpha_p \star \beta) = d\alpha_p \star \beta + (-1)^p \alpha_p \star d\beta$ and is therefore also the $\star$-exterior derivative. This is so, because the exterior derivative commutes with the Lie derivative, i.e., with the Hopf algebra action.

One can endow [1] the module underlying the algebra $U\Xi^{\mathcal{F}} \simeq U\Xi[[\nu]]$ itself with the $\star$-product; the new algebra $U\Xi_\star$ endowed by suitable coproduct, counit, antipode becomes a Hopf algebra isomorphic to $U\Xi^{\mathcal{F}}$, whereby it is manifest that the above differential calculus is bicovariant in the sense of Woronowicz [60]. Ξ is closed under the $\star$-Lie bracket

$$[X, Y]_\star := X \star Y - (\mathcal{R}_2 \triangleright Y) \star (\mathcal{R}_1 \triangleright X) = \left[\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y\right] = \mathcal{L}_{\overline{\mathcal{F}}_1 \triangleright X}\left(\overline{\mathcal{F}}_2 \triangleright Y\right). \tag{38}$$

The action $\mathcal{L}^\star_X$ of $U\Xi_\star$ on $\mathcal{T}$ ($\star$-Lie derivative) is defined by

$$\mathcal{L}^\star_X(T) = (\overline{\mathcal{F}}_1 \triangleright X) \triangleright \left[\overline{\mathcal{F}}_2 \triangleright T\right], \qquad X \in U\Xi. \tag{39}$$

2.2.1 $\star$-Pairing between 1-forms and vector fields, twisted Cartan calculus

Denoting $\langle\, ,\, \rangle$ the commutative pairing between vector fields and 1-forms, the $\star$-pairing is defined as $\langle\, ,\, \rangle_\star := \langle\, ,\, \rangle \circ \overline{\mathcal{F}}(\triangleright \otimes \triangleright) : \Xi_\star \otimes_{\mathbb{C}} \Omega_\star \mapsto \mathcal{X}_\star$, namely

$$(X, \omega) \mapsto \langle X, \omega\rangle_\star := \left\langle \overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright \omega\right\rangle\,. \tag{40}$$

The $\star$-pairing is actually a map $\langle\, ,\, \rangle_\star : \Xi_\star \otimes_\star \Omega_\star \mapsto \mathcal{X}_\star$, as it satisfies the $\mathcal{X}_\star$-linearity properties

$$\begin{aligned} \langle X, h \star \omega\rangle_\star &= \langle X \star h, \omega\rangle_\star, \\ \langle h \star X, \omega \star k\rangle_\star &= h \star \langle X, \omega\rangle_\star \star k\,. \end{aligned} \tag{41}$$

with $h, k \in \mathcal{X}_\star$. From $\langle X, dh\rangle = X(h)$, $g \triangleright dh = d(g \triangleright h)$ and (40) it follows that

$$\langle X, dh\rangle_\star = \left(\overline{\mathcal{F}}_1 \triangleright X\right)\left(\overline{\mathcal{F}}_2 \triangleright h\right) =: X_\star(h). \tag{42}$$

$X_\star$ is a twisted derivation, i.e., fulfills the deformed Leibniz rule

$$X_\star(h\star h') = X_\star(h)\star h' + [\mathcal{R}_2 \triangleright h]\star[(\mathcal{R}_1 \triangleright X)(h')]; \tag{43}$$

the quickest way to prove the latter is by the Leibniz rule for d and (42), (41), (25). The compatibility $X \triangleright \langle Y, \omega\rangle = \left\langle X_{(1)} \triangleright Y, X_{(2)} \triangleright \omega\right\rangle$ of $\langle\ ,\ \rangle$ with the Lie derivative (which expresses the diffeomorphism-invariance of the pairing) implies

$$X \triangleright \langle Y, \omega\rangle_\star = \left\langle X_{\widehat{(1)}} \triangleright Y, X_{\widehat{(2)}} \triangleright \omega\right\rangle_\star . \tag{44}$$

In the commutative case, for any local moving frame (vielbein) $\{e_i\}$ we can build a dual frame of 1-forms $\{\omega^i\}$, $\langle e_i, \omega^j\rangle = \delta_i^j$, and conversely; in particular $\langle \partial_\mu, dx^\lambda\rangle = \delta_\mu^\lambda$. The exterior derivative decomposes as $d = \omega^i e_i$. It is the same in the noncommutative case. The $\star$-dual frame $\{\theta^i\}$ of $\{e_i\}$,

$$\langle e_i, \theta^j\rangle_\star = \delta_i^j\ , \tag{45}$$

can be obtained from $\{\omega^i\}$ via a $\mathcal{X}_\star$-linear transformation that is the identity at zero order in ν [1], and the exterior derivative decomposes also as $d = \theta^i \star e_{i\star}$. Using the $\star$-pairing we can associate to any 1-form ω the left $\mathcal{X}_\star$-linear map $\langle\ , \omega\rangle_\star : \Xi_\star \to \mathcal{X}_\star$, and to any vector field X the right $\mathcal{X}_\star$-linear map $\langle X,\ \rangle_\star : \Omega_\star \to \Omega_\star$. The maps $\mathrm{i}_X := \langle X,\ \rangle_\star$, $\mathrm{i}_\omega := \langle\ , \omega\rangle_\star$ are the simplest twisted insertions (interior products) of a vector field in a form and of a 1-form in a multivector field, respectively. Using the exterior derivative and the twisted insertion, Lie bracket, and Lie derivative one can develop [58] a twisted Cartan calculus in complete analogy with the usual one (see also the thesis [59] for more details). As one can extend the commutative pairing to higher tensor powers setting

$$\langle X_p \otimes \cdots \otimes X_1, \omega_1 \otimes \cdots \otimes \omega_p \otimes \tau\rangle := \langle X_1, \omega_1\rangle \dots \langle X_p, \omega_p\rangle\, \tau \tag{46}$$

for all $X_i \in \Xi, \omega_i \in \Omega$, so can one extend $\langle X,\ \rangle_\star$ to the corresponding twisted tensor powers using the same formula (40). Properties (41), (44) are preserved. There is a $\star$-pairing $\langle\ ,\ \rangle'_\star : \Omega_\star \otimes_\star \Xi_\star \to \mathcal{X}_\star$ with forms on the left and vector fields on the right. It is related to the previous $\star$-pairing via $\langle \omega, X\rangle'_\star = \langle \overline{\mathcal{R}}_1 \triangleright X, \overline{\mathcal{R}}_2 \triangleright \omega\rangle_\star$ for all $\omega \in \Omega_\star$ and $X \in \Xi_\star$. It is left and right $\mathcal{X}_\star$-linear and satisfies $\langle \omega\star h, X\rangle'_\star = \langle \omega, h\star X\rangle'_\star$ for all $h \in \mathcal{X}_\star$. As in the case of $\langle\ ,\ \rangle_\star$ there is an extension of $\langle\ ,\ \rangle'_\star$ to higher twisted tensor powers.

2.2.2 Covariant derivative, torsion, curvature, metric

In [1,2], a *twisted covariant derivative* (or, synonymously, *twisted connection*) $\nabla^{\mathcal{F}}$ is defined as a collection of maps $\nabla^{\mathcal{F}} : \Xi_\star \otimes_{\mathbb{C}[[\nu]]} \mathcal{T}_\star^{p,q} \to \mathcal{T}_\star^{p,q}$, $X \otimes T \mapsto \nabla_X^{\mathcal{F}} T$

$(p, q \in \mathbb{N}_0)$ fulfilling the properties

$$\nabla^{\mathcal{F}}_X h = \mathcal{L}^\star_X(h), \tag{47}$$
$$\nabla^{\mathcal{F}}_{h\star X + h'\star Y} T = h\star\nabla^{\mathcal{F}}_X T + h'\star\nabla^{\mathcal{F}}_Y T, \tag{48}$$
$$\nabla^{\mathcal{F}}_X (T + T') = \nabla^{\mathcal{F}}_X T + \nabla^{\mathcal{F}}_X T', \tag{49}$$
$$\begin{aligned}\nabla^{\mathcal{F}}_X (T \otimes_\star T') = &\left(\overline{\mathcal{R}}_1 \rhd \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}'_2\rhd X}(\overline{\mathcal{R}}''_2 \rhd T)\right) \otimes_\star ((\overline{\mathcal{R}}_2\overline{\mathcal{R}}'_1\overline{\mathcal{R}}''_1) \rhd T') \\ &+ (\overline{\mathcal{R}}_1 \rhd T) \otimes_\star \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\rhd X} T',\end{aligned} \tag{50}$$
$$\nabla^{\mathcal{F}}_X \langle Y, \omega\rangle_\star = \langle\overline{\mathcal{R}}_1 \rhd \left(\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}'_2\rhd X}(\overline{\mathcal{R}}''_2 \rhd Y)\right), (\overline{\mathcal{R}}_2\overline{\mathcal{R}}'_1\overline{\mathcal{R}}''_1) \rhd \omega\rangle_\star + \langle\overline{\mathcal{R}}_1 \rhd Y, \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\rhd X}\omega\rangle_\star \tag{51}$$

for all $X, Y \in \Xi_\star \equiv \mathcal{T}^{0,1}_\star$, $h, h' \in \mathcal{X}_\star \equiv \mathcal{T}^{0,0}_\star$, $T, T' \in \mathcal{T}_\star$. On functions the twisted covariant and Lie derivatives along X coincide, by (47). Equation (51) amounts to the compatibility of the action of $\nabla^{\mathcal{F}}_X$ on 1-forms with the pairing of the latter with vector fields. $\nabla^{\mathcal{F}}$ is left $\mathcal{X}_\star$-linear in the first argument, by (48); it is only $\mathbb{C}[[\nu]]$-linear in the second argument, by (49) and (50) with $T = c \in \mathbb{C}[[\nu]] \subset \mathcal{X}_\star$. Relation (50) for $T = h \in \mathcal{X}_\star$ becomes [by (47)] the deformed Leibniz rule

$$\nabla^{\mathcal{F}}_X (h\star T') = \mathcal{L}^\star_X(h)\star T' + (\overline{\mathcal{R}}_1 \rhd h)\star\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\rhd X} T'; \tag{52}$$

this holds in particular on vector fields $T' = Y$. Actually, the knowledge of $\nabla^{\mathcal{F}}$ just for $(p, q) = (0, 0)$ (i.e. on functions) and $(p, q) = (0, 1)$ (i.e., on vector fields), determines its unique extension to all the $(p, q) \in \mathbb{N}^2_0$: Eq. (51) determines the action of $\nabla^{\mathcal{F}}_X$ on 1-forms, while (50) allows to extend $\nabla^{\mathcal{F}}_X$ recursively to all the $\mathcal{T}^{p,q}_\star$'s, which consist of combinations of tensor products of 1-forms and vector fields.

The torsion $\mathsf{T}^{\mathcal{F}}_\star$ and the curvature $\mathsf{R}^{\mathcal{F}}_\star$ associated with a connection $\nabla^{\mathcal{F}}$ are left $\mathcal{X}_\star$-linear maps $\mathsf{T}^{\mathcal{F}}_\star : \Xi_\star \otimes_\star \Xi_\star \to \Xi_\star$, $\mathsf{R}^{\mathcal{F}}_\star : \Xi_\star \otimes_\star \Xi_\star \otimes_\star \Xi_\star \to \Xi_\star$ defined by

$$\mathsf{T}^{\mathcal{F}}_\star(X, Y) := \nabla^{\mathcal{F}}_X Y - \nabla^{\mathcal{F}}_{\mathcal{R}_2\rhd Y}(\mathcal{R}_1 \rhd X) - [X, Y]_\star\,, \tag{53}$$
$$\mathsf{R}^{\mathcal{F}}_\star(X, Y, Z) := \nabla^{\mathcal{F}}_X\nabla^{\mathcal{F}}_Y Z - \nabla^{\mathcal{F}}_{\mathcal{R}_2\rhd Y}\nabla^{\mathcal{F}}_{\mathcal{R}_1\rhd X} Z - \nabla^{\mathcal{F}}_{[X,Y]_\star} Z\,, \tag{54}$$

for all $X, Y, Z \in \Xi_\star$. They fulfill $\mathsf{T}^{\mathcal{F}}_\star(X, h\star Y) = \mathsf{T}^{\mathcal{F}}_\star(X\star h, Y)$ (and similarly for the curvature), and the antisymmetry property

$$\mathsf{T}^{\mathcal{F}}_\star(X, Y) = -\,\mathsf{T}^{\mathcal{F}}_\star(\mathcal{R}_2 \rhd Y, \mathcal{R}_1 \rhd X)\,, \tag{55}$$
$$\mathsf{R}^{\mathcal{F}}_\star(X, Y, Z) = -\,\mathsf{R}^{\mathcal{F}}_\star(\mathcal{R}_2 \rhd Y, \mathcal{R}_1 \rhd X, Z)\,. \tag{56}$$

Thus, one can regard torsion and curvature as elements of the following $\star$-tensor spaces:

$$\mathsf{T}^{\mathcal{F}} \in \Omega_\star \wedge_\star \Omega_\star \otimes_\star \Xi_\star, \qquad \mathsf{R}^{\mathcal{F}} \in \Omega_\star \otimes_\star \Omega_\star \wedge_\star \Omega_\star \otimes_\star \Xi_\star, \tag{57}$$

acting on vector fields through the twisted pairing (40) applied to higher tensor powers, see (46). We omit the $\star$ in the subscript of the elements (57) in order to distinguish them from the corresponding maps. In other words, for all $X, Y, Z \in \Xi_\star = \Xi[[\nu]]$

$$\mathsf{T}^{\mathcal{F}}_\star(X, Y) = \langle X \otimes_\star Y, \mathsf{T}^{\mathcal{F}} \rangle_\star \quad \text{and} \quad \mathsf{R}^{\mathcal{F}}_\star(X, Y, Z) = \langle X \otimes_\star Y \otimes_\star Z, \mathsf{R}^{\mathcal{F}} \rangle_\star. \tag{58}$$

A *metric* is defined as a non-degenerate element $\mathbf{g}$ in the module $\Omega_\star \otimes_\star \Omega_\star = \Omega \otimes \Omega[[\nu]]$ that is symmetric, i.e., invariant under the flip $\tau(\alpha \otimes \beta) := \beta \otimes \alpha$. Clearly, the two decompositions $\mathbf{g} = \mathbf{g}^\alpha \otimes \mathbf{g}_\alpha = \mathbf{g}^A \otimes_\star \mathbf{g}_A$ (sum over repeated indices) are related by $\mathbf{g}^\alpha \otimes \mathbf{g}_\alpha = \overline{\mathcal{F}}_1 \triangleright \mathbf{g}^A \otimes \overline{\mathcal{F}}_2 \triangleright \mathbf{g}_A$. $\mathbf{g}$ determines the map $\mathbf{g}_\star : \Xi_\star \otimes_\star \Xi_\star \to \mathcal{X}_\star$ defined by

$$\mathbf{g}_\star(X, Y) := \left\langle X, \left\langle Y, \mathbf{g}^A \right\rangle_\star \star \mathbf{g}_A \right\rangle_\star; \tag{59}$$

this fulfills $\mathbf{g}_\star(h \star X, Y) = h \star \mathbf{g}_\star(X, Y)$ (left $\mathcal{X}_\star$-linearity in X) and $\mathbf{g}_\star(X \star h, Y) = \mathbf{g}_\star(X, h \star Y)$. The twisted *Levi–Civita* (LC) connection $\nabla^{\mathcal{F}}$ is a connection fulfilling $\mathsf{T}^{\mathcal{F}} = 0$ and $\nabla^{\mathcal{F}}_X \mathbf{g} = 0$ for all $X \in \Xi_\star$, or equivalently, for all $Y, Z \in \Xi_\star$

$$\begin{aligned}\mathcal{L}^\star_X\big[\mathbf{g}_\star(Y, Z)\big] = &\ \mathbf{g}_\star\Big(\overline{\mathcal{R}}_1 \triangleright (\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}'_2 \triangleright X}(\overline{\mathcal{R}}''_2 \triangleright Y)), (\overline{\mathcal{R}}_2 \overline{\mathcal{R}}'_1 \overline{\mathcal{R}}''_1) \triangleright Z\Big) \\ &+ \mathbf{g}_\star\Big(\overline{\mathcal{R}}_1 \triangleright Y, \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2 \triangleright X} Z\Big).\end{aligned} \tag{60}$$

If a twisted LC connection exists, it is unique by [2] Theorem 5. For equivariant metrics, there is an existence and uniqueness theorem (c.f. [58] Lemma 3.12) of an equivariant twisted LC connection. If $\mathcal{F} = \mathbf{1} \otimes \mathbf{1}$ (whereby $\star$ becomes the ordinary product, and $\mathcal{R} = \mathbf{1} \otimes \mathbf{1}$), the above formulae reduce to the notions and properties of ordinary connection, torsion, curvature, metric, etc. In particular, we recover the characterization of a LC connection:

$$\begin{aligned}&\text{torsion-free, i.e.,} && \mathsf{T} := \nabla_X Y - \nabla_Y X - [X, Y] = 0 \qquad \forall X, Y \in \Xi, \\ &\text{metric-compatible, i.e.,} && \mathcal{L}_Z\big[\mathbf{g}(X,Y)\big] - \mathbf{g}(\nabla_Z X, Y) - \mathbf{g}(X, \nabla_Z Y) = 0 \quad \forall X, Y, Z \in \Xi.\end{aligned} \tag{61}$$

In the commutative case, the Ricci tensor is a contraction of the curvature tensor, $\mathsf{Ric}_{jk} = \mathsf{R}_{ijk}{}^i$. The twisted Ricci map is defined as the following contraction of the curvature:

$$\mathsf{Ric}^{\mathcal{F}}_\star(X, Y) := \langle \theta^i, \mathsf{R}^{\mathcal{F}}_\star(e_i, X, Y) \rangle'_\star, \tag{62}$$

where sum over i and (45) are understood. $\langle\ ,\ \rangle'_\star$ is a contraction between forms on the *left* and vector fields on the *right*, see Sect. 2.2.1. Recall that it is defined by the pairing

$$\langle \omega\,,\ X \rangle'_\star = \left\langle \overline{\mathcal{F}}_1 \triangleright \omega\,,\ \overline{\mathcal{F}}_2 \triangleright X \right\rangle = \langle \mathcal{R}_2 \triangleright X\,,\ \mathcal{R}_1 \triangleright \omega \rangle_\star \tag{63}$$

and has the $\mathcal{X}_\star$-linearity properties

$$\langle h\star\omega, X\star k\rangle'_\star = h\star\langle\omega, X\rangle'_\star\star k\ , \qquad \langle\omega, h\star X\rangle'_\star = \langle\omega\star h, X\rangle'_\star\ . \tag{64}$$

The twisted Ricci map is well defined because (62) is independent of the choice of the frame $\{e_i\}$ (and of the dual frame $\{\theta^i\}$), and because it is defined as a $\mathcal{X}_\star$-linear map:

$$\mathsf{Ric}^{\mathcal{F}}_\star(h\star X, Y) = h\star\mathsf{Ric}^{\mathcal{F}}_\star(X, Y)\ , \qquad \mathsf{Ric}^{\mathcal{F}}_\star(X, h\star Y) = \mathsf{Ric}^{\mathcal{F}}_\star(X\star h, Y)\ . \tag{65}$$

Evaluating the Ricci map on the dual metric $\mathbf{g}^{-1} = \mathbf{g}^{-1\alpha}\otimes\mathbf{g}^{-1}{}_\alpha = \mathbf{g}^{-1A}\otimes_\star\mathbf{g}^{-1}{}_A$ yields the Ricci scalar:

$$\mathfrak{R}^{\mathcal{F}} = \mathsf{Ric}^{\mathcal{F}}_\star\left(\mathbf{g}^{-1A}, \mathbf{g}^{-1}{}_A\right). \tag{66}$$

We now show how to construct nontrivial twisted deformations $\nabla^{\mathcal{F}}$ of ∇. First, we need some preliminary result in ordinary differential geometry. It is easy to check that

$$\mathfrak{e} := \left\{g\in\Xi \quad|\quad [g, \nabla_X Y] = \nabla_{[g,X]}Y + \nabla_X[g, Y] \quad \forall X, Y\in\Xi\right\} \tag{67}$$

is a Lie subalgebra of Ξ; we shall name it the *equivariance Lie algebra of* ∇. It follows

$$g\triangleright(\nabla_X Y) = \nabla_{g_{(1)}\triangleright X}\left(g_{(2)}\triangleright Y\right) \qquad \forall g\in U\mathfrak{e}. \tag{68}$$

Proposition 2 *Given a connection ∇ on M and the associated equivariance Lie algebra $\mathfrak{e}$, setting*

$$\nabla^{\mathcal{F}}_X T := \nabla_{\overline{\mathcal{F}}_1\triangleright X}(\overline{\mathcal{F}}_2\triangleright T) \tag{69}$$

with $\mathcal{F}\in U\mathfrak{e}\otimes U\mathfrak{e}[[\nu]]$ defines a twisted connection along $X\in\Xi_\star$. It is $U\mathfrak{e}^{\mathcal{F}}$-equivariant, i.e.,

$$g\triangleright\nabla^{\mathcal{F}}_X Y = \nabla^{\mathcal{F}}_{g_{\widehat{(1)}}\triangleright X}(g_{\widehat{(2)}}\triangleright Y) \tag{70}$$

and satisfies the additional deformed Leibniz rule (with functions multiplying from the right)

$$\nabla^{\mathcal{F}}_X(T\star h) = (\nabla^{\mathcal{F}}_X T)\star h + (\overline{\mathcal{R}}_1\triangleright T)\star(\mathcal{L}^\star_{\overline{\mathcal{R}}_2\triangleright X}(h)) \tag{71}$$

for all $g\in U\mathfrak{e}^{\mathcal{F}}$, $h\in\mathcal{X}_\star$, $T\in\mathcal{T}_\star$ and $X, Y\in\Xi_\star$. Furthermore, Eqs. (50), (51) *boil down to*

$$\nabla^{\mathcal{F}}_X(T\otimes_\star T') = \nabla^{\mathcal{F}}_X T\otimes_\star T' + \overline{\mathcal{R}}_1\triangleright T\otimes_\star\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\triangleright X}T', \tag{72}$$

$$\nabla^{\mathcal{F}}_X\langle Y, \omega\rangle_\star = \langle\nabla^{\mathcal{F}}_X(Y), \omega\rangle_\star + \langle\overline{\mathcal{R}}_1\triangleright Y, \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\triangleright X}\omega\rangle_\star. \tag{73}$$

Of course, nontrivial deformations of this kind are possible only if $\mathfrak{e} \neq \{0\}$.
We recall that $Z \in \Xi$ is a *Killing vector field* of a (pseudo)Riemannian manifold $(M, \mathbf{g})$ if

$$\mathcal{L}_Z\big[\mathbf{g}(X, Y)\big] - \mathbf{g}\big([Z, X], Y\big) - \mathbf{g}\big(X, [Z, Y]\big) = 0 \qquad \forall X, Y \in \Xi, \tag{74}$$

or equivalently if $\mathbf{g}\big(\nabla_X Z, Y\big) + \mathbf{g}\big(X, \nabla_Y Z\big) = 0$.[6] The Killing vector fields close a Lie subalgebra $\mathfrak{k} \subset \Xi$; this is the Lie algebra of the group of isometries of $(M, \mathbf{g})$ if M is complete.

