Hölder estimates for magnetic Schrödinger semigroups in $\mathbb{R}^d$ from mirror coupling

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Abstract

We use the mirror coupling of Brownian motion to show that under a $\beta \in (0, 1)$ -dependent Kato-type assumption¹ on the possibly nonsmooth electromagnetic potential, the corresponding magnetic Schrödinger semigroup in $\mathbb{R}^d$ has a global L^p -to- $C^{0,\beta}$ Hölder smoothing property for all $p \in [1, \infty]$; in particular, his all eigenfunctions are uniformly β -Hölder continuous. This result shows that the eigenfunctions of the Hamilton operator of a molecule in a magnetic field are uniformly β -Hölder continuous under weak L^q -assumptions on the magnetic potential.

Keywords Brownian Motion · Magnetic Schrödinger semigroups · Hölder smoothing · Kato potentials

Mathematics Subject Classification 47D08 · 35B65 · 35J10 · 60J70

1 Introduction

Kato [7] has shown that each eigenfunction Ψ of a multi-particle Schrödinger operator $H = -\Delta + W$ in $L^2(\mathbb{R}^{3m})$ with a potential $W : \mathbb{R}^{3m} \rightarrow \mathbb{R}$ of the form

$$W(x) = \sum_{1 \leq j \leq m} w_j(\mathbf{x}_j) + \sum_{1 \leq j < k \leq m} w_{jk}(\mathbf{x}_j - \mathbf{x}_k),$$

with $w_j, w_{jk} \in L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for some $p \geq 2$

¹ Which is satisfied under a suitable L^q -assumption on the electromagnetic potential, where q depends on β and the dimension d .

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is uniformly β -Hölder continuous for all $0 < \beta < 2 - 3/p$, that is,

$$\sup_{x, y \in \mathbb{R}^{3m}, x \neq y} \frac{|\Psi(x) - \Psi(y)|}{|x - y|^\beta} < \infty,$$

where we have written points in $\mathbb{R}^{3m}$ in the form $x = (\mathbf{x}_1, \dots, \mathbf{x}_m)$, where $\mathbf{x}_j \in \mathbb{R}^3$ for $j = 1, \dots, m$. In particular, an application of this result to multi-particle Coulomb-type potentials shows that all molecular Hamilton operators (in the infinite mass limit) are uniformly α -Hölder continuous for all $0 < \alpha < 1$. Kato's proof relies on the Fourier transform and so does not apply directly to magnetic Schrödinger operators (even if one assumes a Coulomb gauge). The aim of this paper is to use probabilistic techniques to find a variant of Kato's regularity result that applies to a magnetic Schrödinger operator $H(A, V)$ with magnetic potential $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and electric potential $V : \mathbb{R}^d \rightarrow \mathbb{R}$. To this end, we prove the following smoothing result (cf. Theorem 2.5):

Let $\beta \in (0, 1)$ and let $C^{0,\beta}(\mathbb{R}^d)$ denote the space of uniformly β -Hölder continuous functions on $\mathbb{R}^d$, with its seminorm given by

$$\|f\|_{C^{0,\beta}} := \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\beta}, \quad (1)$$

and consider for $q \in [1, \infty]$ the Banach space $C^{0,\beta}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ with its norm

$$\|f\|_{C^{0,\beta} \cap L^q} := \|f\|_{C^{0,\beta}} + \|f\|_{L^q}.$$

Then for all Borel functions $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $V : \mathbb{R}^d \rightarrow \mathbb{R}$ with

$$\max(|A|^{\frac{2}{1-\beta}}, |\operatorname{div}(A)|^{\frac{1}{1-\beta}}) \in \mathcal{K}(\mathbb{R}^d), \quad V \in \mathcal{K}^\beta(\mathbb{R}^d),$$

and all $t > 0$, $1 \leq p \leq q \leq \infty$ one has

$$e^{-tH(A,V)} : L^p(\mathbb{R}^d) \longrightarrow C^{0,\beta}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d), \quad (2)$$

and the norm of this operator can be estimated explicitly.

Above, $\mathcal{K}^\beta(\mathbb{R}^d)$, $\beta \in [0, 1]$, denotes the β -Kato class (cf. Definition 2.1) of Borel functions $\mathbb{R}^d \rightarrow \mathbb{R}$ which has been introduced in [5], so that $\mathcal{K}(\mathbb{R}^d) := \mathcal{K}^0(\mathbb{R}^d)$ is the classical Kato class [1] and one has $\mathcal{K}^\beta(\mathbb{R}^d) \subset \mathcal{K}^\alpha(\mathbb{R}^d)$ if $\beta \geq \alpha$. Note also that $H(A, V)\Psi = \theta\Psi$ implies $e^{-tH(A,V)}\Psi = e^{-t\theta}\Psi$, so that one also obtains global β -Hölder regularity for eigenfunctions.

The mapping property

$$e^{-tH(A,V)} : L^p(\mathbb{R}^d) \longrightarrow L^q(\mathbb{R}^d) \cap C(\mathbb{R}^d)$$

is well known [2] and only requires a local Kato assumption on $|A|^2$, $|\operatorname{div}(A)|$ and the positive part of V , and a global Kato assumption on the negative part of V .

The proof of (2) uses Brownian mirror coupling techniques (cf. Sect. 2 for the basic definitions) to deal with the magnetic potential A . Let us mention here that the use of Brownian coupling techniques in the context of Hölder estimates for semigroups that generate diffusion (which in our case would correspond to taking $V = 0$, $A = 0$) has a long history, also on Riemannian manifolds (cf. [3, 9, 10] for some classical results).

Our main tool (cf. Theorem 2.3) for the proof of (2) is provided by the following estimate:

There exists a universal constant $c_0 < \infty$, such that for every $q \in (1, \infty)$, every Borel function $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with

$$\max(|A|^{2q}, |\operatorname{div}(A)|^q) \in \mathcal{K}(\mathbb{R}^d), \quad (3)$$

every $t > 0$, $x \neq y$ in $\mathbb{R}^d$, and every mirror coupling (X, Y) of Brownian motions from (x, y) one has

$$\mathbb{E} \left(\left| e^{-\mathcal{S}_t(A|X)} - e^{-\mathcal{S}_t(A|Y)} \right| \right) \leq c_0 C(A, t, q) t^{-\frac{1}{2q^*}} |x - y|^{\frac{1}{q^*}},$$

where for any Brownian motion Z , the process $\mathcal{S}_t(A|Z)$ denotes the magnetic Euclidean action functional (cf. (7)) which appears in the Feynman–Kac–Itô formula and where the constant $C(A, t, q) < \infty$ can be computed explicitly.

This estimate is then combined with the Feynman–Kac–Itô formula (and perturbation theory to deal with V) to finally obtain (2).

