



Weak coupling limit for Schrödinger-type operators with degenerate kinetic energy for a large class of potentials

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Abstract

We improve results by Frank, Hainzl, Naboko, and Seiringer (J Geom Anal 17(4):559–567, 2007) and Hainzl and Seiringer (Math Nachr 283(3):489–499, 2010) on the weak coupling limit of eigenvalues for Schrödinger-type operators whose kinetic energy vanishes on a codimension one submanifold. The main technical innovation that allows us to go beyond the potentials considered in Frank, Hainzl, Naboko, and Seiringer (2007), Hainzl and Seiringer (2010) is the use of the Tomas–Stein theorem.

Keywords Degenerate kinetic energy · Weak coupling limit · Eigenvalue asymptotics · Fourier restriction · Tomas–Stein theorem

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1 Introduction and main results

There has been recent interest in Schrödinger-type operators of the form

$$H_\lambda = T(-i\nabla) - \lambda V \quad \text{in } L^2(\mathbb{R}^d), \quad (1.1)$$

where the kinetic energy $T(\xi)$ vanishes on a submanifold of codimension one, V is a real-valued potential, and $\lambda > 0$ is a coupling constant. We are interested in the weak coupling limit $\lambda \rightarrow 0$ for potentials that decay slowly in some L^p sense

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to be made precise. Operators of this type appear in many areas of mathematical physics [5,9,10,12,15,18–20,23,31,32,35,37,43]. The goal of [20] was to generalize the results and techniques of [12] and [19] to a large class of kinetic energies. Our goal, complementary to [20], is to relax the conditions on the potential. To keep technicalities to a minimum, we state our result for $T(-i\nabla) = |\Delta + 1|$. This was one of the main motivations to study operators of the form (1.1), due to their role in the BCS theory of superconductivity [12,18]. As in previous works [12,20] a key role is played by an operator \mathcal{V}_S on the unit sphere $S \subset \mathbb{R}^d$, whose convolution kernel is given by the Fourier transform of V . The potentials we consider here need not be in $L^1(\mathbb{R}^d)$, but \mathcal{V}_S may be defined as a norm limit of a regularized version (see Sect. 2.2 for details). The potential V is assumed to belong to the amalgamated space $\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$, where the first space measures global (average) decay and the second measures local regularity (see (2.1)). We note that $L^{\frac{d+1}{2}} \cup L^{\frac{d}{2}} \subseteq \ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$ by Jensen’s inequality.

Theorem 1.1 *Let $d \geq 3$ and $H_\lambda = |\Delta + 1| - \lambda V$. If $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$, then for every eigenvalue $a_S^j > 0$ of \mathcal{V}_S in (2.5), counting multiplicity, and every $\lambda > 0$, there is an eigenvalue $-e_j(\lambda) < 0$ of H_λ with weak coupling limit*

$$e_j(\lambda) = \exp\left(-\frac{1}{\lambda a_S^j}(1 + o(1))\right) \quad \text{as } \lambda \rightarrow 0. \tag{1.2}$$

For simplicity we stated the result for $d \geq 3$, but it easily transpires from the proof that it also holds in $d = 2$ for $V \in \ell^{\frac{d+1}{2}} L^{1+\varepsilon}$ for arbitrary $\varepsilon > 0$. All other possible negative eigenvalues (not corresponding to \mathcal{V}_S) satisfy $e_j(\lambda) \leq \exp(-c/\lambda^2)$. The statement in [20] about the convergence of eigenfunctions also holds for the potentials considered here. Since the proofs are completely analogous we will not discuss them.

In previous works [12,20] it was assumed that $V \in L^1(\mathbb{R}^d) \cap L^{\frac{d}{2}}(\mathbb{R}^d)$. Our main contribution is to remove the L^1 assumption, allowing for potentials with slower decay. The main new idea is to use the Tomas–Stein theorem (see Sect. 2.2 and formulae (2.7) and (2.8)). In view of its sharpness, our result is optimal in the sense that the exponent $(d + 1)/2$ in our class of admissible potentials cannot be increased, unless one imposes further (symmetry) restrictions on V , see also the discussion below. Moreover, the use of amalgamated spaces allows us to relax the global regularity to the local condition $V \in L^{\frac{d}{2}}_{\text{loc}}(\mathbb{R}^d)$ which just suffices to guarantee that H_λ is self-adjoint.

The idea of applying the Tomas–Stein theorem and related results such as [29] to problems of mathematical physics is not new, see, e.g., [11,27]. The validity of the Tomas–Stein theorem crucially depends on the curvature of the underlying manifold. A slight modification of our proof (see, e.g., [7,8]) shows that the result of Theorem 1.1 continues to hold for general Schrödinger-type operators (with a suitable modification of the local regularity assumption) of the form (1.1) as long as the Fermi surface $S = \{\xi \in \mathbb{R}^d : T(\xi) = 0\}$ is smooth and has everywhere non-vanishing Gaussian curvature. For example, if T is elliptic at infinity of order $2d/(d + 1) \leq s < d$, then the assumption on the potential becomes $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{s}}$. This is outlined in Theorem 4.2 and improves [20, Theorem 2.1]. The moment-type condition on the potential in that

theorem is unnecessary, regardless of whether the kinetic energy is radial or not. A straightforward generalization to the case where S has at least k non-vanishing principal curvatures can be obtained from the results of [8,17]. In that case the global decay assumption has to be strengthened to $V \in \ell^{\frac{k+2}{2}} L^{\frac{d}{s}}$. Sharp restriction theorems for surfaces with degenerate curvature are available in the three-dimensional case [25].

Based on the results of [4,14,42], if the potential V is radial, one might be able to relax the assumption in Theorem 1.1 to $V \in \ell^d L^{\frac{d}{2}}$. This naive belief is supported by the discussion in Appendix B, see especially Theorem B.8 where we generalize Theorem 1.1 to spherically symmetric potentials with almost L^d decay.

For long-range potentials the weak coupling limit (1.2) does not hold in general. Gontier, Hainzl, and Lewin [15] showed $\exp(-C_1/\sqrt{\lambda}) \leq e_1(\lambda) \leq \exp(-C_2/\sqrt{\lambda})$ for the Coulomb potential $V = |x|^{-1}$ in $d = 3$.

The key estimate (3.2) is a consequence of the Tomas–Stein theorem. The remainder of the proof is standard first-order perturbation theory that is done in exactly the same way as in [12,20]. In a similar manner—again following [19,20]—we will carry out higher order perturbation theory in Sect. 4.4 and show how one may in principle obtain any order in the asymptotic expansion of $e_j(\lambda)$ at the cost of restricting the class of admissible potentials. For instance, our methods allow us to derive the second order for $V \in L^{\frac{d+1}{2}-\varepsilon}$ and some $\varepsilon \in (0, 1/2]$. For $V \in L^1 \cap L^{d/2}$ this was first carried out in [19,20]. Furthermore, we will give an alternative proof for the existence of eigenvalues of H_λ based on Riesz projections in Sect. 4.2. This approach allows us to handle complex-valued potentials¹ on the same footing as real-valued ones. The former play a role, e.g., in the theory of resonances, but are also of independent interest.

We use the following notations: For two non-negative numbers a, b the statement $a \lesssim b$ means that $a \leq Cb$ for some universal constant C . If the estimate depends on a parameter τ , we indicate this by writing $a \lesssim_\tau b$. The dependence on the dimension d is always suppressed. We will assume throughout the article that the (asymptotic) scales e and λ are positive, sufficiently small, and that $\lambda \ln(1/e)$ remains uniformly bounded from above and below. The symbol $o(1)$ stands for a constant that tends to zero as λ (or equivalently e) tends to zero. We set $\langle \nabla \rangle = (\mathbf{1} - \Delta)^{1/2}$.

2 Preliminaries

2.1 Potential class

Let $\{Q_s\}_{s \in \mathbb{Z}^d}$ be a collection of axis-parallel unit cubes such that $\mathbb{R}^d = \bigcup_s Q_s$. We then define the norm

$$\|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}} := \left[\sum_s \|V\|_{L^{\frac{d}{2}}(Q_s)}^{\frac{d+1}{2}} \right]^{\frac{2}{d+1}}. \quad (2.1)$$

¹ In this case, a transformation of statements about non-self-adjoint operators into those about a self-adjoint operator as in the proof of Theorem 1.1 seems impossible.

The exponent $(d + 1)/2$ is natural (cf. [30,34]) in view of the Tomas–Stein theorem. This is the assertion that the Fourier transforms of $L^p(\mathbb{R}^d)$ functions indeed belong to $L^2(S)$ whenever $p \in [1, \kappa]$ where $\kappa = 2(d + 1)/(d + 3)$ denotes the ‘‘Tomas–Stein exponent’’. We discuss this theorem and a certain extension thereof in more detail in the next subsection. Observe that $1/\kappa - 1/\kappa' = 2/(d + 1)$. The following lemma is a straightforward generalization of [27, Lemma 6.1].

Lemma 2.1 *Let $s \geq 2d/(d + 1)$ and $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{s}}$. Then*

$$\| |V|^{1/2} \langle \nabla \rangle^{-\left(\frac{s}{2} - \frac{d}{d+1}\right)} \varphi \|_{L^2} \lesssim \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{s}}}^{1/2} \|\varphi\|_{L^{\kappa'}}. \tag{2.2}$$

Proof We abbreviate $\alpha = s/2 - d/(d + 1) \geq 0$ and first note that, by duality, the assertion is equivalent to

$$\| \langle \nabla \rangle^{-\alpha} |V|^{1/2} \varphi \|_{L^{\kappa}} \lesssim \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{s}}}^{1/2} \|\varphi\|_{L^2}.$$

If $\alpha = 0$, the claim follows from Hölder’s inequality, $d/s = (d + 1)/2$ in this case, and $\ell^p L^p = L^p$ for all $p \in [1, \infty]$. On the other hand, if $\alpha \geq d$ we use the fact that $\langle \nabla \rangle^{-\gamma}$ is L^p bounded for all $p \in (1, \infty)$ and $\gamma \geq 0$ (by the Hörmander–Mihlin multiplier theorem, cf. [16, Theorem 6.2.7]). Thus we shall show

$$\| |V|^{1/2} \langle \nabla \rangle^{-\alpha} \varphi \|_{L^2} \lesssim \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{s}}}^{1/2} \|\varphi\|_{L^{\kappa'}}$$

for $\alpha = s/2 - d/(d + 1)$ with $s \geq 2d/(d + 1)$ such that $\alpha \in (0, d)$. Let $\{Q_s\}_{s \in \mathbb{Z}^d}$ be the above family of axis-parallel unit cubes tiling \mathbb{R}^d , i.e., for $s \in \mathbb{Z}^d$ let $Q_s = \{x \in \mathbb{R}^d : \max_{j=1, \dots, d} |x_j - s_j| \leq 1/2\}$. Next, recall that for $\alpha \in (0, d)$, we have for any $N \in \mathbb{N}_0$

$$| \langle \nabla \rangle^{-\alpha} \varphi(x) | \lesssim_{\alpha, N} |\varphi| * W_\alpha(x)$$

where

$$W_\alpha(x) = |x|^{-(d-\alpha)} \mathbf{1}_{\{|x| \leq 1\}} + |x|^{-N} \mathbf{1}_{\{|x| \geq 1\}}. \tag{2.3}$$