Proposition 3 *The Killing vector fields $\mathfrak{k} \subset \Xi$ form a Lie subalgebra of the equivariance Lie algebra $\mathfrak{e}$ of the Levi–Civita connection ∇ on a (pseudo)Riemannian manifold $(M, \mathbf{g})$. For all twists $\mathcal{F} \in U\mathfrak{k} \otimes U\mathfrak{k}[[\nu]]$, the map $\mathbf{g}_\star$ is also right $\mathcal{X}_\star$-linear in the second argument and related to the undeformed one $\mathbf{g} : \Xi \otimes \Xi[[\nu]] \to \mathcal{X}[[\nu]]$, $\mathbf{g}(X, Y) := \langle X, \langle Y, \mathbf{g}^\alpha \rangle\, \mathbf{g}_\alpha \rangle$, by*

$$\mathbf{g}_\star(X, Y) = \mathbf{g}\left(\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y\right), \tag{75}$$

and $\nabla^{\mathcal{F}}_X$ is the unique twisted Levi–Civita connection corresponding to $\mathbf{g}_\star$. Torsion and curvature of the twisted Levi–Civita connection remain undeformed as elements of the tensor spaces

$$0 = \mathsf{T}^{\mathcal{F}} = \mathsf{T} \in \Omega \wedge \Omega \otimes \Xi[[\nu]], \qquad \mathsf{R}^{\mathcal{F}} = \mathsf{R} \in \Omega \otimes \Omega \wedge \Omega \otimes \Xi[[\nu]] \tag{76}$$

and the associated maps $\mathsf{T}^{\mathcal{F}}_\star$, $\mathsf{R}^{\mathcal{F}}_\star$ are also right $\mathcal{X}_\star$-linear in the last argument. Equation (60) *boils down to*

$$\mathcal{L}^\star_X\big[\mathbf{g}_\star(Y, Z)\big] = \mathbf{g}_\star(\nabla^{\mathcal{F}}_X Y, Z) + \mathbf{g}_\star\Big(\overline{\mathcal{R}}_1 \triangleright Y, \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2 \triangleright X} Z\Big). \tag{77}$$

The proofs of these propositions are in “Appendix”. The existence and uniqueness of the twisted Levi–Civita connection of Proposition 3 was proven in [2] Theorem 6 and Theorem 7. In NCG right function-linearity of the curvature in the last argument is in general not true, see, e.g., [24,29]. In Sect. 4, we will find nontrivial $\mathfrak{k}$ for suitably symmetric quadrics in $\mathbb{R}^3$.

3 Twisted smooth submanifolds of $\mathbb{R}^n$

3.1 Differential geometry of manifolds embedded in $\mathbb{R}^n$

We develop some theoretical tools for the $(n-k)$-dimensional submanifolds $M_c \subseteq \mathcal{D}_f \subseteq \mathbb{R}^n$ defined by Eq. (1). Recall definitions (4), (10). We can identify $\Xi_M \subset \Xi$

[6] The latter condition is obtained taking the difference of $(61)_2$, (74), using the bilinearity of $\mathbf{g}$ and $(61)_1$.

with the Lie subalgebra of smooth vector fields tangent to M at all points, and $\Xi_t \subset \Xi$ with the Lie subalgebra of smooth vector fields tangent to *all* M_c $(c \in f(\mathcal{D}_f))$ at all points, because $X(f^a)=0$ implies $X(f^a_c)=0$. Decomposing $X=X^i\partial_i$ and abbreviating $f^a_i := \partial_i(f^a)$, $X \in \Xi_t$ amounts to $X^i f^a_i = 0$ for all $a=1,\ldots,k$; as the Jacobian matrix $J=(f^a_i)$ has by assumption rank k, $\dim(\Xi_t)=n-k=:m$. Henceforth, Ω will stand for the $\mathcal{X}$-bimodule of differential 1-forms on $\mathcal{D}_f$, i.e., the dual one of Ξ. We also define a $\mathcal{X}$-subbimodule $\Omega_\perp \subset \Omega$ of 1-forms by

$$\Omega_\perp := \{\omega \in \Omega \mid \langle \Xi_t, \omega\rangle = 0\}. \tag{78}$$

Let $\Xi^\bullet_t = \bigwedge^\bullet \Xi_t$, $\Xi^\bullet_M = \bigwedge^\bullet \Xi_M$, $\Omega^\bullet_\perp = \bigwedge^\bullet \Omega_\perp$ be the corresponding exterior algebras.

Proposition 4 *The Ξ_t, $\Xi_{\mathcal{C}}$, Ξ_M defined in (4), (10) are Lie subalgebras of Ξ. $\Xi_{\mathcal{C}\mathcal{C}}$ is an ideal in $\Xi_{\mathcal{C}}$. $\Xi^\bullet_t$, $\Xi^\bullet_M$, $\Omega^\bullet_\perp$ are $U\Xi_t$-equivariant $\mathcal{X}$-bimodules. $\Omega_\perp$ can be explicitly decomposed as*

$$\Omega_\perp = \bigoplus_{a=1}^{k} \mathcal{X} \mathrm{d} f^a = \bigoplus_{a=1}^{k} \mathrm{d} f^a \mathcal{X}. \tag{79}$$

Proof $\omega=\omega_a df^a$ implies $\langle X,\omega\rangle = \langle X, df^a\rangle\omega_a = X(f^a)\omega_a = 0$. Conversely, in any basis $\{X_\alpha = X^i_\alpha\partial_i\}_{\alpha=1}^m$ of Ξ_t the $n\times m$ matrix (X^i_α) has rank m and fulfills $f^a_i X^i_\alpha = 0$; decomposing $\omega=\omega_i dx^i$, $\langle\Xi_t,\omega\rangle=0$ amounts to $\langle X_\alpha,\omega\rangle = X^i_\alpha\omega_i = 0$ for all α, and this linear system of m independent equations admits only solutions $\omega_i = f^a_i\omega_a$, $\omega_a \in \mathcal{X}$, whence $\omega=\omega_a df^a$. This proves (79). For all $X, W \in \Xi_t$, $Y \in \Xi_{\mathcal{C}}$, $Z \in \Xi_{\mathcal{C}\mathcal{C}}$, $a=1,\ldots,k$, $h\in\mathcal{X}$, $\omega\in\Omega_\perp$ we find

$$\begin{aligned}
&(X \triangleright W)(f^a) = [X,W](f^a) = X\big(W(f^a)\big) - W\big(X(f^a)\big) = 0 && \Rightarrow X \triangleright W \in \Xi_t,\\
&(X \triangleright Y)(f^a) = [X,Y](f^a) = X\big(Y(f^a)\big) - Y\big(X(f^a)\big) = X\big(Y(f^a)\big) \in \mathcal{C} && \Rightarrow X \triangleright Y \in \Xi_{\mathcal{C}},\\
&(X \triangleright Z)(h) = [X,Z](h) = X\big(Z(h)\big) - Z\big(X(h)\big) = X\big(Z(h)\big) \in \mathcal{C} && \Rightarrow X \triangleright Z \in \Xi_{\mathcal{C}\mathcal{C}},\\
&X \triangleright \omega = X(\omega_a)\, df^a + \omega_a\, X \triangleright \big(df^a\big) = X(\omega_a)\, df^a + \omega_a\, d[X(f^a)] && = X(\omega_a)\, df^a \in \Omega_\perp.
\end{aligned}$$

This implies in turn that $g \triangleright W \in \Xi_t$, $g \triangleright Y \in \Xi_{\mathcal{C}}$, $g \triangleright Z \in \Xi_{\mathcal{C}\mathcal{C}}$, $g \triangleright \omega \in \Omega_\perp$ for all $g \in U\Xi_t$, so that Ξ_t, $\Xi_{\mathcal{C}}$, $\Xi_{\mathcal{C}\mathcal{C}}$, Ξ_M, $\Omega_\perp$ are $U\Xi_t$-equivariant $\mathcal{X}$-bimodules, and also $\Xi^\bullet_t$, $\Xi^\bullet_M$, $\Omega^\bullet_\perp$ are. □

3.1.1 Metric, Levi–Civita connection, intrinsic and extrinsic curvatures

We now discuss Ξ_t, $\Omega_\perp$ as addends in the decomposition of Ξ, Ω with respect to a metric. Consider a (non-degenerate) metric $\mathbf{g} \equiv \mathbf{g}^\alpha \otimes \mathbf{g}_\alpha \in \Omega\otimes\Omega$ on $\mathcal{D}_f$ (actually, the following discussion is valid on any smooth manifold) and its dual $\mathbf{g}^{-1} = \mathbf{g}^{-1\alpha}\otimes\mathbf{g}^{-1}_\alpha \in \Xi\otimes\Xi$. We recall that

$$\mathcal{G}: \Xi \to \Omega, \qquad X \mapsto \omega_X = \langle X, \mathbf{g}^\alpha\rangle \mathbf{g}_\alpha \tag{80}$$

is an isomorphism of $\mathcal{X}$-bimodules with inverse given by $\omega \mapsto X_\omega = \langle \mathbf{g}^{-1\alpha}, \omega\rangle \mathbf{g}^{-1}_\alpha$. In fact $X_{\omega_X} = \langle \mathbf{g}^{-1\alpha}, \omega_X\rangle \mathbf{g}^{-1}_\alpha = \langle \mathbf{g}^{-1\alpha}, \langle X, \mathbf{g}^\beta\rangle \mathbf{g}_\beta\rangle \mathbf{g}^{-1}_\alpha = X$ for all $X \in \Xi$ and

$\omega_{X_\omega} = \langle X_\omega, \mathbf{g}^\alpha\rangle\mathbf{g}_\alpha = \langle\langle\mathbf{g}^{-1\beta}, \omega\rangle\mathbf{g}_\beta^{-1}, \mathbf{g}^\alpha\rangle\mathbf{g}_\alpha = \omega$ for all $\omega \in \Omega$. It follows that for all $Y \in \Xi, \alpha \in \Omega$,

$$\mathbf{g}(Y, X) = \langle Y, \omega\rangle, \qquad \mathbf{g}^{-1}(\omega, \alpha) = \langle X, \alpha\rangle, \tag{81}$$

whenever $\omega = \mathcal{G}(X)$, or equivalently $X = \mathcal{G}^{-1}(\omega)$. Let us now introduce the $\mathcal{X}$-subbimodules

$$\Xi_\perp := \{X \in \Xi \mid \mathbf{g}(X, \Xi_t) = 0\}, \qquad \Omega_t := \{\omega \in \Omega \mid \mathbf{g}^{-1}(\omega, \Omega_\perp) = 0\}, \tag{82}$$

and let $\mathcal{D}'_f \subseteq \mathcal{D}_f$ be the open subset where the restriction

$$\mathbf{g}_\perp^{-1} := \mathbf{g}^{-1}|_{\Omega_\perp\otimes\Omega_\perp} : \Omega_\perp \otimes \Omega_\perp \to \mathcal{X} \tag{83}$$

is non-degenerate. If $\mathbf{g}$ is Riemannian, then $\mathcal{D}'_f = \mathcal{D}_f$. For simplicity, henceforth we shall denote the restrictions of Ξ, Ξ_t, $\Xi_\perp$ Ω, $\Omega_\perp$, Ω_t to $\mathcal{D}'_f$ by the same symbols, and by $\mathfrak{k} \subset \Xi_t$ the Lie subalgebra of Killing vector fields of $\mathbf{g}$ that are also tangent to the submanifolds $M_c \subset \mathcal{D}'_f$.

Proposition 5 *The Lie algebra Ξ of smooth vector fields and the $\mathcal{X}$-bimodule Ω of 1-forms on $\mathcal{D}'_f$ split into the direct sums of $\mathcal{X}$-subbimodules*

$$\Xi = \Xi_t \oplus \Xi_\perp, \qquad \Omega = \Omega_t \oplus \Omega_\perp, \tag{84}$$

orthogonal with respect to the metric $\mathbf{g}$ and $\mathbf{g}^{-1}$, respectively. Ξ_t is a Lie subalgebra of Ξ. Ω_t is orthogonal to $\Xi_\perp$ with respect to the pairing: $\Omega_t = \{\omega \in \Omega \mid \langle\Xi_\perp, \omega\rangle = 0\}$. Also the restrictions of $\mathbf{g}^{-1}$ to the tangent *forms and of $\mathbf{g}$ to the* tangent *and* normal *vector fields*

$$\mathbf{g}_t^{-1} := \mathbf{g}^{-1}|_{\Omega_t\otimes\Omega_t} : \Omega_t \otimes \Omega_t \to \mathcal{X}, \qquad \mathbf{g}_\perp := \mathbf{g}|_{\Xi_\perp\otimes\Xi_\perp} : \Xi_\perp \otimes \Xi_\perp \to \mathcal{X}, \tag{85}$$

$$\mathbf{g}_t := \mathbf{g}|_{\Xi_t\otimes\Xi_t} : \Xi_t \otimes \Xi_t \to \mathcal{X} \tag{86}$$

are non-degenerate. The orthogonal projections $\mathrm{pr}_\perp : \Xi \to \Xi_\perp$, $\mathrm{pr}_t : \Xi \to \Xi_t$, $\mathrm{pr}_\perp : \Omega \to \Omega_\perp$, $\mathrm{pr}_t : \Omega \to \Omega_t$ are uniquely extended as projections to the bimodules of multivector fields and higher rank forms through the rules $\mathrm{pr}_\perp(\omega\omega') = \mathrm{pr}_\perp(\omega)\mathrm{pr}_\perp(\omega')$, $\mathrm{pr}_t(\omega\omega') = \mathrm{pr}_t(\omega)\mathrm{pr}_t(\omega')$, ...:

$$\begin{aligned} &\mathrm{pr}_\perp : \Omega^p \to \Omega_\perp^p, \quad \mathrm{pr}_t : \Omega^p \to \Omega_t^p, \\ &\mathrm{pr}_\perp : \bigwedge^p \Xi \to \bigwedge^p \Xi_\perp, \quad \mathrm{pr}_t : \bigwedge^p \Xi \to \bigwedge^p \Xi_t. \end{aligned} \tag{87}$$

Ξ_t, $\Xi_\perp$, Ω_t, $\Omega_\perp$, their exterior powers and the projections $\mathrm{pr}_\perp$, pr_t are $U\mathfrak{k}$-equivariant.

Proof On $\mathcal{D}'_f$ one can build unique projections $\mathrm{pr}_\perp : \omega \in \Omega \to \omega_\perp \in \Omega_\perp$, $\mathrm{pr}_t = id - \mathrm{pr}_\perp : \omega \in \Omega \to \omega_t \in \Omega_t$, $\mathrm{pr}_\perp : X \in \Xi \to X_\perp \in \Xi_\perp$, $\mathrm{pr}_t = id - \mathrm{pr}_\perp :$

$X \in \Xi \to X_t \in \Xi_t$ such that the decompositions (84) hold, see Sect. 3.1.2. By Proposition 4, Ξ_t, $\Omega_\perp$ are in particular $U\mathfrak{k}$-equivariant $\mathcal{X}$-subbimodules. Also $\Xi_\perp$, Ω_t are $U\mathfrak{k}$-equivariant $\mathcal{X}$-subbimodules, by the $U\mathfrak{k}$-equivariance and $\mathcal{X}$-linearity in both arguments of $\mathbf{g}(\cdot,\cdot)$ and of $\langle\cdot,\cdot\rangle$: if $X \in \Xi_\perp$, then $\mathbf{g}(\xi \triangleright X, Y) = \xi_{(1)} \triangleright \mathbf{g}(X, S(\xi_{(2)}) \triangleright Y) = 0$ for all $\xi \in U\mathfrak{k}$ and $Y \in \Xi_t$ by the $U\mathfrak{k}$-equivariance of Ξ_t; hence, $\xi \triangleright X \in \Xi_\perp$ and $\Xi_\perp$ is $U\mathfrak{k}$-equivariant. Similarly, one shows that Ω_t is $U\mathfrak{k}$-equivariant. Consequently, also pr_t, $\mathrm{pr}_\perp$ acting on Ξ, Ω, as well as their extensions to $\bigwedge^\bullet \Xi$, $\Omega^\bullet$, are $\mathcal{X}$-linear in all arguments and $U\mathfrak{k}$-equivariant; for instance, the $U\mathfrak{k}$-equivariance on Ω follows from $\mathrm{pr}_t(\xi \triangleright \omega) = \mathrm{pr}_t(\xi \triangleright \omega_t + \xi \triangleright \omega_\perp) = \xi \triangleright \omega_t = \xi \triangleright \mathrm{pr}_t(\omega)$ for all $\xi \in U\mathfrak{k}$ and $\omega = \omega_t + \omega_\perp \in \Omega$. Now, note that by $(81)_1$ $\mathcal{G}, \mathcal{G}^{-1}$ map $\Xi_\perp$, $\Omega_\perp$ into each other and Ξ_t, Ω_t into each other. In fact,

- $X \in \Xi_\perp$ implies $\langle \Xi_t, \omega\rangle = \mathbf{g}(\Xi_t, X) = 0$, whence $\omega \in \Omega_\perp$; and vice versa.
- $X \in \Xi_t$ implies $\langle \Xi_\perp, \omega\rangle = \mathbf{g}(\Xi_\perp, X) = 0$, whence$\omega \in \Omega_t$; and vice versa.

Then, by $(81)_2$, if $\alpha \in \Omega_t$, then for all $X \in \Xi_\perp$ it is $\langle X, \alpha\rangle = \mathbf{g}^{-1}(\omega_X, \alpha) = 0$, because $\omega_X \in \Omega_\perp$; conversely, if $\langle \Xi_\perp, \alpha\rangle = 0$, then for all $\omega \in \Omega_\perp$ it is $\mathbf{g}^{-1}(\omega, \alpha) = \langle X_\omega, \alpha\rangle = 0$, because $X_\omega \in \Xi_\perp$. So we have proved that $\Omega_t = \{\omega \in \Omega \,|\, \langle \Xi_\perp, \omega\rangle = 0\}$. Next, let $\omega \in \Omega_t$; then $X_\omega \in \Xi_t$. By $(81)_2$, $\mathbf{g}_t^{-1}(\omega, \Omega_t) = 0$ implies $\langle X_\omega, \Omega_t\rangle = 0$ and therefore also $\langle X_\omega, \Omega\rangle = 0$, whence by the non-degeneracy of the pairing, $X_\omega = 0$, and in turn $\omega = 0$, namely $\mathbf{g}_t^{-1}$ is non-degenerate. Since $\mathbf{g}$ is non-degenerate, for all $X \in \Xi_t$ there is $Y \in \Xi$, and hence also $Y_t \in \Xi_t$, such that $0 \neq \mathbf{g}(X, Y) = \mathbf{g}(X, Y_t)$: $\mathbf{g}_t$ is non-degenerate. Similarly, one proves that also $\mathbf{g}_\perp$ is. □

Remarks (i) The non-degeneracy of $\mathbf{g}_\perp^{-1}$ (or, equivalently, of $\mathbf{g}_t^{-1}$) is not only sufficient, but also necessary to ensure that $\Omega_\perp \cap \Omega_t = \{0\}$. In fact, if $\mathbf{g}_\perp^{-1}$ is degenerate, there is a nonzero $\omega \in \Omega_\perp$ such that $0 = \mathbf{g}_\perp^{-1}(\omega, \Omega_\perp) = \mathbf{g}^{-1}(\omega, \Omega_\perp)$; hence, ω belongs to Ω_t as well. (ii) Similarly, the non-degeneracy of $\mathbf{g}_\perp$ (or, equivalently, of $\mathbf{g}_t$) is necessary for $\Xi_\perp \cap \Xi_t = \{0\}$. (iii) While Ξ_t is a Lie subalgebra of Ξ, in general $\Xi_\perp$ is not. (iv) In general, $\Xi_\perp$, Ω_t, and therefore also the orthogonal projections $\mathrm{pr}_\perp$, pr_t, are not $U\Xi_t$-equivariant; for this reason in Sect. 3.2.1 we are able to deform (pseudo)Riemannian geometry only via twists based on $\mathfrak{k} \subset \Xi_t$. (v) We refer to elements of $\Xi_\perp$, $\Omega_\perp$ and Ω_t as *normal* vector fields, *normal* 1-forms and *tangent* 1-forms.

As said, we identify $\Xi_t \subset \Xi$ with the Lie subalgebra of smooth vector fields tangent to *all* M_c $(c \in f(\mathcal{D}_f))$ at all points, because $X(f^a) = 0$ implies $X(f^a_c) = 0$; and $\Xi_M \subset \Xi$ defined in (4) with the Lie subalgebra of smooth vector fields tangent to M at all points. Similarly, we can identify Ω_t with the subbimodule of Ω tangent to *all* M_c $(c \in f(\mathcal{D}_f))$ at all points. We find $\Omega_t \subset \Omega_\mathcal{C} := \{\omega \in \Omega \;\mid\; \langle \Xi_\perp, \omega\rangle \subset \mathcal{C}\}$. Let $\Omega_{\mathcal{C}\mathcal{C}} := \bigoplus_{a=1}^k f^a\Omega = \bigoplus_{a=1}^k \Omega f^a \subset \Omega_\mathcal{C}$. It fulfills $\langle \Xi, \Omega_{\mathcal{C}\mathcal{C}}\rangle \subset \mathcal{C}$. We can identify the $\mathcal{X}^M$-bimodule of 1-forms Ω_M on M with the quotient

$$\Omega_M = \Omega_\mathcal{C}/\Omega_{\mathcal{C}\mathcal{C}} = \{[\omega] = \omega + \Omega_{\mathcal{C}\mathcal{C}} \;\mid\; \omega \in \Omega_\mathcal{C}\}\,. \tag{88}$$

Proposition 6 *For all $X \in \Xi_\mathcal{C}$, $\omega \in \Omega_\mathcal{C}$, the tangent projections $X_t \in \Xi_t$, $\omega_t \in \Omega_t$ belong to $[X] \in \Xi_M$ and $[\omega] \in \Omega_M$, respectively; similarly for multivector fields and higher-rank forms.*

Consequently, we can represent every element of Ξ_M, Ω_M, or more generally Ξ_{M_c}, Ω_{M_c}, resp. by an element of Ξ_t, Ω_t; etc. In "Appendix," we prove Proposition 6, as well as the relations

$$\begin{aligned} &\Omega_\perp = \{\omega \in \Omega \mid \langle \Xi_t, \omega\rangle = 0\}, \qquad \Omega_\perp \subset \Omega_\square, \qquad \text{where} \\ &\Omega_\square := \{\omega \in \Omega \mid \langle \Xi_{\mathcal{C}}, \omega\rangle \in \mathcal{C}\} = \{\omega \in \Omega \mid \langle \Xi_t, \omega\rangle \in \mathcal{C}\}. \end{aligned} \tag{89}$$

We call the restriction $\mathbf{g}_t$ in (86) of the metric map $\mathbf{g}$ *first fundamental form* for the family of manifolds $M_c \subset \mathcal{D}'_f$, $c \in f\big(\mathcal{D}'_f\big)$. It is $\mathcal{X}$-linear in both arguments and further satisfies $\mathbf{g}_t(X \cdot h, Y) = \mathbf{g}_t(X, h \cdot Y)$ for all $X, Y \in \Xi_t$ and $h \in \mathcal{X}$ (*middle-linearity*). Since $\mathbf{g}_t$ is uniquely determined (via the pairing) by the tangent projection $\tilde{\mathbf{g}}_t = (\mathrm{pr}_t \otimes \mathrm{pr}_t)(\mathbf{g}) \in \Omega_t \otimes \Omega_t$ of the metric $\mathbf{g} \in \Omega \otimes \Omega$, when there is no risk of confusion we will drop the tilde and with a slight abuse of notation denote $\tilde{\mathbf{g}}_t$ by $\mathbf{g}_t$. It is a symmetric element, i.e., $\tau(\mathbf{g}_t) = \mathbf{g}_t$. The *first fundamental form* (induced metric) *on M* is obtained by the further projection $\mathcal{X} \to \mathcal{X}^M$, which amounts to choosing the $c = 0$ manifold M out of the family. The same prescription will hold for the Levi–Civita connection, curvature, etc., on M. Applying the decomposition of Ξ in tangent and normal vector fields to the restriction of the Levi–Civita connection

$$\nabla|_{\Xi_t \otimes \Xi_t} = \nabla_t + II \colon \Xi_t \otimes \Xi_t \to \Xi \tag{90}$$

we obtain the *projected Levi–Civita connection* and the *second fundamental form* for the family of manifolds M_c:

$$\nabla_t := \mathrm{pr}_t \circ \nabla|_{\Xi_t \otimes \Xi_t} \colon \Xi_t \otimes \Xi_t \to \Xi_t, \qquad II := \mathrm{pr}_\perp \circ \nabla|_{\Xi_t \otimes \Xi_t} \colon \Xi_t \otimes \Xi_t \to \Xi_\perp. \tag{91}$$

Proposition 7 *The first fundamental form $\mathbf{g}_t$, the second fundamental form II and the projected Levi–Civita covariant derivative ∇_t are $U\mathfrak{k}$-equivariant maps.*

Proof As compositions of $U\mathfrak{k}$-equivariant maps, $\mathbf{g}_t$, ∇_t and II are $U\mathfrak{k}$-equivariant. □

By the Leibniz rule for ∇ and the $\mathcal{X}$-linearity of pr_t, $\mathrm{pr}_t(hZ) = h\,\mathrm{pr}_t(Z)$ for all $h \in \mathcal{X}$, $Z \in \Xi$, ∇_t is $\mathcal{X}$-linear in the first argument, $\nabla_{t,hX}Y = h\nabla_{t,X}Y$, and fulfills the Leibniz rule

$$\nabla_{t,X}(hY) = \mathrm{pr}_t\big[X(h)Y + h\nabla_X Y\big] = X(h)\mathrm{pr}_t(Y) + h\mathrm{pr}_t(\nabla_X Y) = X(h)Y + h\nabla_{t,X}Y \tag{92}$$

in the second argument, for all $h \in \mathcal{X}$ and $X, Y \in \Xi_t$. Similarly, we find that II is $\mathcal{X}$-linear in both arguments. By applying the further projection $\mathcal{X} \to \mathcal{X}^M$, which amounts to choosing the $c = 0$ manifold M out of the M_c family, one finally obtains the expected $\mathcal{X}^M$-linearity of the first and second fundamental form on M, as well as the expected $\mathcal{X}^M$-linearity in the first argument and Leibniz rule in the second for the

Levi–Civita connection on M (see, e.g., [43] Chapter 3). Clearly, if $\mathbf{g}$ is Riemannian also the first fundamental form on M is.