Let us mention that locally uniform β -Hölder continuity results for nonmagnetic Schrödinger eigenfunctions under L^q -assumptions on V have also been obtained in [8] (cf. Theorem 11.7 therein) using straightforward Sobolev embedding techniques. In addition, in [11] (cf. Theorem B.3.5 therein) it is shown that β -dependent (Kato-type) assumptions in V leading to locally uniform β -Hölder smoothing results for nonmagnetic Schrödinger semigroups. In the latter case, the ultimate argument relies on potential theory, while Brownian motion only enters through the Feynman–Kac formula in order to show that the Schrödinger semigroup is L^∞ -smoothing.

Using L^p -criteria for the β -Kato class (cf. Remark 2.2), we show that our result directly implies the following generalization of Kato's result for multi-particle Schrödinger operators in $\mathbb{R}^{3n}$ to magnetic multi-particle Schrödinger operators:

Assume there exists $\beta \in (0, 1)$, $l \in \mathbb{N}$ and Borel functions $a : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $v_i, v_{ij} : \mathbb{R}^3 \rightarrow \mathbb{R}$ with

$$|a|^{2/(1-\beta)}, |\operatorname{div}(a)|^{1/(1-\beta)} \in L^s(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \quad \text{for some } s > 3/2, \quad (4)$$

$$v_i, v_{ij} \in L^s(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \quad \text{for some } s > \frac{3}{2(1-\beta/2)}, \quad (5)$$

where div is defined in the distributional sense, and define a vector potential, resp. a magnetic potential on $\mathbb{R}^{3n}$ through

$$A(x) := \sum_{i=1}^n a(\mathbf{x}_i), \quad V(x) = \sum_{1 \leq i < j \leq n} v_{ij}(\mathbf{x}_i - \mathbf{x}_j) + \sum_{i=1}^n v_i(\mathbf{x}_i).$$

Then for all $t > 0$ and $p \in [1, \infty]$ one has

$$e^{-tH(A,V)} : L^p(\mathbb{R}^d) \longrightarrow C^{0,\beta}(\mathbb{R}^d).$$

To the best of our knowledge, this is the first global Hölder-regularity result for multi-particle magnetic Schrödinger operators.

Let us finally explain how this result applies to molecules in a magnetic field: Given $R \in \mathbb{R}^{3n}$, $l \in \mathbb{N}$, $Z \in [0, \infty)^l$, consider the potential

$$V_{R,Z} : \mathbb{R}^{3n} \longrightarrow \mathbb{R}, \quad V_{R,Z}(\mathbf{x}_1, \dots, \mathbf{x}_n) := - \sum_{i=1}^n \sum_{j=1}^l \frac{Z_j}{|\mathbf{x}_i - \mathbf{R}_j|} + \sum_{1 \leq i < j \leq n} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}.$$

Given $a : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ (sufficiently well behaved), set as above $A(x) := \sum_{i=1}^n a(\mathbf{x}_i)$. Then the operator

$$H_{A,R,Z} := H(A, V_{R,Z})$$

is the Hamilton operator corresponding to a molecule (in the infinite mass limit) with l protons and with n electrons, where the j th nucleus is located in $\mathbf{R}_j$, and has Z_j protons, and the electrons interact with the magnetic field induced by A . Then given an arbitrary $\beta \in (0, 1)$, one has (5) for

$$v_{ij}(\mathbf{x}) := 1/|\mathbf{x}|, \quad v_i(\mathbf{x}) := - \sum_{j=1}^l \frac{Z_j}{|\mathbf{x} - \mathbf{R}_j|},$$

so that the previous result gives that for all $t > 0$ and $p \in [1, \infty]$ one has

$$e^{-tH(A,V)} : L^p(\mathbb{R}^{3n}) \longrightarrow C^{0,\beta}(\mathbb{R}^{3n}),$$

as long as

$$|a|^{2/(1-\beta)}, |\operatorname{div}(a)|^{1/(1-\beta)} \in L^s(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \quad \text{for some } s > 3/2.$$

2 Main results

We start by recalling the definition of the mirror coupling of Brownian motions as presented in [6] and follow their exposition (pages 1-3 therein) closely before presenting our main results.

A continuous process (X, Y) with values in $\mathbb{R}^d \times \mathbb{R}^d$ is called a *coupling of Brownian motions* from $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, if X and Y are Brownian motions starting in x and y , respectively. Then, with the coupling time

$$\tau(X, Y) := \inf\{t > 0 : X_s = Y_s \text{ for all } s > t\},$$

the coupling (X, Y) is said to be *maximal*, if for all $t > 0$ one has

$$\mathbb{P}(\tau(X, Y) \geq t) = \frac{1}{2} \int_{\mathbb{R}^d} |\rho(t, x, z) - \rho(t, y, z)| dz, \quad (6)$$

with

$$(t, b) \mapsto \rho(t, a, b) = (2\pi t)^{-d/2} e^{-\frac{|a-b|^2}{2t}}$$

the transition density of Brownian motion starting in a . The reason for this notion of maximality is that for an arbitrary coupling of Brownian motions one has $\leq$ in (6). Let x and y be two distinct points of $\mathbb{R}^d$. Then

$$N_{x,y} := \left\{ v \in \mathbb{R}^d : \langle v - (x + y)/2, x - y \rangle = 0 \right\}$$

is the hyperplane orthogonal on and bisecting the segment $\underline{xy}$. Furthermore, define the affine map

$$R_{x,y} : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad R_{x,y}v := v - 2 \langle v - (x + y)/2, (x - y) | x - y |^{-1} \rangle (x - y) | x - y |^{-1}.$$

This is the reflection at the hyperplane $N_{x,y}$. Let $L_{x,y}$ be the linear part of $R_{x,y}$. Note that $L_{x,y}$ is symmetric and idempotent.

A coupling (X, Y) of Brownian motions from (x, y) is called a *mirror coupling*, if

$$Y_t = \begin{cases} R_{x,y}X_t, & t \in [0, \tau_{x,y}(X)], \\ X_t & t \in (\tau_{x,y}(X), \infty), \end{cases}$$

where

$$\tau_{x,y}(X) := \inf \{ t \geq 0 : X_t \in N_{x,y} \}$$

is the hitting time of X with respect to $N_{x,y}$. In other words, Y is equal to the reflection of X at $N_{x,y}$ before X hits $N_{x,y}$, and is then equal to X . It follows that $\tau(X, Y) = \tau_{x,y}(X)$, which by an explicit calculation of $\mathbb{P}(t \leq \tau_{x,y}(X))$ implies that every mirror coupling is maximal.