(For a proof of these facts, see, e.g., [38, p. 132].) Abbreviating further $q_0 = d/s$, we obtain

$$\begin{aligned} \| |V|^{1/2} \langle \nabla \rangle^{-\alpha} \varphi \|_{L^2}^2 &\lesssim_{\alpha, N} \sum_{s \in \mathbb{Z}^d} \int_{Q_s} |V(x)| [(|\varphi| * W_\alpha)(x)]^2 dx \\ &\leq \sum_{s \in \mathbb{Z}^d} \|V\|_{L^{q_0}(Q_s)} \cdot \| |\varphi| * W_\alpha \|_{L^{2q'_0}(Q_s)}^2 \\ &\leq \sum_{s \in \mathbb{Z}^d} \|V\|_{L^{q_0}(Q_s)} \left[\sum_{s' \in \mathbb{Z}^d} \| (\mathbf{1}_{Q_{s'}} |\varphi|) * W_\alpha \|_{L^{2q'_0}(Q_s)} \right]^2 \end{aligned}$$

$$\begin{aligned} &\lesssim \sum_{s \in \mathbb{Z}^d} \|V\|_{L^{q_0}(Q_s)} \left[\sum_{s' \in \mathbb{Z}^d} \|\varphi\|_{L^{k'}(Q_{s'})} (1 + |s - s'|)^{-N} \right]^2 \\ &\lesssim N \left[\sum_{s \in \mathbb{Z}^d} \|V\|_{L^{q_0}(Q_s)}^{(d+1)/2} \right]^{2/(d+1)} \|\varphi\|_{L^{k'}}^2 \end{aligned}$$

where we used Hölder's inequality in the second line, the Hardy–Littlewood–Sobolev inequality in the penultimate line, and Hölder's and Young's inequality in the last line. This concludes the proof. \square

2.2 Definition of \mathcal{V}_S

As observed in [31], the weak coupling limit of $e_j(\lambda)$ is determined by the behavior of the potential on the zero energy surface of the kinetic energy, i.e., on the unit sphere S . We denote the Lebesgue measure on S by $d\omega$. For $V \in L^1(\mathbb{R}^d)$ we consider the self-adjoint operator $\mathcal{V}_S : L^2(S) \rightarrow L^2(S)$, defined by

$$(\mathcal{V}_S u)(\xi) = \int_S \widehat{V}(\xi - \eta) u(\eta) d\omega(\eta), \quad u \in L^2(S), \quad (2.4)$$

see, e.g., [12, Formula (2.2)]. Here we have absorbed the prefactors in the definition of the Fourier transform, i.e., we use the convention

$$\widehat{V}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} V(x) dx.$$

Our definition of \mathcal{V}_S differs from that of [12,20] by a factor of 2; this is reflected in the formula (1.2). Since $V \in L^1(\mathbb{R}^d)$, its Fourier transform is a bounded continuous function by the Riemann–Lebesgue lemma and is therefore defined pointwise. The Tomas–Stein theorem allows us to extend the definition of \mathcal{V}_S to a larger potential class. To this end we observe that the operator in (2.4) can be written as

$$\mathcal{V}_S = \mathcal{F}_S V \mathcal{F}_S^*, \quad (2.5)$$

where $\mathcal{F}_S : \mathcal{S}(\mathbb{R}^d) \rightarrow L^2(S)$, $\varphi \mapsto \widehat{\varphi}|_S$ is the Fourier restriction operator (here \mathcal{S} is the Schwartz space on \mathbb{R}^d). Its adjoint, the Fourier extension operator $\mathcal{F}_S^* : L^2(S) \rightarrow \mathcal{S}'(\mathbb{R}^d)$, is given by

$$(\mathcal{F}_S^* u)(x) = \int_S u(\xi) e^{2\pi i x \cdot \xi} d\omega(\xi). \quad (2.6)$$

A fundamental question in harmonic analysis is to find optimal sufficient conditions for κ such that \mathcal{F}_S is an $L^\kappa \rightarrow L^q$ bounded operator. By the Hausdorff–Young inequality, the case $\kappa = 1$ is trivial. On the other hand, the Knapp example (see, e.g.,

[40, p. 387–388]) and the decay of the Fourier transform of the surface measure [21] show that $\kappa < 2d/(d + 1)$ and $(d + 1)/\kappa' \leq (d - 1)/q$ are necessary conditions. The content of the Tomas–Stein theorem (unpublished, but see, e.g., Stein [39, Theorem 3] and Tomas [41]) is that, for $q = 2$, these conditions are indeed also sufficient. Concretely, the estimate

$$\|\mathcal{F}_S \varphi\|_{L^2(S)} \lesssim \|\varphi\|_{L^p(\mathbb{R}^d)}, \quad p \in [1, \kappa], \quad \kappa = 2(d + 1)/(d + 3) \quad (2.7)$$

holds for all $d \geq 2$, whenever S is a smooth and compact hypersurface with everywhere nonzero Gaussian curvature. In particular, this estimate is applicable to the Fermi surfaces that we consider later in Sect. 4.1. Moreover, by Hölder’s inequality it follows that $|V|^{1/2} \mathcal{F}_S^*$ is an $L^2(S) \rightarrow L^2(\mathbb{R}^d)$ bounded operator, whenever $V \in L^q(\mathbb{R}^d)$ and $q \in [1, (d + 1)/2]$. In this case, \mathcal{V}_S is of course $L^2(S)$ bounded as well with

$$\|\mathcal{V}_S\| \lesssim \|V\|_{L^q}, \quad q \in [1, (d + 1)/2]. \quad (2.8)$$

In the following, we will often refer to this estimate as the Tomas–Stein theorem. Recently, Frank and Sabin [13, Theorem 2] extended (2.8) and showed

$$\|W_1 \mathcal{F}_S^* \mathcal{F}_S W_2\|_{\mathfrak{S}^{\frac{(d-1)q}{d-q}}} \lesssim_q \|W_1\|_{L^{2q}} \|W_2\|_{L^{2q}}, \quad W_1, W_2 \in L^{2q}(\mathbb{R}^d), \quad q \in [1, (d+1)/2] \quad (2.9)$$

where $\mathfrak{S}^q(L^2)$ denotes the q -th Schatten space over L^2 . Observe that the Schatten exponent is monotonously increasing in q . In particular, taking $q = (d + 1)/2$, $W_1 = |V|^{1/2}$, and $W_2 = V^{1/2}$ where $V^{1/2} = |V|^{1/2} \text{sgn}(V)$ with $\text{sgn}(V(x)) = 1$ whenever $V(x) = 0$, shows that \mathcal{V}_S belongs to $\mathfrak{S}^{d+1}(L^2(S))$ with

$$\|\mathcal{V}_S\|_{\mathfrak{S}^{d+1}} \lesssim \|V\|_{L^{(d+1)/2}}. \quad (2.10)$$

We will now extend the definition of (2.5) to incorporate potentials in the larger class $\ell^{(d+1)/2} L^{d/2}$ that appears in our main result.

Proposition 2.2 *Let $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$. Then (2.5) defines a bounded operator on $L^2(S)$. Moreover, if $(V_n)_n$ is a sequence of Schwartz functions converging to V in $\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$ and $\mathcal{V}_S^{(n)}$ are the corresponding operators in (2.4), then \mathcal{V}_S is the norm limit of the $\mathcal{V}_S^{(n)}$.*

Proof We first assume that $V \in L^{\frac{d+1}{2}}(\mathbb{R}^d)$. It follows from the above discussion that \mathcal{V}_S is the norm limit of the $\mathcal{V}_S^{(n)}$. To extend the definition to all $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$, we prove

$$\|\mathcal{F}_S V \mathcal{F}_S^*\| \lesssim \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}}. \quad (2.11)$$

To this end we use the following observation. For $u \in L^2(S)$ and $\xi \in S$ we write

$$(\mathcal{V}_S^{(n)} u)(\xi) = \int_S (\widehat{V_n \varphi})(\xi - \eta) u(\eta) \, d\omega(\eta), \quad (2.12)$$

where φ is a bump function that equals 1 in $B(0, 2)$. This has the same effect as replacing V_n by $\varphi^\vee * V_n$. (Here, $\varphi^\vee(x) := \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} \varphi(\xi) d\xi$ denotes the inverse Fourier transform.) Since (2.11) is equivalent to the bound

$$\|\sqrt{|V|} \mathcal{F}_S^* \mathcal{F}_S \sqrt{|V|}\| \lesssim \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}}, \tag{2.13}$$

where $V^{1/2} = |V|^{1/2} \text{sgn}(V)$ and $\text{sgn}(V)$ is a unitary multiplication operator, we may assume without loss of generality that $V \geq 0$. Passing to a subsequence, we may also assume that $(V_n)_n$ converges to V almost everywhere. By Fatou’s lemma, for any $u \in L^2(S)$,

$$\begin{aligned} \langle \mathcal{F}_S^* u, V \mathcal{F}_S^* u \rangle &\leq \liminf_{n \rightarrow \infty} \langle \mathcal{F}_S^* u, V_n \mathcal{F}_S^* u \rangle \leq \liminf_{n \rightarrow \infty} \|(\varphi^\vee * V_n)(\mathcal{F}_S^* u)\|_{L^\kappa} \|u\|_2 \\ &\lesssim \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}} \|u\|_2^2, \end{aligned} \tag{2.14}$$

where the penultimate inequality follows from the Tomas–Stein theorem (2.8) and the last inequality from the bound

$$\|(\varphi^\vee * V)(\mathcal{F}_S^* u)\|_{L^\kappa} \leq \|\varphi^\vee * V\|_{L^{\frac{d+1}{2}}} \|F_S^* u\|_{L^{\kappa'}} \lesssim_\varphi \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}} \|u\|_{L^2} \tag{2.15}$$

whose proof is similar to that of Lemma 2.1 since the convolution kernel of φ^\vee is a Schwartz function, i.e., in particular $|\varphi^\vee(x)| \lesssim_N (1 + |x|)^{-N}$ for any $N \in \mathbb{N}$. More precisely, for the same family $\{Q_s\}_{s \in \mathbb{Z}^d}$ of axis-parallel unit cubes tiling \mathbb{R}^d that we used in the proof of Lemma 2.1, we have for any $N > 0$,

$$\begin{aligned} \|\varphi^\vee * V\|_{\frac{d+1}{2}} &= \left\| \sum_s \mathbf{1}_{Q_s} (\varphi^\vee * V) \right\|_{\frac{d+1}{2}} = \sum_s \|\varphi^\vee * \left(\sum_{s'} V \mathbf{1}_{Q_{s'}} \right)\|_{L^{\frac{d+1}{2}}(Q_s)} \\ &\leq \sum_s \left[\sum_{s'} \|\varphi^\vee * (V \mathbf{1}_{Q_{s'}})\|_{L^{\frac{d+1}{2}}(Q_s)} \right]^{\frac{d+1}{2}} \\ &\lesssim_N \sum_s \left[\sum_{s'} (1 + |s - s'|)^{-N} \|V\|_{L^{\frac{d}{2}}(Q_{s'})} \right]^{\frac{d+1}{2}} \lesssim_N \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}} \end{aligned} \tag{2.16}$$

where we used Young’s inequality in the last two estimates. This concludes the proof. □

2.3 Compactness of \mathcal{V}_S

We show that \mathcal{V}_S belongs to a certain Schatten space $\mathfrak{S}^p(L^2(S))$ and is thus a compact operator. In particular, the spectrum of \mathcal{V}_S is compact and countable with accumulation point 0. The nonzero elements are eigenvalues of finite multiplicity. That 0 is in the spectrum follows from the fact that $L^2(S)$ is infinite-dimensional.