Of course, one can do the same for any other M_c by a different choice of c. The second fundamental form yields the extrinsic curvature of the M_c's. The intrinsic curvature R_t is related to the curvature R of ∇ on $\mathbb{R}^n$ by the Gauss equation (valid for all $X, Y, Z, W \in \Xi_t$)

$$\mathbf{g}\big(\mathsf{R}(X,Y)Z, W\big) = \mathbf{g}(\mathsf{R}_t(X,Y)Z, W) \\ + \mathbf{g}\big(II(X,Z), II(Y,W)\big) - \mathbf{g}\big(II(Y,Z), II(X,W)\big). \tag{93}$$

3.1.2 Decompositions in bases of $\boldsymbol{\Omega}$, $\boldsymbol{\Xi}$; Euclidean, Minkowski metrics

In this section, we explicitly determine the geometry (in particular, the decompositions (84) and the associated projections $\mathrm{pr}_t, \mathrm{pr}_\perp$) in terms of bases of $\Omega, \Omega_\perp, \Omega_t$ and $\Xi, \Xi_\perp, \Xi_t$ for a generic metric $\mathbf{g}$, specializing to the Euclidean and Minkowski metrics at the end.

Let $(x^1, \ldots, x^n)$ be a n-ple of Cartesian coordinates; we lower and raise indices $i, j, \ldots$ using the metric components $g_{ij} := \mathbf{g}(\partial_i, \partial_j)$ and the dual ones $g^{ij} = {g^{-1}}_{ij} = \mathbf{g}^{-1}(dx^i, dx^j)$, respectively: $dx_i = g_{ij}dx^j$, $Y_i = g_{ij}Y^j$, $\partial^i = g^{ij}\partial_j$, etc. Thus, we can write the metric and its dual in the form

$$\mathbf{g} = dx^i \otimes dx_i, \qquad \mathbf{g}^{-1} = \partial^i \otimes \partial_i, \tag{94}$$

implying, for all vector fields $X = X^i\partial_i$, $Y = Y^i\partial_i$ and 1-forms $\alpha = \alpha_i dx^i$, $\omega = \omega_i dx^i$,

$$\mathbf{g}(X, Y) = X^i Y_i, \qquad \mathbf{g}^{-1}(\alpha, \omega) = \alpha_i \omega^i. \tag{95}$$

On $\mathcal{D}'_f \subseteq \mathcal{D}_f$ the $k \times k$ matrix defined by $E^{ab} = \mathbf{g}_\perp^{-1}(df^a, df^b)$ ($E^{ab} = f^{ai} f_i^b = \big(Jg^{-1}J^T\big)^{ab}$, in terms of Cartesian coordinates) is symmetric and invertible, by (79), (83); we denote its inverse by $K := E^{-1}$. If the metric $\mathbf{g}$ is *Riemannian*, then E is also positive-definite on $\mathcal{D}'_f = \mathcal{D}_f$. Let

$$N_\perp^a := K^{ab}\ \mathbf{g}^{-1}(df^b, dx^i)\, \partial_i = K^{ab} f^{bi} \partial_i \tag{96}$$

and, for all $\omega \in \Omega$, $X \in \Xi$,

$$\omega_\perp := df^a\, K^{ab}\ \mathbf{g}^{-1}(df^b, \omega), \qquad X_\perp := \mathbf{g}(X, N_\perp^a)\, E^{ab} N_\perp^b, \tag{97}$$

or, explicitly in terms of the decompositions $\omega = \omega_i dx^i$, $X = X^i\partial_i$,

$$\omega_\perp = df^a\, K^{ab} f^{bh}\omega_h, \qquad X_\perp = X^i f_i^a\, N_\perp^a \tag{98}$$

(sum over repeated indices: $h, i, j, \ldots$ run over $1, \ldots, n$, while $a, b, c, d, \ldots$ run over $1, \ldots, k$).

Proposition 8 $\mathcal{N}_\perp := \{N_\perp^a\}_{a=1}^k$, $\mathcal{B}_\perp := \{df^a\}_{a=1}^k$ *are bases resp. of* $\Xi_\perp, \Omega_\perp$ *dual to each other, in the sense*

$$\langle N_\perp^a, df^b\rangle = N_\perp^a(f^b) = \delta^{ab}, \qquad a, b \in \{1, \dots, k\}. \tag{99}$$

$\mathbf{g}^{-1}(df^a, df^b) = E^{ab}$, $\mathbf{g}(N_\perp^a, N_\perp^b) = K^{ab}$. *The* df^a, $N_\perp^a$ *as well the* E^{ab}, K^{ab} *are* $\mathfrak{k}$*-invariant. On* $X \in \Xi$, $\omega \in \Omega$ *the action of the projections* $\mathrm{pr}_\perp, \mathrm{pr}_t$ *explicitly reads* $\mathrm{pr}_\perp(X) = X_\perp$, $\mathrm{pr}_t(X) = X_t := X - X_\perp$, $\mathrm{pr}_\perp(\omega) = \omega_\perp$, $\mathrm{pr}_t(\omega) = \omega_t := \omega - \omega_\perp$.

Proof We have already proved in Proposition 4 that $\mathcal{B}_\perp$ is a basis of $\Omega_\perp$. As a consequence, $\omega_\perp \in \Omega_\perp$. From the definition, we find $\mathbf{g}(X, N_\perp^a) = K^{ab}X^i f_i^b = K^{ab}X(f^b) = 0$ for all $X \in \Xi_t$ and $a = 1, \dots, k$, whence $N_\perp^a \in \Xi_\perp$; moreover, $N_\perp^a(f^b) = K^{ac}f^{ci}\partial_i(f^a) = K^{ac}E^{cb} = \delta^{ab}$, and $\mathcal{N}_\perp$ is the basis of $\Xi_\perp$ dual to $\mathcal{B}_\perp$. As a consequence, $X_\perp \in \Xi_\perp$. $g \triangleright df^a = 0$ for all $g \in \Xi_t$ holds in particular for $g \in \mathfrak{k}$. By Proposition 7 $g \triangleright N_\perp^a \in \Xi_\perp$ for all $g \in \mathfrak{k}$, and therefore $g \triangleright N_\perp^a = C_c^a(g)N_\perp^c$ with some coefficients $C_c^a(g)$. Applying $g\triangleright$ to both sides of (99) and using the Ξ-equivariance of the pairing, we thus find the $\mathfrak{k}$-invariance also of the $N_\perp^a$:

$$\langle g \triangleright N_\perp^a, df^b\rangle = 0 \Rightarrow 0 = C_c^a(g)\langle N_\perp^c, df^b\rangle = C_b^a(g) \quad \forall a, b \Rightarrow g \triangleright N_\perp^a = 0.$$

Checking $\mathbf{g}^{-1}(df^a, df^b) = E^{ab}$, $\mathbf{g}(N_\perp^a, N_\perp^b) = K^{ab}$ is a straightforward computation; their $\mathfrak{k}$-invariance follows from that of df^a and the $U\mathfrak{k}$-equivariance of $\mathbf{g}$; in fact, $\forall g \in U\mathfrak{k}$

$$\begin{aligned} g \triangleright E^{ab} &= g \triangleright \mathbf{g}^{-1}(df^a, df^b) = \mathbf{g}^{-1}\Big(g_{(1)} \triangleright df^a, g_{(2)} \triangleright df^b\Big) \\ &= \varepsilon(g)\,\mathbf{g}^{-1}(df^a, df^b) = \varepsilon(g)E^{ab}. \end{aligned}$$

The linear maps $X \mapsto X_\perp \in \Xi_\perp$, $\omega \mapsto \omega_\perp \in \Omega_\perp$ indeed realize the projection $\mathrm{pr}_\perp$, because

$$\begin{aligned} &(X_\perp)_\perp = \mathbf{g}(X_\perp, N_\perp^a)\, E^{ab}N_\perp^b = \mathbf{g}(X, N_\perp^c)\, E^{cd}\mathbf{g}(N_\perp^d, N_\perp^a)\, E^{ab}N_\perp^b \\ &\quad = \mathbf{g}(X, N_\perp^a)\, E^{ab}N_\perp^b = X_\perp, \\ &\quad (\omega_\perp)_\perp = df^a K^{ab}\mathbf{g}^{-1}(df^b, \omega_\perp) = df^a K^{ab}\mathbf{g}^{-1}(df^b, df^c)K^{cd}\mathbf{g}^{-1}(df^d, \omega) \\ &\quad = df^a K^{ab}\mathbf{g}^{-1}(df^b, \omega) = \omega_\perp; \end{aligned}$$

hence also the linear maps $X \mapsto X_t := X - X_\perp$, $\omega \mapsto \omega_t := \omega - \omega_\perp$ realize the projection pr_t. □

Remark If $\mathbf{g}$ is *Riemannian*, setting $\mathcal{H} := E^{-1/2}$, $\theta^a := \mathcal{H}^{ab}df^b$, $U_\perp^a := \mathcal{H}^{ab}f^{bi}\partial_i$, one finds that $\{U_\perp^a\}_{a=1}^k$, $\{\theta^a\}_{a=1}^k$ are orthonormal bases of $\Xi_\perp, \Omega_\perp$, respectively, and are dual to each other, i.e.,

$$\mathbf{g}(U_\perp^a, U_\perp^b) = \delta^{ab}, \qquad \mathbf{g}^{-1}(\theta^a, \theta^b) = \delta^{ab}, \qquad \langle U_\perp^a, \theta^b\rangle = \delta^{ab}. \tag{100}$$

The $\mathfrak{k}$-invariance of $\theta^a, U_\perp^a$ follows from that of $df^a, N_\perp^a$ and of E. In terms of the bases $\{U_\perp^a\}_{a=1}^k$, $\{\theta^a\}_{a=1}^k$ the normal components of $X \in \Xi$, $\omega \in \Omega$ read

$$\omega_\perp = \theta^a\, \mathbf{g}^{-1}(\theta^a, \omega), \qquad X_\perp = \mathbf{g}(X, U_\perp^a)\, U_\perp^a. \tag{101}$$

Even if $\mathbf{g}$ is *not Riemannian*, one can find in $\mathcal{D}'_f$ a $k \times k$ symmetric matrix $\mathcal{H}$, such that $\theta^a := \mathcal{H}^{ab} df^b$ $U_\perp^a := \mathcal{H}^{ab} f^{bi} \partial_i$ are $\mathfrak{k}$-invariant, make up bases $\{U_\perp^a\}_{a=1}^k$, $\{\theta^a\}_{a=1}^k$ of $\Xi_\perp$, $\Omega_\perp$, respectively, that are orthonormal up to suitable signs $\epsilon_a = \pm 1$ and dual to each other, in the sense

$$\mathbf{g}(U_\perp^a, U_\perp^b) = \zeta^{ab}, \qquad \mathbf{g}^{-1}(\theta^a, \theta^b) = \zeta^{ab}, \qquad \langle U_\perp^a, \theta^b \rangle = \delta^{ab}, \tag{102}$$

where $\zeta^{ab} = \zeta_{ab} := \epsilon_a \delta^{ab}$ (no sum over a). The normal components of $X \in \Xi, \omega \in \Omega$ read

$$\omega_\perp = \theta^a \zeta_{ab}\, \mathbf{g}^{-1}(\theta^b, \omega), \qquad X_\perp = \mathbf{g}(X, U_\perp^a)\, \zeta_{ab} U_\perp^b. \tag{103}$$

If $\mathbf{g}$ is the *Euclidean metric* ($g_{ij} = \delta_{ij}$), the associated Levi–Civita connection on $\mathbb{R}^n$ is

$$\nabla = dx^i \otimes \mathcal{L}_{\partial_i}, \qquad \text{e.g.,} \qquad \nabla_X Y = X^i \partial_i(Y^j)\partial_j. \tag{104}$$

We endow $M \subset \mathbb{R}^n$ with the induced metric $\mathbf{g}_t$. Using $X, Y, Z \in \Xi_t$ as representatives of elements of Ξ_M, the Levi–Civita connection on $(M, \mathbf{g}_t)$ is $\nabla_{t,X} Y := (\nabla_X Y)_t$: (61), (74) hold with $\mathbf{g}$, ∇, T, R replaced by $\mathbf{g}_t$, ∇_t, T_t, R_t. Deriving the identities $Y(f_c) = Y^j f_j^a = 0$ we find that $\partial_i(Y^j) f_j^a = -Y^j f_{ij}^a$, where we have abbreviated $f_{ij}^a := \partial_i\big(\partial_j(f^a)\big)$; thus, the second fundamental form $II(X, Y) := (\nabla_X Y)_\perp$ takes the explicit form

$$II(X, Y) = X^i \partial_i(Y^j) f_j^a N_\perp^a = -X^i Y^j f_{ij}^a N_\perp^a. \tag{105}$$

Replacing this result and $\mathsf{R} = 0$ in the Gauss equation (93), we find for the intrinsic curvature

$$[\mathsf{R}_t(X,Y)Z)]^m\, W^m = f_{ij}^a K^{ab} f_{lm}^b (Y^i X^l - X^i Y^l) Z^j W^m$$

on all $X, Y, Z, W \in \Xi_t$. Finally, $Z \in \Xi_t$ is a Killing vector field on $(M, \mathbf{g}_t)$ if[7]

$$\begin{aligned} Z\Big(\mathbf{g}(X, Y)\Big) - \mathbf{g}\big([Z, X], Y\big) - \mathbf{g}\big(X, [Z, Y]\big) &= X^h Y^i (\partial_h Z_i + \partial_i Z_h) \\ &= 0 \qquad \forall X, Y \in \Xi_t. \end{aligned} \tag{106}$$

[7] In fact, $lhs = Z^h \partial_h \left(X^i Y_i\right) - \left[Z^h \partial_h(X^i) - X^h \partial_h(Z^i)\right] Y_i - \left[Z^h \partial_h(Y^i) - Y^h \partial_h(Z^i)\right] X_i = [X^h Y_i + X_i Y^h]\, \partial_h(Z^i) = rhs$.

In fact, this condition guarantees that Z is Killing on $(M_c, \mathbf{g}_t)$ for all c. The Killing vector fields close the Lie algebra $\mathfrak{k} = \mathfrak{h} \cap \Xi_t$ of the group of isometries $\mathfrak{K}$ of the M_c's; $\mathfrak{K}$ is a subgroup of the group $\mathfrak{H}$ of isometries of $\mathbb{R}^n$, i.e., of the Euclidean group (every element of $\mathfrak{H}$ is a composition of a rotation, a translation and possibly an inversion of axis).

If $\mathbf{g}$ is the *Minkowski metric* $[g_{ij} = g^{ij} = \eta_{ij} = \mathrm{diag}(1, \dots, 1, -1)]$, the associated Levi–Civita connection on $\mathbb{R}^n$ is again as in (104). Endowing $M_c \subset \mathcal{D}'_f$ with the induced metric $\mathbf{g}_t$ and using $X, Y, Z \in \Xi_t$ as representatives of elements of Ξ_{M_c}, the Levi–Civita connection on $(M_c, \mathbf{g}_t)$ is again $\nabla_{t,X} Y := (\nabla_X Y)_t$: (61), (74) hold with $\mathbf{g}, \nabla, \mathsf{T}, \mathsf{R}$ replaced by $\mathbf{g}_t, \nabla_t, \mathsf{T}_t, \mathsf{R}_t$. In terms of components the condition for $Z \in \Xi_t$ to be a Killing vector field on $(M_c, \mathbf{g}_t)$ remains (106).

Bases and complete sets of Ξ_t, Ω_t

As seen, $\mathcal{B}_\perp := \{df^a\}_{a=1}^k$, $\mathcal{N}_\perp := \left\{N_\perp^a\right\}_{a=1}^k$ are globally defined bases of the $\mathcal{X}$-bimodules $\Omega_\perp$, $\Xi_\perp$, respectively. Also the $V_\perp^a := f^{ai}\partial_i = E^{ab} N_\perp^b$ make a basis of $\Xi_\perp$. The globally defined sets

$$\Theta_t := \left\{\vartheta^j\right\}_{j=1}^n, \quad S_W := \left\{W_j\right\}_{j=1}^n, \qquad \text{with} \quad \vartheta^j := \mathrm{pr}_t(\xi^j), \quad W_j := \mathrm{pr}_t(\partial_j), \tag{107}$$

are, respectively, complete in Ω_t, Ξ_t, but are not bases, because of the linear dependence relations

$$\vartheta^j f_j^a = 0, \qquad f^{aj} W_j = 0, \qquad a = 1, \dots, k. \tag{108}$$

The above definition of $\mathcal{B}_\perp$ does not involve any specific metric, as the Definition (78) of $\Omega_\perp$ itself. Similarly, as the definition (10) of Ξ_t does not involve any metric, there should be some alternative complete set in Ξ_t with the same feature. To determine it we start with the case $k = 1$, i.e., with a $(n-1)$-dimensional (hyper)surface $M \subset \mathcal{D}_f$ determined by a single equation

$$f(x) = 0. \tag{109}$$

Rescaling S_W by the factor $f^i f_i$ we obtain another complete set: $S_V := \left\{V_j\right\}_{j=1}^n$, with $V_j := (f^i f_i)\partial_j - f_j V_\perp$. A third complete set (of globally defined vector fields) in Ξ_t is

$$S_L := \left\{L_{ij}\right\}_{i,j=1}^n, \qquad L_{ij} := f_i \partial_j - f_j \partial_i. \tag{110}$$

In fact, L_{ij} annihilate f; S_L is complete because $V_j = f^i L_{ij}$. This is the searched set, because its definition does not involve the metric. Clearly $L_{ij} = -L_{ji}$, so at most $n(n-1)/2$ of the L_{ij} (e.g., those with $i < j$) are linearly independent over $\mathbb{R}$. Obviously, both S_V, S_L are of rank $n-1$ over $\mathcal{X}$; they are, respectively, characterized by the dependence relations

$$f^i V_i = 0, \qquad f_{[i} L_{jk]} = 0 \tag{111}$$

(here and below square brackets enclosing indices mean a complete antisymmetrization of the latter). As known, if M is not parallelizable, there is no basis (i.e., complete set of just $(n-1)$ elements) of Ξ_t consisting of globally defined vector fields: redundancy is unavoidable. In the case of spheres $f \equiv (x^i x^i - R^2)/2 = 0$ the $n(n-1)/2$ $L_{ij} := x^i \partial_j - x^j \partial_i$ $(i < j)$ are the usual generators of rotations (angular momentum components), i.e., span $so(n)$. The L_{ij} are antihermitian under the $*$-structure (116), namely $L_{ij}^* = -L_{ij}$.

By an explicit computation, we find that their Lie brackets are

$$[L_{ij}, L_{hk}] = f_{jh} L_{ik} - f_{ih} L_{jk} - f_{jk} L_{ih} + f_{ik} L_{jh}. \tag{112}$$

Now we consider the general k case. The globally defined vector fields

$$L_{i_1 i_2 \ldots i_{k+1}} := f^1_{[i_1} f^2_{i_2} \ldots f^k_{i_k} \partial_{i_{k+1}]} \tag{113}$$

are antihermitian, fulfill $L_{i_1 i_2 \ldots i_{k+1}} f^a = 0$ for all $a = 1, \ldots, k$, are completely antisymmetric with respect to $(i_1, i_2, \ldots, i_{k+1})$, and make up a set S_L complete (over $\mathcal{X}$) in Ξ_t, independently of the metric. The $L_{i_1 i_2 \ldots i_{k+1}}$ with $i_1 < i_2 < \cdots < i_{k+1}$, or a subset thereof, is linearly independent over $\mathbb{C}$. Even the latter may be linearly dependent over $\mathcal{X}$, because $f^a{}_{[j} L_{i_1 i_2 \ldots i_{k+1}]} = 0$ for all a. We do not compute their Lie brackets here.

3.1.3 Differential calculus algebras $\mathcal{Q}^\bullet$, $\mathcal{Q}^\bullet_{M_c}$ on $\mathbb{R}^n$, M_c

Henceforth, we abbreviate $\xi^i := dx^i$. Let $S = \{e_\alpha\}_{\alpha=1}^A$ be a set of vector fields, globally defined on $\mathcal{D}_f$ that is complete in Ξ. The e_α, ξ^i fulfill relations of the type

$$\begin{aligned} &\textstyle\sum_{\alpha=1}^A t_l^\alpha \, e_\alpha = 0, \qquad l = 1, \ldots, A-n, \\ &e_\alpha e_\beta - e_\beta e_\alpha - C^\gamma_{\alpha\beta} \, e_\gamma = 0, \\ &e_\alpha \xi^i - \xi^i e_\alpha = 0, \qquad \xi^i \xi^j + \xi^j \xi^i = 0 \end{aligned} \tag{114}$$

(with suitable $t_l^a, C^\gamma_{\alpha\beta} \in \mathcal{X}$). The first line contains possible linear dependence relations among the e_α, like (111). If we choose $S = \{\partial_1, \ldots, \partial_n\}$ this is empty, while in the second line $C^\gamma_{\alpha\beta} \equiv 0$. Clearly, the coefficients in the decomposition $X = X^\alpha e_\alpha \in \Xi$ are defined up to shifts $X^\alpha \mapsto X^\alpha + \sum_l h^l t_l^\alpha$, with $h^l \in \mathcal{X}$. Consider the unital algebra $\mathcal{Q}^\bullet$ over $\mathbb{C}$ consisting of polynomials in ξ^i, e_α with (left or right) coefficients in $\mathcal{X}$, modulo relations (114) and the ones

$$h\xi^i - \xi^i h = 0, \qquad e_\alpha h - h\, e_\alpha - e_\alpha(h) = 0 \qquad \forall h \in \mathcal{X}; \tag{115}$$

$\mathcal{Q}^\bullet$ is a $U\Xi$-equivariant $\mathcal{X}$-bimodule. It is easy to check that a different choice of S changes (114, 115), but leads to an equivalent definition of $\mathcal{Q}^\bullet$. (One could choose also a different basis of 1-forms, but we will not consider this here.) We shall name $\mathcal{Q}^\bullet$ *differential calculus algebra on* $\mathcal{D}_f$. The elements of $\mathcal{Q}^\bullet$ can be considered as differential-operator-valued inhomogeneous forms. Relations (114, 115) encode all the information about the differential calculus and allow to order the ξ^i, e_α in any prescribed way, with the coefficient functions at the left, center, or right—as one wishes. $\mathcal{Q}^\bullet$ admits $\mathcal{X}$, $\Omega^\bullet$, $\mathcal{H}$ as subalgebras; the *enlarged Heisenberg algebra* $\mathcal{H}$ is the component of form degree zero. While $\mathcal{Q}^\bullet$, $\Omega^\bullet$ are graded by the form degree, $\mathcal{Q}^\bullet$, $\mathcal{H}$ are filtered by the degree r in the e_α; r gives the order of an element of $\mathcal{H}$ seen as a differential operator on $\mathcal{X}$. Note that within $\mathcal{Q}^\bullet$ also the action of a generic vector field $X = X^\alpha e_\alpha$ on a function h can be expressed as a commutator: $[X, h] = [X^\alpha e_\alpha, h] = X^\alpha[e_\alpha, h] = X(h)$. In the $\mathcal{Q}^\bullet$ framework $Xh = hX + X(h)$ is the inhomogeneous first-order differential operator sum of a first-order part (the vector field hX) and a zero-order part (the multiplication operator by $X(h)$); it must not be confused with the product of X by h from the right, which is equal to hX and in the previous sections has been denoted in the same way. In the $\mathcal{Q}^\bullet$ framework we denote the latter by $X \triangleleft h$ (of course $(X \triangleleft h)(h') = X(h')h = hX(h')$, $X \triangleleft (hh') = hh'X$ remain valid). We endow $\mathcal{Q}^\bullet$ with the natural $*$-structure defined by

$$f^*(x) = \overline{f(x)}, \qquad \partial_i^* = -\partial_i, \qquad \xi^{i*} = \xi^i. \tag{116}$$

If one chooses S so that a subset $S_t := \{e_\alpha\}_{\alpha=1}^B$ ($B := A-k$) is complete in Ξ_t (e.g., it consists of the $L_{i_1 i_2 \ldots i_{k+1}}$), while $e_{B+a} := s_a^b N_\perp^b$, with some matrix $s_a^b(x)$ ($a, b \in \{1, \ldots, k\}$) invertible everywhere, then if $\alpha, \beta \le B$ the sum in $(114)_2$ is extended over $\gamma \le B$. The differential calculus algebra $\mathcal{Q}_{M_c}^\bullet$ on M_c is the $\mathcal{X}^{M_c}$-bimodule generated by the $\xi^1, \ldots, \xi^n, e_1, \ldots, e_B$ modulo the relations (114, 115) (with $\alpha, \beta \le B$) and the ones

$$f_c^a \equiv f^a - c^a \mathbf{1} = 0, \qquad df^a \equiv \xi^h f_h^a = 0, \qquad a = 1, \ldots, k. \tag{117}$$

3.2 Twisted differential geometry of manifolds embedded in $\mathbb{R}^n$

Using a twist $\mathcal{F} \in \big(U\Xi_t \otimes U\Xi_t\big)[[\nu]]$ and following the general twisting approach, we deform the differential geometry on $\mathcal{D}_f$ in a way compatible with the embeddings, i.e., so that it projects to the twist deformation of the differential geometry on the submanifolds M_c, $c \in \mathbb{R}^n$. Equivalently, we deform the differential calculus algebra $\mathcal{Q}^\bullet$ on $\mathcal{D}_f$ into an associated $\mathcal{Q}_\star^\bullet$ in a way compatible with the embeddings, i.e., encoding through projections all deformations $\mathcal{Q}_{M_c}^\bullet \rightsquigarrow \mathcal{Q}_{M_c\star}^\bullet$. Unless explicitly stated, we still denote by $X \star h = (\overline{\mathcal{R}}_1 \triangleright h) \star (\overline{\mathcal{R}}_2 \triangleright X)$ the vector field that is $\star$-product of the one X by the function h from the right, as done so far.