Whenever well defined, we consider the following action functional on the paths of any Brownian motion Z , which depends on a sufficiently regular function $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$:

$$\mathcal{I}_t(A|Z) := i \int_0^t \langle A(Z_s), dZ_s \rangle + \frac{i}{2} \int_0^t \operatorname{div}(A)(Z_s) ds, \quad t \geq 0. \quad (7)$$

Above, i is the imaginary unit, $\operatorname{div}(A)$ denotes the divergence of A understood in the distributional sense and the stochastic integral is understood in Itô's sense.

Let $\mathbb{P}_a$ denote the law of Brownian motion starting in a , which is considered as a probability measure on the space of continuous paths $\omega : [0, \infty) \rightarrow \mathbb{R}^d$. Generalizing the Kato class, the following hierarchy of Kato classes has been introduced in [5]:

Definition 2.1 Given $\alpha \in [0, 1]$, a Borel function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be in the α -Kato class $\mathcal{K}^\alpha(\mathbb{R}^d)$, if

$$\lim_{t \rightarrow 0^+} \sup_{z \in \mathbb{R}^d} \int_0^t s^{-\alpha/2} \int |f(\omega(s))| \mathbb{P}_z(d\omega) ds = 0.$$

Note that

$$\begin{aligned} \int |f(\omega(s))| \mathbb{P}_z(d\omega) &= \int |f(\omega(s))| \mathbb{P}_z(d\omega) = \int_{\mathbb{R}^d} \rho(t, z, y) |f(y)| dy \\ &= \int_{\mathbb{R}^d} \rho(t, z, y) |f(y)| dy = (2\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{|z-y|^2}{2t}} |f(y)| dy. \end{aligned} \quad (8)$$

Remark 2.2 1) Each $\mathcal{K}^\alpha(\mathbb{R}^d)$ is a linear space and $\mathcal{K}(\mathbb{R}^d) := \mathcal{K}^0(\mathbb{R}^d)$ is the usual Kato class.

2) One trivially has

$$\mathcal{K}^\alpha(\mathbb{R}^d) \subset \mathcal{K}^\beta(\mathbb{R}^d), \quad \text{if } \alpha \geq \beta,$$

and using

$$\int_{\mathbb{R}^d} \rho(t, z, y) dy = 1,$$

one gets

$$L^\infty(\mathbb{R}^d) \subset \mathcal{K}^\alpha(\mathbb{R}^d).$$

3) For all $q \in [1, \infty)$ with $q > d/(2 - \alpha)$, one has

$$L^q(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \subset \mathcal{K}^\alpha(\mathbb{R}^d),$$

which follows straightforwardly (cf. Lemma 3.9 in [5]) from

$$\rho(t, z, y) \leq C t^{-d/2}.$$

4) For every natural $D \geq d$, every linear surjective map $\pi : \mathbb{R}^D \rightarrow \mathbb{R}^d$, and every $f \in \mathcal{K}^\alpha(\mathbb{R}^d)$, one has $f \circ \pi \in \mathcal{K}^\alpha(\mathbb{R}^D)$, cf. [5].

5) For every $W \in \mathcal{K}(\mathbb{R}^d)$, $z \in \mathbb{R}^d$, $t > 0$, one has (cf. Lemma 3.9 in [5])

$$\int_0^t |W(\omega(s))| ds < \infty \quad \mathbb{P}_z \text{ a.s. for all } t > 0, \quad (10)$$

and if $W \in \mathcal{K}^\beta(\mathbb{R}^d)$, then also (cf. the proof of Theorem 3.10 in [5])

$$\sup_{z \in \mathbb{R}^d} \int_0^t s^{-\beta/2} \int |W(\omega(s))| \mathbb{P}_z(d\omega) ds < \infty \quad \text{for all } t > 0.$$

The following probabilistic estimate is our main technical result:

Theorem 2.3 *There exists a universal constant $c_0 < \infty$, such that for every $q \in (1, \infty)$, every Borel function $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with*

$$\max(|A|^{2q}, |\operatorname{div}(A)|^q) \in \mathcal{K}(\mathbb{R}^d), \quad (11)$$

every $t > 0$, $x \neq y$ in $\mathbb{R}^d$, and every mirror coupling (X, Y) of Brownian motions from (x, y) one has

$$\mathbb{E} \left(\left| e^{-\mathcal{S}_t(A|X)} - e^{-\mathcal{S}_t(A|Y)} \right| \right) \leq c_0 C(A, t, q) t^{-\frac{1}{2q^*}} |x - y|^{\frac{1}{q^*}}, \quad (12)$$

where $1/q^* + 1/q = 1$ and

$$\begin{aligned} C(A, t, q) := & \left(\sup_{z \in \mathbb{R}^d} \int_0^t \int |A(\omega(s))|^{2q} \mathbb{P}_z(d\omega) ds \right)^{\frac{1}{q}} \\ & + \left(\sup_{z \in \mathbb{R}^d} \int_0^t \int \left| \frac{i}{2} \operatorname{div}(A)(\omega(s)) \right|^q \mathbb{P}_z(d\omega) ds \right)^{\frac{1}{q}} < \infty. \end{aligned}$$

Remark 2.4 1) As every Kato function is locally integrable (cf. Lemma VI.5 c) in [4]), $\operatorname{div}(A)$ exists as a distribution in the above situation.

2) Remark 2.2.5 easily shows that $C(A, t, q) < \infty$ under the assumptions of Theorem 2.3 and that $\int_0^\cdot \langle A(Z_s), dZ_s \rangle$ is a continuous L^2 -martingale for every Brownian motion Z having a deterministic initial value. In particular, the process $\mathcal{S}(A|Z)$ is a continuous semimartingale.

3) The function $t \mapsto C(A, t, q)$ is locally bounded under the assumptions of Theorem 2.3: The easiest way to see this is to refer to Khashminski's lemma, which implies that for every $W \in \mathcal{K}(\mathbb{R}^d)$ one has

$$\sup_{z \in \mathbb{R}^d} \int e^{\int_0^t W(\omega(s)) ds} \mathbb{P}_z(d\omega) \leq C_W e^{C_W t} \quad \text{for all } t > 0,$$

and so trivially

$$\sup_{z \in \mathbb{R}^d} \int \int_0^t W(\omega(s)) ds \mathbb{P}_z(d\omega) \leq C_W e^{C_W t} \quad \text{for all } t > 0.$$

Proof of Theorem 2.3 Let $x \neq y$ in $\mathbb{R}^d$ and $t > 0$ be fixed. We set

$$\tau := \tau(X, Y) = \tau_{x,y}(X), \quad L := L_{x,y}, \quad R := R_{x,y}.$$

Given a Brownian motion Z , we split

$$\mathcal{S}_t(Z) := \mathcal{S}_t(A|Z)$$

into

$$\begin{aligned}\mathcal{S}_t(Z) &= \mathcal{S}_t^{\text{It}\hat{o}}(Z) + \mathcal{S}_t^{\text{Leb}}(Z), \\ \mathcal{S}_t^{\text{It}\hat{o}}(Z) &:= \int_0^t \langle A(Z_s), dZ_s \rangle, \\ \mathcal{S}_t^{\text{Leb}}(Z) &:= \frac{i}{2} \int_0^t \text{div}(A)(Z_s) ds.\end{aligned}$$

