The Coxeter relations and KP map for non-commuting symbols

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Abstract

We give an action of the symmetric group on non-commuting indeterminates in terms of series in the corresponding Mal'cev–Newmann division ring. The action is constructed from the non-Abelian Hirota–Miwa (discrete KP) system. The corresponding companion map, which gives generators of the action, is discussed in the generic case, and the corresponding explicit formulas have been found in the periodic reduction. We discuss also briefly connection of the companion to the KP map with context-free languages.

Keywords Discrete integrable systems · Non-commutative Hirota–Miwa equation · Yang–Baxter maps · Symmetric group · Mal'cev–Neumann series · Context-free languages

Mathematics Subject Classification 37K10 · 37K60 · 16T25 · 39A14 · 14E07 · 12E15

1 Introduction

Non-commutative extensions of integrable systems are of growing interest in mathematical physics [6,13–15,18,24,27,37,41,50,52]. They may be considered as a useful platform to more thorough understanding of integrable quantum or statistical mechanics lattice systems, where the quantum Yang–Baxter equation [3,38] plays a role. Recently, integrable maps with variables satisfying various commutation relations

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(including the canonical Weyl relations and those coming from the quantum group theory) have been studied in [4,21,28,42,57,60].

Such a direction from commutative to non-commutative objects in the theory of integrable systems is not an isolated issue. One can say it is in mainstream of current developments in contemporary mathematics. The origins of this trend are rooted back in quantum physics, where the idea of replacing functions by not necessarily commuting operators appeared. There is a topology of non-commutative spaces, a non-commutative integration theory, a non-commutative differential geometry [11], non-commutative probability theory [63], etc. The present research aims to make a step in direction to create a kind of free integrability. The advantage of working with totally non-commutative variables has been advocated also in [26]. As the geometric research [14] suggests, the algebraic environment of our work will be the theory of Mal'cev–Neumann division rings constructed from free groups.

In this paper, we study maps obeying the Coxeter relations, which are obtained from the non-commutative discrete KP system of equations (called originally in [52] the non-Abelian Hirota–Miwa system). In the commutative case, such maps were studied in [23,34,59], see also [17] for a version with certain commutativity restrictions. We remark that Hirota's discrete KP equation [31] gives as reductions majority of known integrable systems [40], both discrete and continuous. The A -type Coxeter relations [33] considered in the paper are closely related to the notion of Yang–Baxter maps (originally [22] called the set-theoretical solutions of quantum Yang–Baxter equation). We will use, well established by now [2], correspondence between the Yang–Baxter maps and the multidimensional consistency property of integrable equations on quad-graphs [1,49].

The paper is organized as follows. In the next section, we first collect relevant results about the Mal'cev–Neumann construction of division rings. Then, we discuss the Coxeter relations in connection to the multidimensional consistency. We close the preparatory Sect. 2 with discussion of properties of a map arising from the non-Abelian Hirota–Miwa (discrete KP) system. In Sect. 3, we study the companion map to the above one. The corresponding explicit formulas in the case of periodic reduction, given in Sect. 4, generalize to fully non-commutative level the maps studied in [17,34]. We discuss also in Sect. 5 relation of the series representing the maps to context-free languages. A surprising consequence is that, although the original non-commutative KP map is given by a rational expression, the series which give the companion map are not rational.

2 Preliminaries

2.1 The Mal'cev–Neumann construction and rational series over free group

Let $\mathbb{k}$ be a field (in our case the field of rational numbers $\mathbb{Q}$), and let G be a group. Consider the $\mathbb{k}$ -vector space $\mathbb{k}[[G]]$ of formal series over G with coefficients in $\mathbb{k}$, where a series $S \in \mathbb{k}[[G]]$ is just a map $G \ni g \mapsto (S, g) \in \mathbb{k}$ represented by

$$S = \sum_{g \in G} (S, g)g.$$

The support of the series S is then the set $\text{supp}(S) = \{g \in G \mid (S, g) \neq 0\}$. Series with finite support can be multiplied using the formula

$$(ST, h) = \sum_{fg=h} (S, f)(T, g), \quad (2.1)$$

and form the group algebra $\mathbb{k}[G]$.