To state the twisted analog of Proposition 4, we first define a $\mathcal{X}_\star$-subbimodule $\Omega_{\perp\star} \subset \Omega_\star$:

$$\Omega_{\perp\star} := \{\omega \in \Omega_\star \mid \langle \Xi_{t\star}, \omega \rangle_\star = 0\}. \tag{118}$$

Proposition 9 *Equipped with the* $\star$*-Lie bracket* $[\ ,\ ]_\star$ $\Xi_{t\star}, \Xi_{\mathcal{C}\star}$ *are* $\star$*-Lie subalgebras of* $\Xi_\star$*, and* $\Xi_{\mathcal{C}\mathcal{C}\star}$ *is an ideal of* $\Xi_{\mathcal{C}\star}$*. Another* $\star$*-Lie algebra is thus*

$$\Xi_{M\star} := \Xi_{\mathcal{C}\star}/\Xi_{\mathcal{C}\mathcal{C}\star} \equiv \left\{ [X] := X + \Xi_{\mathcal{C}\mathcal{C}\star} \mid X \in \Xi_{\mathcal{C}\star} \right\}. \tag{119}$$

Moreover, $\Xi_{t\star}, \Xi_{\mathcal{C}\star}, \Xi_{\mathcal{C}\mathcal{C}\star}, \Xi_{M\star}, \Omega_{\perp\star}$ *resp. coincide with* $\Xi_t[[\nu]], \Xi_{\mathcal{C}}[[\nu]], \Xi_{\mathcal{C}\mathcal{C}}[[\nu]], \Xi_M[[\nu]], \Omega_\perp[[\nu]]$ *as* $\mathbb{C}[[\nu]]$*-modules.* $\Xi_{t\star}, \Xi_{M\star}, \Omega_{\perp\star}$ *and the corresponding exterior algebras* $\Xi^\bullet_{t\star} = \bigwedge^\bullet_\star \Xi_{t\star}$*,* $\Xi^\bullet_{M\star} = \bigwedge^\bullet_\star \Xi_M$*,* $\Omega^\bullet_{\perp\star} = \bigwedge^\bullet_\star \Omega_{\perp\star}$ *are* $U\Xi^{\mathcal{F}}_t$*-equivariant* $\mathcal{X}_\star$*-bimodules.* $\Omega_{\perp\star}$ *can be explicitly decomposed as*

$$\Omega_{\perp\star} = \bigoplus_a \mathcal{X}_\star \star \mathrm{d}f^a = \bigoplus_a \mathrm{d}f^a \star \mathcal{X}_\star. \tag{120}$$

Proof These are direct consequences of the following properties. By Proposition 4, for all $h \in \mathcal{X}[[\nu]],\ g \in U\Xi_t[[\nu]],\ X, X' \in \Xi_t[[\nu]],\ Y, Y' \in \Xi_{\mathcal{C}}[[\nu]], Z \in \Xi_{\mathcal{C}\mathcal{C}}[[\nu]]$, $\omega = \omega_a df^a \in \Omega_\perp[[\nu]]$:

- $h\star X,\ \ X\star h,\ \ g \triangleright X$ and $[X, X']_\star$ belong to $\Xi_t[[\nu]]$.
- $h\star Y,\ \ Y\star h,\ \ g \triangleright Y$ and $[Y, Y']_\star$ belong to $\Xi_{\mathcal{C}}[[\nu]]$.
- $h\star Z,\ \ Z\star h,\ \ g \triangleright Z$ and $[Y, Z]_\star$ belong to $\Xi_{\mathcal{C}\mathcal{C}}[[\nu]]$ (because $\Xi_{\mathcal{C}\mathcal{C}}$ is an ideal in $\Xi_{\mathcal{C}}$).
- $h\star[Y],\ \ [Y]\star h,\ \ g \triangleright [Y]$ and $\big[[Y], [Y']\big]_\star$ belong to $\Xi_M[[\nu]]$.
- $h\star df^a = h df^a = (df^a)\star h$ and $g \triangleright \omega = (g \triangleright \omega_a) df^a$ belong to $\Omega_\perp[[\nu]]$, by (14) and the relation $g \triangleright df^a = \varepsilon(g) df^a$.
- $\langle X, \omega\rangle_\star = \left\langle \overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright \omega \right\rangle = 0$, because $\overline{\mathcal{F}}_1 \triangleright X \in \Xi_t[[\nu]]$ and $\overline{\mathcal{F}}_2 \triangleright \omega \in \Omega_\perp[[\nu]]$. □

This means in particular that taking the quotient commutes with twisting. To build explicit examples of twist-deformed submanifolds, we recall that several known types of Drinfel'd twists (as the ones mentioned in Sect. 2.1.1) are based on finite-dimensional Lie algebras. When does the infinite-dimensional Ξ_t admit a finite-dimensional Lie subalgebra $\mathfrak{g}$ over $\mathbb{R}$, so that we can choose $\mathcal{F} \in (U\mathfrak{g} \otimes U\mathfrak{g})[[\nu]]$? Given a set S of vector fields that is complete in Ξ_t, the question is which combinations (with coefficients in $\mathcal{X}$) of them, if any, close a finite-dimensional Lie algebra $\mathfrak{g}$. An easy answer is available for the quadrics in $\mathbb{R}^n$, see Sect. 4. If $\mathbb{R}^n$ endowed with a metric admits a family M_c of (pseudo)Riemannian submanifolds manifestly symmetric under a Lie group $\mathfrak{K}$ (its group of isometries),[8] then a nontrivial $\mathfrak{g}$ exists and contains the (Killing) Lie algebra $\mathfrak{k}$ of $\mathfrak{K}$ (if M_c is maximally symmetric, then $\mathfrak{k}$ is even complete—over $\mathcal{X}$—in Ξ_t). In the next subsections, we consider such a case and stick to deformations induced by a twist $\mathcal{F}$ based on $\mathfrak{k} \subset \Xi_t$; under these assumptions the deformation is compatible with the geometry. $\Xi_{t\star}, \Omega_{\perp\star}$ appear as addends in the decomposition of $\Xi_\star$ in tangent and orthogonal vector fields. In Sect. 3.2.2, we first give explicit results for a generic metric and then specialize the discussion to the Euclidean and Minkowski metric.

[8] For instance, the sphere S^{n-1} is $SO(n)$ invariant; a cylinder in $\mathbb{R}^3$ is invariant under $SO(2) \times \mathbb{R}$; the hyperellipsoid of equation $(x^1)^2 + (x^2)^2 + 2[(x^3)^2 + (x^4)^2] = 1$ is invariant under $SO(2) \times SO(2)$; etc.

3.2.1 Twisted metric, Levi–Civita connection, intrinsic and extrinsic curvatures

As seen in Sect. 3.1.1, endowing $\mathcal{D}_f \subseteq \mathbb{R}^n$ with a (non-degenerate) metric $\mathbf{g}$ makes all the $M_c \subset \mathcal{D}'_f$ into (pseudo)Riemannian submanifolds; $\mathcal{D}'_f \subseteq \mathcal{D}_f$ is where the restriction $\mathbf{g}_\perp^{-1}$ is non-degenerate. For a generic twist $\mathcal{F} \in (U\Xi_t \otimes U\Xi_t)[[\nu]]$, we introduce the $\mathcal{X}$-subbimodules

$$\Xi_{\perp\star} := \{X \in \Xi_\star \mid \mathbf{g}_\star(X, \Xi_{t\star}) = 0\}, \qquad \Omega_{t\star} := \{\omega \in \Omega_\star \mid \mathbf{g}_\star^{-1}(\omega, \Omega_{\perp\star}) = 0\}, \tag{121}$$

Again, let $\mathfrak{k} \subset \Xi_t$ the Lie subalgebra of Killing vector fields of $\mathbf{g}$ that are also tangent to the submanifolds $M_c \subset \mathcal{D}'_f$. The twisted version of Proposition 5 reads

Proposition 10 *If $\mathcal{F} \in (U\mathfrak{k} \otimes U\mathfrak{k})[[\nu]]$ the $\star$-Lie algebra $\Xi_\star$ of smooth vector fields and the $\mathcal{X}_\star$-bimodule $\Omega_\star$ of 1-forms on $\mathcal{D}'_f$ split into the direct sums of $\mathcal{X}_\star$-subbimodules*

$$\Xi_\star = \Xi_{t\star} \oplus \Xi_{\perp\star}, \qquad \Omega_\star = \Omega_{t\star} \oplus \Omega_{\perp\star} \tag{122}$$

orthogonal with respect to the twisted metrics $\mathbf{g}_\star$ and $\mathbf{g}_\star^{-1}$, respectively. $\Xi_{t\star}$ is a $\star$-Lie subalgebra of $\Xi_\star$. $\Omega_{t\star}$, $\Xi_{\perp\star}$ are orthogonal with respect to the $\star$-pairing, $\Omega_{t\star} = \{\omega \in \Omega_\star \mid \langle \Xi_{\perp\star}, \omega\rangle_\star = 0\}$. Also the restrictions of $\mathbf{g}_\star^{-1}$ (resp. $\mathbf{g}$) to the tangent *and* normal *1-forms (resp. vector fields)*

$$\begin{aligned}
\mathbf{g}_{\perp\star}^{-1} &:= \mathbf{g}_\star^{-1}|_{\Omega_{\perp\star}\otimes_\star\Omega_{\perp\star}} : \Omega_{\perp\star} \otimes_\star \Omega_{\perp\star} \to \mathcal{X}_\star,\\
\mathbf{g}_{t\star}^{-1} &:= \mathbf{g}_\star^{-1}|_{\Omega_{t\star}\otimes_\star\Omega_{t\star}} : \Omega_{t\star} \otimes_\star \Omega_{t\star} \to \mathcal{X}_\star,
\end{aligned} \tag{123}$$

$$\begin{aligned}
\mathbf{g}_{\perp\star} &:= \mathbf{g}_\star|_{\Xi_{\perp\star}\otimes_\star\Xi_{\perp\star}} : \Xi_{\perp\star} \otimes_\star \Xi_{\perp\star} \to \mathcal{X}_\star,\\
\mathbf{g}_{t\star} &:= \mathbf{g}_\star|_{\Xi_{t\star}\otimes_\star\Xi_{t\star}} : \Xi_{t\star} \otimes_\star \Xi_{t\star} \to \mathcal{X}_\star
\end{aligned} \tag{124}$$

are non-degenerate. $\Xi_{t\star}$, $\Omega_{\perp\star}$, $\Xi_{\perp\star}$, $\Omega_{t\star}$ resp. coincide with $\Xi_t[[\nu]]$, $\Omega_\perp[[\nu]]$, $\Xi_\perp[[\nu]]$, $\Omega_t[[\nu]]$ as $\mathbb{C}[[\nu]]$-modules. Similarly, for $\star$-tensor powers of the former. The orthogonal projections $\mathrm{pr}_{\perp\star} : \Xi_\star \to \Xi_{\perp\star}$, $\mathrm{pr}_{t\star} : \Xi_\star \to \Xi_{t\star}$, $\mathrm{pr}_{\perp\star} : \Omega_\star \to \Omega_{\perp\star}$, $\mathrm{pr}_{t\star} : \Omega_\star \to \Omega_{t\star}$ are uniquely extended as projections to the bimodules of multivector fields and higher rank forms through the rules $\mathrm{pr}_{\perp\star}(\omega\star\omega') = \mathrm{pr}_{\perp\star}(\omega)\star\mathrm{pr}_\perp(\omega')$, $\mathrm{pr}_{t\star}(\omega\star\omega') = \mathrm{pr}_{t\star}(\omega)\star\mathrm{pr}_{t\star}(\omega')$, …:

$$\begin{aligned}
&\mathrm{pr}_{\perp\star} : \Omega_\star^p \to \Omega_{\perp\star}^p, \quad \mathrm{pr}_{t\star} : \Omega_\star^p \to \Omega_{t\star}^p,\\
&\mathrm{pr}_{\perp\star} : \bigwedge\nolimits_\star^p \Xi_\star \to \bigwedge\nolimits_\star^p \Xi_{\perp\star}, \quad \mathrm{pr}_{t\star} : \bigwedge\nolimits_\star^p \Xi_\star \to \bigwedge\nolimits_\star^p \Xi_{t\star}.
\end{aligned} \tag{125}$$

$\mathrm{pr}_{\perp\star}, \mathrm{pr}_{t\star}$ are the $\mathbb{C}[[\nu]]$-linear extensions of $\mathrm{pr}_\perp, \mathrm{pr}_t$. $\Xi_{t\star}$, $\Xi_{\perp\star}$, $\Omega_{t\star}$, $\Omega_{\perp\star}$, their $\star$-exterior powers and the projections $\mathrm{pr}_{\perp\star}, \mathrm{pr}_{t\star}$ are $U\mathfrak{k}^{\mathcal{F}}$-equivariant.

Again we stress that, while $\Xi_{t\star}$ is a $\star$-Lie subalgebra of $\Xi_\star$, in general $\Xi_{\perp\star}$ is not. Furthermore, as $\Xi_{\perp\star}$, $\Omega_{t\star}$ are not $U\Xi_t^{\mathcal{F}}$-equivariant, also the orthogonal projections $\mathrm{pr}_{\perp\star}, \mathrm{pr}_{t\star}$ are not.

Proof By Proposition 9 $\Xi_{t\star}$ is a $\star$-Lie subalgebra of $\Xi_\star$ and a $U\Xi_t^{\mathcal{F}}$-equivariant $\mathcal{X}_\star$-subbimodule; in particular, it is $U\mathfrak{k}^{\mathcal{F}}$-equivariant. Moreover, according to Proposition 3,

$$\mathbf{g}_\star(X,Y)=\mathbf{g}(\overline{\mathcal{F}}_1\triangleright X,\overline{\mathcal{F}}_2\triangleright Y)=\mathbf{g}(X,Y)+\mathcal{O}(\nu)\text{ for all }X,Y\in\Xi.$$

If $X\in\Xi_\perp$ (i.e., $\mathbf{g}(X,Y)=0$ for all $Y\in\Xi_t$), it follows that

$$\mathbf{g}_\star(X,Y)=\mathbf{g}(\underbrace{\overline{\mathcal{F}}_1\triangleright X}_{\in\Xi_\perp[[\nu]]},\underbrace{\overline{\mathcal{F}}_2\triangleright Y}_{\in\Xi_t[[\nu]]})=0$$

for all $Y\in\Xi_t$, i.e., $\Xi_\perp[[\nu]]\subseteq\Xi_{\perp\star}$. On the other hand, for every $X=\sum_{n=0}^\infty\nu^nX_n\in\Xi_{\perp\star}$ with $X_n\in\Xi$ it follows that $0=\mathbf{g}_\star(X,Y)=\mathbf{g}(X_0,Y)+\mathcal{O}(\nu)$ for all $Y\in\Xi_t$, i.e., $\mathbf{g}(X_0,Y)=0$ for all $Y\in\Xi_t$. In other words, $X_0\in\Xi_\perp$. Also $X_1\in\Xi_\perp$, since

$$0=\mathbf{g}_\star(X,Y)=\underbrace{\mathbf{g}(X_0,Y)}_{=0}+\nu\left(\mathbf{g}(X_1,Y)+\underbrace{\mathbf{g}(\overbrace{\overline{F}_1^1\triangleright X_0}^{\in\Xi_\perp},\overbrace{\overline{F}_2^1\triangleright Y}^{\in\Xi_t})}_{=0}\right)+\mathcal{O}(\nu^2)$$

for all $Y\in\Xi_t$, where $\overline{\mathcal{F}}=\sum_{n=0}^\infty\nu^n\overline{F}_1^n\otimes\overline{F}_2^n$ and $\overline{F}_1^n\otimes\overline{F}_2^n\in U\mathfrak{k}\otimes U\mathfrak{k}$. Inductively $X_n\in\Xi_\perp$ for all $n\geq0$, implying $\Xi_\perp[[\nu]]=\Xi_{\perp\star}$, as claimed. This also implies the equality

$$\mathrm{pr}_{\perp\star}=\mathrm{pr}_\perp\colon\Xi[[\nu]]\to\Xi_\perp[[\nu]].\tag{126}$$

Note that $\mathbf{g}_\star^{-1}(\omega,\alpha)=\mathbf{g}^{-1}(\overline{\mathcal{F}}_1\triangleright\omega,\overline{\mathcal{F}}_2\triangleright\alpha)$ for all $\omega,\alpha\in\Omega$, since $\mathcal{F}$ is based on Killing vector fields. Now assume that $X\in\Xi_{\perp\star}$ (resp. $X\in\Xi_{t\star}$) fulfills $\mathbf{g}_{\perp\star}(X,\Xi_{\perp\star})=0$ (resp. $\mathbf{g}_{t\star}(X,\Xi_{t\star})=0$). Expanding X and $\mathbf{g}_{\perp\star}$ (resp. $\mathbf{g}_{t\star}$) in ν-powers and arguing as above, we find $X=0$, whence the non-degeneracy of $\mathbf{g}_{\perp\star}$ (resp. $\mathbf{g}_{t\star}$). By employing (40), Proposition 5 and the equivariance of the $\star$-pairing and $\mathbf{g}_\star^{-1}$, one similarly proves that $\Omega_{t\star}=\Omega_t[[\nu]]$, $\Omega_{\perp\star}=\Omega_\perp[[\nu]]$ and the non-degeneracy of $\mathbf{g}_{\perp\star}^{-1}$, $\mathbf{g}_{t\star}^{-1}$ on $\Omega_\star$. Let $X\in\Xi_{\perp\star}$, $\omega\in\Omega_{t\star}$, $\xi\in U\mathfrak{k}^{\mathcal{F}}$. Then,

$$\begin{aligned}\mathbf{g}_\star(\xi\triangleright X,Y)&=\xi_{\widehat{(1)}}\triangleright\mathbf{g}_\star(X,S_{\mathcal{F}}(\xi_{\widehat{(2)}})\triangleright Y)=0\quad\forall Y\in\Xi_{t\star}\Rightarrow\xi\triangleright X\in\Xi_{\perp\star}\\ \langle X,\xi\triangleright\omega\rangle_\star&=\xi_{\widehat{(1)}}\triangleright\langle S_{\mathcal{F}}(\xi_{\widehat{(2)}})\triangleright X,\omega\rangle_\star=0\quad\forall X\in\Xi_{\perp\star}\Rightarrow\xi\triangleright\omega\in\Omega_{t\star}\end{aligned}$$

since $\mathbf{g}_\star$ and the $\star$-pairing are equivariant under the action of $U\mathfrak{k}^{\mathcal{F}}$ and $\Xi_{t\star}$ is a $U\mathfrak{k}^{\mathcal{F}}$-equivariant $\mathcal{X}_\star$-bimodule. This proves that also $\Xi_{\perp\star}$, $\Omega_{t\star}$ are $U\mathfrak{k}$-equivariant $\mathcal{X}$-bimodules. To verify that $\mathrm{pr}_{\perp\star}$ is $U\mathfrak{k}^{\mathcal{F}}$-equivariant let $X=X_{t\star}+X_{\perp\star}\in\Xi_\star$ be the decomposition (122) with $X_{t\star}\in\Xi_{t\star}$ and $X_{\perp\star}\in\Xi_{\perp\star}$. Then $\xi\triangleright X=\xi\triangleright X_{t\star}+\xi\triangleright X_{\perp\star}$ and according to the $U\mathfrak{k}^{\mathcal{F}}$-invariance of $X_{t\star}$, $X_{\perp\star}$

$$\mathrm{pr}_{\perp\star}(\xi\triangleright X)=\mathrm{pr}_{\perp\star}(\xi\triangleright X_{t\star}+\xi\triangleright X_{\perp\star})=\xi\triangleright X_{\perp\star}=\xi\triangleright\mathrm{pr}_{\perp\star}(X)$$

for all $\xi \in U\mathfrak{k}^{\mathcal{F}}$. Similarly, one argues with $\mathrm{pr}_{t\star}$ on $\Omega_\star$ and on the $\star$-exterior powers of $\Xi_\star, \Omega_\star$. Finally, $\Omega_{t\star} = \{\omega \in \Omega_\star \mid \langle \Xi_{\perp\star}, \omega\rangle_\star = 0\}$ follows from its undeformed counterpart and the previous results. □

As in the undeformed case, we identify $\Xi_{t\star} \subset \Xi_\star$ with the $\star$-Lie subalgebra of smooth vector fields tangent to *all* M_c ($c \in f(\mathcal{D}'_f)$) at all points, because $X(f^a) = 0$ implies $X(f^a_c) = 0$; and $\Xi_{M\star} \subset \Xi_\star$ defined in (119) with the twisted Lie subalgebra of smooth vector fields tangent to M at all points. Similarly, we identify $\Omega_{t\star}$ with the subbimodule of $\Omega_\star$ tangent to *all* M_c ($c \in f(\mathcal{D}'_f)$) at all points. We find $\Omega_{t\star} \subset \Omega_{\mathcal{C}\star} := \{\omega \in \Omega_\star \mid \langle \Xi_{\perp\star}, \omega\rangle_\star \subset \mathcal{C}[[\nu]]\}$. Let $\Omega_{\mathcal{C}\mathcal{C}\star} := \bigoplus_{a=1}^k f^a \star \Omega_\star = \bigoplus_{a=1}^k \Omega_\star \star f^a \subset \Omega_{\mathcal{C}\star}$. It fulfills $\langle \Xi_\star, \Omega_{\mathcal{C}\mathcal{C}\star}\rangle_\star \subset \mathcal{C}[[\nu]]$. We can identify the $\mathcal{X}^M_\star$-bimodule of 1-forms $\Omega_{M\star}$ on M with the quotient

$$\Omega_{M\star} = \Omega_{\mathcal{C}\star}/\Omega_{\mathcal{C}\mathcal{C}\star} = \{[\omega] = \omega + \Omega_{\mathcal{C}\mathcal{C}\star} \mid \omega \in \Omega_{\mathcal{C}\star}\}. \tag{127}$$

Proposition 11 *For all* $X \in \Xi_{\mathcal{C}\star}$, $\omega \in \Omega_{\mathcal{C}\star}$ *the tangent projections* $X_{t\star} := \mathrm{pr}_{t\star}(X) \in \Xi_{t\star}$, $\omega_{t\star} := \mathrm{pr}_{t\star}(\omega) \in \Omega_{t\star}$, *respectively, belong to* $[X] \in \Xi_{M\star}$ *and* $[\omega] \in \Omega_{M\star}$

Consequently, we can represent every element of $\Xi_{M\star}$ (resp. $\Omega_{M\star}$) by an element of $\Xi_{t\star}$ (resp. $\Omega_{t\star}$). Similarly, for multivector fields and higher rank forms.