Clearly we a.s. have

$$\begin{aligned}I_t &:= \int_0^t \mathbb{1}_{\{s < \tau\}} \left(\frac{i}{2} \text{div}(A)(X_s) - \frac{i}{2} \text{div}(A)(RX_s) \right) ds \\ &= \mathcal{S}_t^{\text{Leb}}(X) - \mathcal{S}_t^{\text{Leb}}(Y).\end{aligned}$$

Likewise, heuristically, for $s < \tau$ one has $dY_s = LdX_s$, while for $s \geq \tau$ one has $dY_s = dX_s$, and we therefore expect that

$$\mathcal{S}_t^{\text{It}\hat{o}}(X) - \mathcal{S}_t^{\text{It}\hat{o}}(Y) = M_t \tag{13}$$

holds a.s., where

$$\begin{aligned}\tilde{A}(x) &:= A(x) - LA(Rx), \\ M_t &:= \int_0^t \mathbb{1}_{\{s < \tau\}} \langle \tilde{A}(X_s), dX_s \rangle.\end{aligned}$$

To show that Eq. (13) holds, by replacing

$$A = (A_1, \dots, A_d)$$

with the sequence

$$A_n := (\max(A_1, n), \dots, \max(A_d, n)), \quad n \in \mathbb{N},$$

and using the Itô isometry and dominated convergence, we can assume that A is bounded. By Theorem 6.5 in [12] we have the L^2 -convergence of the dyadic approximations

$$\mathcal{S}_t^{\text{It}\hat{o}}(X) - \mathcal{S}_t^{\text{It}\hat{o}}(Y) = \lim_{n \rightarrow \infty} \sum_{i=1}^{2^n-1} \frac{2^n}{t} \int_{t_{i-1}}^{t_i} (\langle A(X_u), X_{t_{i+1}} - X_{t_i} \rangle - \langle A(Y_u), Y_{t_{i+1}} - Y_{t_i} \rangle) du,$$

$$M_t = \lim_{n \rightarrow \infty} \sum_{i=1}^{2^n-1} \frac{2^n}{t} \int_{t_{i-1}}^{t_i} \langle \mathbb{1}_{\{u < \tau\}} \tilde{A}(X_u), X_{t_{i+1}} - X_{t_i} \rangle du,$$

where $t_i := \frac{it}{2^n}$ for $i = 0, \dots, 2^n$. We immediately note that in case $t < \tau$, we have $Y_s = RX_s$ on $[0, t]$; hence, in that case by the above limits we conclude that:

$$\mathcal{S}_t^{\text{It}\hat{0}}(X) - \mathcal{S}_t^{\text{It}\hat{0}}(Y) = M_t, \quad \text{for } t < \tau.$$

If we now assume that $\tau \in (t_k, t_{k+1}]$ for some $k = 0, \dots, 2^n - 1$, we get the following expressions for the summands in the above limits:

For $i \leq k - 1$:

$$\begin{aligned} & \int_{t_{i-1}}^{t_i} (\langle A(X_u), X_{t_{i+1}} - X_{t_i} \rangle - \langle A(Y_u), Y_{t_{i+1}} - Y_{t_i} \rangle) du \\ &= \int_{t_{i-1}}^{t_i} (\langle A(X_u), X_{t_{i+1}} - X_{t_i} \rangle - \langle A(RX_u), RX_{t_{i+1}} - RX_{t_i} \rangle) du \\ &= \int_{t_{i-1}}^{t_i} \langle \tilde{A}(X_u), X_{t_{i+1}} - X_{t_i} \rangle du. \end{aligned}$$

In the last step, we have used that L is self-adjoint and $Rv - Rw = L(v - w)$.

For $i = k$:

$$\begin{aligned} & \int_{t_{k-1}}^{t_k} (\langle A(X_u), X_{t_{k+1}} - X_{t_k} \rangle - \langle A(Y_u), Y_{t_{k+1}} - Y_{t_k} \rangle) du \\ &= \int_{t_{k-1}}^{t_k} (\langle A(X_u), X_{t_{k+1}} - X_{t_k} \rangle - \langle A(RX_u), X_{t_{k+1}} - RX_{t_k} \rangle) du. \end{aligned}$$

For $i = k + 1$:

$$\begin{aligned} & \int_{t_k}^{t_{k+1}} (\langle A(X_u), X_{t_{k+2}} - X_{t_{k+1}} \rangle - \langle A(Y_u), Y_{t_{k+2}} - Y_{t_{k+1}} \rangle) du \\ &= \int_{t_k}^{\tau} \langle A(X_u) - A(RX_u), X_{t_{k+2}} - X_{t_{k+1}} \rangle du, \\ & \int_{t_k}^{t_{k+1}} \langle \mathbb{1}_{\{u < \tau\}} \tilde{A}(X_u), X_{t_{k+2}} - X_{t_{k+1}} \rangle du \\ &= \int_{t_k}^{\tau} \langle A(X_u) - LA(RX_u), X_{t_{k+2}} - X_{t_{k+1}} \rangle du. \end{aligned}$$

For $i \geq k + 2$, the summands vanish. Compiling these equations allows us to make the following estimates,

$$\mathbb{E} \left(\left| \mathcal{S}_t^{\text{It}\hat{0}}(X) - \mathcal{S}_t^{\text{It}\hat{0}}(Y) - M_t \right|^2 \right) = \mathbb{E} \left(\mathbb{1}_{\{t \geq \tau\}} \left| \mathcal{S}_t^{\text{It}\hat{0}}(X) - \mathcal{S}_t^{\text{It}\hat{0}}(Y) - M_t \right|^2 \right)$$