Lemma 2.3 *Let $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$. Then $\mathcal{V}_S \in \mathfrak{S}^{d+1}(L^2(S))$ and*

$$\|\mathcal{V}_S\|_{\mathfrak{S}^{d+1}} \lesssim \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}}.$$

Proof We recycle the proof of Proposition 2.2 and suppose $V \geq 0$ without loss of generality again. We apply the Tomas–Stein theorem (2.10) for trace ideals with V replaced by $\varphi^\vee * V$ where φ is the same bump function as in that proof. Note that, by (2.12), this replacement does not affect the value of $\|\mathcal{V}_S\|_{\mathfrak{S}^{d+1}}$ since the eigenvalues remain the same. Thus, by (2.16), $\|\mathcal{V}_S\|_{\mathfrak{S}^{d+1}} \lesssim \|\varphi^\vee * V\|_{L^{(d+1)/2}} \lesssim \|V\|_{\ell^{(d+1)/2} L^{d/2}}. \square$

2.4 Birman–Schwinger operator

As in [12,20], our proof is based on the well-known Birman–Schwinger principle. This is the assertion that, if

$$BS(e) := \sqrt{|\bar{V}|}(T + e)^{-1}\sqrt{\bar{V}} \tag{2.17}$$

with $e > 0$, then

$$-e \in \text{spec}(H_\lambda) \iff \frac{1}{\lambda} \in \text{spec}(BS(e)). \tag{2.18}$$

Here $\sqrt{\bar{V}} := \text{sgn}(V)\sqrt{|V|}$ and $T = |\Delta + 1|$. Thus, (1.2) would follow from

$$\ln(1/e)a_S^j(1 + o(1)) \in \text{spec}(BS(e)) \tag{2.19}$$

for every eigenvalue $a_S^j > 0$ of \mathcal{V}_S . We note that since V and the symbol of $(T + e)^{-1}$ both vanish at infinity, $BS(e)$ is a compact operator, see, e.g., [36, Chapter 4]. Moreover, we have the following operator norm bound.

Lemma 2.4 *Let $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$. Then*

$$\|BS(e)\| \lesssim \ln(1/e)\|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}}$$

for all $e \in (0, 1/2)$.

Proof The proof follows from (3.3) and (3.7) below. □

3 Proof of Theorem 1.1

3.1 Outline of the proof

We briefly sketch the strategy of the proof of (2.19). We first split the Birman–Schwinger operator into a sum of high and low energy pieces

$$BS(e) = BS^{\text{low}}(e) + BS^{\text{high}}(e).$$

More precisely, we fix $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on the unit ball. We also fix $0 < \tau < 1$ and set

$$BS^{\text{low}}(e) = \sqrt{|V|} \chi(T/\tau) (T + e)^{-1} \sqrt{V}.$$

As we will see in (3.3), the high energy piece is harmless. The low energy piece is split further into a singular and a regular part,

$$BS^{\text{low}}(e) = BS_{\text{sing}}^{\text{low}}(e) + BS_{\text{reg}}^{\text{low}}(e),$$

where the singular part is defined as

$$BS_{\text{sing}}^{\text{low}}(e) = \ln(1 + \tau/e) \sqrt{|V|} \mathcal{F}_S^* \mathcal{F}_S \sqrt{V}. \quad (3.1)$$

Note that $\sqrt{|V|} \mathcal{F}_S^* \mathcal{F}_S \sqrt{V}$ is isospectral to \mathcal{V}_S . As already mentioned in the introduction and the previous section, Theorem 1.1 would follow from standard perturbation theory if we could show the key bound

$$\lambda \|BS_{\text{reg}}^{\text{low}}(e)\| = o(1) \quad (3.2)$$

for $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$, as long as $\lambda \ln(1/e)$ remains uniformly bounded from above and below.

3.2 Bound for $BS^{\text{high}}(e)$

Here we prove that

$$\|BS^{\text{high}}(e)\| \lesssim_\tau \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}} \quad (3.3)$$

Proof By a trivial L^2 -bound we have

$$\|BS^{\text{high}}(e)\| \lesssim_\tau \| |V|^{1/2} \langle \nabla \rangle^{-1} \|^2. \quad (3.4)$$

The TT^* version of Lemma 2.1 for $s = 2$,

$$\| \langle \nabla \rangle^{-\frac{1}{d+1}} V \langle \nabla \rangle^{-\frac{1}{d+1}} \varphi \|_{L^\kappa} \lesssim \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}} \|\varphi\|_{L^{\kappa'}},$$

together with Sobolev embedding $H^{\frac{d}{d+1}}(\mathbb{R}^d) \subset L^{\kappa'}(\mathbb{R}^d)$ yields

$$\| \langle \nabla \rangle^{-1} V \langle \nabla \rangle^{-1} \varphi \|_{L^2} \lesssim \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}} \|\varphi\|_{L^2}.$$

Combining the last inequality with (3.4) yields the claim. \square

3.3 Bound for $BS^{\text{low}}(e)$

The Fermi surface of T at energy $t \in (0, \tau]$ consists of two connected components $S_t^\pm = (1 \pm t)^{1/2}S$. The spectral measure E_T of T is given by

$$dE_T(t) = \sum_{\pm} \mathcal{F}_{S_t^\pm}^* \mathcal{F}_{S_t^\pm} \frac{dt}{2\sqrt{1 \pm t}}. \tag{3.5}$$

in the sense of Schwartz kernels, see, e.g., [24, Chapter XIV]. By the spectral theorem, (3.5) implies that

$$BS^{\text{low}}(e) = \sum_{\pm} \int_0^\tau \frac{\sqrt{|V|} \mathcal{F}_{S_t^\pm}^* \mathcal{F}_{S_t^\pm} \sqrt{V}}{t + e} \frac{dt}{2\sqrt{1 \pm t}}. \tag{3.6}$$

Together with the proof of Lemma 2.3 this yields

$$\|BS^{\text{low}}(e)\|_{\mathfrak{S}^{d+1}} \lesssim_\tau \ln(1/e) \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}}. \tag{3.7}$$

3.4 Proof of the key bound (3.2)

From (3.6) and the definition of $BS^{\text{low}}_{\text{sing}}(e)$ (see (3.1)) we infer that

$$BS^{\text{low}}_{\text{reg}}(e) = \sum_{\pm} \int_0^\tau \frac{\sqrt{|V|} (\mathcal{F}_{S_t^\pm}^* \mathcal{F}_{S_t^\pm} - \sqrt{1 \pm t} \mathcal{F}_S^* \mathcal{F}_S) \sqrt{V}}{t + e} \frac{dt}{2\sqrt{1 \pm t}}. \tag{3.8}$$

If V were a strictly positive Schwartz function, then by the Sobolev trace theorem, the map $t \mapsto \sqrt{|V|} \mathcal{F}_{S_t^\pm}^* \mathcal{F}_{S_t^\pm} \sqrt{V}$ would be Lipschitz continuous in operator norm, see, e.g., [44, Chapter 1, Proposition 6.1], [33, Theorem IX.40]. Hence, we would obtain a stronger bound than (3.2) in this case. Using (3.5) and observing that

$$\mathcal{F}_{\mu S}^* \mathcal{F}_{\mu S}(x, y) = \mu^{d-1} \int_S e^{2\pi i \mu(x-y) \cdot \xi} d\omega(\xi)$$

for $\mu > 0$, it is not hard to see that Lipschitz continuity even holds in the Hilbert–Schmidt norm. Since $\mathfrak{S}^2 \subseteq \mathfrak{S}^{d+1}$ we conclude that, if V were Schwartz, we would get

$$\lambda \|BS^{\text{low}}_{\text{reg}}(e)\|_{\mathfrak{S}^{d+1}} = o(1). \tag{3.9}$$

We now prove that (3.9) (and hence also (3.2)) holds for the potentials considered in Theorem 1.1.

Lemma 3.1 *If $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$, then (3.9) holds as $\lambda \rightarrow 0$ and $\lambda \ln(1/e)$ remains bounded.*

Proof Without loss of generality we may again assume $V \geq 0$. Let $V_n^{1/2}$ be strictly positive Schwartz functions converging to $V^{1/2}$ in $\ell^{d+1}L^d$. We use that the bound (2.13) is locally uniform in t and can be upgraded to a Schatten bound as in Lemma 2.3. That is, for fixed τ , we have the bound

$$\sup_{t \in [0, \tau]} \|\sqrt{V} \mathcal{F}_{S_t}^* \mathcal{F}_{S_t} \sqrt{V}\|_{\mathfrak{S}^{d+1}} \lesssim_{\tau} \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}}. \quad (3.10)$$

Since we have already proved (3.9) for such V_n , we may thus estimate

$$\lambda \|B S_{\text{reg}}^{\text{low}}(e)\|_{\mathfrak{S}^{d+1}} \lesssim_{\tau} \lambda \ln(1/e) \|\sqrt{V} - \sqrt{V_n}\|_{\ell^{d+1}L^d} \|\sqrt{V}\|_{\ell^{d+1}L^d} + o(1).$$

Since $\lambda \ln(1/e)$ is bounded, (3.9) follows upon letting $n \rightarrow \infty$. \square

4 Further results

The purpose of this subsection is fourfold. First we outline how our main result, Theorem 1.1, can be generalized to treat operators whose kinetic energy vanishes on other smooth, curved surfaces. Second, we provide an alternative proof (to that of [12,20]), based on Riesz projections, that weakly coupled bound states of $H_{\lambda} = |\Delta + 1| - \lambda V$ actually exist, provided \mathcal{V}_S has at least one positive eigenvalue. This follows from standard perturbation theory [28, Sections IV.3.4–5], but the argument is robust enough to handle complex-valued potentials. In fact, we do not know how the arguments in [12,20] could be adapted to treat such potentials as the Birman–Schwinger operator cannot be transformed to a self-adjoint operator anymore. Third, we give two examples of (real-valued) potential classes for which the operator \mathcal{V}_S does have at least one positive eigenvalue. In both examples the potentials are neither assumed to be integrable, nor positive. Fourth, we derive the second order in the asymptotic expansion of $e_j(\lambda)$ in Theorem 4.4 for $V \in L^{\frac{d+1}{2}-\varepsilon}$ and $\varepsilon \in (0, 1/2]$.

4.1 Generalization to other kinetic energies

As the Tomas–Stein theorem holds for arbitrary compact, smooth, curved surfaces (cf. [39, Theorem 3] and [13, Theorem 2]) it is not surprising that Theorem 1.1 continues to hold for more general symbols $T(\xi)$. In what follows, we assume that $T(\xi)$ satisfies the geometric and analytic assumptions stated in [20]—that we recall in a moment—and a certain curvature assumption. First, we assume that $T(\xi)$ attains its minimum, which we set to zero for convenience, on a manifold

$$S = \{\xi \in \mathbb{R}^d : T(\xi) = 0\} \quad (4.1)$$

of codimension one. Next, we assume that S consists of finitely many connected and compact components and that there exists a $\delta > 0$ and a compact neighborhood $\Omega \subseteq \mathbb{R}^d$ of S containing S with the property that the distance of any point in S to the complement of Ω is at least δ .