Proof By Propositions 9, 10 the twist-deformed spaces can be identified with formal power series of the undeformed ones and the twisted projections are given by the $[[\nu]]$-linear extensions of the undeformed ones. The claim follows as a corollary of Proposition 6. □

Motivated from the classical situation we define the *twisted first* and *second fundamental form* on the family of submanifolds M_c by

$$\begin{aligned} \mathbf{g}_{t\star} &:= \mathbf{g}_\star|_{\Xi_{t\star}\otimes_\star\Xi_{t\star}} \colon \Xi_{t\star}\otimes_\star\Xi_{t\star} \to \mathcal{X}_\star, \\ II^{\mathcal{F}}_\star &:= \mathrm{pr}_{\perp\star} \circ \nabla^{\mathcal{F}}|_{\Xi_{t\star}\otimes_\star\Xi_{t\star}} \colon \Xi_{t\star} \otimes_\star \Xi_{t\star} \to \Xi_{\perp\star}, \end{aligned} \tag{128}$$

as well as the *twisted projected Levi–Civita connection* on the family of submanifolds M_c

$$\nabla^{\mathcal{F}}_t := \mathrm{pr}_{t\star} \circ \nabla^{\mathcal{F}}|_{\Xi_{t\star}\otimes\Xi_{t\star}} \colon \Xi_{t\star}\otimes\Xi_{t\star} \to \Xi_{t\star}. \tag{129}$$

In the following proposition, we clarify the relation of these objects to their classical counterparts. In particular, that twist deformation and projection to the submanifold commute.

Proposition 12 $\nabla^{\mathcal{F}}_t$ *is a twisted covariant derivative on the family of submanifolds* M_c. *The twisted first fundamental form* $\mathbf{g}_{t\star}$ *is a metric on the family, with corresponding twisted Levi–Civita covariant derivative* $\nabla^{\mathcal{F}}_t$. *They, as well as the second fundamental form, are* $U\mathfrak{k}^{\mathcal{F}}$*-equivariant. In terms of the undeformed objects, we obtain*

$$\mathbf{g}_{t\star}(X, Y) = \mathbf{g}_t(\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y), \tag{130}$$

$$II_\star^{\mathcal{F}}(X,Y)=II(\overline{\mathcal{F}}_1\triangleright X,\overline{\mathcal{F}}_2\triangleright Y), \tag{131}$$

and

$$\nabla_{t,X}^{\mathcal{F}}Y=\nabla_{t,\overline{F}_1\triangleright X}(\overline{\mathcal{F}}_2\triangleright Y) \tag{132}$$

for all $X,Y\in\Xi_{t\star}=\Xi_t[[\nu]]$. *Furthermore,*

$$\nabla_X^{\mathcal{F}}=\nabla_t^{\mathcal{F}}+II_\star^{\mathcal{F}}:\ \Xi_{t\star}\otimes\Xi_{t\star}\to\Xi_\star. \tag{133}$$

Proof As a composition of $U\mathfrak{k}^{\mathcal{F}}$-equivariant maps, $\mathbf{g}_{t\star}$, $II_\star^{\mathcal{F}}$ and $\nabla_t^{\mathcal{F}}$ also are. Equation (130) follows from (75). Since $\nabla_X^{\mathcal{F}}=\nabla_{\overline{\mathcal{F}}_1\triangleright X}\overline{\mathcal{F}}_2\triangleright$ for all $X\in\Xi$ we find (131) and (132) (see also [2] eq. 129). Then, it follows from Proposition 7 that $\mathbf{g}_{t\star}$ is a (non-degenerate) metric on the M_c's with twisted Levi–Civita covariant derivative given by $\nabla_t^{\mathcal{F}}$. □

A generalization of Proposition 12 to braided commutative geometry is in [58] Proposition 4.4.

The twisted second fundamental form (128) yields the twisted extrinsic curvature of M. The twisted intrinsic curvature $\mathsf{R}_{t\star}^{\mathcal{F}}$ is related to the twisted curvature $\mathsf{R}_\star^{\mathcal{F}}$ of $\nabla^{\mathcal{F}}$ on $\mathbb{R}^n$ by the following quantum analogue of the Gauss equation (see "Appendix" 6.7 for the proof):

Proposition 13 *For all* $X,Y,Z,W\in\Xi_{t\star}$, *the following twisted Gauss equation holds:*

$$\begin{aligned}\mathbf{g}_\star(R_\star^{\mathcal{F}}(X,Y)Z,W)&=\mathbf{g}_\star(R_{t\star}^{\mathcal{F}}(X,Y)Z,W)+\mathbf{g}_\star(II_\star^{\mathcal{F}}(X,\overline{\mathcal{R}}_1\triangleright Z),II_\star^{\mathcal{F}}(\overline{\mathcal{R}}_2\triangleright Y,W))\\&\quad-\mathbf{g}_\star(II_\star^{\mathcal{F}}(\overline{\mathcal{R}}_{1\widehat{(1)}}\triangleright Y,\overline{\mathcal{R}}_{1\widehat{(2)}}\triangleright Z),II_\star^{\mathcal{F}}(\overline{\mathcal{R}}_2\triangleright X,W)).\end{aligned} \tag{134}$$

The twisted first and second fundamental forms, Levi–Civita connection, curvature tensor, Ricci tensor, Ricci scalar *on* M are finally obtained from the above objects by applying the further projection $\mathcal{X}_\star\to\mathcal{X}_\star^M$, which amounts to choosing the $c=0$ manifold M out of the M_c family. Of course, by a different choice of c one can do the same on any other M_c.

3.2.2 Decompositions in bases of $\Omega_\star$, $\Xi_\star$; Euclidean, Minkowski metrics

In this section, we explicitly determine the twisted geometry induced by a twist $\mathcal{F}\in U\mathfrak{k}\otimes U\mathfrak{k}[[\nu]]$ (in particular, the decompositions (122)) in terms of bases of $\Omega_\star$, $\Omega_{\perp\star}$, $\Omega_{t\star}$ and $\Xi_\star$, $\Xi_{\perp\star}$, $\Xi_{t\star}$ for a generic metric $\mathbf{g}$ on $\mathbb{R}^n$, specializing to the Euclidean and Minkowski metric at the end, as done in Sect. 3.1.2.

By Proposition 10 $\Xi_{t\star}$, $\Omega_{\perp\star}$, $\Xi_{\perp\star}$, $\Omega_{t\star}$ are $U\mathfrak{k}^{\mathcal{F}}$-equivariant, and the projections $\mathrm{pr}_{\perp\star}$, $\mathrm{pr}_{t\star}$ are $U\mathfrak{k}^{\mathcal{F}}$-equivariant. Here is the twisted analogue of Proposition 8 and the remark following it:

Proposition 14 *$df^a, N_\perp^a, \theta^a, U_\perp^a$ are $U\mathfrak{k}^{\mathcal{F}}$-invariant. $\mathcal{N}_\perp := \{N_\perp^a\}_{a=1}^k$, $\mathcal{B}_\perp := \{df^a\}_{a=1}^k$ are $\star$-dual bases of $\Xi_{\perp\star}, \Omega_{\perp\star}$, respectively:*

$$\langle N_\perp^a, df^b\rangle_\star = \delta^{ab}, \qquad a, b \in \{1, \dots, k\}. \tag{135}$$

$\mathbf{g}_\star^{-1}(df^a, df^b) = E^{ab}$, $\mathbf{g}_\star(N_\perp^a, N_\perp^b) = K^{ab}$. $\{U_\perp^a\}_{a=1}^k$, $\{\theta^a\}_{a=1}^k$ are $\star$-dual, orthonormal (possibly up to signs $\epsilon_a = \pm 1$) bases of $\Xi_{\perp\star}, \Omega_{\perp\star}$, respectively, in the sense

$$\mathbf{g}_\star(U_\perp^a, U_\perp^b) = \epsilon_a \delta^{ab}, \qquad \mathbf{g}_\star^{-1}(\theta^a, \theta^b) = \epsilon_a \delta^{ab}, \qquad \langle U_\perp^a, \theta^b\rangle_\star = \delta^{ab}. \tag{136}$$

On $X \in \Xi_\star$, $\omega \in \Omega_\star$ the actions of the projections $\mathrm{pr}_{\perp\star}, \mathrm{pr}_{t\star}$ explicitly read $\mathrm{pr}_{\perp\star}(X) = X_\perp$, $\mathrm{pr}_{\perp\star}(\omega) = \omega_\perp$, and $\mathrm{pr}_{t\star}(X) = X_t = X - X_\perp$, $\mathrm{pr}_{t\star}(\omega) = \omega_t = \omega - \omega_\perp$; the normal components explicitly read

$$\begin{aligned} \omega_\perp &= df^a \star K^{ab} \star \mathbf{g}_\star^{-1}(df^b, \omega) = \mathbf{g}_\star^{-1}(\omega, df^a) \star K^{ab} \star df^b, \\ X_\perp &= \mathbf{g}_\star(X, N_\perp^a) \star E^{ab} \star N_\perp^b = N_\perp^a \star E^{ab} \star \mathbf{g}_\star(N_\perp^b, X) \end{aligned} \tag{137}$$

in terms of the mentioned bases, twisted product and metric.

Proof All statements but the last one are straightforward consequences of the choice of the twist and of Propositions 5, 8. As $\mathrm{pr}_{\perp\star}, \mathrm{pr}_{t\star}$ are just the $\mathbb{C}[[\nu]]$-linear extensions of $\mathrm{pr}_\perp, \mathrm{pr}_t$ (see Proposition 10), then $\mathrm{pr}_{\perp\star}(\omega) = \omega_\perp$, $\mathrm{pr}_{\perp\star}(X) = X_\perp$, with the right-hand sides as defined in (97), and $\mathrm{pr}_{t\star}(X) = X_t = X - X_\perp$, $\mathrm{pr}_{t\star}(\omega) = \omega_t = \omega - \omega_\perp$. Equation (137) holds because any twist $\star$-product boils down to an ordinary product if one of the two factors is $U\mathfrak{k}$-invariant, and similarly $\mathbf{g}_\star^{-1}(\omega, \omega') = \mathbf{g}^{-1}(\omega, \omega')$, and $\mathbf{g}_\star(X, X') = \mathbf{g}(X, X')$, if one of the arguments is $U\mathfrak{k}$-invariant, by Eqs. (23), (75) (14); the order of the factors and of the arguments of $\mathbf{g}_\star, \mathbf{g}_\star^{-1}$ can be freely changed, for the same reason and the symmetry of metric. □

An equivalent alternative to (137) is

$$\begin{aligned} \omega_\perp &= \theta^a \star \zeta_{ab} \mathbf{g}_\star^{-1}(\theta^b, \omega) = \mathbf{g}_\star^{-1}(\omega, \theta^a) \zeta_{ab} \star \theta^b, \\ X_\perp &= \mathbf{g}_\star(X, U_\perp^a) \zeta_{ab} \star U_\perp^b = U_\perp^a \star \zeta_{ab} \mathbf{g}_\star(U_\perp^b, X), \end{aligned} \tag{138}$$

where $\zeta_{ab} = \epsilon_a \delta^{ab}$. By the $\star$-bilinearity of $\mathbf{g}_\star$, the above equations imply in particular

$$\begin{aligned} \omega_\perp &= df^a \star K^{ab} \star \mathbf{g}_\star^{-1}(df^b, dx^i) \star \check{\omega}_i = \hat{\omega}_i \star \mathbf{g}_\star^{-1}(dx^i, df^a) \star K^{ab} \star df^b, \\ X_\perp &= \hat{X}^i \star \mathbf{g}_\star(\partial_i, N_\perp^a) \star E^{ab} \star N_\perp^b = N_\perp^a \star E^{ab} \star \mathbf{g}_\star(N_\perp^b, \partial_i) \star \check{X}^i, \end{aligned} \tag{139}$$

in terms of the left and right decompositions $\omega = \hat{\omega}_i \star dx^i = dx^i \star \check{\omega}_i \in \Omega_\star$, $X = \hat{X}^i \star \partial_i = \partial_i \star \check{X}^i \in \Xi_\star$ in the bases $\{dx^i\}_{i=1}^n$, $\{\partial_i\}_{i=1}^n$. In the latter formulae, one can decompose $df^a, N_\perp^a, \theta^a, U_\perp^a$ themselves in the same way, if one wishes.

By the previous propositions, every complete set of Ξ_t, e.g., Θ_t, is also a complete set of $\Xi_{t\star}$; similarly, every complete set of Ω_t, e.g., S_V, or S_L, is also a complete set of $\Omega_{t\star}$.

If the metric is Euclidean ($g_{ij} = \delta_{ij}$) or Minkowski [$g_{ij} = g^{ij} = \eta_{ij} = \mathrm{diag}(1, \dots, 1, -1)$]

$$\begin{aligned}
&\mathbf{g}_\star^{-1}(dx^i, df^a) = \mathbf{g}^{-1}(dx^i, df^a) = f^{ai}, \qquad \mathbf{g}_\star^{-1}(df^a, dx^i) = f^{ai},\\
&\mathbf{g}_\star(\partial_i, N_\perp^a) = \mathbf{g}(\partial_i, N_\perp^a) = K^{ab} f_i^b = K^{ab}\star f_i^b,\ \mathbf{g}_\star(N_\perp^a, \partial_i) = K^{ab}\star f_i^b;
\end{aligned} \tag{140}$$

replacing the right-hand sides in (139) makes the latter more explicit.

3.2.3 Twisted differential calculus algebras $\mathcal{Q}_\star^\bullet$, $\mathcal{Q}_{M_c\star}^\bullet$

The twist deformation of the differential calculus algebra $\mathcal{Q}^\bullet$ on $\mathbb{R}^n$ introduced in Sect. 3.1.3 gives the one $\mathcal{Q}_\star^\bullet$, with the same generators e_α, ξ^i and relations

$$\begin{aligned}
&\textstyle\sum_{\alpha=1}^{A} (\mathcal{F}_1 \triangleright t_l^\alpha)\star(\mathcal{F}_2 \triangleright e_\alpha) = 0, \qquad l = 1, \dots, A-n,\\
&e_\alpha \star e_\beta - (\overline{\mathcal{R}}_1 \triangleright e_\beta)\star(\overline{\mathcal{R}}_2 \triangleright e_\alpha) - C^\gamma_{\star\alpha\beta}\star e_\gamma = 0,\\
&e_\alpha \star \xi^i - (\overline{\mathcal{R}}_1 \triangleright \xi^i)\star(\overline{\mathcal{R}}_2 \triangleright e_\alpha) = 0,\\
&\xi^i \star \xi^j + (\overline{\mathcal{R}}_1 \triangleright \xi^j)\star(\overline{\mathcal{R}}_2 \triangleright \xi^i) = 0,\\
&h \star \xi^i - (\overline{\mathcal{R}}_1 \triangleright \xi^i)\star(\overline{\mathcal{R}}_2 \triangleright h) = 0,
\end{aligned} \tag{141}$$

$$e_\alpha \star h - \underbrace{(\overline{\mathcal{R}}_1 \triangleright h)\star(\overline{\mathcal{R}}_2 \triangleright e_\alpha)}_{e_\alpha \triangleleft_\star h} - e_{\alpha\star}(h) = 0, \qquad \forall h \in \mathcal{X}_\star, \tag{142}$$

where $C^\gamma_{\star\alpha\beta} \in \mathcal{X}_\star$ are defined by the decomposition $\left[e_\alpha, e_\beta\right]_\star \equiv \left[\overline{\mathcal{F}}_1 \triangleright e_\alpha, \overline{\mathcal{F}}_2 \triangleright e_\beta\right] = C^\gamma_{\star\alpha\beta}\star e_\gamma$. $\mathcal{Q}_\star^\bullet$ is a $U\Xi^{\mathcal{F}}$-equivariant $\mathcal{X}_\star$-bimodule. We endow $\mathcal{Q}_\star^\bullet$ with the $*_{\mathcal{F}}$-structure.

Note the change of notation: in the $\mathcal{Q}_\star^\bullet$ framework $X\star h = (\overline{\mathcal{R}}_1 \triangleright h)\star(\overline{\mathcal{R}}_2 \triangleright X) + X_\star(h)$, hence $X\star h$ has a different meaning with respect to the previous sections, where it stood just for the first term at the right-hand side, i.e., for the $\star$-product of the vector field X by the function h from the right; in the $\mathcal{Q}_\star^\bullet$ framework, we denote the latter by $X \triangleleft_\star h := (\overline{\mathcal{R}}_1 \triangleright h)\star(\overline{\mathcal{R}}_2 \triangleright X)$, so that we can abbreviate $X\star h = X \triangleleft_\star h + X_\star(h)$. Of course $(X \triangleleft_\star h)_\star(h') = [X_\star(\overline{\mathcal{R}}_1 \triangleright h')]\star(\overline{\mathcal{R}}_2 \triangleright h)$, $(X \triangleleft_\star h) \triangleleft_\star h' = X \triangleleft_\star (h\star h')$ remain valid. Analogues of relations (141-142) for q-deformed differential calculi on $\mathbb{R}^n_q$, $\mathbb{C}^n_q$ that are equivariant under *quasitriangular* (rather than triangular) Hopf algebras can be found e.g. in [32].

If one chooses S so that a subset $S_t := \{e_\alpha\}_{\alpha=1}^B$ $(B := A-k)$ is complete in $\Xi_{t\star}$ (e.g., it consists of the $L_{i_1 i_2 \dots i_{k+1}}$), while $e_{B+a} := V_\perp^a$, then, if $\alpha, \beta \le B$, the sum in $(141)_2$ is extended over $\gamma \le B$. The twisted differential calculus algebra $\mathcal{Q}_{M_c\star}^\bullet$ on M_c is the $\mathcal{X}^{M_c\star}$-bimodule generated by the $\xi^1, \dots, \xi^n, e_1, \dots, e_B$, modulo the relations (141-142) with $\alpha, \beta \le B$ and the ones

$$\begin{aligned}
f_c^a &\equiv f^a - c^a\mathbf{1} = 0,\\
df^a &\equiv \xi^h \star f_h^a = 0,
\end{aligned} \qquad a = 1, \dots, k. \tag{143}$$

4 Examples of twisted algebraic submanifolds of $\mathbb{R}^3$

We can apply the whole machinery developed in the previous two sections to twist deform algebraic manifolds embedded in $\mathbb{R}^n$, provided we adopt $\mathcal{X} = \mathrm{Pol}^\bullet(\mathbb{R}^n)$, etc., everywhere. We can assume without loss of generality that the f^a be irreducible polynomial functions.[9] Following Sect. 3.2, it is interesting to ask for which algebraic submanifolds $M_c \subset \mathbb{R}^n$ the infinite-dimensional Lie algebra Ξ_t admits a nontrivial finite-dimensional subalgebra $\mathfrak{g}$ over $\mathbb{R}$, so that we can build concrete examples of twisted M_c by choosing a twist $\mathcal{F} \in (U\mathfrak{g} \otimes U\mathfrak{g})[[\nu]]$ of a known type. As said, manifestly symmetric M_c are of this type.

We can easily answer this question when $k = 1$ and the L_{ij} themselves close a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. This means that in (112) f_{ij} =const; hence, $f(x)$ is a quadratic polynomial, and M is either a quadric or the union of two hyperplanes (reducible case); moreover, $\mathfrak{g}$ is a Lie subalgebra of the affine Lie algebra $aff(n)$ of $\mathbb{R}^n$. More explicitly, if

$$f(x) \equiv \frac{1}{2}a_{ij}x^i x^j + a_{0i}x^i + \frac{1}{2}a_{00} = 0 \tag{144}$$

with some real constants $a_{\mu\lambda} = a_{\lambda\mu}$ $(\mu, \lambda = 0, 1, \ldots, n)$, $f_i = a_{ij}x^j + a_{i0}$, $f_{ij} = a_{ij}$ are constant, and (112) has already the desired form

$$[L_{ij}, L_{hk}] = a_{jh}L_{ik} - a_{ih}L_{jk} - a_{jk}L_{ih} + a_{ik}L_{jh}; \tag{145}$$

$L_{ih}\triangleright$ act as linear transformations of the coordinates x^k:

$$L_{ij} \triangleright x^h = (a_{ik}x^k + a_{0i})\delta^h_j - (a_{jx}x^k + a_{j0})\delta^h_i. \tag{146}$$

By an Euclidean transformation (this is also an affine one), one can always make the x^i canonical coordinates for the quadric, so that $a_{ij} = a_i\delta_{ij}$ (no sum over i), $b_i := a_{0i} = 0$ if $a_i \neq 0$. In [38] the authors first classify $\mathfrak{g}$ and derive some general results from the only assumptions $\mathcal{X} = \mathrm{Pol}^\bullet(\mathbb{R}^n)$ and $\mathfrak{g} \subset aff(n)$; in particular, that the *global* description of differential geometry on $\mathbb{R}^n$, M_c in terms of generators and relations extends to their twist deformations, in such a way to preserve the subspaces of the differential calculus algebras consisting of polynomials of any fixed degrees in the coordinates x^i, differential dx^i and vector fields chosen as generators. Then, they analyze in detail the twisted quadrics embedded in $\mathbb{R}^3$.

Here, we just present two families of the latter as examples of applications of the formalism developed in the previous sections. We analyze (referring for details to

[9] If for some $a = 1, \ldots, k$ f^a is reducible, i.e., $f^a(x) = g^a(x)h^a(x)$ with $g^a, h^a \in \mathcal{X}$ of positive degree, then $M = M^g \cup M^h$, where the manifold M^g is defined by the equations $g^a(x) = 0$ and $f^h(x) = 0$ if $h \neq a$, while M^h is defined by $h^a(x) = 0$ and $f^h(x) = 0$ if $h \neq a$. If $k = 1$, $f(x) = g(x)h(x)$, we find $L_{ij} = h(x)[g_i\partial_j - g_j\partial_i] + g(x)[h_i\partial_j - h_j\partial_i]$; on M_g the second term vanishes and the first is tangent to M_g, as it must be; and similarly on M_h. Having assumed the Jacobian everywhere of maximal rank M_g, M_h have empty intersection and can be analyzed separately. Otherwise, L_{ij} vanishes on $M_g \cap M_h \neq \emptyset$ (the singular part of M), so that on the latter a twist built using the L_{ij} will reduce to the identity, and the $\star$-product to the pointwise product (see the conclusions).

	a_1	a_2	a_3	a_{03}	a_{00}	r	quadric	$\mathfrak{g} \simeq$	Abelian	Jordanian
(a)	+	+	0	0	−	3	elliptic cylinder	$\mathfrak{so}(2) \ltimes \mathbb{R}^2$	Yes	No
								$\mathfrak{so}(2) \times \mathbb{R}$	Yes	No
(b)	+	+	−	0	−	4	1-sheet hyperboloid	$\mathfrak{so}(2,1)$	No	Yes
(c)	+	+	−	0	0	3	elliptic cone	$\mathfrak{so}(2,1) \times \mathbb{R}$	Yes	Yes
(d)	+	+	−	0	+	4	2-sheet hyperboloid	$\mathfrak{so}(2,1)$	No	Yes

Fig. 1 Signs of the coefficients of the canonical equations, rank, associated symmetry Lie algebra $\mathfrak{g}$, type of twist deformation. The cone (c) consists of two open components (nappes) disconnected by the apex (a singular point); we build an abelian twist for it using also the generator of dilatations

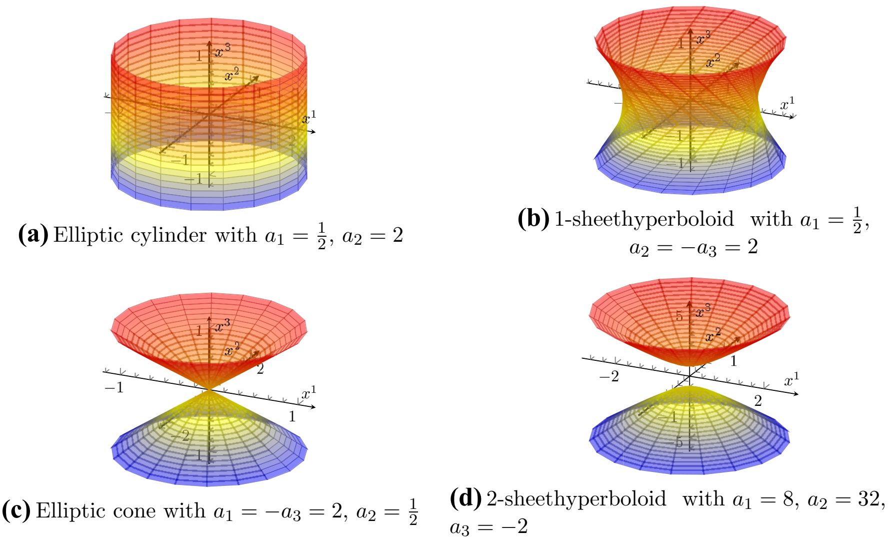
[38]) few twist deformations of the following classes of quadrics in $\mathbb{R}^3$: (a) elliptic cylinders; (c) elliptic cone; (b) 1-sheet and (d) 2-sheet hyperboloids. As usual, we identify two quadric surfaces if they can be translated into each other via an Euclidean transformation. By a suitable one, we can make the equation $f(x) = 0$ take a canonical (i.e., simplest) form, which we use to identify the class. In Fig. 1, we summarize the characterizing signs, rank, associated symmetry Lie algebra $\mathfrak{g}$, and type of twist deformation that we perform; an example in each class is plotted in Fig. 2. The elliptic cylinders of class (a) make up a family M_c parametrized by $c > 0$ (the axis of the cylinder M_0 is the $\mathcal{E}_f$ of the family), while classes (b), (c), (d) altogether give a single family M_c parametrized by $c \equiv -a_{00} \in \mathbb{R}$. This splits into a class of connected manifolds (the 1-sheet hyperboloids) and a class of disconnected ones (the 2-sheet hyperboloids and the cone, which has two nappes separated by the apex—a singular point); all are closed, except the cone, whose apex gives the $\mathcal{E}_f$ of the family. In either case $\mathcal{E}_f$ is an algebraic variety. We note that since the $L_{ij} = f_i \partial_j - f_j \partial_i \in \mathfrak{g}$ involved in the twist vanish on $\mathcal{E}_f$, the deformation automatically disappears on it, and the twisted algebraic variety is well defined as the undeformed. For other examples of submanifold algebras that are not algebras of functions on smooth manifolds, we refer the reader to the recent paper [18]. We devote a subsection to each family and a proposition to each twist deformation; propositions are proved in [38], where twist deformations also of the other classes of quadrics are discussed in detail. Throughout this section the $\star$-product $X \star h$ of a vector field X by a function h from the right is understood in the $\mathcal{Q}_\star$, $\mathcal{Q}_{M_c \star}$ sense $X \star h = X \triangleleft_\star h + X_\star(h)$ (see Sect. 3.2.3).