$$\begin{aligned}
&\leq \limsup_{n \rightarrow \infty} \mathbb{E} \left(\mathbb{1}_{\{\tau \geq \tau\}} \left| \sum_{i=1}^{2^n-1} \frac{2^n}{t} \int_{t_{i-1}}^{t_i} \left(\langle A(X_u) 2 - \mathbb{1}_{\{s < \tau\}} \tilde{A}(X_u), X_{t_{i+1}} - X_{t_i} \rangle \right. \right. \right. \\
&\quad \left. \left. \left. - \langle A(Y_u), Y_{t_{i+1}} - Y_{t_i} \rangle \right) du \right|^2 \right) \\
&= \limsup_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \mathbb{E} \left(\mathbb{1}_{\{\tau \in (t_k, t_{k+1}]\}} \left| \sum_{i=1}^{2^n-1} \frac{2^n}{t} \int_{t_{i-1}}^{t_i} \left(\langle A(X_u) - \mathbb{1}_{\{s < \tau\}} \tilde{A}(X_u), X_{t_{i+1}} - X_{t_i} \rangle \right. \right. \right. \\
&\quad \left. \left. \left. - \langle A(Y_u), Y_{t_{i+1}} - Y_{t_i} \rangle \right) du \right|^2 \right) \\
&= \limsup_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \mathbb{E} \left(\mathbb{1}_{\{\tau \in (t_k, t_{k+1}]\}} \left| \frac{2^n}{t} \left(\int_{t_{k-1}}^{t_k} \langle A(RX_u), (R - \mathbb{1})X_{t_{k+1}} \rangle du \right. \right. \right. \\
&\quad \left. \left. \left. + \int_{t_k}^{\tau} \langle \gamma(X_u), X_{t_{k+2}} - X_{t_{k+1}} \rangle du \right) \right|^2 \right),
\end{aligned}$$

where $\gamma(z) := LA(Rz) - A(Rz)$. Note that

$$(R - \mathbb{1})X_{t_{k+1}} = L(X_{t_{k+1}} - X_{\tau}) - (X_{t_{k+1}} - X_{\tau}),$$

because $RX_{\tau} = X_{\tau}$. In particular, since L is self-adjoint and idempotent,

$$|(R - \mathbb{1})X_{t_{k+1}}|^2 = |X_{t_{k+1}} - X_{\tau}|^2.$$

Since A is bounded by some $\kappa > 0$, and so $|\gamma| \leq 2\kappa$, using

$$(a + b)^2 \leq 2a^2 + 2b^2, \quad t_{j+1} - t_j = \frac{t}{2^n},$$

we can thus estimate as follows,

$$\begin{aligned}
&\mathbb{E} \left(\left| \mathcal{S}_t^{\text{It}\hat{0}}(X) - \mathcal{S}_t^{\text{It}\hat{0}}(Y) - M_t \right|^2 \right) \\
&\leq \limsup_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \frac{2^{n+2}}{t^2} \mathbb{E} \left(\mathbb{1}_{\{\tau \in (t_k, t_{k+1}]\}} \left(\int_{t_{k-1}}^{t_k} |A(RX_u)| |(R - \mathbb{1})X_{t_{k+1}}| du \right)^2 \right) \\
&\quad + \limsup_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \frac{2^{n+2}}{t^2} \mathbb{E} \left(\mathbb{1}_{\{\tau \in (t_k, t_{k+1}]\}} \left(\int_{t_k}^{\tau} |\gamma(X_u)| |X_{t_{k+2}} - X_{t_{k+1}}| du \right)^2 \right) \\
&\leq \limsup_{n \rightarrow \infty} 4\kappa^2 \sum_{k=0}^{2^n-1} \mathbb{E} \left(\mathbb{1}_{\{\tau \in (t_k, t_{k+1}]\}} |(R - \mathbb{1})X_{t_{k+1}}|^2 \right) \\
&\quad + 16\kappa^2 \limsup_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \mathbb{E} \left(\mathbb{1}_{\{\tau \in (t_k, t_{k+1}]\}} |X_{t_{k+2}} - X_{t_{k+1}}|^2 \right) \\
&\leq 16\kappa^2 \limsup_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \left(\mathbb{E} \left(\mathbb{1}_{\{\tau \in (t_k, t_{k+1}]\}} |X_{t_{k+2}} - X_{t_{k+1}}|^2 \right) + \mathbb{E} \left(\mathbb{1}_{\{\tau \in (t_k, t_{k+1}]\}} |X_{t_{k+1}} - X_{\tau}|^2 \right) \right).
\end{aligned}$$

Since τ is an X -stopping time, we conclude by the Markov property of X , using

$$\int |\omega(r) - \omega(s)|^2 \mathbb{P}_z(d\omega) = r - s, \quad r > s > 0,$$

and once more $t_{j+1} - t_j = \frac{t}{2^n}$, that

$$\begin{aligned} & \mathbb{E} \left(\left| \mathcal{S}_t^{\text{Itô}}(X) - \mathcal{S}_t^{\text{Itô}}(Y) - M_t \right|^2 \right) \\ & \leq 16\kappa^2 \limsup_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \left(\frac{t}{2^n} \mathbb{P}(\tau \in (t_k, t_{k+1}]) \right) + \int_{t_k}^{t_{k+1}} \mathbb{E} \left(|X_{t_{k+1}} - X_u|^2 \mid \tau = u \right) \tau_* \mathbb{P}(du) \\ & \leq 16\kappa^2 \limsup_{n \rightarrow \infty} \frac{t}{2^n} \mathbb{P}(\tau \geq \tau) + 16\kappa^2 \limsup_{n \rightarrow \infty} \frac{t}{2^n} \sum_{k=0}^{2^n-1} \int_{t_k}^{t_{k+1}} \tau_* \mathbb{P}(du) \\ & = \limsup_{n \rightarrow \infty} \frac{32\kappa^2 t}{2^n} \mathbb{P}(\tau \geq \tau) = 0. \end{aligned}$$

Altogether, we have found that under the assumptions of the theorem one has

$$\mathcal{S}_t(X) - \mathcal{S}_t(Y) = M_t + I_t \quad \text{a.s.} \quad (14)$$

We are now going to estimate the L^1 -norms of M_t and I_t . Let us start with $\mathbb{E}(|I_t|)$: setting

$$w := \frac{i}{2} \operatorname{div}(A), \quad p := q^*,$$

we have

$$\begin{aligned} \mathbb{E}(|I_t|) & \leq \int_0^t \mathbb{E} \left(\left| \mathbb{1}_{\{s < \tau\}} (w(X_s) - w(RX_s)) \right| \right) ds \\ & \leq \left(\int_0^t \mathbb{P}(s < \tau) ds \right)^{\frac{1}{p}} \left(\int_0^t \mathbb{E}(|w(X_s) - w(RX_s)|^q) ds \right)^{\frac{1}{q}} \\ & \leq 2 \left(\int_0^t \mathbb{P}(s < \tau) ds \right)^{\frac{1}{p}} C_{w,t,q}, \end{aligned}$$

where

$$C_{w,t,q} := \left(\sup_{z \in \mathbb{R}^d} \int_0^t \int |w(\omega(s))|^q \mathbb{P}_z(d\omega) ds \right)^{\frac{1}{q}}.$$