We now make some analytic assumptions on the symbol $T(\xi)$. We assume that

1. there exists a measurable, locally bounded function $P \in C^\infty(\Omega)$ such that $T(\xi) = |P(\xi)|$,
2. $|\nabla P(\xi)| > 0$ for all $\xi \in \Omega$, and
3. there exist constants $C_1, C_2 > 0$ and $s \in [2d/(d + 1), d)$ such that $T(\xi) \geq C_1|\xi|^s + C_2$ for $\xi \in \mathbb{R}^d \setminus \Omega$.

Since S is the zero set of the function $P \in C^\infty(\Omega)$ and $\nabla P \neq 0$, it is a compact C^∞ submanifold of codimension one. Finally, we also assume that S has everywhere nonzero Gaussian curvature². Note that this assumption was not needed in [20].

Next, we redefine the singular part of the Birman–Schwinger operator (2.4), namely

$$(\mathcal{V}_S u)(\xi) = \int_S \widehat{V}(\xi - \eta)u(\eta) \, d\sigma_S(\eta), \quad u \in L^2(S, d\sigma_S). \tag{4.2}$$

Here, $d\sigma_S(\xi) := |\nabla P(\xi)|^{-1}d\omega(\xi)$ where $d\omega$ denotes the euclidean (Lebesgue) surface measure on S . In particular, the elementary volume $d\xi$ in \mathbb{R}^d satisfies $d\xi = dr \, d\sigma_S(\xi)$ where dr is the Lebesgue measure on \mathbb{R} . In what follows, we abbreviate the notation and write $L^2(S)$ instead of $L^2(S, d\sigma_S)$.

The new definition (4.2) of \mathcal{V}_S now does not differ anymore from that of [12,20] by a factor of 2. Similarly as before, (4.2) can be written as

$$\mathcal{V}_S = \mathcal{F}_S V \mathcal{F}_S^*, \tag{4.3}$$

where $\mathcal{F}_S : \mathcal{S}(\mathbb{R}^d) \rightarrow L^2(S)$, $\varphi \mapsto \widehat{\varphi}|_S$ is the Fourier restriction operator and its adjoint, the Fourier extension operator $\mathcal{F}_S^* : L^2(S) \rightarrow \mathcal{S}'(\mathbb{R}^d)$, is now given by

$$(\mathcal{F}_S^* u)(x) = \int_S u(\xi)e^{2\pi i x \cdot \xi} \, d\sigma_S(\xi). \tag{4.4}$$

Recall that the Tomas–Stein theorem asserts that \mathcal{F}_S is a $L^p(\mathbb{R}^d) \rightarrow L^2(S)$ bounded operator for all $p \in [1, \kappa]$. In particular, the extension to trace ideals [13, Theorem 2] continues to hold, i.e., $\|\mathcal{V}_S\|_{\mathfrak{S}^{d+1}} \lesssim \|V\|_{L^{(d+1)/2}}$. By Sobolev embedding and $s < d$, the operator $T(-i\nabla) - \lambda V$ can be meaningfully defined if $V \in L^{d/s}(\mathbb{R}^d)$. By the assumption $s \geq 2d/(d + 1)$, we have $(d + 1)/2 \geq d/s$.

We will now outline the necessary changes in the proof of Theorem 1.1 for T as above and $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{s}}$. First, the corresponding analogs of Proposition 2.2 and Lemma 2.3 follow immediately from the Tomas–Stein theorem that we just discussed, and the analog of (2.16) (using $(d + 1)/2 \geq d/s$).

Next, the splitting of $BS(e)$ is the same as in Sect. 3. There we have the analogous bound (3.3), i.e., $\|BS^{\text{high}}(e)\| \lesssim_\tau \|V\|_{\ell^{(d+1)/2} L^{d/s}}$ by the same arguments of that proof (cf. Lemma 2.1). Next the Fermi surface of T at energy $t \in (0, \tau]$ again consists of

² The precise definition of Gaussian curvature can be found, e.g., in [39, p. 321–322].

two connected components S_t^\pm . Using the above definition of $\mathcal{F}_{S_t^\pm}$, we observe that the spectral measure E_T of T is now given by

$$dE_T(t) = \sum_{\pm} \mathcal{F}_{S_t^\pm}^* \mathcal{F}_{S_t^\pm} dt.$$

Thus, by the spectral theorem,

$$BS^{\text{low}}(e) = \sum_{\pm} \int_0^\tau \frac{\sqrt{|V|} \mathcal{F}_{S_t^\pm}^* \mathcal{F}_{S_t^\pm} \sqrt{V}}{t+e} dt$$

and by the proof of the analog of Lemma 2.3 (i.e., the Tomas–Stein theorem and the analog of (2.16)), we again obtain $\|BS^{\text{low}}(e)\|_{\mathfrak{S}^{d+1}} \lesssim_\tau \ln(1/e) \|V\|_{\ell^{(d+1)/2} L^{d/s}}$. Thus, we are left to prove the analog of the key bound (3.2). But this just follows from the proof of Lemma 3.1 and the fact that the Tomas–Stein estimate (3.10) is valid locally uniform in t for surfaces S_t that we discuss here. In turn, by [2, Theorem 1.1], this is a consequence of the following assertion.

Proposition 4.1 *Assume $T(\xi)$ satisfies the assumptions stated at the beginning of this section. Then for fixed $\tau > 0$, one has $\sup_{t \in [0, \tau]} |(d\sigma_{S_t^\pm})^\vee(x)| \lesssim_\tau (1 + |x|)^{-\frac{d-1}{2}}$.*

Proof For $t = 0$ this estimate is well known, see, e.g., [39, Theorem 1]. Now let $t \in (0, \tau]$. First note that $S_t = S_t^+ \cup S_t^-$ where S_t^+ , S_t^- lie outside, respectively, inside S . In the following we treat S_t^+ and abuse notation by writing $S_t \equiv S_t^+$. The arguments for S_t^- are completely analogous. We will now express $d\sigma_{S_t}$ in terms of $d\sigma_S$. To that end we follow [44, Chapter 2, Section 1]. Let $\psi(t) : S \rightarrow S_t$ be the diffeomorphism³ defined by the formula

$$\psi(t)\zeta = \xi(t), \quad \zeta \in S$$

where $\xi(t)$ solves the differential equation

$$\begin{cases} \frac{d\xi(t)}{dt} = j(\xi(t)) \\ \xi(0) = \zeta \in S \end{cases}$$

with

$$j(\xi) := \frac{\nabla P(\xi)}{|\nabla P(\xi)|^2} \in C^\infty(P^{-1}[0, t]),$$

i.e., $j(\xi(t))$ is the vector field generating the flow $\xi(t)$ along the normals of S_t . Next,

$$\tau(t, \xi) = \frac{d\sigma_{S_t}(\psi(t)\xi)}{d\sigma_S(\xi)}, \quad \xi \in S$$

³ Its construction is carried out in [44, p. 112–113] and requires actually only $P \in C^2$. However, we need the smoothness of P to obtain the claimed decay of $(d\sigma_{S_t})^\vee$ by means of a stationary phase argument.

is the Radon–Nikodým derivative of the preimage of the measure $d\sigma_S$, under the mapping $\psi(t)$ with respect to the measure $d\sigma_S$. By [44, Chapter 2, Lemma 1.9] it is given by

$$\tau(t, \xi) = \exp \left(\int_0^t (\operatorname{div} j)(\psi(\mu)\xi) \, d\mu \right), \quad \xi \in S.$$

Thus, we have

$$\begin{aligned} (d\sigma_{S_t})^\vee(x) &= \int_S d\sigma_S(\xi) e^{2\pi i x \cdot \psi(t)\xi} \exp \left(\int_0^t \operatorname{div} j(\psi(\mu)\xi) \, d\mu \right) \\ &\equiv \int_S d\sigma_S(\xi) e^{2\pi i x \cdot \xi} F_{t,x}(\xi) \end{aligned} \tag{4.5}$$

with

$$F_{t,x}(\xi) := e^{2\pi i x \cdot (\psi(t)\xi - \xi)} \exp \left(\int_0^t \operatorname{div} j(\psi(\mu)\xi) \, d\mu \right)$$

which depends smoothly on ξ . Thus, we are left to show that the absolute value of the right side of (4.5) is bounded by $C_\tau(1 + |x|)^{-(d-1)/2}$ for all $t \in (0, \tau]$. Decomposing $F_{t,x}(\xi)$ on S smoothly into (sufficiently small) compactly supported functions, say $\{F_{t,x}(\xi)\chi_\kappa(\xi)\}_{\kappa=1}^K$ for a finite, smooth partition of unity $\{\chi_\kappa\}_{\kappa=1}^K$ subordinate to S , shows that there is for every $x \in \mathbb{R}^d \setminus \{0\}$, at most one point $\bar{\xi}(x) \in S$ with a normal pointing in the direction of x . Then, by the stationary phase method, Hlawka [22] and Herz [21] (see also Stein [40, p. 360]) already showed that the leading order in the asymptotic expansion (as $|x| \rightarrow \infty$) of (4.5) with the cut off amplitude $F_{t,x}\chi_\kappa$ is given by

$$|x|^{-(d-1)/2} F_{t,x}(\bar{\xi}(x))\chi_\kappa(\bar{\xi}(x))|K(\bar{\xi}(x))|^{-1/2} e^{i\pi n/4 + 2\pi i x \cdot \bar{\xi}(x)}.$$

Here, $|K(\xi)|$ is the absolute value of the Gaussian curvature of S at $\xi \in S$ which is, by assumption, strictly bigger than zero, and n is the excess of the number of positive curvatures over the number of negative curvatures in the direction x . But since $|F_{t,x}(\xi)| \lesssim_\tau 1$ for all $t \in (0, \tau]$, $x \in \mathbb{R}^d$, and $\xi \in S$, this concludes the proof. \square

We summarize the findings of this subsection as follows.

Theorem 4.2 *Let $d \geq 3$, $s \in [2d/(d + 1), d)$, and assume $T(\xi)$ satisfies the assumptions stated at the beginning of this subsection. If $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{s}}$, then for every eigenvalue $a_j^i > 0$ of \mathcal{V}_S in (4.3), counting multiplicity, and every $\lambda > 0$, there is an eigenvalue $-e_j(\lambda)$ of $T(-i\nabla) - \lambda V$ with weak coupling limit*

$$e_j(\lambda) = \exp \left(-\frac{1}{2\lambda a_j^i} (1 + o(1)) \right) \quad \text{as } \lambda \rightarrow 0.$$

4.2 Alternative proof and complex-valued potentials

We first consider the case where V is real-valued and then indicate how to modify the proof in the complex-valued case. For simplicity, we even assume $V \geq 0$ so that the Birman–Schwinger operator is automatically self-adjoint. The case where V does not have a sign could also be treated by the methods of [12], but here it follows from the general case considered later.