4.1 (a) Family of elliptic cylinders in Euclidean $\mathbb{R}^3$

Their equations in canonical form (with $a_1 = 1$, $a_3 = a_{0i} = 0$) are parametrized by $c \equiv -a_{00} > 0$, $a \equiv a_2 > 0$ and read

$$f_c(x) := \frac{1}{2}\left[(x^1)^2 + a(x^2)^2\right] - c = 0. \tag{147}$$

For every $a > 0$, $\{M_c\}_{c \in \mathbb{R}^+}$ is a foliation of $\mathbb{R}^3 \backslash \vec{z}$, where $\vec{z}$ is the axis of equations $x^1 = x^2 = 0$. The vector fields $L_{12} = x^1 \partial_2 - a x^2 \partial_1$, $L_{13} = x^1 \partial_3$, $L_{23} = a x^2 \partial_3$ generate $\mathfrak{g} \simeq \mathfrak{so}(2)1\mathbb{R}^2$:



(a) Elliptic cylinder with $a_1 = \frac{1}{2}$, $a_2 = 2$

(b) 1-sheethyperboloid with $a_1 = \frac{1}{2}$, $a_2 = -a_3 = 2$

(c) Elliptic cone with $a_1 = -a_3 = 2$, $a_2 = \frac{1}{2}$

(d) 2-sheethyperboloid with $a_1 = 8$, $a_2 = 32$, $a_3 = -2$

Fig. 2 The irreducible quadric surfaces of $\mathbb{R}^3$ that we twist-deform here

$$[L_{12}, L_{13}] = -L_{23}, \qquad [L_{12}, L_{23}] = aL_{13}, \qquad [L_{13}, L_{23}] = 0. \tag{148}$$

The commutation relations $[L_{ij}, x^h] = L_{ij} \triangleright x^h$, $[L_{ij}, \partial_h] = L_{ij} \triangleright \partial_h$, $[L_{ij}, \xi^h] = 0$ hold in $\mathcal{Q}^\bullet$.

Proposition 15 *$\mathcal{F} = \exp(i\nu L_{13} \otimes L_{23})$ is a unitary abelian twist inducing a twist deformation of $U\mathfrak{g}$, of $\mathcal{Q}^\bullet$ on $\mathbb{R}^3$ and of $\mathcal{Q}^\bullet_{M_c}$ on the elliptic cylinders (147). The basic relations characterizing the $U\mathfrak{g}^{\mathcal{F}}$-module $*$-algebra $\mathcal{Q}^\bullet_{M_c\star}$ keep the same form, in particular*

$$f_c(x) \equiv \frac{1}{2}(x^1 \star x^1 + ax^2 \star x^2) - c = 0, \quad df \equiv \xi^1 \star x^1 + a\,\xi^2 \star x^2 = 0, \quad \epsilon^{ijk} f_i \star L_{jk} = 0. \tag{149}$$

Alternatively, as a complete set in Ξ_t instead of $\{L_{12}, L_{13}, L_{23}\}$ we can use $S_t = \{L_{12}, \partial_3\}$, which is actually a basis of Ξ_t; the Lie algebra $\mathfrak{g} \simeq \mathfrak{so}(2) \times \mathbb{R}$ generated by the latter is abelian.

Proposition 16 *$\mathcal{F} = \exp(i\nu\partial_3 \otimes L_{12})$ is a unitary abelian twist inducing a twist deformation of $U\mathfrak{g}$, of $\mathcal{Q}^\bullet$ on $\mathbb{R}^3$ and of $\mathcal{Q}^\bullet_{M_c}$ on the elliptic cylinders (147). The basic relations characterizing the $U\mathfrak{g}^{\mathcal{F}}$-module $*$-algebra $\mathcal{Q}^\bullet_{M_c\star}$ keep the same form, in particular*

$$f_c(x) \equiv \frac{1}{2}(x^1 \star x^1 + ax^2 \star x^2) - c = 0, \quad df_c \equiv \xi^1 \star x^1 + a\,\xi^2 \star x^2 = 0, \quad \epsilon^{ijk} f_i \star L_{jk} = 0. \tag{150}$$

In [38], the reader can find the relations characterizing these two deformations in detail.

In the case $a_1 = a_2 = 1$ (circular cylinder of radius $R = \sqrt{2c}$ embedded in the Euclidean $\mathbb{R}^3$) $S := \{L, \partial_3, N_\perp\}$ is an orthonormal basis of Ξ alternative to $S' := \{\partial_1, \partial_2, \partial_3\}$ and such that $S_t := \{L, \partial_3\}$, $S_\perp := \{N_\perp\}$ are orthonormal bases of Ξ_t, $\Xi_\perp$, respectively; here $L := L_{12}/R$, $N_\perp = f^i\partial_i/R = (x^1\partial_1 + x^2\partial_2)/R$. The Killing Lie algebra $\mathfrak{k}$ is abelian and spanned (over $\mathbb{R}$) by S_t. $\nabla_X Y = 0$ for all $X, Y \in S'$, whereas the only nonzero $\nabla_X Y$, with $X, Y \in S$ are

$$\nabla_L L = -\frac{1}{R}N_\perp, \qquad \nabla_L N_\perp = \frac{1}{R}L, \qquad \nabla_{N_\perp} L = \frac{1}{R}L, \qquad \nabla_{N_\perp} N_\perp = \frac{1}{R}N_\perp. \tag{151}$$

The second fundamental form is explicitly given by $II(X,Y) = (\nabla_X Y)_\perp = -\frac{\tilde{X}\tilde{Y}}{R}N_\perp$, for all $X, Y \in \Xi_t$; here we are using the decomposition $Z = \tilde{Z}L + Z^3\partial_3$ of the generic $Z \in \Xi_t$. Thus, II is diagonal in the basis S_t, with diagonal elements (i.e., principal curvatures) $\kappa_1 = 0, \kappa_2 = -1/R$. Hence, the Gauss (i.e., intrinsic) curvature $K = \kappa_1\kappa_2$ vanishes; $\mathsf{R}_t = 0$ easily follows also from $\mathsf{R} = 0$ using the Gauss theorem. The mean (i.e., extrinsic) curvature is $H = (\kappa_1 + \kappa_2)/2 = -1/2R$. The Levi–Civita covariant derivative $\nabla_{t,X}$ on M_c is

$$\nabla_{t,X} Y = \mathrm{pr}_t(\nabla_X Y) = \nabla_X Y - II(X,Y) = \nabla_X Y + \tilde{X}\,\tilde{Y}\,N_\perp/R.$$

The deformation via the abelian twist $\mathcal{F} = \exp(i\nu\partial_3 \otimes L_{12})$ yields $\nabla^{\mathcal{F}}_X = \nabla_X$ for all $X \in S \cup S' = \{\partial_1, \partial_2, \partial_3, L, N_\perp\}$, $\nabla^{\mathcal{F}}_{t,X} Y = \mathrm{pr}_t(\nabla_X Y) = \nabla_{t,X} Y$ for all $X, Y \in \{\partial_3, L\}$, because ∂_3 commutes with all such X, so that $\overline{\mathcal{F}}_1 \triangleright X \otimes \overline{\mathcal{F}}_2 = X \otimes \mathbf{1}$, and the projections $\mathrm{pr}_\perp, \mathrm{pr}_t$, stay undeformed, as shown in Proposition 10. These results determine $\nabla^{\mathcal{F}}_X Y$ for all $X, Y \in \Xi_\star$ and $\nabla^{\mathcal{F}}_{t,X} Y = \nabla_{t,X} Y$ for all $X, Y \in \Xi_{t\star}$ via the function left $\star$-linearity in X and the deformed Leibniz rule for Y. The twisted curvatures $\mathsf{R}^{\mathcal{F}}$, $\mathsf{R}^{\mathcal{F}}_t$ vanish. Furthermore, for all $X, Y \in S_t$

$$II^{\mathcal{F}}_\star(X,Y) \overset{(131)}{=} II(\mathcal{F}_1^{-1} \triangleright X, \mathcal{F}_2^{-1} \triangleright Y) = \mathbf{g}(\nabla_{\mathcal{F}_1^{-1}\triangleright X}(\mathcal{F}_2^{-1} \triangleright Y), N_\perp)N_\perp = II(X,Y), \tag{152}$$

i.e., the principal curvatures $\kappa_1 = 0, \kappa_2 = 1/R$, Gauss and mean curvatures are undeformed.

4.2 (b-c-d) Family of hyperboloids and cone in Minkowski $\mathbb{R}^3$

Their equations in canonical form (with $a_1 = 1$) are parametrized by $a = a_2 > 0$, $b = -a_3 > 0$, $c = -a_{00}$ ($c > 0$, $c < 0$ resp. for the 1-sheet and the 2-sheet hyperboloids, $c = 0$ for the cone) and read

$$f_c(x) := \frac{1}{2}[(x^1)^2 + a(x^2)^2 - b(x^3)^2] - c = 0. \tag{153}$$

For all $a, b > 0$, $\{M_c\}_{c\in\mathbb{R}\setminus\{0\}}$ is a foliation of $\mathbb{R}^3\setminus M_0$, where M_0 is the cone of equation $f_0 = 0$. The Lie algebra $\mathfrak{g}$ is spanned by $L_{12} = x^1\partial_2 - ax^2\partial_1$, $L_{13} = x^1\partial_3 + bx^3\partial_1$, $L_{23} = ax^2\partial_3 + bx^3\partial_2$, which fulfill $[L_{12}, L_{13}] = -L_{23}$, $[L_{12}, L_{23}] = aL_{13}$, $[L_{13}, L_{23}] = bL_{12}$. Setting $H := \frac{2}{\sqrt{b}}L_{13}$, $E := \frac{1}{\sqrt{a}}L_{12} + \frac{1}{\sqrt{ab}}L_{23}$ and $E' := \frac{1}{\sqrt{a}}L_{12} - \frac{1}{\sqrt{ab}}L_{23}$, we obtain

$$[H, E] = 2E, \qquad [H, E'] = -2E', \qquad [E, E'] = -H, \tag{154}$$

showing that the corresponding symmetry Lie algebra is $\mathfrak{g} \simeq \mathfrak{so}(2, 1)$. The commutation relations $[L_{ij}, x^h] = L_{ij} \triangleright x^h$, $[L_{ij}, \partial_h] = L_{ij} \triangleright \partial_h$, $[L_{ij}, \xi^h] = 0$ hold in $\mathcal{Q}^\bullet$. To compute the action of $\mathcal{F}$ on functions, it is convenient to adopt as new coordinates the eigenvectors of $H\triangleright$

$$y^1 = x^1 + \sqrt{b}x^3, \qquad y^2 = x^2, \qquad y^3 = x^1 - \sqrt{b}x^3; \tag{155}$$

with eigenvalues $\lambda_1 = 2$, $\lambda_2 = 0$ and $\lambda_3 = -2$. Also $\eta^i \equiv dy^i$, $\tilde{\partial}_i \equiv \frac{\partial}{\partial y^i}$ are eigenvectors of $H\triangleright$.

Proposition 17 $\mathcal{F} = \exp(H/2 \otimes \log(\mathbf{1} + i\nu E))$ *is a unitary twist inducing a twist deformation of* $U\mathfrak{g}$, *of* $\mathcal{Q}^\bullet$ *on* $\mathbb{R}^3$ *and of* $\mathcal{Q}^\bullet_{M_c}$ *on the elliptic hyperboloids and cone (153). The basic relations characterizing the* $U\mathfrak{g}^{\mathcal{F}}$*-module* $*$*-algebra* $\mathcal{Q}^\bullet_{M_c\star}$ *are*

$$\begin{aligned} 0 &= f_c(y) \equiv \tfrac{1}{2}y^3\star y^1 + \tfrac{a}{2}y^2\star y^2 - c,\\ 0 &= \mathrm{d}f = \tfrac{1}{2}(y^3\star\eta^1 + \eta^3\star y^1) + ay^2\star\eta^2,\\ 0 &= y^3\star E - y^1\star E' - \sqrt{a}\, y^2\star H + i\nu y^1\star H - 2i\nu(1 + i\nu)y^1\star E. \end{aligned} \tag{156}$$

In Ref. [38], the reader can find, first the detailed actions of H, E, E' on y^i, ∂_i, η^i, then the twisted coproducts and antipodes of H, E, E', the star products and commutation relations among the generators $H, E, E', y^i, \partial_i, \eta^i$ arising from this twist.

Let us now focus on the case $1 = a_1 = a = b$, i.e., $f_c(x) = \frac{1}{2}((x^1)^2 + (x^2)^2 - (x^3)^2) - c$. This covers the circular cone, the circular hyperboloid of one and two sheets. We endow $\mathbb{R}^3$ with the Minkowski metric $\mathbf{g} := \eta_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 - \mathrm{d}x^3 \otimes \mathrm{d}x^3$, whence $\mathbf{g}(\partial_i, \partial_j) = \eta_{ij}$. $\mathbf{g}$ is equivariant with respect to $U\mathfrak{g}$, where $\mathfrak{g}$ is the Lie *-algebra spanned by the vector fields L_{ij}, tangent to M_c. $E = \mathbf{g}^{-1}(df, df) = x^i x_i = 2c$, which vanishes for $c = 0$. Therefore, $\mathcal{D}'_f = \mathbb{R}^3\setminus M_0$ (M_0 is the cone). The induced metric on the remaining $M_c \subset \mathcal{D}'_f$ (or first fundamental form) $\mathbf{g}_t$ makes M_c Riemannian if $c < 0$, Lorentzian if $c > 0$ (whereas it is degenerate on the cone M_0); moreover, in any basis $S_t := \{v_1, v_2\}$ of Ξ_t we find $II(v_\alpha, v_\beta) = -g_{\alpha\beta}\, V_\perp/2c$, where $\alpha, \beta \in \{1, 2\}$, $g_{\alpha\beta} := \mathbf{g}(v_\alpha, v_\beta)$, $V_\perp = f_j\eta^{ji}\partial_i = x^i\partial_i$. We find the components of the curvature, Ricci tensors, the Ricci scalar (or *Gauss curvature*) on M_c

$$\mathsf{R}_{t\ \alpha\beta\gamma}^{\ \delta} = \frac{g_{\alpha\gamma}\delta^\delta_\beta - g_{\beta\gamma}\delta^\delta_\alpha}{2c}, \qquad \mathsf{Ric}_{t\,\beta\gamma}$$

$$= \mathsf{R}_t{}^{\alpha}_{\ \alpha\beta\gamma} = -\frac{g_{\beta\gamma}}{2c}, \qquad \mathfrak{R}_t = \mathsf{Ric}_{t\,\beta\gamma}\, g^{\beta\gamma} = -\frac{1}{c} \tag{157}$$

applying the Gauss theorem. All diverge as $c \to 0$ (i.e., in the cone M_0 limit). M_c is therefore anti-de Sitter space AdS_2 if $c > 0$, the union of two copies of de Sitter space dS_2 if $c < 0$.

Under twist deformation the curvature (and Ricci) tensor on $\mathbb{R}^3$ remain zero. By Propositions 10, 3, 12 on M_c the first and second fundamental forms, as well as the curvature and Ricci tensors, stay undeformed as elements of the corresponding tensor spaces: $\mathbf{g}_t^{\mathcal{F}} = \mathbf{g}_t \in \Omega_t \otimes \Omega_t[[\nu]], \ldots$

Only the associated multilinear maps of twisted tensor products $\mathbf{g}_{t\star} : \Xi_{t\star} \otimes_\star \Xi_{t\star} \to \mathcal{X}_\star$, ..., 'feel' the twist; they are related to the undeformed maps through formulae (75), (131) and

$$\mathsf{Ric}_{t\star}^{\mathcal{F}}(X, Y) = \mathsf{Ric}_t\left(\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y\right) \tag{158}$$

[compare also to [2] Theorem 7 and eq. (6.138)], and similarly for $\mathsf{R}_{t\star}^{\mathcal{F}}$. Also the Ricci scalar (or Gauss curvature) $\mathfrak{R}_t^{\mathcal{F}}$ remains the undeformed one $-1/c$.

Finally, one can elaborate [38] also abelian twist deformations for the elliptic cone, (153) with $c = 0$, enlarging the Lie algebra $\mathfrak{g}$ by a generator $D = x^i \partial_i = y^i \tilde{\partial}_i$ of dilatations, which commutes with all L_{ij} and is also tangent to the cone (only), as $D(f) = 2f$: $D \in \Xi_M$.

5 Outlook and final remarks

Considering a generic embedded submanifold $M \subset \mathbb{R}^n$ that consists of the solutions x of a set of k equations $f^a(x) = 0$ $(a = 1, \ldots, k)$, where $f : \mathcal{D}_f \subset \mathbb{R}^n \to \mathbb{R}^k$ is a k-ple of smooth functions with Jacobian matrix of rank k, in this work we have explicitly built its noncommutative analogue in the framework of Drinfel'd (cocycle) twist [22] deformation of differential geometry [1,2]. This can be considered as a successful result, also in the broader framework of deformation quantization [6,7,44,56], in the sense that the deformed algebra $\mathcal{X}_\star$ of functions on $\mathcal{D}_f$ and the one $\mathcal{X}_\star^M$ of functions on M both coincide with their undeformed counterparts $\mathcal{X}[[\nu]]$, $\mathcal{X}^M[[\nu]]$ as $\mathbb{C}[[\nu]]$-modules (ν is the deformation parameter), because the same occurs for the ideal $\mathcal{C}$ of functions vanishing on M and its deformed counterpart, $\mathcal{C}_\star$, $\mathcal{C}_\star = \mathcal{C}[[\nu]]$; only the pointwise product is deformed into a (in general noncommutative) one, the $\star$-product. In other words, taking the quotient and performing the deformation commute: $\mathcal{X}_\star^M = (\mathcal{X}/\mathcal{C})_\star = \mathcal{X}_\star/\mathcal{C}_\star$. The key point has been to perform the deformation using a Drinfeld twist $\mathcal{F}$ based on the Lie subalgebra Ξ_t (10) of vector fields on $\mathcal{D}_f$ that are tangent to *every* manifold M_c of the family of level sets of f (the latter is parametrized by $c \in f\left(\mathcal{D}_f\right) \subset \mathbb{R}^k$; M_c consists of the solutions of $f^a(x) - c^a = 0$, $a = 1, \ldots, k$), rather than on the Lie algebra Ξ_M of vector fields tangent to M only; this has given for free the deformation of the whole family by the same twist. Every vector field in the $\star$-Lie algebra $\Xi_{M\star}$ can be represented by an element of the $\star$-Lie algebra $\Xi_{t\star}$, as it occurs in the undeformed case. The whole twisted Cartan calculus is automatically equivariant under the non-cocommutative Hopf algebra $U\Xi_t^{\mathcal{F}}$; the latter may be interpreted as the quantum group of (small) diffeomorphisms of the deformed submanifolds. The dimensions of $\Xi_\star$, $\Xi_{t\star}$ as $\mathcal{X}_\star$ bimodules, as well as of their duals $\Omega_\star$, $\Omega_{t\star}$, remain undeformed, contrary to what happens to the quantum group bicovariant or equivariant differential calculi mentioned in the introduction. This is because we consider 2-cocycles twists, but could change with more general twists leading to quasitriangular Hopf algebras or quasi-Hopf algebras, or twists in the category of bialgebroids. We have also shown that, when $\mathbb{R}^n$ is endowed with a connection ∇, taking the tangent projection (from $\mathbb{R}^n$ to M) of ∇ and the associated torsion, curvature, commutes with performing the deformation, provided $\mathcal{F}$ is based on the equivariance Lie subalgebra $\mathfrak{e} \subset \Xi_t$ [see (67)]. When $\mathbb{R}^n$ is endowed with a metric $\mathbf{g}$, the same holds for $\mathbf{g}$ itself, the associated Levi–Civita connection, the intrinsic and extrinsic curvatures (while the torsions remain zero), only if $\mathcal{F}$ is based on the Lie algebra $\mathfrak{k} \subset \Xi_t$ of Killing vector fields of the metric.

All our results are global, in that we have determined global (i.e., defined on all of M) bases—or complete sets—of all the relevant $\mathcal{X}_\star$- and $\mathcal{X}_\star^M$-bimodules from their undeformed counterparts: $\mathcal{C}_\star$ is spanned by the globally defined functions f^a, $\Xi_{t\star}$ by some complete set $\{e_\alpha\}$ of globally defined vector fields [e.g., (113)]; these fulfill some linear dependence relations], the $\mathcal{X}_\star$-bimodule $\Xi_{\perp\star} \subset \Xi_\star$ of twisted vector fields normal to the M_c's (with respect to the metric $\mathbf{g}$) is spanned by the globally defined vector fields (96), and similarly the dual ones $\Omega_{t\star}$, $\Omega_{\perp\star}$ of 1-forms, their tensor or wedge powers,.... This means that both in the undeformed and deformed context these bimodules/algebras can be formulated in terms of (the mentioned) generators and polynomial relations, with elements in $\mathcal{X}$, $\mathcal{X}_\star$ as coefficients.