In view of

$$\begin{aligned} \mathbb{P}(\tau > s) & \leq \frac{1}{2} \int_{\mathbb{R}^d} |\rho(s, x, z) - \rho(s, y, z)| dz = \frac{2}{\sqrt{2\pi s}} \int_0^{\frac{|x-y|}{2}} e^{-\frac{u^2}{2s}} du \\ & \leq (2\pi)^{-1/2} |x - y| s^{-1/2}, \end{aligned}$$

we conclude

$$\mathbb{E}(|I_t|) \leq CC_{w,t,q} t^{-\frac{1}{2p}} |x - y|^{\frac{1}{p}},$$

where from here on $C < \infty$ denotes a universal constant whose value may change from line to line. Now let us turn to the estimate for $\mathbb{E}(|M_t|)$: Define

$$h(r) := \begin{cases} \frac{1}{12} (2 - |r|)^3 + |r|, & |r| \leq 2, \\ |r|, & |r| > 2. \end{cases}$$

We note that $h(r) \geq |r|$ and that h is in $C^2(\mathbb{R})$ with

$$h''(r) = \begin{cases} 1 - \frac{1}{2} |r|, & |r| \leq 2, \\ 0, & |r| > 2. \end{cases}$$

In particular, we have $|h''| \leq \mathbb{1}_{[-2,2]}$ and $|h'| \leq 1$. We conclude by Itô's formula

$$h(M_t) = \frac{1}{2} \int_0^t h''(M_s) \mathbb{1}_{\{s < \tau\}} |\tilde{A}(X_s)|^2 ds + \tilde{M}_t,$$

where

$$\tilde{M}_t = \int_0^t h'(M_s) \mathbb{1}_{\{s < \tau\}} \langle \tilde{A}(X_s), dX_s \rangle$$

is an (L^2) -martingale, as follows from the assumption on A and the boundedness of h' . Thus,

$$\mathbb{E}(\tilde{M}_t) = \mathbb{E}(\tilde{M}_0) = 0,$$

and we have

$$\begin{aligned} \mathbb{E}(|M_t|) &\leq \mathbb{E}(h(M_t)) \\ &= \frac{1}{2} \mathbb{E} \left(\int_0^t h''(M_s) \mathbb{1}_{\{s < \tau\}} |\tilde{A}(X_s)|^2 ds \right) \\ &\leq \frac{1}{2} \mathbb{E} \left(\int_0^t \mathbb{1}_{\{s < \tau\}} |\tilde{A}(X_s)|^2 ds \right) \\ &\leq \frac{1}{2} \left(\int_0^t \mathbb{P}(s < \tau) ds \right)^{\frac{1}{p}} \left(\int_0^t \mathbb{E} \left(|\tilde{A}(X_s)|^{2q} \right) ds \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^t \mathbb{P}(s < \tau) ds \right)^{\frac{1}{p}} \left(\sup_{z \in \mathbb{R}^d} \int_0^t \int |A(\omega(s))|^{2q} \mathbb{P}_z(\omega) ds \right)^{\frac{1}{q}}. \end{aligned}$$

Hence, similarly to the Lebesgue integral I_t , we conclude

$$\mathbb{E}(|M_t|) \leq C C_{A,t,q} t^{-\frac{1}{2p}} |x - y|^{\frac{1}{p}},$$

where

$$C_{A,t,q} := \left(\sup_{z \in \mathbb{R}^d} \int_0^t \int |A(\omega(s))|^{2q} \mathbb{P}_z(d\omega) ds \right)^{\frac{1}{q}}.$$

Thus, we have shown

$$\mathbb{E}(|\mathcal{S}_t(X) - \mathcal{S}_t(Y)|) \leq C(C_{w,t,q} + C_{A,t,q}) t^{-\frac{1}{2p}} |x - y|^{\frac{1}{p}}.$$

Finally, noting that for all purely imaginary z, z' one has the elementary estimate

$$|e^z - e^{z'}| \leq C |z - z'|,$$

and $\Re(\mathcal{S}_t(Z)) = 0$, the proof is complete. $\square$

If $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $V : \mathbb{R}^d \rightarrow \mathbb{R}$ are Borel functions with

$$\max(|A|^2, |\operatorname{div}(A)|, |V|) \in \mathcal{K}(\mathbb{R}^d), \quad (15)$$

then the symmetric sesquilinear form

$$(\Psi_1, \Psi_2) \mapsto \frac{1}{2} \int_{\mathbb{R}^d} ((i\nabla + A)\Psi_1, (i\nabla + A)\nabla\Psi_2) + \int_{\mathbb{R}^d} V \cdot \overline{\Psi_1} \cdot \Psi_2 \in \mathbb{C}$$

in $L^2(\mathbb{R}^d)$ with domain of definition $C_c^\infty(\mathbb{R}^d)$ is semibounded from below and closable [2]. Thus, the closure of this form induces a self-adjoint semibounded from below operator $H(A, V)$ in $L^2(\mathbb{R}^d)$. The corresponding magnetic Schrödinger semigroup is given by the Feynman–Kac–Itô formula [2]

$$e^{-tH(A,V)}\Psi(x) = \mathbb{E} \left(e^{-\mathcal{S}_t(A|Z(x)) - \int_0^t V(Z)ds} \Psi(Z_t(x)) \right), \quad \Psi \in L^2(\mathbb{R}^d),$$

where $Z(x)$ is an arbitrary Brownian motion in $\mathbb{R}^d$ starting in $x \in \mathbb{R}^d$. Using the Feynman–Kac–Itô formula for $V = 0$ with Theorem 2.3 to deal with the magnetic potential A , and perturbation theory to deal with the electric potential V , we can now establish:

Theorem 2.5 *Let $\beta \in (0, 1)$, let $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $V : \mathbb{R}^d \rightarrow \mathbb{R}$ be Borel functions which satisfy*

$$\max(|A|^{\frac{2}{1-\beta}}, |\operatorname{div}(A)|^{\frac{1}{1-\beta}}) \in \mathcal{K}(\mathbb{R}^d), \quad V \in \mathcal{K}^\beta(\mathbb{R}^d), \quad (16)$$

and let $t > 0$, $1 \leq p \leq q \leq \infty$. Then one has

$$e^{-tH(A,V)} : L^p(\mathbb{R}^d) \longrightarrow C^{0,\beta}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$$

continuously, and there exists a universal constant $c_0 < \infty$ and a constant $C_V < \infty$ which only depends on V , such that

$$\begin{aligned} & \left\| e^{-tH(A,V)} \right\|_{L^p \rightarrow C^{0,\beta} \cap L^q} \\ & \leq C_V t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} e^{C_V t} + \left(c_0 C(A, t/2, (1-\beta)^{-1}) \left(\frac{t}{2} \right)^{-\frac{\beta}{2}} + c_0 \left(\frac{t}{2} \right)^{-\frac{\beta}{2}} \right. \\ & \quad \left. + C_V e^{C_V \frac{t}{2}} \int_0^{t/2} (C(A, s/2, (1-\beta)^{-1}) s^{-\frac{\beta}{2}} + s^{-\frac{\beta}{2}}) D_V(s/2) ds \right) C_V \left(\frac{t}{2} \right)^{-\frac{d}{2}(\frac{1}{p})} e^{C_V \frac{t}{2}}, \end{aligned}$$

where

$$(0, \infty) \ni s \longrightarrow C(A, s, (1-\beta)^{-1}) \in [0, \infty)$$

is the locally bounded function from Theorem 2.3, and

$$D_V : (0, \infty) \longrightarrow [0, \infty], \quad D_V(s) := \sup_{z \in \mathbb{R}^d} \int |V(\omega(s))| \mathbb{P}_z(d\omega).$$