For $V \geq 0$ we have, by self-adjointness,

$$\|(BS_{\text{sing}}^{\text{low}}(e) - z)^{-1}\| \leq 1/\min_j |z - z_j(e)|, \quad (4.6)$$

where $z_j(e) = \ln(1 + \tau/e) a_S^j$ are the eigenvalues of $BS_{\text{sing}}^{\text{low}}(e)$. Fixing an integer i and a range for e such that $\lambda \ln(1/e)$ is bounded by an absolute constant from above and below, it follows that if γ is a circle of radius $c \ln(1/e)$ around the eigenvalue $z_i(e)$, with c a sufficiently small positive number, then there are no other eigenvalues in the interior of γ , and

$$\max_{z \in \gamma} \|(BS_{\text{sing}}^{\text{low}}(e_i(\lambda)) - z)^{-1}\| \leq 1/(c \ln(1/e)).$$

Hence, by (3.3) and (3.2), if we set $C(z) = (BS_{\text{sing}}^{\text{low}}(e) - z)^{-1}(BS(e) - BS_{\text{sing}}^{\text{low}}(e))$, then

$$r^{-1} := \max_{z \in \gamma} \|C(z)\| \leq c^{-1}o(1), \quad (4.7)$$

and this is < 1 for λ small enough. It follows from a Neumann series argument that γ is contained in the resolvent set of the family $T(\kappa) = BS_{\text{sing}}^{\text{low}}(e) + \kappa(BS(e) - BS_{\text{sing}}^{\text{low}}(e))$ for $|\kappa| < r$ and that $(T(\kappa) - z)^{-1}$ is continuous in $|\kappa| < r$, $z \in \gamma$. This implies that the Riesz projection

$$P(\kappa) = -\frac{1}{2\pi i} \oint_{\gamma} (T(\kappa) - z)^{-1} dz$$

has constant rank for $|\kappa| < r$. In particular, $\text{rank } P(0) = \text{rank } P(1)$, which means that $BS_{\text{sing}}^{\text{low}}(e)$ and $BS(e)$ have the same number of eigenvalues in the interior of γ . Hence $BS(e)$ has exactly one (real) eigenvalue $w_i(e)$ at a distance $\leq c \ln(1/e)$ from $z_i(e)$. Since c can be chosen arbitrarily small, it follows that $w_i(e) = z_i(e)(1 + o(1))$. By the Birman–Schwinger principle this implies (1.2).

We now drop the assumption that V is real-valued. By inspection of the proof, it is evident that Lemma 2.4 and (3.9) continue to hold for complex-valued V and e if $\ln(1/e)$ is replaced by its absolute value. We assume here that $e \in \mathbb{C} \setminus (-\infty, 0]$ and take the branch of the logarithm that agrees with the real logarithm on the positive real line. We also replace our standing assumption by requiring that $|e|, \lambda > 0$ are sufficiently small and $\lambda |\ln(e)|$ remains uniformly bounded from above and below. The additional difficulty in the present case is that the bound for the inverse (4.6) fails

in general. We use the following replacement, which is a consequence of [3, Theorem 4.1],

$$\|(BS_{\text{sing}}^{\text{low}}(e) - z)^{-1}\| \leq \frac{1}{d(e; z)} \exp\left(a \frac{\|BS_{\text{sing}}^{\text{low}}(e)\|_{\mathfrak{S}^{d+1}}^{d+1}}{d(e; z)^{d+1}} + b\right),$$

where $d(e; z) = \text{dist}(z, \text{spec}(BS_{\text{sing}}^{\text{low}}(e)))$ and $a, b > 0$. Note that $\|BS_{\text{sing}}^{\text{low}}(e)\|_{\mathfrak{S}^{d+1}} \lesssim |\ln(1/e)| \|V\|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}}$ by Lemma 2.3. Thus, for a similar circle γ of radius $c|\ln(1/e)|$ around $z_i(e)$, we find that (4.7) holds with an additional factor of $\exp(a/c^{d+1} + b)$ on the right, and hence we conclude $\text{rank } P(0) = \text{rank } P(1)$ as before.

4.3 Existence of positive eigenvalues of \mathcal{V}_S

It is well known that operators of the form (1.1) have at least one negative eigenvalue if either $V \in L^1(\mathbb{R}^d)$ and $\int V > 0$ or if $V \geq 0$ and not almost everywhere vanishing [12,20,23,31]. In the latter case there are even infinitely many negative eigenvalues [20, Corollary 2.2]. By Theorem 1.1, H_λ has at least as many negative eigenvalues as $-\mathcal{V}_S$. We will therefore restrict our attention to this operator. By a slight modification of the following two examples (where the trial state is an approximation of the identity in Fourier space to a thickened sphere), this result may also be obtained without reference to Theorem 1.1.

Since $\mathcal{F}_S^* \varphi = (\varphi d\omega)^\vee$ it follows from (2.5) that

$$\langle \varphi, \mathcal{V}_S \varphi \rangle = \int_{\mathbb{R}^d} V(x) |(\varphi d\omega)^\vee(x)|^2 dx, \quad \varphi \in L^2(S). \tag{4.8}$$

If φ is a radial function, then so is $(\varphi d\omega)^\vee$. In particular, for $\varphi \equiv 1$ we get

$$\langle \varphi, \mathcal{V}_S \varphi \rangle = \int_0^\infty dr r^{d-1} |(d\omega)^\vee(r)|^2 \left(\int_S V(r\omega) d\omega \right).$$

Standard stationary phase computations show that $(d\omega)^\vee(r) = \mathcal{O}((1+r)^{-(d-1)/2})$ and that it oscillates on the unit scale; in fact, it is proportional to the Bessel function $J_{\frac{d-2}{2}}$, see, e.g., [16, Appendix B.5]. The integral is convergent if the spherical average of V is in $L^1(\mathbb{R}_+, \min\{r^{d-1}, 1\} dr)$. This condition is satisfied, e.g., if V is short range, $|V(x)| \lesssim (1+|x|)^{-1-\varepsilon}$ for some $\varepsilon > 0$. If the integral is positive, then \mathcal{V}_S has a positive eigenvalue.

For the second example we take φ as a normalized bump function adapted to a spherical cap of diameter $R^{-1/2}$ with $R > 1$; this is called a Knapp example in the context of Fourier restriction theory. Then $(\varphi d\omega)^\vee$ will be a Schwartz function concentrated on a tube $T = T_R$ of length R and radius $R^{1/2}$, centered at the origin. More precisely, let

$$\varphi(\xi) = R^{\frac{d-1}{4}} \widehat{\chi}(R(\xi_1 - 1), R^{1/2} \xi') \tag{4.9}$$

where $\xi_1 = \sqrt{1 - |\xi'|^2}$ and $\widehat{\chi}$ is a bump function. We write $\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1}$ and similarly for x here. We may choose $\chi \geq 0$ and such that $\chi \geq \mathbf{1}_{B(0,1)}$. Indeed, if g is an even bump function, then we can take $\widehat{\chi}(\xi) = A^d B(g * g)(A\xi)$ for some $A > 1, B > 0$. Then the $L^2(S)$ -norm of φ is bounded from above and below uniformly in R and

$$(\varphi d\omega)^\vee(x) = R^{-\frac{d-1}{4}} e^{2\pi i x_1} \chi_T(x),$$

where χ_T is a Schwartz function concentrated on

$$T = \{x \in \mathbb{R}^d : |x_1| \leq R, |x'| \leq R^{1/2}\}, \tag{4.10}$$

i.e., a tube pointing in the x_1 direction. We can also take linear combinations of the wave packets (4.9) to obtain real-valued trial functions. Indeed, choosing χ symmetric and setting $\psi(\xi) = [\varphi(\xi_1, \xi') + \varphi(-\xi_1, \xi')]/2$, we get

$$(\psi d\omega)^\vee(x) = R^{-\frac{d-1}{4}} \cos(2\pi x_1) \chi_T(x),$$

with a slightly different χ_T . Without loss of generality we may assume that $\chi_T(x) \geq 1$ for $x \in T$. By (4.8), if $V \in L^1_{\text{loc}}(\mathbb{R}^d)$ and of tempered growth, then

$$\langle \psi, \mathcal{V}_S \psi \rangle = R^{-\frac{d-1}{2}} \int_{\mathbb{R}^d} V(x) \cos^2(2\pi x_1) |\chi_T(x)|^2 dx.$$

In particular, this holds for $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$, which we assume from now on. By Hölder and the rapid decay of χ_T away from T , we have that, for any $M, N > 1$,

$$\begin{aligned} \left| \int_{\mathbb{R}^d \setminus MT} V(x) |\chi_T(x)|^2 dx \right| &\leq \| \mathbf{1}_{\mathbb{R}^d \setminus MT} \chi_T^2 \|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}} \| V \|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}} \\ &\lesssim_N M^{-N} R^{\frac{d-1}{2}} \| V \|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}}. \end{aligned}$$

It follows that for $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$,

$$\langle \psi, \mathcal{V}_S \psi \rangle \geq R^{-\frac{d-1}{2}} \int_{MT} V(x) \cos^2(2\pi x_1) |\chi_T(x)|^2 dx - C_N M^{-N} \| V \|_{\ell^{\frac{d+1}{2}} L^{\frac{d}{2}}}. \tag{4.11}$$

If the first term on the right is positive and bounded from below by, say, a fixed power of M^{-1} , then this expression is positive for large R . As a concrete example, consider the potential

$$V(x) = \frac{\cos(4\pi x_1)}{(1 + |x_1| + |x'|^2)^{1+\varepsilon}},$$

with $\varepsilon > 0$ (see also [6,14,26] for related examples). A straightforward calculation shows that $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$. Since the average of $\cos^2(2\pi x_1) \cos(4\pi x_1)$ over a full period of $\cos(4\pi x_1)$ is always $\gtrsim 1$ and $|\chi_T|^2$ is approximately constant on the unit scale, with ≥ 1 on T , a computation shows that the first term on the right side of (4.11) is bounded from below by $MR^{-\varepsilon}$. Taking $M = R^\varepsilon$ yields positivity of the whole expression for sufficiently large R . Therefore, $-\mathcal{V}_S$, and hence H_λ , has a negative eigenvalue. This example has a straightforward generalization to more than one eigenvalue. Let $(\kappa_j)_{j=1}^K$ be mutually disjoint spherical caps of diameter $R^{-1/2}$ and let φ_j be normalized bump functions adapted to κ_j , similar to (4.9). Note that $K \lesssim R^{-\frac{d-1}{2}}$ since the caps are disjoint. If the condition following (4.11) is satisfied for all tubes T_j corresponding to the caps κ_j (these are dual to the caps and centered at the origin), then the expression (4.11) is positive (for large R) for every φ_j . Since the φ_j are orthogonal (by Plancherel), it follows that \mathcal{V}_S has at least K positive eigenvalues.

4.4 Higher orders in the eigenvalue asymptotics

Hainzl and Seiringer carried out the higher order asymptotic expansion of the eigenvalues $e_j(\lambda)$ in [19, Formula (16)] and [20, Theorem 2.7] under the assumption that V has an L^1 tail. Similarly as in Theorems 1.1 and 4.2, the purpose of this section is to show that their findings in fact hold for potentials decaying substantially slower. For the sake of simplicity and concreteness, we again only consider $T = |\Delta + 1|$ here.