In the polynomial setting, if the polynomial functions $f^a(x)$ fulfill suitable irreducibility conditions, then also $\mathcal{X}$, $\mathcal{X}_\star$, $\mathcal{X}^M$, $\mathcal{X}_\star^M$ can be defined in terms of generators x^i (the Cartesian coordinates) and polynomial relations [38]. The procedure can be potentially applied to a large number of algebraic manifolds, starting from algebraic hypersurfaces ($k = 1$), in particular quadrics; one can use the examples of cocycle twists available in the literature (typically based on finite-dimensional Lie algebras $\mathfrak{g}$) to build concrete deformations of these submanifolds. In [38] the authors discuss in detail deformations of all families of quadric surfaces embedded in $\mathbb{R}^3$ that are induced by unitary twists of the abelian [54] or Jordanian [52,53] type, except the ellipsoids. Here (Sect. 4) we have only presented the results for the elliptic (in particular, circular) cylinders, hyperboloids and cone. Endowing $\mathbb{R}^3$ with the Euclidean (resp. Minkowski) metric we have found twisted circular (i.e., maximally symmetric) cylinders, hyperboloids and cone M_c that are (pseudo)Riemannian and equivariant under a non-trivial Hopf algebra $U\mathfrak{k}^{\mathcal{F}}$ ("quantum group of isometries"); the twisted Levi–Civita connection on all M_c equals the projection of the twisted Levi–Civita connection on $\mathbb{R}^3$ (the exterior derivative), while the twisted intrinsic curvature can be expressed in terms of the twisted second fundamental form (or extrinsic curvature) via the twisted Gauss theorem; the twisted curvatures are the same constants as their undeformed counterparts. The twisted hyperboloids with $c < 0$ (resp. $c > 0$) can be thus considered as twisted de Sitter spaces dS_2 (resp. anti-de Sitter spaces AdS_2).

We recall that the higher-dimensional generalizations of the latter manifolds play a prominent role in present cosmology and theoretical physics as maximally symmetric cosmological solutions to the Einstein field equations of general relativity with a nonzero cosmological constant Λ; in particular, de Sitter spacetime ($\Lambda > 0$) can describe a universe with accelerating expansion rate (see, e.g., [19]), while anti-de Sitter spacetimes ($\Lambda < 0$) are at the base of the so-called Ads/CFT correspondence [49]. Interpreting Minkowski $\mathbb{R}^{2+1}$ as a relativistic momentum, rather than position, space (x^1, x^2 playing the role of components of the momentum, x^3 of energy), the equations (153) as dispersion relations for relativistic particles, and performing the deformations, we should regard the $x^3 > 0$ component of the twisted 2-sheet hyperboloid ($c < 0$) as the twisted mass shell of a particle of mass $\sqrt{|2c|}$; similarly, the $x^3 > 0$ nappe of the cone $c = 0$ would do for a massless particle, while $c > 0$ would do for a tachyon.

Generalizing the framework to submanifolds of $\mathbb{C}^n$ looks straightforward and should make things even simpler, as we drop $*$-structures and the related constraints on the twist. For instance, there are no abelian twist deformations of the ellipsoids in $\mathbb{R}^3$, because the corresponding $\mathfrak{g} \simeq \mathfrak{so}(3)$ is simple; neither are there Jordanian ones, because $\mathfrak{so}(3)$ (over $\mathbb{R}$) contains no elements E, H fulfilling $[H, E] = 2E$. If we considered $f(x) \equiv x^i x^i - 1$ as a polynomial function $f : \mathbb{C}^3 \to \mathbb{C}$, then such $E, H \in \mathfrak{g} \simeq \mathfrak{so}(3, \mathbb{C})$ would exist, and we could perform a Jordanian deformation also of the complex ellipsoid $M \subset \mathbb{C}^3$ solution of $f(x) = 0$.

Finally, in [35–37,39] an alternative approach to introduce noncommutative (fuzzy) embedded submanifolds S in $\mathbb{R}^n$ was proposed and applied to the spheres; it is obtained projecting the algebra of observables of a quantum particle in $\mathbb{R}^n$, in a confining potential with a very sharp minimum on S, to the Hilbert subspace with energy below a certain cutoff.

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6 Appendix

6.1 Proof of Proposition 1

As the inclusion $\mathcal{C} \supset \bigoplus_{a=1}^{k} \mathcal{X} f^a$ is trivial, we need to prove the converse one. For all $\bar{x} \in M$ we can find a local smooth change of coordinates $\phi : x \in V_{\bar{x}} \mapsto z \in U_{\bar{z}}$ of the form $\phi(x) = (f, y) \equiv (f^1, \ldots, f^k, y^1, \ldots, y^{n-k})$, where $\bar{z} \equiv \phi(\bar{x}) = (0_k, \bar{y})$ (0_k stands for the row with k zeroes), $U_{\bar{z}} \subset \mathbb{R}^n$ is an open ball with center $\bar{z}$, and $V_{\bar{x}} = \phi^{-1}(U_{\bar{z}}) \subset \mathbb{R}^n$; one can choose the extra coordinates y^h, e.g., as a subset of the x^j themselves[10]. For all $h \in \mathcal{X}$ the function defined on $U_{\bar{z}}$ by $\hat{h}(z) = h(x)$ is smooth as well. In terms of the new coordinates the points of $V_{\bar{x}} \cap M$ belong to the hyperplane $z^1 = \cdots = z^k = 0$. For all $z = (c, y) \in U_{\bar{z}}$ we denote as $z' := (0_k, y)$ its projection on this hyperplane; the segment zz' is contained in the ball $U_{\bar{z}}$. Applying Hadamard's lemma to the dependence of $\hat{h}(z)$ on the first k coordinates [considering y as parameters] we find $\hat{h}(z) = \hat{h}(z') + \sum_{a=1}^{k} c^a \hat{h}^a(z)$ in $U_{\bar{z}}$, with smooth $\hat{h}^a$; more explicitly,

$$\hat{h}^a(z) = \int_0^1 \frac{\partial \hat{h}}{\partial z^a}(tc, y)\, \mathrm{d}t.$$

Equivalently, $h(x) = h(x') + \sum_{a=1}^{k} f^a(x) h^a_{\bar{x}}(x)$ in $V_{\bar{x}}$, where $x' = \phi^{-1}(z') \in V_{\bar{x}} \cap M$ and $h^a_{\bar{x}}$ are defined by $h^a_{\bar{x}}(x) = \hat{h}^a(z)$ and smooth in $V_{\bar{x}}$. If $h \in \mathcal{C}$ then $h(x') = 0$, and $h(x) = \sum_{a=1}^{k} f^a(x) h^a_{\bar{x}}(x)$. This is the desired decomposition, but only locally. To make it global, consider the open cover of $\mathcal{D}_f$

$$\mathcal{O} = \mathcal{O}' \cup \{V_1\} \cup \cdots \cup \{V_k\}, \qquad \mathcal{O}' := \{V_{\bar{x}} \mid \bar{x} \in M\}, \quad V_a := \mathcal{D}_f \backslash \overline{M_a},$$

where $\overline{M_a}$ is the closure of the hypersurface M_a, which is the level set of f^a ($M = \bigcap_{a=1}^{k} M_a$). Since $\mathcal{D}_f$ is paracompact (as so is the metric space $\mathbb{R}^n$), there is a smooth partition of unity subordinated to $\mathcal{O}$, i.e., there exist: a function ρ_a with support contained in V_a, for all $a \in \{1, \ldots, k\}$, and a function $\rho_{\bar{x}} \in \mathcal{X}$ with support contained

[10] Consider in fact the set of equations in the variables $(x, c) \in \mathbb{R}^{n+k}$

$$l^a(x, c) := f^a(x) - c^a = 0, \qquad a = 1, 2, \ldots, k < n. \tag{159}$$

The Jacobian matrix of $l = (l^1, \ldots, l^k)$ is the $k \times (n+k)$-matrix $(J| - I_k)$, where $J = \partial f / \partial x$ has rank k. M consists of the points x such that $(x, 0_k)$ solves (159). Fixed a $\bar{x} \in M$, we can always permute the coordinates so that the $k \times k$-matrix $A := \left(\partial f^a / \partial x^b\right)_{a,b=1}^{k}$ is invertible in $\bar{x}$. By the implicit function theorem there exists an open ball $U_{\bar{z}} \subset \mathbb{R}^n$ centered at $\bar{z} := (0_k, \bar{x}^{k+1}, \ldots, \bar{x}^n)$ and smooth functions $x^a(z)$ of $z := (c^1, \ldots, c^k, x^{k+1}, \ldots, x^n) \in U_{\bar{z}}$ such that $x^a(\bar{z}) = \bar{x}^a$, and $l\big(x^1(z), \ldots, x^k(z), x^{k+1}, \ldots, x^n, c^1, \ldots, c^k\big) = 0$; thus we can set $y^1 = x^{k+1}, \ldots, y^{n-k} = x^n$.

in $V_{\bar{x}}$, for all $\bar{x} \in M$, such that for all $x \in \mathcal{D}_f$ $\sum_{\bar{x}\in M} \rho_{\bar{x}}(x) + \sum_{a=1}^k \rho_a(x) = 1$, with only a finite number of non-zero terms in the sum. The functions

$$\tilde{h}^a_{\bar{x}}(x) := \begin{cases} h^a_{\bar{x}}(x)\rho_{\bar{x}}(x) & \text{if } x \in V_{\bar{x}}, \\ 0 & \text{if } x \in \mathbb{R}^n \backslash V_{\bar{x}}, \end{cases} \qquad \tilde{h}^a(x) := \begin{cases} h(x)\dfrac{\rho_a(x)}{f^a(x)} & \text{if } x \in V_a, \\ 0 & \text{if } x \in \mathbb{R}^n \backslash V_a, \end{cases}$$

belong to $\mathcal{X}$ and fulfill $\sum_{a=1}^k f^a(x)\tilde{h}^a_{\bar{x}}(x) = h(x)\rho_{\bar{x}}(x)$, $f^a(x)\tilde{h}^a(x) = h(x)\rho_a(x)$; hence $h^a := \sum_{\bar{x}\in M} \tilde{h}^a_{\bar{x}} + \tilde{h}^a \in \mathcal{X}$ are the coefficients needed for (3) to hold. In fact, for all $x \in \mathcal{D}_f$

$$\sum_{a=1}^k h^a(x) f^a(x) = \sum_{a=1}^k f^a(x) h^a(x) = h(x) \left[\sum_{\bar{x}\in M} \rho_{\bar{x}}(x) + \sum_{a=1}^k \rho_a(x) \right] = h(x).$$

6.2 More on twists

We write in a compact notation the inverse of (15) and its consequences

$$\begin{aligned} &\overline{\mathcal{F}}_{(12)3}\overline{\mathcal{F}}_{12} = \overline{\mathcal{F}}_{1(23)}\overline{\mathcal{F}}_{23}, && \overline{\mathcal{F}}_{(123)4}\overline{\mathcal{F}}_{(12)3} = \overline{\mathcal{F}}_{(12)(34)}\overline{\mathcal{F}}_{34}, \\ &\overline{\mathcal{F}}_{(123)4}\overline{\mathcal{F}}_{1(23)} = \overline{\mathcal{F}}_{1(234)}\overline{\mathcal{F}}_{(23)4}, && \overline{\mathcal{F}}_{(12)(34)}\overline{\mathcal{F}}_{12} = \overline{\mathcal{F}}_{1(234)}\overline{\mathcal{F}}_{2(34)}, \end{aligned} \tag{160}$$

obtained applying Δ on the first, second, third tensor factor and recalling that Δ is cocommutative; the bracket encloses tensor factors obtained from one by application of Δ. To denote the decomposition of $\mathcal{F}_{(12)3}$ we have used a Sweedler-type notation $\mathcal{F}_{(12)3} \equiv (\Delta\otimes id)(\mathcal{F}) = \mathcal{F}_{1(1)}\otimes\mathcal{F}_{1(2)}\otimes\mathcal{F}_2$, and similarly for $\mathcal{F}_{1(23)}, \overline{\mathcal{F}}_{(12)3}\ldots$. Several proofs are based on these formulae.

6.3 Isomorphism of twisted Hopf ∗-algebras for unitary twists

We use the notation of Sect. 2.1. Assume that $(H, *)$ is a Hopf ∗-algebra. We now prove

Proposition 18 *If $\mathcal{F}$ is unitary, then $D : (H_\star, *_\star) \to (H^{\mathcal{F}}, *)$ is an isomorphism of triangular Hopf ∗-algebras; in particular, $D \circ *_\star = * \circ D$.*

Proof Via (160) one can prove the relation (see, e.g., Lemma 2.2. in [42] or eq. (126) in [34])

$$\Delta(\beta) = \mathcal{F}^{-1}(\beta\otimes\beta)[(S\otimes S)\mathcal{F}_{21}^{-1}] = \mathcal{F}_{21}^{-1}(\beta\otimes\beta)[(S\otimes S)\mathcal{F}^{-1}]. \tag{161}$$

$D \circ *_\star = * \circ D$ is almost the same as eq. (31) in [34]. We prove it again using (161):

$$D(\xi^{*_\star}) = D\left[S(\beta) \triangleright \xi^*\right] = D\left[S\left(\beta_{(1)}\right) \xi^* \beta_{(2)}\right]$$

$$
\begin{aligned}
&= \mathcal{F}_1 S\left(\beta_{(1)}\right) \xi^* \beta_{(2)} S(\mathcal{F}_2)\beta^{-1} \overset{(161)}{=} S(\beta) S(\overline{\mathcal{F}}_2)\, \xi^* \, \overline{\mathcal{F}}_1 \\
&= S(\beta)\,[\mathcal{F}_1 \, \xi \, S(\mathcal{F}_2)]^* = S(\beta)\,[D(\xi)\beta]^* = S(\beta) S\big(\beta^{-1}\big)\,[D(\xi)]^* = [D(\xi)]^* .
\end{aligned} \tag{162}
$$

As particular consequences, $\Delta_\star \circ *_\star = (*_\star \otimes *_\star) \circ \Delta_\star$, $S_\star \circ *_\star \circ S_\star \circ *_\star = id$ follow from $\Delta_{\mathcal{F}} \circ * = (*{\otimes}*) \circ \Delta_{\mathcal{F}}$, $S_{\mathcal{F}} \circ * \circ S_{\mathcal{F}} \circ * = id$. □

6.4 Proof of Proposition 2

First of all, $\nabla^{\mathcal{F}}$ is well-defined, since $U\mathfrak{g}[[\nu]] \triangleright \mathcal{T}_\star^{p,q} \subseteq \mathcal{T}_\star^{p,q}$. Equation (47) easily follows from the properties of the classical covariant derivative: $\nabla^{\mathcal{F}}_X h = \nabla_{\overline{\mathcal{F}}_1 \triangleright X}(\overline{\mathcal{F}}_2 \triangleright h) = \mathcal{L}_{\overline{\mathcal{F}}_1 \triangleright X}(\overline{\mathcal{F}}_2 \triangleright h) = \mathcal{L}^\star_X h$. Furthermore, for every $g \in U\mathfrak{e}^{\mathcal{F}}$ we obtain

$$
\begin{aligned}
g \triangleright (\nabla^{\mathcal{F}}_X T) &= \nabla_{(g_{(1)}\overline{\mathcal{F}}_1) \triangleright X}((g_{(2)}\overline{\mathcal{F}}_2) \triangleright T) \\
&= \nabla_{(\overline{\mathcal{F}}_1 g_{\widehat{(1)}}) \triangleright X}((\overline{\mathcal{F}}_2 g_{\widehat{(2)}}) \triangleright T) = \nabla^{\mathcal{F}}_{g_{\widehat{(1)}} \triangleright X}(g_{\widehat{(2)}} \triangleright T),
\end{aligned}
$$

where $X \in \Xi_\star$ and $T \in \mathcal{T}_\star$ are arbitrary. In other words, $\nabla^{\mathcal{F}}$ is $U\mathfrak{e}^{\mathcal{F}}$-equivariant. If $\mathcal{F}$ is based on $U\mathfrak{e}$, $\nabla^{\mathcal{F}}$ is equivariant with respect to the action of any leg of $\mathcal{F}$ or $\mathcal{R}$ (and their inverses). By the linearity properties of the classical covariant derivative and (160) we obtain

$$
\begin{aligned}
\nabla^{\mathcal{F}}_{h\star X} T &= \nabla_{\overline{\mathcal{F}}_1 \triangleright ((\overline{\mathcal{F}}'_1 \triangleright h)(\overline{\mathcal{F}}'_2 \triangleright X))}(\overline{\mathcal{F}}_2 \triangleright T) = ((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1) \triangleright h) \nabla_{(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2) \triangleright X}(\overline{\mathcal{F}}_2 \triangleright T) \\
&= (\overline{\mathcal{F}}_1 \triangleright h) \nabla_{(\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_1) \triangleright X}((\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_2) \triangleright T) \\
&= (\overline{\mathcal{F}}_1 \triangleright h)\Big(\overline{\mathcal{F}}_2 \triangleright \Big(\nabla_{\overline{\mathcal{F}}'_1 \triangleright X}(\overline{\mathcal{F}}'_2 \triangleright T)\Big)\Big) = h \star \nabla^{\mathcal{F}}_X T,
\end{aligned}
$$

which, together with $\nabla^{\mathcal{F}}_{Z+Z'} T = \nabla_{\overline{\mathcal{F}}_1 \triangleright Z + \overline{\mathcal{F}}_1 \triangleright Z'}(\overline{\mathcal{F}}_2 \triangleright T) = \nabla_{\overline{\mathcal{F}}_1 \triangleright Z}(\overline{\mathcal{F}}_2 \triangleright T) + \nabla_{\overline{\mathcal{F}}_1 \triangleright Z'}(\overline{\mathcal{F}}_2 \triangleright T) = \nabla^{\mathcal{F}}_Z T + \nabla^{\mathcal{F}}_{Z'} T$ gives (48). By the Leibniz rule of the classical covariant derivative, the inverse 2-cocycle property, the equivariance of ∇ and the Lie derivative we obtain (52). The rule (71)

$$
\begin{aligned}
\nabla^{\mathcal{F}}_X (T \star h) &= (\mathcal{L}^\star_X(\overline{\mathcal{R}}_1 \triangleright h)) \star (\overline{\mathcal{R}}_2 \triangleright T) + ((\overline{\mathcal{R}}'_1 \overline{\mathcal{R}}_1) \triangleright h) \star \Big(\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}'_2 \triangleright X}(\overline{\mathcal{R}}_2 \triangleright T)\Big) \\
&= ((\overline{\mathcal{R}}'_1 \overline{\mathcal{R}}_2) \triangleright T) \star (\overline{\mathcal{R}}'_2 \triangleright \mathcal{L}^\star_X(\overline{\mathcal{R}}_1 \triangleright h)) + \Big(\overline{\mathcal{R}}''_1 \triangleright \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}'_2 \triangleright X}(\overline{\mathcal{R}}_2 \triangleright T)\Big) \star ((\overline{\mathcal{R}}''_2 \overline{\mathcal{R}}'_1 \overline{\mathcal{R}}_1) \triangleright h) \\
&= ((\overline{\mathcal{R}}'_1 \overline{\mathcal{R}}_2) \triangleright T) \star \Big(\mathcal{L}^\star_{\overline{\mathcal{R}}'_{2(1)} \triangleright X}((\overline{\mathcal{R}}'_{2(2)} \overline{\mathcal{R}}_1) \triangleright h)\Big) \\
&\quad + \Big(\nabla^{\mathcal{F}}_{(\overline{\mathcal{R}}''_{1(1)} \overline{\mathcal{R}}'_2) \triangleright X}((\overline{\mathcal{R}}''_{1(2)} \overline{\mathcal{R}}_2) \triangleright T)\Big) \star ((\overline{\mathcal{R}}''_2 \overline{\mathcal{R}}'_1 \overline{\mathcal{R}}_1) \triangleright h) \\
&= ((\overline{\mathcal{R}}'_1 \overline{\mathcal{R}}''_1 \overline{\mathcal{R}}_2) \triangleright T) \star \Big(\mathcal{L}^\star_{\overline{\mathcal{R}}'_2 \triangleright X}((\overline{\mathcal{R}}''_2 \overline{\mathcal{R}}_1) \triangleright h)\Big) \\
&\quad + \Big(\nabla^{\mathcal{F}}_{(\overline{\mathcal{R}}'''_1 \overline{\mathcal{R}}'_2) \triangleright X}((\overline{\mathcal{R}}''_1 \overline{\mathcal{R}}_2) \triangleright T)\Big) \star ((\overline{\mathcal{R}}''_2 \overline{\mathcal{R}}'''_2 \overline{\mathcal{R}}'_1 \overline{\mathcal{R}}_1) \triangleright h) \\
&= (\overline{\mathcal{R}}'_1 \triangleright T) \star \Big(\mathcal{L}^\star_{\overline{\mathcal{R}}'_2 \triangleright X}(h)\Big) + \big(\nabla^{\mathcal{F}}_X T\big) \star h
\end{aligned}
$$

holds for all $X \in \Xi$, $T \in \mathcal{T}^{p,q}$ and $h \in \mathcal{X}$. The proof of (50, 51) is

$$\begin{aligned}
\nabla^{\mathcal{F}}_X(T \otimes_\star T') &= \nabla_{\overline{\mathcal{F}}_1 \triangleright X}(((\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_1) \triangleright T) \otimes ((\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_2) \triangleright T'))\\
&= \left(\nabla_{(\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1) \triangleright X}((\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2) \triangleright T)\right) \otimes (\overline{\mathcal{F}}_2 \triangleright T')\\
&\quad + ((\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_1) \triangleright T) \otimes \left(\nabla_{\overline{\mathcal{F}}_1 \triangleright X}((\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_2) \triangleright T')\right)\\
&= \left(\overline{\mathcal{F}}_1 \triangleright \nabla_{\overline{\mathcal{F}}'_1 \triangleright X}(\overline{\mathcal{F}}'_2 \triangleright T)\right) \otimes (\overline{\mathcal{F}}_2 \triangleright T')\\
&\quad + \left((\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2\mathcal{R}'_1\overline{\mathcal{R}}_1) \triangleright T\right) \otimes \left((\nabla_{\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1\mathcal{R}'_2\overline{\mathcal{R}}_2) \triangleright X}(\overline{\mathcal{F}}_2 \triangleright T')\right)\\
&= \nabla^{\mathcal{F}}_X(T) \otimes_\star T' + ((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1\overline{\mathcal{R}}_1) \triangleright T) \otimes \left((\nabla_{\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2\overline{\mathcal{R}}_2) \triangleright X}(\overline{\mathcal{F}}_2 \triangleright T')\right)\\
&= \nabla^{\mathcal{F}}_X(T) \otimes_\star T' + (\overline{\mathcal{R}}_1 \triangleright T) \otimes_\star \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2 \triangleright X} T' \quad (72)\\
&= \nabla^{\mathcal{F}}_{(\overline{\mathcal{R}}_1\overline{\mathcal{R}}'_2) \triangleright X}((\overline{\mathcal{R}}'''_1\overline{\mathcal{R}}''_2) \triangleright T) \otimes_\star ((\overline{\mathcal{R}}'''_2\overline{\mathcal{R}}_2\overline{\mathcal{R}}'_1\overline{\mathcal{R}}''_1) \triangleright T') + (\overline{\mathcal{R}}_1 \triangleright T) \otimes_\star \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2 \triangleright X} T'\\
&= \nabla^{\mathcal{F}}_{(\overline{\mathcal{R}}_{1\widehat{(1)}}\overline{\mathcal{R}}'_2) \triangleright X}((\overline{\mathcal{R}}_{1\widehat{(2)}}\overline{\mathcal{R}}''_2) \triangleright T) \otimes_\star ((\overline{\mathcal{R}}_2\overline{\mathcal{R}}'_1\overline{\mathcal{R}}''_1) \triangleright T') + (\overline{\mathcal{R}}_1 \triangleright T) \otimes_\star \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2 \triangleright X} T'\\
&= \left(\overline{\mathcal{R}}_1 \triangleright \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}'_2 \triangleright X}(\overline{\mathcal{R}}''_2 \triangleright T)\right) \otimes_\star ((\overline{\mathcal{R}}_2\overline{\mathcal{R}}'_1\overline{\mathcal{R}}''_1) \triangleright T') + (\overline{\mathcal{R}}_1 \triangleright T) \otimes_\star \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2 \triangleright X} T'
\end{aligned}$$

and (in complete analogy)

$$\begin{aligned}
\nabla^{\mathcal{F}}_X \langle Y, \omega\rangle_\star &= \langle \nabla^{\mathcal{F}}_X(Y), \omega\rangle_\star + \langle \overline{\mathcal{R}}_1 \triangleright Y, \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2 \triangleright X}\omega\rangle_\star \quad (73)\\
&= \langle \overline{\mathcal{R}}_1 \triangleright \left(\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}'_2 \triangleright X}(\overline{\mathcal{R}}''_2 \triangleright Y)\right), (\overline{\mathcal{R}}_2\overline{\mathcal{R}}'_1\overline{\mathcal{R}}''_1) \triangleright \omega\rangle_\star + \langle \overline{\mathcal{R}}_1 \triangleright Y, \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2 \triangleright X}\omega\rangle_\star,
\end{aligned}$$

for all $X, Y \in \Xi$, $\omega \in \Omega^1$ and $T, T \in \mathcal{T}^{p,q}$. In particular, we proved (72) and (73) on the way. Note that we further used the cocommutativity of Δ, the equivariance property (68), the (inverse) 2-cycle condition, as well as the relations $(\Delta_{\mathcal{F}} \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$ and $\mathcal{R}^{-1} = \mathcal{R}_{21}$.