Remark 2.6 1) Using monotone convergence, one finds

$$\int_0^t s^{-\beta/2} \sup_{z \in \mathbb{R}^d} \int |V(\omega(s))| \mathbb{P}_z(d\omega) ds \leq \sup_{z \in \mathbb{R}^d} \int_0^t s^{-\beta/2} \int |V(\omega(s))| \mathbb{P}_z(d\omega) ds,$$

which is finite for all $t > 0$ by Remark 2.2.5), so that a posteriori one also has $D_V < \infty$ a.e.

2) As our proof shows, the constant C_V can be chosen to be any constant which satisfies that for all $1 \leq p \leq q \leq \infty$, $r > 0$ one has

$$\left\| e^{-rH(A,V)} \right\|_{L^p \rightarrow L^q} \leq C_V r^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} e^{C_V r}.$$

The existence of such a uniform constant has been shown in [2].

3) Using $\Psi = e^{t\mathbb{R}^d} e^{-tH(A,V)} \Psi$ for eigenfunctions Ψ of $H(A, V)$, one obtains explicit $L^r \rightarrow C^{0,\beta}$ -estimates for eigenfunctions.

Proof of Theorem 2.5 We start by noting that the assumptions on A together with Jensen's inequality, and that $\mathcal{K}^\beta(\mathbb{R}^d) \subset \mathcal{K}(\mathbb{R}^d)$ shows that the pair (A, V) satisfies (15).

Set $q := 1/(1 - \beta) \in (1, \infty)$ so that $q^* = 1/\beta$ and pick a mirror coupling (X, Y) from $(x, y) \in (\mathbb{R}^d \times \mathbb{R}^d) \setminus \text{diag}(\mathbb{R}^d)$ and set $\tau := \tau(X, Y)$. Then, given $r > 0$, $\Phi \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ we can estimate as follows,

$$\begin{aligned}
 & \left| e^{-rH(A,0)} \Phi(x) - e^{-rH(A,0)} \Phi(y) \right| \\
 & \leq \mathbb{E} \left(\left| e^{-\mathcal{J}_r(A|X)} - e^{-\mathcal{J}_r(A|Y)} \right| |\Phi(X_r)| \right) + \mathbb{E} \left(\left| e^{-\mathcal{J}_r(A|X)} \right| |\Phi(X_r) - \Phi(Y_r)| \right) \\
 & = \mathbb{E} \left(\left| e^{-\mathcal{J}_r(A|X)} - e^{-\mathcal{J}_r(A|Y)} \right| |\Phi(X_r)| \right) + \mathbb{E} \left(1_{\{r < \tau\}} \left| e^{-\mathcal{J}_r(A|X)} \right| |\Phi(X_r) - \Phi(Y_r)| \right) \\
 & \leq \|\Phi\|_\infty \mathbb{E} \left(\left| e^{-\mathcal{J}_r(A|X)} - e^{-\mathcal{J}_r(A|Y)} \right| \right) + \mathbb{E} \left(1_{\{r < \tau\}} |\Phi(X_r) - \Phi(Y_r)| \right) \\
 & \leq C(A, r, q) r^{-\frac{1}{2q^*}} |x - y|^{\frac{1}{q^*}} \|\Phi\|_\infty + 2\mathbb{P}(r < \tau) \|\Phi\|_\infty \\
 & \leq C(A, r, q) r^{-\frac{1}{2q^*}} |x - y|^{\frac{1}{q^*}} \|\Phi\|_\infty + 2\mathbb{P}(r < \tau)^{1/q^*} \|\Phi\|_\infty \\
 & \leq C(A, r, q) r^{-\frac{1}{2q^*}} |x - y|^{\frac{1}{q^*}} \|\Phi\|_\infty + c_0 r^{-\frac{1}{2q^*}} |x - y|^{\frac{1}{q^*}} \|\Phi\|_\infty,
 \end{aligned}$$

where $c_0 < \infty$ is a universal constant. Thus, we have shown

$$\left\| e^{-rH(A,0)} \right\|_{L^\infty \rightarrow C^{0,\beta}} \leq c_0 C(A, r, 1/(1 - \beta)) r^{-\frac{\beta}{2}} + c_0 r^{-\frac{\beta}{2}}.$$

Duhamel's formula² states that

$$e^{-tH(A,V)} \Phi = e^{-tH(A,0)} \Phi + \int_0^t e^{-\frac{s}{2}H(A,0)} e^{-\frac{s}{2}H(A,0)} V e^{-(t-s)H(A,V)} \Phi ds,$$

and so

$$\begin{aligned}
 & \left\| e^{-tH(A,V)} \Phi \right\|_{C^{0,\beta}} \leq \left\| e^{-tH(A,0)} \Phi \right\|_{C^{0,\beta}} \\
 & + \int_0^t \left\| e^{-\frac{s}{2}H(A,0)} \right\|_{L^\infty \rightarrow C^{0,\beta}} \left\| e^{-\frac{s}{2}H(A,0)} V \right\|_{L^\infty \rightarrow L^\infty} \left\| e^{-(t-s)H(A,V)} \Phi \right\|_{L^\infty} ds.
 \end{aligned} \tag{17}$$

There exists [2] a constant C_V such that for all $1 \leq p \leq q \leq \infty$, $r > 0$ one has

$$\left\| e^{-rH(A,V)} \right\|_{L^p \rightarrow L^q} \leq C_V r^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} e^{C_V r}, \tag{18}$$

so that

$$\left\| e^{-(t-s)H(A,V)} \Phi \right\|_\infty \leq C_V e^{C_V t} \|\Phi\|_\infty. \tag{19}$$

² In principle, one should be more careful here as V is not bounded, but for the purpose of proving the estimate from Theorem 2.5 one can approximate V with a sequence of bounded potentials V_n and take $n \rightarrow \infty$ in the end (cf. the proof Theorem 3.10 in [5]).