Let $BS_{\text{reg}}(e) = BS(e) - BS_{\text{sing}}^{\text{low}}(e)$ and recall that if $1 + \lambda BS_{\text{reg}}(e)$ is invertible, then the Birman–Schwinger principle (2.18) asserts that H_λ has a negative eigenvalue $-e$ if and only if the operator

$$\frac{\lambda}{1 + \lambda BS_{\text{reg}}(e)} BS_{\text{sing}}^{\text{low}}(e) \tag{4.12}$$

has an eigenvalue -1 . The following is a simple but useful observation which follows from a Neumann series argument and the fact that (4.12) is isospectral to

$$\ln(1 + \tau/e) \mathcal{F}_S V^{1/2} \frac{\lambda}{1 + \lambda BS_{\text{reg}}(e)} |V|^{1/2} \mathcal{F}_S^*.$$

Lemma 4.3 *Let $e, \lambda > 0$ and suppose V is real-valued and such that*

$$\lambda \|BS_{\text{reg}}(e)\| < 1. \tag{4.13}$$

Then H_λ has an eigenvalue $-e$ if and only if

$$\lambda \ln(1 + \tau/e) \mathcal{F}_S V^{1/2} \left(\sum_{n \geq 0} (-1)^n (\lambda BS_{\text{reg}}(e))^n \right) |V|^{1/2} \mathcal{F}_S^* \tag{4.14}$$

has an eigenvalue -1 .

Recall that assumption (4.13) is satisfied for $V \in \ell^{\frac{d+1}{2}} L^{\frac{d}{2}}$ (cf. (3.3) and Lemma 3.1), i.e., in particular for $V \in L^{\frac{d+1}{2}-\varepsilon}$ with $\varepsilon \in (0, 1/2]$. In fact, combining Lemma 3.1 for $BS_{\text{reg}}^{\text{low}}(e)$ and the Seiler–Simon inequality (cf. [36, Theorem 4.1]) for $BS^{\text{high}}(e)$ shows⁴

$$\begin{aligned} \|BS_{\text{reg}}(e)\|_{\mathfrak{S}^{\frac{(d-1)((d+1)/2-\varepsilon)}{(d-1)/2+\varepsilon}}} &\leq \|BS_{\text{reg}}^{\text{low}}(e)\|_{\mathfrak{S}^{\frac{(d-1)((d+1)/2-\varepsilon)}{(d-1)/2+\varepsilon}}} + \|BS^{\text{high}}(e)\|_{\mathfrak{S}^{\frac{(d-1)((d+1)/2-\varepsilon)}{(d-1)/2+\varepsilon}}} \\ &= o_V(\ln(1/e)), \quad V \in L^{\frac{d+1}{2}-\varepsilon}, \quad \varepsilon \in (0, 1/2). \end{aligned} \tag{4.15}$$

We will now use (4.14) to compute the eigenvalue asymptotics of $e_j(\lambda)$ to second order. To that end, we define

$$\mathcal{W}_S(e) := \mathcal{F}_S V^{1/2} BS_{\text{reg}}(e) |V|^{1/2} \mathcal{F}_S^* \tag{4.16}$$

which is, modulo the $-\lambda^2 \ln(1 + \tau/e)$ prefactor, just the second summand in (4.14). Note that due to the additional operators $\mathcal{F}_S V^{1/2}$ on the left and $|V|^{1/2} \mathcal{F}_S^*$ on the right of $BS_{\text{reg}}(e)$, estimate (2.9), and $\lambda \|BS_{\text{reg}}(e)\| = o_V(1)$, we infer

$$\|\mathcal{W}_S(e)\|_{\mathfrak{S}^{\frac{(d-1)((d+1)/2-\varepsilon)}{(d-1)/2+\varepsilon}}} = o_V(\ln(1/e)), \quad V \in L^{\frac{d+1}{2}-\varepsilon}, \quad \varepsilon \in (0, 1/2]. \tag{4.17}$$

We will momentarily show the existence of $\mathcal{W}_S(0)$ and the limit $\lim_{e \searrow 0} \mathcal{W}_S(e) = \mathcal{W}_S(0)$ in operator norm for $V \in L^{\frac{d+1}{2}-\varepsilon}$. Let $b_S^j(\lambda) < 0$ denote the negative eigenvalues of

$$\mathcal{B}_S(\lambda) := \mathcal{V}_S - \lambda \mathcal{W}_S(0) \quad \text{on } L^2(S) \tag{4.18}$$

and recall that $\mathcal{V}_S \in \mathfrak{S}^{\frac{(d-1)((d+1)/2-\varepsilon)}{d-(d+1)/2+\varepsilon}}$ if $V \in L^{\frac{d+1}{2}-\varepsilon}$ by (2.9). This and (4.17) show that $\mathcal{B}_S(\lambda)$ is a compact operator as well. Note that, by the definition of $BS_{\text{reg}}(e)$, the operator $\mathcal{B}_S(\lambda)$ has at least one negative eigenvalue if \mathcal{V}_S has a zero-eigenvalue. The asymptotic expansion of $e_j(\lambda)$ to second order then reads as follows.

Theorem 4.4 *Let $d \geq 3$ and $V \in L^{\frac{d+1}{2}-\varepsilon}$ for some $\varepsilon \in (0, 1/2]$. If $\lim_{\lambda \searrow 0} b_S^j(\lambda) < 0$ then H_λ has, for small λ , a corresponding negative eigenvalue $-e_j(\lambda) < 0$ that satisfies*

$$\lim_{\lambda \rightarrow 0} \left(\ln(1 + 1/e_j(\lambda)) + \frac{1}{\lambda b_S^j(\lambda)} \right) = 0. \tag{4.19}$$

⁴ Using Cwikel’s inequality [36, Theorem 4.2], one obtains $\|BS_{\text{reg}}(e)\|_{\mathfrak{S}^{\frac{(d-1)((d+1)/2-\varepsilon)}{(d-1)/2+\varepsilon}, \infty}} = o_V(\ln(1/e))$ for $\varepsilon = 1/2$.

The proof of Theorem 4.4 relies on the fact that $|V|^{1/2}(\mathcal{F}_{S_t^\pm}^* \mathcal{F}_{S_t^\pm} - \sqrt{1 \pm t} \mathcal{F}_S^* \mathcal{F}_S) V^{1/2}$ is Hölder continuous in $\mathcal{B}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$ for $t \leq \tau \in (0, 1)$. We already saw in Sect. 3.4 that this is true for $V \in \mathcal{S}(\mathbb{R}^d)$ (or more generally V satisfying $|V(x)| \lesssim (1 + |x|)^{-1-\varepsilon}$) because of Hölder continuity of the Sobolev trace theorem. The following proposition, whose proof is deferred to Appendix A, yields Hölder continuity of the (non-endpoint) Tomas–Stein theorem.

Proposition 4.5 *Let $0 < \tau < 1$, $1 \leq p < \kappa$, $1/q = 1/p - 1/p'$, i.e., $1 \leq q < (d + 1)/2$, and $0 < \alpha < \min\{(d + 1)/2 - q, q\}$. Then*

$$\sup_{t \in (0, \tau)} \|\mathcal{F}_{S_t^\pm}^* \mathcal{F}_{S_t^\pm} - \sqrt{1 \pm t} \mathcal{F}_S^* \mathcal{F}_S\|_{L^p \rightarrow L^{p'}} \lesssim_{\alpha, q, \tau} t^{\alpha/q}. \tag{4.20}$$

Proof of Theorem 4.4 Recall that $V \in L^{\frac{d+1}{2}-\varepsilon}$ satisfies the assumption of Lemma 4.3. Thus, H_λ has an eigenvalue $-e_j(\lambda) < 0$ if and only if

$$\lambda \ln(1 + \tau/e_j(\lambda)) \left[\mathcal{B}_S(\lambda) + \lambda(\mathcal{W}_S(0) - \mathcal{W}_S(e_j(\lambda))) + \mathcal{F}_S V^{1/2} \left(\sum_{n \geq 2} (-1)^n (\lambda B S_{\text{reg}}(e_j(\lambda)))^n \right) |V|^{1/2} \mathcal{F}_S^* \right] \tag{4.21}$$

has an eigenvalue -1 . Thus, our claim is established, once we show $\lim_{e \rightarrow 0} \mathcal{W}_S(e) = \mathcal{W}_S(0)$ in operator norm topology. In turn, by the definition of $\mathcal{W}_S(e)$, this follows once we show the existence of $\lim_{e \rightarrow 0} B S_{\text{reg}}(e) = B S_{\text{reg}}(0)$ since $|V|^{1/2} \mathcal{F}_S^*$ and $\mathcal{F}_S V^{1/2}$ are bounded by the Tomas–Stein theorem (2.8). We decompose $B S_{\text{reg}}(e) = B S^{\text{high}}(e) + B S_{\text{reg}}^{\text{low}}(e)$ and observe that $B S^{\text{high}}(e) \rightarrow B S^{\text{high}}(0)$ (e.g., by Plancherel and dominated convergence). On the other hand, Proposition 4.5 shows that the difference

$$B S_{\text{reg}}^{\text{low}}(e) - B S_{\text{reg}}^{\text{low}}(0) = \sum_{\pm} \int_0^\tau \left(\sqrt{|V|} (\mathcal{F}_{S_t^\pm}^* \mathcal{F}_{S_t^\pm} - \sqrt{1 \pm t} \mathcal{F}_S^* \mathcal{F}_S) \sqrt{V} \right) \left(\frac{1}{t+e} - \frac{1}{t} \right) \frac{dt}{2\sqrt{1 \pm t}}$$

vanishes in operator norm as $e \rightarrow 0$. This concludes the proof. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Appendix A. Hölder continuity of the Tomas–Stein theorem

In this section we prove Proposition 4.5 on the Hölder continuity of the Tomas–Stein theorem for the sphere. The arguments can be generalized to treat arbitrary smooth, curved, and compact hypersurfaces by refining the analysis in the proof of Proposition 4.1. However, for the sake of simplicity, we restrict ourselves to the unit sphere $S = \mathbb{S}^{d-1}$.

Lemma A.1 *Let $\tau \in (0, 1)$, $0 < \beta \leq (d-1)/2$, $p_\circ = 2(1+\beta)/(2+\beta) \in (1, 2)$, and $1 \leq p < p_\circ$, and denote $1/q := 1/p - 1/p'$ and $\widehat{d\omega}^\pm := \widehat{d\omega}_{S_t^\pm} - \widehat{d\omega}_S$. If there is $\alpha \in (0, \min\{\beta + 1 - q, q\})$ such that*

$$|\widehat{d\omega}^\pm(x)| \leq c_\tau t^\alpha (1 + |x|)^{\alpha-\beta} \quad (\text{A.1})$$

holds for some $c_\tau > 0$ and all $t \in (0, \tau)$, then

$$\sup_{t \in (0, \tau)} \|\mathcal{F}_{S_t^\pm}^* \mathcal{F}_{S_t^\pm} - \sqrt{1 \pm t} \mathcal{F}_S^* \mathcal{F}_S\|_{L^p \rightarrow L^{p'}} \lesssim_{\alpha, q, \tau} t^{\alpha/q}. \quad (\text{A.2})$$