6.5 Proof of Proposition 3

First of all we prove that $\mathfrak{k}$ is a Lie subalgebra of the equivariance Lie algebra (67) of ∇, which implies that $\nabla^{\mathcal{F}}$ is a well-defined covariant derivative according to Proposition 2. Let $\xi \in \mathfrak{k}$. Then, making use of the Koszul formula, we obtain

$$\begin{aligned}
&2\mathbf{g}(\nabla_{\xi_{(1)} \triangleright X}(\xi_{(2)} \triangleright Y), Z)\\
&= (\xi_{(1)} \triangleright X)(\mathbf{g}(\xi_{(2)} \triangleright Y, Z)) + (\xi_{(2)} \triangleright Y)(\mathbf{g}(Z, \xi_{(1)} \triangleright X)) - Z(\mathbf{g}(\xi_{(1)} \triangleright X, \xi_{(2)} \triangleright Y))\\
&\quad - \mathbf{g}(\xi_{(1)} \triangleright X, [\xi_{(2)} \triangleright Y, Z]) + \mathbf{g}(\xi_{(2)} \triangleright Y, [Z, \xi_{(1)} \triangleright X]) + \mathbf{g}(Z, [\xi_{(1)} \triangleright X, \xi_{(2)} \triangleright Y])\\
&= \xi_{(1)} \triangleright \left\{X(\mathbf{g}(Y, S(\xi_{(2)}) \triangleright Z)) + Y(\mathbf{g}(S(\xi_{(2)}) \triangleright Z, X)) - (S(\xi_{(2)}) \triangleright Z)(\mathbf{g}(X, Y))\right.\\
&\quad \left. -\mathbf{g}(X, [Y, S(\xi_{(2)}) \triangleright Z]) + \mathbf{g}(Y, [S(\xi_{(2)}) \triangleright Z, X]) + \mathbf{g}(S(\xi_{(2)}) \triangleright Z, [X, Y])\right\}\\
&= \xi_{(1)} \triangleright (2\mathbf{g}(\nabla_X Y, S(\xi_{(2)}) \triangleright Z)) = 2\mathbf{g}(\xi \triangleright \nabla_X Y, Z)
\end{aligned}$$

for all $X, Y, Z \in \Xi$, where we further employed the $U\mathfrak{k}$-equivariance of $\mathbf{g}$ and of the pairing of vector fields with forms, as well as the cocommutativity of $U\mathfrak{k}$ and the antipode properties. Since $\mathbf{g}$ is non-degenerate it follows that $\xi \triangleright \nabla_X Y = \nabla_{\xi_{(1)}\triangleright X}(\xi_{(2)} \triangleright Y)$, i.e., ξ is an element of the equivariance Lie algebra of ∇. Thus we have shown the inclusion $\mathfrak{k} \subset \mathfrak{g}$. If $\mathcal{F} \in U\mathfrak{k} \otimes U\mathfrak{k}[[\nu]]$, then $\overline{\mathcal{F}}_2 \triangleright \mathbf{g} = \varepsilon(\overline{\mathcal{F}}_2)\mathbf{g}$, and using (14), (27) we immediately find (75). In fact

$$\begin{aligned}\mathbf{g}_\star(X,Y) &= \langle X, \langle Y, \mathbf{g}^A\rangle_\star \star \mathbf{g}_A\rangle_\star = \langle X, \langle Y, \mathbf{g}^A\rangle \mathbf{g}_A\rangle_\star = \langle \overline{\mathcal{F}}_1 \triangleright X, \langle \overline{\mathcal{F}}_2 \triangleright Y, \mathbf{g}^A\rangle \mathbf{g}_A\rangle \\ &= \left\langle \overline{\mathcal{F}}_1 \triangleright (X \otimes_\star Y), \overline{\mathcal{F}}_2 \triangleright \mathbf{g} \right\rangle = \langle\, X \otimes_\star Y, \mathbf{g}\,\rangle,\end{aligned}$$

where in the last two equations the pairing is extended to double tensor products, see (46). This reduces to the undeformed $\mathbf{g}(X,Y)$ if $\overline{\mathcal{F}} = \mathbf{1}\otimes\mathbf{1}$. The twisted metric $\mathbf{g}_\star$ is right $\mathcal{X}_\star$-linear, since

$$\begin{aligned}\mathbf{g}_\star(X, Y\star f) &= \langle X, \langle Y\star f, \mathbf{g}^A\rangle_\star \star \mathbf{g}_A\rangle_\star = \langle X, \langle Y, \overline{\mathcal{R}}_1 \triangleright \mathbf{g}^A\rangle_\star \star (\overline{\mathcal{R}}'_1 \triangleright \mathbf{g}_A)\star((\overline{\mathcal{R}}'_2\overline{\mathcal{R}}_2) \triangleright f)\rangle_\star \\ &= \langle X, \langle Y, \mathbf{g}^A\rangle_\star \star \mathbf{g}_A\rangle_\star \star f = \mathbf{g}_\star(X,Y)\star f\end{aligned}$$

for all $f \in \mathcal{X}_\star$. Next we prove that $\nabla^{\mathcal{F}}$ is torsion-free with respect to the twisted torsion and metric compatible with respect to the twisted metric. The first property holds since

$$\begin{aligned}\mathsf{T}^{\mathcal{F}}_\star(X,Y) &= \nabla_{\overline{\mathcal{F}}_1\triangleright X}(\overline{\mathcal{F}}_2 \triangleright Y) - \nabla_{(\overline{\mathcal{F}}_1\mathcal{R}_2)\triangleright Y}((\overline{\mathcal{F}}_2\mathcal{R}_1)\triangleright X) - [\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y] \\ &= \nabla_{\overline{\mathcal{F}}_1\triangleright X}(\overline{\mathcal{F}}_2 \triangleright Y) - \nabla_{\overline{\mathcal{F}}_2\triangleright Y}(\overline{\mathcal{F}}_1 \triangleright X) - [\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y] \\ &= \mathsf{T}_\star(\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y) = 0,\end{aligned}$$

while the second one holds because

$$\begin{aligned}&\mathcal{L}^\star_X \mathbf{g}_\star(Y,Z) = (\overline{\mathcal{F}}_1 \triangleright X)(\mathbf{g}((\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_1)\triangleright Y, (\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_2)\triangleright Z)) \\ &= \mathbf{g}(\nabla_{\overline{\mathcal{F}}_1\triangleright X}((\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_1)\triangleright Y), (\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_2)\triangleright Z) + \mathbf{g}((\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_1)\triangleright Y, \nabla_{\overline{\mathcal{F}}_1\triangleright X}((\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_2)\triangleright Z)) \\ &= \mathbf{g}(\nabla_{(\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1)\triangleright X}((\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2)\triangleright Y), \overline{\mathcal{F}}_2 \triangleright Z) + \mathbf{g}((\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2)\triangleright Y, \nabla_{(\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1)\triangleright X}(\overline{\mathcal{F}}_2 \triangleright Z)) \\ &= \mathbf{g}(\overline{\mathcal{F}}_1 \triangleright (\nabla_{\overline{\mathcal{F}}'_1\triangleright X}(\overline{\mathcal{F}}'_2 \triangleright Y)), \overline{\mathcal{F}}_2 \triangleright Z) + \mathbf{g}((\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_1\mathcal{R}_2)\triangleright Y, \nabla_{(\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_2\mathcal{R}_1)\triangleright X}(\overline{\mathcal{F}}_2 \triangleright Z)) \\ &= \mathbf{g}_\star(\nabla^{\mathcal{F}}_X Y, Z) + \mathbf{g}((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1\mathcal{R}_2)\triangleright Y, \nabla_{(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2\mathcal{R}_1)\triangleright X}(\overline{\mathcal{F}}_2 \triangleright Z)) \\ &= \mathbf{g}_\star(\nabla^{\mathcal{F}}_X Y, Z) + \mathbf{g}((\overline{\mathcal{F}}_1\mathcal{R}_2)\triangleright Y, \nabla_{(\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_1\mathcal{R}_1)\triangleright X}((\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_2)\triangleright Z)) \\ &= \mathbf{g}_\star(\nabla^{\mathcal{F}}_X Y, Z) + \mathbf{g}((\overline{\mathcal{F}}_1\mathcal{R}_2)\triangleright Y, \overline{\mathcal{F}}_2 \triangleright (\nabla_{(\overline{\mathcal{F}}'_1\mathcal{R}_1)\triangleright X}(\overline{\mathcal{F}}'_2 \triangleright Z))) \\ &= \mathbf{g}_\star(\nabla^{\mathcal{F}}_X Y, Z) + \mathbf{g}_\star(\mathcal{R}_2 \triangleright Y, \nabla^{\mathcal{F}}_{\mathcal{R}_1\triangleright X} Z)\end{aligned}$$

for all $X, Y, Z \in \Xi$, which is equivalent to $\nabla^{\mathcal{F}}_X \mathbf{g} = 0$. The statements about the twisted curvature and torsion are proven in [2] Theorem 7, while the uniqueness of $\nabla^{\mathcal{F}}$ is given by [2] Theorem 5. We prove that the twisted torsion and curvature are right $\star$-linear in the last argument if $\mathcal{F}$ is based on Killing vector fields. Let $X, Y, Z \in \Xi$ and $h \in \mathcal{X}$. Then

$$\begin{aligned}
\mathsf{T}^{\mathcal{F}}_{\star}(X, Y{\star}h) &= \nabla^{\mathcal{F}}_X(Y{\star}h) - \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_1\rhd(Y\star h)}(\overline{\mathcal{R}}_2 \rhd X) - [X, Y{\star}h] \\
&= \nabla^{\mathcal{F}}_X(Y){\star}h + (\overline{\mathcal{R}}_1 \rhd Y){\star}\mathcal{L}^{\star}_{\overline{\mathcal{R}}_2\rhd X}(h) - \nabla^{\mathcal{F}}_{(\overline{\mathcal{R}}_{1(1)}\rhd Y)\star(\overline{\mathcal{R}}_{1(2)}\rhd h)}(\overline{\mathcal{R}}_2 \rhd X) \\
&\quad - [X, Y]{\star}h - (\overline{\mathcal{R}}_1 \rhd Y){\star}\mathcal{L}^{\star}_{\overline{\mathcal{R}}_2\rhd X}(h) \\
&= \nabla^{\mathcal{F}}_X(Y){\star}h - \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_1\rhd Y}((\overline{\mathcal{R}}'_1\overline{\mathcal{R}}''_2\overline{R}_2) \rhd X){\star}((\overline{\mathcal{R}}'_2\overline{\mathcal{R}}''_1) \rhd h) \\
&\quad - [X, Y]{\star}h\mathsf{T}^{\mathcal{F}}_{\star}(X, Y){\star}h
\end{aligned}$$

proves right $\mathcal{X}_\star$-linearity of the twisted torsion. Finally,

$$\begin{aligned}
&\mathsf{R}^{\mathcal{F}}_{\star}(X, Y, Z{\star}h) = \nabla^{\mathcal{F}}_X\nabla^{\mathcal{F}}_Y(Z{\star}h) - \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_1\rhd Y}\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\rhd X}(Z{\star}h) - \nabla^{\mathcal{F}}_{[X,Y]}(Z{\star}h) \\
&= \nabla^{\mathcal{F}}_X\Big((\nabla^{\mathcal{F}}_Y Z){\star}h + (\overline{\mathcal{R}}_1 \rhd Y){\star}\Big(\mathcal{L}^{\star}_{\overline{\mathcal{R}}_2\rhd X}h\Big)\Big) - \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_1\rhd Y}\Big(\Big(\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\rhd X}Z\Big){\star}h \\
&\quad + (\overline{\mathcal{R}}'_1 \rhd Z){\star}\Big(\mathcal{L}^{\star}_{(\overline{\mathcal{R}}'_2\overline{\mathcal{R}}_2)\rhd X}h\Big)\Big) - (\nabla^{\mathcal{F}}_{[X,Y]}Z){\star}h - (\overline{\mathcal{R}}_1 \rhd Z){\star}\Big(\mathcal{L}^{\star}_{\overline{\mathcal{R}}_2\rhd[X,Y]}h\Big) \\
&= \mathsf{R}^{\mathcal{F}}_{\star}(X, Y, Z){\star}h + (\overline{\mathcal{R}}_1 \rhd (\nabla^{\mathcal{F}}_Y Z)){\star}\Big(\mathcal{L}^{\star}_{\overline{\mathcal{R}}_2\rhd X}h\Big) + \nabla^{\mathcal{F}}(\overline{\mathcal{R}}_1 \rhd Y){\star}\Big(\mathcal{L}^{\star}_{\overline{\mathcal{R}}_2\rhd X}h\Big) \\
&\quad + ((\overline{\mathcal{R}}'_1\overline{\mathcal{R}}_1) \rhd Y)\Big(\mathcal{L}^{\star}_{\overline{\mathcal{R}}'_2\rhd X}\Big(\mathcal{L}^{\star}_{\overline{\mathcal{R}}_2\rhd X}h\Big)\Big) - \Big(\overline{\mathcal{R}}'_1 \rhd \Big(\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2\rhd X}Z\Big)\Big){\star}\Big(\mathcal{L}^{\star}_{(\overline{\mathcal{R}}'_2\overline{\mathcal{R}}_1)\rhd Y}h\Big) \\
&\quad - \Big(\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_1\rhd Y}\Big(\overline{\mathcal{R}}'_1 \rhd Z\Big)\Big){\star}\Big(\mathcal{L}^{\star}_{(\overline{\mathcal{R}}'_2\overline{\mathcal{R}}_2)\rhd X}h\Big) - \Big((\overline{\mathcal{R}}''_1\overline{\mathcal{R}}'_1) \rhd Z\Big){\star}\mathcal{L}^{\star}_{(\overline{\mathcal{R}}''_2\overline{\mathcal{R}}_1)\rhd Y}\Big(\mathcal{L}^{\star}_{(\overline{\mathcal{R}}'_2\overline{\mathcal{R}}_2)\rhd X}h\Big) \\
&\quad - (\overline{\mathcal{R}}_1 \rhd Z){\star}\Big(\mathcal{L}^{\star}_{\overline{\mathcal{R}}_2\rhd[X,Y]}h\Big) = \mathsf{R}^{\mathcal{F}}_{\star}(X, Y, Z){\star}h,
\end{aligned}$$

where in the last equation the eighth term cancels with the fourth and seventh term, the second and sixth cancel each other, and so do the third and fifth terms.

6.6 Proof of Proposition 6 and Eq. (89)

Decompose $X = X_t + X_\perp$, $X_\perp = X^a N^a_\perp$. Then $X(f^b) = X^a N^a_\perp(f^b) = X^a K^{ac}(f^{ci} f^b_i) = X^b$ must belong to $\mathcal{C}$ for all $b = 1, .., k$, i.e., must be of the form $X^a = f^b X^a_b$, for some $X^a_b \in \mathcal{X}$. Hence $X_\perp = f^b(X^a_b N^a_\perp)$ belongs to $\Xi_{\mathcal{CC}}$, and $X_t \in [X]$. Decompose $\omega = \omega_t + \omega_\perp$. One can find an atlas of $\mathcal{D}_f$, with a pair $\{e_i\}, \{\theta^i\}$ of dual frames in each chart, such that $\{e_\alpha\}_{\alpha=1}^{n-k}$ is a basis of Ξ_t and $\{\theta^\alpha\}_{\alpha=1}^{n-k}$ is a basis of Ω_t. Then $\omega_t = \omega_\alpha\theta^\alpha$, and for all $X = X^\alpha e_\alpha \in \Xi_t$ it is $\langle X, \omega\rangle = \langle X, \omega_t\rangle = X^\alpha\omega_\alpha$; by Theorem 1, this belongs to $\mathcal{C}$ for all (X^α) if and only if $\omega_\alpha = f^a\omega^a_\alpha$, for some $\omega^a_\alpha \in \mathcal{X}$. Hence $\omega_\perp = f^a\omega^a_\alpha\theta^\alpha$ belongs to $\Omega_{\mathcal{CC}}$, and $\omega_t \in [\omega]$.

In Proposition 4 we have shown that $\Omega_\perp \subseteq \Omega'_\perp := \{\omega \in \Omega \mid \langle \Xi_t, \omega\rangle = 0\}$. Conversely, for any $\omega \in \Omega'_\perp$ we have $0 = \langle X, \omega\rangle = \langle X, \omega_t\rangle$ for all $X \in \Xi_t$, whence $\omega_t = 0$ and $\omega = \omega_\perp \in \Omega_\perp$. This proves the first equality in (89). To prove the last equality, decompose $X = X_t + X_\perp$; this belongs to $\Xi_{\mathcal{C}}$ if and only if $X_\perp$ is of the form $X_\perp = f^b X_{ba} N^a_\perp$, whence $\langle X, \omega\rangle = \langle X_t, \omega\rangle + f^b X_{ba}\langle N^a_\perp, \omega\rangle$ belongs to $\mathcal{C}$ iff $\langle X_t, \omega\rangle$ does, for all $X_t \in \Xi_t$.

6.7 Proof of Proposition 13

We reduce Eq. (134) to the ν-linear extension of Eq. (93). For $X, Y, Z, W \in \Xi_t$ we obtain

$$
\begin{aligned}
&\mathbf{g}_\star(\mathsf{R}^{\mathcal{F}}_{t\star}(X,Y)Z,W) = \mathbf{g}(\mathsf{R}_t((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_{1(1)}\overline{\mathcal{F}}''_1)\rhd X,(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_{1(2)}\overline{\mathcal{F}}''_2)\rhd Y)((\overline{\mathcal{F}}_{1(3)}\overline{\mathcal{F}}'_2)\rhd Z),\overline{\mathcal{F}}_2\rhd W)\\
&\quad = \mathbf{g}(\mathsf{R}_t((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}''_1)\rhd X,(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}''_2)\rhd Y)((\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_1)\rhd Z),(\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_2)\rhd W)\\
&\quad \mathbf{g}_\star(II^{\mathcal{F}}_\star(X,\overline{\mathcal{R}}_1\rhd Z),II^{\mathcal{F}}_\star(\overline{\mathcal{R}}_2\rhd Y,W))\\
&\quad = \mathbf{g}(II((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1)\rhd X,(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2\overline{\mathcal{R}}_1)\rhd Z),II((\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}''_1\overline{\mathcal{R}}_2)\rhd Y,(\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}''_2)\rhd W))\\
&\quad = \mathbf{g}(II(\overline{\mathcal{F}}_1\rhd X,(\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_1\overline{\mathcal{R}}_1)\rhd Z),II((\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_{2(1)}\overline{\mathcal{F}}''_1\overline{\mathcal{R}}_2)\rhd Y,(\overline{\mathcal{F}}_{2(3)}\overline{\mathcal{F}}'_{2(2)}\overline{\mathcal{F}}''_2)\rhd W))\\
&\quad = \mathbf{g}(II(\overline{\mathcal{F}}_1\rhd X,(\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_{1(1)}\overline{\mathcal{F}}''_1\overline{\mathcal{R}}_1)\rhd Z),II((\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_{1(2)}\overline{\mathcal{F}}''_2\overline{\mathcal{R}}_2)\rhd Y,(\overline{\mathcal{F}}_{2(3)}\overline{\mathcal{F}}'_2)\rhd W))\\
&\quad = \mathbf{g}(II(\overline{\mathcal{F}}_1\rhd X,(\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_{1(2)}\overline{\mathcal{F}}''_2)\rhd Z),II((\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_{1(1)}\overline{\mathcal{F}}''_1)\rhd Y,(\overline{\mathcal{F}}_{2(3)}\overline{\mathcal{F}}'_2)\rhd W))\\
&\quad = \mathbf{g}(II(\overline{\mathcal{F}}_1\rhd X,(\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}'_{2(1)}\overline{\mathcal{F}}''_1)\rhd Z),II((\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}'_1)\rhd Y,(\overline{\mathcal{F}}_{2(3)}\overline{\mathcal{F}}'_{2(2)}\overline{\mathcal{F}}''_2)\rhd W))\\
&\quad = \mathbf{g}(II((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1)\rhd X,(\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}''_1)\rhd Z),II((\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2)\rhd Y,(\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}''_2)\rhd W))\\
&\qquad -\mathbf{g}_\star(II^{\mathcal{F}}_\star(\overline{\mathcal{R}}_{1\widehat{(1)}}\rhd Y,\overline{\mathcal{R}}_{1\widehat{(2)}}\rhd Z),II^{\mathcal{F}}_\star(\overline{\mathcal{R}}_2\rhd X,W))\\
&\quad = -\mathbf{g}(II((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1\overline{\mathcal{R}}_{1\widehat{(1)}})\rhd Y,(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2\overline{\mathcal{R}}_{1\widehat{(2)}})\rhd Z),II((\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}''_1\overline{\mathcal{R}}_2)\rhd X,(\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}''_2)\rhd W))\\
&\quad = -\mathbf{g}(II((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{R}}_{1(1)}\overline{\mathcal{F}}'_1)\rhd Y,(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{R}}_{1(2)}\overline{\mathcal{F}}'_2)\rhd Z),II((\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}''_1\overline{\mathcal{R}}_2)\rhd X,(\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}''_2)\rhd W))\\
&\quad = -\mathbf{g}(II((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}''_{1(1)}\overline{\mathcal{R}}_{1(1)}\overline{\mathcal{F}}'_1)\rhd Y,(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}''_{1(2)}\overline{\mathcal{R}}_{1(2)}\overline{\mathcal{F}}'_2)\rhd Z),II((\overline{\mathcal{F}}_{1(3)}\overline{\mathcal{F}}''_2\overline{\mathcal{R}}_2)\rhd X,\overline{\mathcal{F}}_2\rhd W))\\
&\quad = -\mathbf{g}(II((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}''_{2(1)}\overline{\mathcal{F}}'_1)\rhd Y,(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}''_{2(2)}\overline{\mathcal{F}}'_2)\rhd Z),II((\overline{\mathcal{F}}_{1(3)}\overline{\mathcal{F}}''_1)\rhd X,\overline{\mathcal{F}}_2\rhd W))\\
&\quad = -\mathbf{g}(II((\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}''_{1(2)}\overline{\mathcal{F}}'_2)\rhd Y,(\overline{\mathcal{F}}_{1(3)}\overline{\mathcal{F}}''_2)\rhd Z),II((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}''_{1(1)}\overline{\mathcal{F}}'_1)\rhd X,\overline{\mathcal{F}}_2\rhd W))\\
&\quad = -\mathbf{g}(II((\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2)\rhd Y,(\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}''_1)\rhd Z),II((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1)\rhd X,(\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}''_2)\rhd W)),
\end{aligned}
$$

using the 2-cocycle property of $\overline{\mathcal{F}}$ and its consequences (160), Eq. (17), the definition of $\mathcal{R}$, as well as the $U\mathfrak{k}$-equivariance of $\mathsf{R}^{\mathcal{F}}_\star$, $\mathsf{R}^{\mathcal{F}}_{t\star}$ and $II^{\mathcal{F}}_\star$. The sum of these three terms is the right-hand side (rhs) of (134). By (93) it equals the left-hand side:

$$
\begin{aligned}
\mathrm{rhs}(134) &= \mathbf{g}(\mathsf{R}_t((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1)\rhd X,(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2)\rhd Y)((\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}''_1)\rhd Z),(\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}''_2)\rhd W)\\
&\quad + \mathbf{g}(II((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1)\rhd X,(\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}''_1)\rhd Z),II((\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2)\rhd Y,(\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}''_2)\rhd W))\\
&\quad - \mathbf{g}(II((\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2)\rhd Y,(\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}''_1)\rhd Z),II((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1)\rhd X,(\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}''_2)\rhd W))\\
&= \mathbf{g}(\mathsf{R}((\overline{\mathcal{F}}_{1(1)}\overline{\mathcal{F}}'_1)\rhd X),(\overline{\mathcal{F}}_{1(2)}\overline{\mathcal{F}}'_2)\rhd Y)(\overline{\mathcal{F}}_{2(1)}\overline{\mathcal{F}}''_1)\rhd Z),\overline{\mathcal{F}}_{2(2)}\overline{\mathcal{F}}''_2)\rhd W)\\
&= \mathbf{g}_\star(\mathsf{R}^{\mathcal{F}}_\star(X,Y)Z,W).
\end{aligned}
$$

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