Moreover, by what we have shown above,

$$\left\| e^{-\frac{s}{2}H(A,0)} \right\|_{L^\infty \rightarrow C^{0,\beta}} \leq CC(A, s/2, 1/(1-\beta))s^{-\frac{\beta}{2}} + Cs^{-\frac{\beta}{2}}. \quad (20)$$

Given $f \in L^\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, and a Brownian motion $Z(x)$ in $\mathbb{R}^d$ starting in x we have, using $|e^{-\mathcal{L}_{s/2}(A|Z(x))}| = 1$, the estimate

$$\begin{aligned} \left| e^{-\frac{s}{2}H(A,0)} Vf(x) \right| &= \left| \mathbb{E} \left(e^{-\mathcal{L}_{s/2}(A|Z(x))} V(Z_{s/2}(x)) f(Z_{s/2}(x)) \right) \right| \\ &\leq \mathbb{E} (V(Z_{s/2}(x)) f(Z_{s/2}(x))) = \int |V(\omega(s/2))| \cdot |f(\omega(s/2))| \mathbb{P}_x(d\omega) \\ &\leq \|f\|_\infty D_V(s/2), \end{aligned}$$

so that

$$\left\| e^{-\frac{s}{2}H(A,0)} V \right\|_{L^\infty \rightarrow L^\infty} \leq D_V(s/2). \quad (21)$$

Combining (17), (19), (20), (21), we have shown that for all $\Phi \in L^\infty(\mathbb{R}^d)$,

$$\begin{aligned} &\left| e^{-\frac{t}{2}H(A,V)} \Phi(x) - e^{-\frac{t}{2}H(A,V)} \Phi(y) \right| \\ &\leq \left(c_0 C(A, t/2, (1-\beta)^{-1}) \left(\frac{t}{2} \right)^{-\frac{\beta}{2}} + c_0 \left(\frac{t}{2} \right)^{-\frac{\beta}{2}} \right. \\ &\quad \left. + C_V e^{C_V \frac{t}{2}} \int_0^{t/2} (C(A, s/2, (1-\beta)^{-1}) s^{-\frac{\beta}{2}} + s^{-\frac{\beta}{2}}) D_V(s/2) ds \right) \|\Phi\|_{L^\infty} |x - y|^\beta, \end{aligned}$$

and so

$$\begin{aligned} &\left\| e^{-\frac{t}{2}H(A,V)} \right\|_{L^\infty \rightarrow C^{0,\beta}} \\ &\leq c_0 C(A, t/2, (1-\beta)^{-1}) \left(\frac{t}{2} \right)^{-\frac{\beta}{2}} + c_0 \left(\frac{t}{2} \right)^{-\frac{\beta}{2}} \\ &\quad + C_V e^{C_V \frac{t}{2}} \int_0^{t/2} (C(A, s/2, (1-\beta)^{-1}) s^{-\frac{\beta}{2}} + s^{-\frac{\beta}{2}}) D_V(s/2) ds. \end{aligned}$$

The above estimate together with $\Phi = e^{-\frac{t}{2}H(A,V)} \Psi$ and (18) shows

$$\begin{aligned} &\left\| e^{-tH(A,V)} \right\|_{L^p \rightarrow C^{0,\beta}} \\ &\leq \left\| e^{-\frac{t}{2}H(A,V)} \right\|_{L^\infty \rightarrow C^{0,\beta}} \left\| e^{-\frac{t}{2}H(A,V)} \right\|_{L^p \rightarrow L^\infty} \\ &\leq \left(c_0 C(A, t/2, (1-\beta)^{-1}) \left(\frac{t}{2} \right)^{-\frac{\beta}{2}} + c_0 \left(\frac{t}{2} \right)^{-\frac{\beta}{2}} \right) \end{aligned}$$

$$+C_V e^{C_V \frac{t}{2}} \int_0^{t/2} (C(A, s/2, (1-\beta)^{-1})s^{-\frac{\beta}{2}} + s^{-\frac{\beta}{2}}) D_V(s/2) ds \Big) C_V \left(\frac{t}{2}\right)^{-\frac{d}{2}(\frac{1}{p})} e^{C_V \frac{t}{2}}.$$

Finally, using (18) we end up with

$$\begin{aligned} & \left\| e^{-tH(A,V)} \right\|_{L^p \rightarrow C^{0,\beta} \cap L^q} \\ & \leq C_V t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} e^{C_V t} + \left(c_0 C(A, t/2, (1-\beta)^{-1}) \left(\frac{t}{2}\right)^{-\frac{\beta}{2}} + c_0 \left(\frac{t}{2}\right)^{-\frac{\beta}{2}} \right. \\ & \quad \left. + C_V e^{C_V \frac{t}{2}} \int_0^{t/2} (C(A, s/2, (1-\beta)^{-1})s^{-\frac{\beta}{2}} + s^{-\frac{\beta}{2}}) D_V(s/2) ds \Big) C_V \left(\frac{t}{2}\right)^{-\frac{d}{2}(\frac{1}{p})} e^{C_V \frac{t}{2}}, \end{aligned}$$

which completes the proof. $\square$

For the following result, consider the linear surjective maps

$$\begin{aligned} \pi_j : \mathbb{R}^{3n} &\longrightarrow \mathbb{R}^3, \quad (\mathbf{x}_1, \dots, \mathbf{x}_n) \longmapsto \mathbf{x}_j, \\ \pi_{ij} &:= \pi_i - \pi_j : \mathbb{R}^{3n} \longrightarrow \mathbb{R}^3, \end{aligned}$$

and let $A : \mathbb{R}^{3n} \rightarrow \mathbb{R}^{3n}$, $V : \mathbb{R}^{3n} \rightarrow \mathbb{R}$ be arbitrary functions. Remark 2.2 then shows:

Corollary 2.7 *Let $\beta \in (0, 1)$, $l \in \mathbb{N}$ and let $a : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $v_i, v_{ij} : \mathbb{R}^3 \rightarrow \mathbb{R}$ be Borel functions with*

$$A = \sum_{i=1}^n a \circ \pi_i, \quad V = \sum_{1 \leq i < j \leq n} v_{ij} \circ \pi_{ij} + \sum_{i=1}^n v_i \circ \pi_i,$$

and

$$\begin{aligned} & |a|^{2/(1-\beta)}, |\operatorname{div}(a)|^{1/(1-\beta)} \in L^s(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \quad \text{for some } s > 3/2, \\ & v_i, v_{ij} \in L^s(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \quad \text{for some } s > \frac{3}{2(1-\beta/2)}. \end{aligned}$$

Then for all $t > 0$ and $p \in [1, \infty]$ one has

$$e^{-tH(A,V)} : L^p(\mathbb{R}^{3n}) \longrightarrow C^{0,\beta}(\mathbb{R}^{3n})$$

continuously.

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