Proof In the following, we consider only S_t^+ and write $S_t \equiv S_t^+$ and $d\omega \equiv d\omega^+$. As in Tomas' proof, we decompose

$$\begin{aligned} (\mathcal{F}_{S_t}^* \mathcal{F}_{S_t} - \sqrt{1+t} \mathcal{F}_S^* \mathcal{F}_S) f &= [(\mathcal{F}_{S_t}^* \mathcal{F}_{S_t} - \mathcal{F}_S^* \mathcal{F}_S) - (\sqrt{1+t} - 1) \mathcal{F}_S^* \mathcal{F}_S] f \\ &= \widehat{d\omega} * f - (\sqrt{1+t} - 1) \mathcal{F}_S^* \mathcal{F}_S f \\ &= \sum_{k=0}^{\infty} (\widehat{d\omega} \psi_k) * f - (\sqrt{1+t} - 1) \mathcal{F}_S^* \mathcal{F}_S f \\ &=: \sum_{k=0}^{\infty} T_k f - (\sqrt{1+t} - 1) \mathcal{F}_S^* \mathcal{F}_S f, \end{aligned} \quad (\text{A.4})$$

where $(\psi_k)_k$ is a standard dyadic partition of unity such that ψ_0 is adapted to the unit ball and ψ_k is adapted to the annulus $2^{k-1} \leq |x| \leq 2^{k+1}$. By the Tomas–Stein theorem, the operator norm of the second term on the right side of (A.4) is bounded by a constant times

$$|\sqrt{1+t} - 1| \|\mathcal{F}_S^* \mathcal{F}_S\|_{L^p \rightarrow L^{p'}} \lesssim t.$$

We now focus on the first term on the right side of (A.4). By the triangle inequality, Plancherel, the rapid decay of $\widehat{\psi}_k$, and the fact that $d\omega_{S_t}$ is a $(d-1)$ -dimensional

measure, we estimate

$$\|T_k\|_{2 \rightarrow 2} \lesssim 2^k, \tag{A.5}$$

whereas we use (A.1) to bound

$$\|T_k\|_{1 \rightarrow \infty} \lesssim t^\alpha 2^{-k(\beta-\alpha)}. \tag{A.6}$$

Interpolating between those two bounds yields

$$\|T_k\|_{p \rightarrow p'} \lesssim 2^{k(1-\theta)} \cdot t^{\alpha\theta} 2^{-k(\beta-\alpha)\theta},$$

where $1 - \theta = 2/p'$, i.e., $\theta = 1/q = 1/p - 1/p'$. Thus,

$$\|T_k\|_{p \rightarrow p'} \lesssim t^{\alpha/q} 2^{k\left(1 - \frac{\beta+1-\alpha}{q}\right)}.$$

Since the exponent of 2^k is negative for $\alpha < \beta + 1 - q$, we obtain

$$\left\| \sum_{k \geq 0} T_k \right\|_{p \rightarrow p'} \lesssim t^{\alpha/q} \sum_{k=0}^{\infty} 2^{k\left(1 - \frac{\beta+1-\alpha}{q}\right)} \lesssim t^{\alpha/q},$$

which concludes the proof of (A.2). □

Proof of Proposition 4.5 For $T(\xi) = |\xi^2 - 1|$ we have

$$S_0 = \mathbb{S}^{d-1} := S, \quad S_t = \sqrt{1-t}S \cup \sqrt{1+t}S, \quad 0 < t \leq \tau < 1.$$

Setting $\rho = \sqrt{1 \pm t} \in [\sqrt{1-\tau}, \sqrt{1+\tau}]$, Lemma A.1 with $\beta = (d - 1)/2$, i.e., $p_\circ = \kappa$ shows that (4.20) would follow from

$$|\widehat{d\omega_{\rho S}}(x) - \widehat{d\omega_S}(x)| \lesssim |\rho - 1|^\alpha (1 + |x|)^{\alpha - \frac{d-1}{2}} \tag{A.6}$$

for some $\alpha \in (0, \min\{(d - 1)/2 + 1 - q, q\})$. We have

$$\widehat{d\omega_{\rho S}}(x) = \int_{\rho S} e^{2\pi i x \cdot \xi} d\omega_{\rho S}(\xi) = \rho^{d-1} \widehat{d\omega_S}(\rho x), \quad x \in \mathbb{R}^d. \tag{A.7}$$

The classic stationary phase argument (see, e.g., Stein [39, Theorem 1]) yields

$$|\widehat{d\omega_S}(x)| \lesssim (1 + |x|)^{-\frac{d-1}{2}},$$

with the same bound for $|\nabla \widehat{d\omega_S}(x)|$. Combining this with (A.7) yields (A.6) for $\alpha = 0$ and $\alpha = 1$ and hence for any $\alpha \in [0, 1]$, thereby concluding the proof. □

Appendix B. Further L^2 based restriction estimates

Throughout this appendix we take $T(\xi) = |\xi^2 - 1|$. Our main results crucially relied on the fact that $\mathcal{F}_{S_t}^* \mathcal{F}_{S_t}$ belongs to $\mathcal{B}(L^p \rightarrow L^{p'})$ or $\mathcal{B}(L^2((1+|x|)^{1+\varepsilon} dx) \rightarrow L^2((1+|x|)^{-1-\varepsilon} dx))$ uniformly for $t \in [0, \tau]$ and some fixed $\tau > 0$. Besides the Tomas–Stein estimate or the trace lemma, there exist various other L^2 -based restriction estimates. In this appendix we present two such estimates and apply them to obtain corresponding upper bounds on the Birman–Schwinger operator, whenever the potential belongs to the suitable dual space. It turns out that this space contains spherically symmetric V with almost L^d decay, but see Propositions B.2 and B.7 below for the precise assumptions. By the uniformity of these restriction theorems on small compact sets around \mathbb{S}^{d-1} (see (B.1) and (B.5)) and following the arguments in Sects. 2 and 3, one obtains the analog of Theorem 1.1 for potentials living in the spaces mentioned below. This is the content of Theorem B.8.

B.1 Estimates for potentials in mixed norm spaces

Vega [42] observed that the Tomas–Stein estimate can be enhanced for \mathbb{S}^{d-1} if one replaces the L^p spaces by suitable mixed norm spaces. For $k > 0$ recall the extension operator

$$(\mathcal{F}_{k\mathbb{S}^{d-1}}^* g)(x) = \int_{k\mathbb{S}^{d-1}} g(\xi) e^{2\pi i x \cdot \xi} d\omega_{k\mathbb{S}^{d-1}}(\xi) = k^{d-1} \int_{\mathbb{S}^{d-1}} g(k\xi) e^{2\pi i kx \cdot \xi} d\omega(\xi)$$

where $d\omega$ and $d\omega_{k\mathbb{S}^{d-1}}$ denote the euclidean surface measures on \mathbb{S}^{d-1} and $k\mathbb{S}^{d-1}$, respectively.

Theorem B.1 (Vega [42, Theorem 2]) *Let $d \in \mathbb{N} \setminus \{1\}$ and*

$$1 \leq p < \frac{2d}{d+1} \quad \text{and} \quad \frac{1}{2} \geq \frac{1}{\sigma'} \geq \max \left\{ \frac{1}{p'}, \frac{1}{p'} \frac{2d}{d-2} - \frac{1}{2} \right\}.$$

Then the restriction estimate

$$\begin{aligned} k^{-\frac{d-1}{2}} \|\hat{f}\|_{L^2(k\mathbb{S}^{d-1})} &= \|\hat{f}(k\cdot)\|_{L^2(\mathbb{S}^{d-1})} \lesssim \left[\int_0^\infty \left(\int_{\mathbb{S}^{d-1}} |k^{-d} f(r\omega/k)|^\sigma d\omega \right)^{\frac{p}{\sigma}} r^{d-1} dr \right]^{\frac{1}{p}} \\ &= k^{-d/p'} \|f\|_{L^p(\mathbb{R}_+, r^{d-1} dr; L^\sigma(\mathbb{S}^{d-1}))}, \end{aligned} \tag{B.1}$$

the extension estimate

$$\|\mathcal{F}_{k\mathbb{S}^{d-1}}^* g\|_{L^{p'}(\mathbb{R}_+, r^{d-1} dr; L^{\sigma'}(\mathbb{S}^{d-1}))} \lesssim k^{\frac{d-1}{2} - \frac{d}{p'}} \|g\|_{L^2(k\mathbb{S}^{d-1})},$$

and the combined estimate

$$\| \mathcal{F}_{k\mathbb{S}^{d-1}}^* \mathcal{F}_{k\mathbb{S}^{d-1}} \psi \|_{L^{p'}(\mathbb{R}_+, r^{d-1} dr; L^{\sigma'}(\mathbb{S}^{d-1}))} \lesssim k^{d-1-\frac{2d}{p'}} \| \psi \|_{L^p(\mathbb{R}_+, r^{d-1} dr; L^\sigma(\mathbb{S}^{d-1}))} \tag{B.2}$$

hold for all $k > 0$ and are equivalent to each other.

Recall that the exponent $2d/(d + 1)$ is sharp, i.e., Theorem B.1 cannot hold for $p \geq 2d/(d + 1)$. Moreover, observe that the estimates are uniform in the radius k as long as $k \in [1 - \delta, 1 + \delta]$ for some $0 < \delta < 1$. Theorem B.1 allows us to prove the following bound on the Birman–Schwinger operator.

Proposition B.2 *Let $d \geq 3, 1 \leq p < 2d/(d+1), 1/2 \geq 1/\sigma' \geq \max\{1/p', 2d/(p'(d-2)) - 1/2\}, e > 0$, and assume $T(\xi) = |\xi^2 - 1|$. Suppose $V \in L^{p/(2-p)}(\mathbb{R}_+, r^{d-1} dr : L^{\sigma/(2-\sigma)}(\mathbb{S}^{d-1}))$ and, if $p \leq 2d/(d + 2)$, suppose additionally $V \in L^{d/2}(\mathbb{R}^d)$. Then*

$$\begin{aligned} \|(T + e)^{-1/2} |V|^{1/2}\|^2 &\lesssim g(e) \left[\int_0^\infty \|V(r \cdot)\|_{L^{\sigma/(2-\sigma)}(\mathbb{S}^{d-1})}^{p/(2-p)} r^{d-1} dr \right]^{(2-p)/p} \\ &\quad + \|V\|_{d/2} \theta(2d/(d + 2) - p) \end{aligned}$$

where

$$g(e) = \int_{1/2}^{3/2} \frac{k^{d(\frac{1}{p} - \frac{1}{p'})-1}}{T(k) + e} dk \lesssim \max\{\ln(1/e), 1\}. \tag{B.3}$$

Here, θ denotes the Heaviside function with the convention $\theta(0) = 1$. Taking, e.g., $\sigma = 2$ and $p \rightarrow 2d/(d + 1)$ shows that spherically symmetric $L^d(\mathbb{R}^d)$ potentials are almost admissible.

Proof Let $f = |V|^{1/2} \varphi$ and $\varphi \in L^2(\mathbb{R}^d)$. By Hölder’s inequality, we have

$$\|f\|_{L^p(\mathbb{R}_+, r^{d-1} dr; L^\sigma(\mathbb{S}^{d-1}))} \leq \left[\int_0^\infty \left(\int_{\mathbb{S}^{d-1}} |V(r\omega)|^{\frac{\sigma}{2-\sigma}} d\omega \right)^{\frac{p(2-\sigma)}{\sigma(2-p)}} r^{d-1} dr \right]^{\frac{2-p}{2p}} \|\varphi\|_2.$$

As in Sect. 3, we consider small and large momenta separately and start with the latter.

So let $\chi \in C_c^\infty(\mathbb{R}^d : [0, 1])$ be a radial bump function centered at $|\xi| = 1$ with $\text{supp } \chi \subseteq \{\xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 3/2\}$. If $p \leq 2d/(d + 2)$, the $L^{\frac{2d}{d+2}} \rightarrow L^2$ boundedness of $(T + e)^{-1/2}(1 - \chi(-i\nabla))$ follows from $(T(\xi) + e)^{-1/2}(1 - \chi) \lesssim (1 + \xi^2)^{-1/2}$ and Sobolev embedding. Else, if $p > 2d/(d + 2)$, the $L^p \rightarrow L^2$ boundedness follows from Sobolev embedding, interpolation, and the fact that

$$\|f\|_p \lesssim \|f\|_{L^p(\mathbb{R}_+, r^{d-1} dr; L^\sigma(\mathbb{S}^{d-1}))}$$

by Hölder’s inequality, since $\sigma \geq p$.

Thus, we are left to estimate $\|\chi(T + e)^{-1/2}|V|^{1/2}\|_{2 \rightarrow 2}$ with $p \in [1, 2d/(d + 1))$. By Plancherel, using spherical coordinates, and Vega’s estimate (B.1), we obtain

$$\begin{aligned} \|\chi(T + e)^{-1/2}|V|^{1/2}\varphi\|_2^2 &= \int_0^\infty dk \frac{\chi(k)^2 k^{d-1}}{T(k) + e} \int_{\mathbb{S}^{d-1}} |\hat{f}(k\omega)|^2 d\omega \\ &\lesssim \|f\|_{L^p(\mathbb{R}_+, r^{d-1} dr; L^\sigma(\mathbb{S}^{d-1}))}^2 \int_{1/2}^{3/2} dk \frac{k^{d-1-2d/p'}}{T(k) + e} \\ &\leq g(e) \|V\|_{L^{\frac{p}{2-p}}(\mathbb{R}_+, r^{d-1} dr; L^{\sigma/(2-\sigma)}(\mathbb{S}^{d-1}))} \|\varphi\|_2^2, \end{aligned}$$

which concludes the proof. □

B.2 Estimates for potentials satisfying the MT condition

We finally discuss potentials V satisfying the “radial Mizohata–Takeuchi” condition.

Definition B.3 Let V be a measurable, non-negative function on \mathbb{R}^d and $H(r) := \sup_{\omega \in \mathbb{S}^{d-1}} V(r\omega)$. Then V is said to satisfy the *radial Mizohata–Takeuchi (MT) condition* if

$$\|V\|_{\text{MT}} := \sup_{\mu \geq 0} \int_\mu^\infty \frac{H(r)r}{(r^2 - \mu^2)^{1/2}} dr < \infty. \tag{B.4}$$

Observe that $\|V(\cdot/k)\|_{\text{MT}} = k\|V\|_{\text{MT}}$ for all $k > 0$. We mention some examples of V satisfying this condition.

Example B.4 1. Frank and Simon [14, (4.2)] showed $\|V\|_{\text{MT}} \lesssim \|V\|_{L^{d,1}(\mathbb{R}_+, r^{d-1} dr; L^\infty(\mathbb{S}^{d-1}))}$, where

$$\|V\|_{L^{d,1}(\mathbb{R}_+, r^{d-1} dr; L^\infty(\mathbb{S}^{d-1}))} := \int_0^\infty |\{r > 0 : \text{ess - sup}_{\omega \in \mathbb{S}^{d-1}} |V(r\omega)| > \alpha\}|_d^{1/d} d\alpha$$

and $|\cdot|_d$ denotes the measure $|\mathbb{S}^{d-1}|r^{d-1} dr$.

2. Barcelo, Ruiz, and Vega [4, Proposition 1] showed that for radial $V(x) = V(|x|) \equiv V(r)$, one has $\|V\|_{\text{MT}} \lesssim \|V\|_{D_p}$ for $p > 2$, where

$$\|V\|_{D_p} := \sum_{j=-\infty}^\infty \left(\int_{2^j}^{2^{j+1}} |V(r)|^p r^{p-1} dr \right)^{1/p}.$$

In particular, the functions $r^{-a}\mathbf{1}_{(0,1)}(r) + r^{-b}\mathbf{1}_{[1,\infty)}(r)$ and $r^{-1}(1 + |\log r|)^{-b}$ for $a < 1$ and $b > 1$ have finite $\|\cdot\|_{D_p}$ norm.

Barcelo, Ruiz, and Vega [4] proved the following weighted analog of the classical trace lemma.

Theorem B.5 ([4, Theorem 3]) *Let $d \in \mathbb{N} \setminus \{1\}$ and V be a radial, non-negative function satisfying $\|V\|_{\text{MT}} < \infty$. Then the weighted restriction theorem*

$$k^{-\frac{d-1}{2}} \|\hat{f}\|_{L^2(k\mathbb{S}^{d-1})} = \|\hat{f}(k\cdot)\|_{L^2(\mathbb{S}^{d-1})} \lesssim k^{-\frac{d-1}{2}} \|V\|_{\text{MT}}^{1/2} \|f\|_{L^2(\mathbb{R}^d; V^{-1}(x) dx)}, \tag{B.5}$$

the weighted extension estimate

$$\|\mathcal{F}_{k\mathbb{S}^{d-1}}^* g\|_{L^2(V)} \lesssim \|V\|_{\text{MT}}^{1/2} \|g\|_{L^2(k\mathbb{S}^{d-1})},$$

and the combined estimate

$$\|\mathcal{F}_{k\mathbb{S}^{d-1}}^* \mathcal{F}_{k\mathbb{S}^{d-1}} \psi\|_{L^2(V)} \lesssim \|V\|_{\text{MT}} \|\psi\|_{L^2(V^{-1})} \tag{B.6}$$

hold for all $k > 0$ and are equivalent to each other. Conversely, if one of the above estimates holds, and V is radial and non-negative, then $\|V\|_{\text{MT}} < \infty$.

Remark B.6 Using a result of Agmon and Hörmander [1, Theorem 3.1], Barcelo, Ruiz, and Vega [4, p. 360-361] observed that $V \in C_0(\mathbb{R}^d)$ (vanishing at infinity, but not necessarily radial) has a uniformly bounded X-ray transform, i.e.,

$$\sup \left\{ \int_{\mathbb{R}} V(y + t\omega) dt : \omega \in \mathbb{S}^{d-1}, y \in \mathbb{R}^d \right\} < \infty, \tag{B.7}$$

whenever the restriction estimate (B.5) holds for such V . By adapting their arguments to radial potentials V , they further remark that in this case (B.7) reduces to the MT condition (B.4). We shall, however, not make use of this remarkable fact in the following.

Theorem B.7 enables us to prove the following Birman–Schwinger bound.

Proposition B.7 *Assume $d \in \mathbb{N} \setminus \{1\}$, $e > 0$, $T(\xi) = |\xi^2 - 1|$, and V is a radial, non-negative function that satisfies the MT condition (B.4). Then*

$$\|(T + e)^{-1/2} V^{1/2}\|^2 \lesssim (1 + g_{\text{MT}}(e)) \|V\|_{\text{MT}},$$

where

$$g_{\text{MT}}(e) = \int_{1/2}^{3/2} \frac{1}{T(k) + e} dk \lesssim \max\{\ln(1/e), 1\}. \tag{B.8}$$

Proof Let $f = V^{1/2}\varphi$ and $\varphi \in L^2(\mathbb{R}^d)$, i.e., $\|f\|_{L^2(V^{-1})} = \|\varphi\|_2$ and χ be the same radial bump function in Fourier space as in the proof of Proposition B.2. First observe that large momenta are controlled by

$$\|(1 - \chi)^2 (T + e)^{-s/2} f\|_{L^2(V)} \lesssim \|(1 - \Delta)^{-s/2} f\|_{L^2(V)} \leq \|V\|_{\text{MT}} \|f\|_{L^2(V^{-1})}, \quad s \geq 1$$

where the second estimate is the content of [4, Lemma 4]. Taking $s = 2$ and plugging in $f = V^{1/2}\varphi$ shows

$$\|(1 - \chi)(T + e)^{-1/2} V^{1/2}\|_{L^2 \rightarrow L^2}^2 = \|V^{1/2}(1 - \chi)^2(T + e)^{-1} V^{1/2}\|_{L^2 \rightarrow L^2} \lesssim \|V\|_{\text{MT}}$$

as desired.

For momenta close to \mathbb{S}^{d-1} , we use the restriction estimate (B.5) to obtain

$$\begin{aligned} \|\chi(T + e)^{-1/2} f\|_2^2 &= \int_0^\infty dk \frac{\chi(k)^2 k^{d-1}}{T(k) + e} \int_{\mathbb{S}^{d-1}} |\hat{f}(k\omega)|^2 d\omega \\ &\lesssim \|V\|_{\text{MT}} \|f\|_{L^2(V^{-1})}^2 \int_{1/2}^{3/2} dk \frac{1}{T(k) + e} \\ &= g_{\text{MT}}(e) \|V\|_{\text{MT}} \|\varphi\|_2^2. \end{aligned}$$

This concludes the proof. □

B.3 Weak coupling asymptotics for potentials in mixed norm spaces or satisfying the MT condition

We are now in position to combine the above results in the following theorem.

Theorem B.8 *Let $T(\xi) = |\xi^2 - 1|$, and suppose d and V satisfy the assumptions in Proposition B.2 or B.7. Then for every eigenvalue $a_S^j > 0$ of \mathcal{V}_S in (2.5), counting multiplicity, and every $\lambda > 0$, there is an eigenvalue $-e_j(\lambda)$ of $T(-i\nabla) - \lambda V$ with weak coupling limit*

$$e_j(\lambda) = \exp\left(-\frac{1}{\lambda a_S^j} (1 + o(1))\right) \text{ as } \lambda \rightarrow 0.$$

The notation $X \lesssim_V 1$ in the proof below conceals the more precise estimate $X \lesssim \min\{\|V\|_{\text{MT}}, \|V\|_{L^{p/(2-p)}(\mathbb{R}_+, r^{d-1} dr; L^{\sigma/(2-\sigma)}(\mathbb{S}^{d-1}))} + \|V\|_{d/2\theta(2d/(d+2)-p)}\}$ for p and σ as in Proposition B.2.

Proof We follow the proof of Theorem 1.1. First, the corresponding analog of Proposition 2.2, i.e., L^2 boundedness of \mathcal{V}_S , follows immediately from the restriction theorems B.1 and B.5 above. Next, the splitting of $BS(e)$ is the same as in Sect. 3. For concreteness, suppose that the frequency cutoff is at some $\tau < 1/2$. As we have seen in the proofs of Propositions B.2 and B.7, the high frequencies are harmless, i.e., in both cases we have $\|BS^{\text{high}}(e)\| \lesssim_{\tau, V} 1$, whereas the low frequencies are responsible for $\|BS^{\text{low}}(e)\| \lesssim_{\tau, V} \ln(1/e)$ for $e \in (0, 1)$. Thus, we are left to prove the analog of the key bound (3.2). Recall that the Fermi surface of T at energy $t \in (0, \tau]$ consists of the two connected components $S_t^\pm = \sqrt{t \pm 1} \mathbb{S}^{d-1}$. As explained below (3.8) and in the proof of Lemma 3.1, this merely relies on the validity of $\| |V|^{1/2} \mathcal{F}_{S_t^*}^* \mathcal{F}_{S_t} V^{1/2} \| \lesssim_{\tau, V} 1$ for all $t \in [0, \tau]$. But this is just reflected in (B.2) and (B.6) for $1/2 < k < 3/2$. This concludes the proof. □

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