Assume that G is totally ordered by the order $<$ compatible with its group structure, i.e.,

$$\forall f, g, h \in G, \quad f < g \Rightarrow \begin{cases} fh < gh. \\ hf < hg. \end{cases}$$

Recall that $P \subset G$ is well ordered if every non-empty subset of P admits a smallest element for the order of G . The set $\mathbb{k}((G, <))$ of Mal'cev–Neumann series is the set of the series in $\mathbb{k}[[G]]$ whose supports are well ordered (with respect to the given order $<$). One can equip $\mathbb{k}((G, <))$ with a $\mathbb{k}$ -algebra structure by defining the (Cauchy) product of two Mal'cev–Neumann series S and T by (2.1). The remarkable result by Mal'cev [48], and independently by Neumann [51], is that this algebra is a division ring (skew field). Moreover, the space of Mal'cev–Neumann series is complete in the natural ultra-metric topology [51], where a sequence converges if and only if the coefficient of every monomial stabilizes.

Example 2.1 Let $G = \mathbb{Z}$ with its standard order, then

$$\mathbb{k}((\mathbb{Z}, <)) = \left\{ \sum_{n=d}^{\infty} a_n x^n \mid a_n \in \mathbb{k}, d \in \mathbb{Z} \right\},$$

thus, we obtain the usual formal Laurent series (bounded from below) in one variable over $\mathbb{k}$. If we took the opposite order on $\mathbb{Z}$, we would obtain the Laurent series in x^{-1} over $\mathbb{k}$.

The free group $\Gamma = \Gamma(x_1, x_2, \dots)$ with generators $x_1, x_2, \dots$, possibly an infinite list, can be regarded as the set of equivalence classes of finite words in the letters x_i and their formal inverses x_i^{-1} , where the words are considered equivalent if one can pass from one to the other by removing (or inserting) consecutive letters of the form $x_i x_i^{-1}$ or $x_i^{-1} x_i$. The group operation is concatenation, and the empty word represents the identity element. By fixing a total order $<$ in Γ compatible with its group structure, one can consider the corresponding Mal'cev–Neumann division ring $\mathbb{k}((\Gamma, <))$.

There are number of ways to order free groups [62]—in fact for free groups with more than one generator there are uncountably many. The standard one, described in [5] in terms of the Magnus embedding [47], is as follows. Define the ring

$$\Lambda = \mathbb{Z}[[X_1, X_2, \dots]]$$

to be the ring of formal power series in the non-commuting variables X_i , one for each generator of Γ . If there are infinitely many variables, we only allow for expressions involving a finite set of variables to belong to Λ , so that an element of Λ has only a finite number of terms of a given degree. The advantage of Λ is that the variables have no negative exponents (unlike in Γ) and so it is easier to define an ordering, without having to worry about cancellation problems. Define the (multiplicative) homomorphism $\mu: \Gamma \rightarrow \Lambda$ which on the generators of Γ is given by

$$\begin{aligned}\mu(x_i) &= 1 + X_i, \\ \mu(x_i^{-1}) &= 1 - X_i + X_i^2 - X_i^3 + \cdots.\end{aligned}$$

To compare two elements in Λ write them with terms of lower degree first, and lexicographically order terms of the same degree (this is the so-called shortlex order). Then, the two elements are ordered according to the coefficient of the first term at which they differ. Finally, define the ordering on the free group Γ by declaring

$$v < w \text{ in } \Gamma \Leftrightarrow \mu(v) < \mu(w) \text{ in } \Lambda.$$

Example 2.2 To compare $x_1 x_2^2 x_1^{-1}$ and $x_1^{-1} x_2^2 x_1$, notice that the series

$$\begin{aligned}\mu(x_1 x_2^2 x_1^{-1}) &= 1 + 2X_2 + 2X_1 X_2 - 2X_2 X_1 + X_2^2 + \cdots \\ \mu(x_1^{-1} x_2^2 x_1) &= 1 + 2X_2 - 2X_1 X_2 + 2X_2 X_1 + X_2^2 + \cdots\end{aligned}$$

have different coefficients first at $X_1 X_2$. Because $-2 < 2$ in $\mathbb{Z}$, then we have $x_1^{-1} x_2^2 x_1 < x_1 x_2^2 x_1^{-1}$.

Consider the ordered free group $(\Gamma, <)$ with finite number of generators. The smallest sub-division ring in $\mathbb{k}((\Gamma, <))$ containing the group algebra $\mathbb{k}[\Gamma]$ is called the division ring of fractions. It turns out that the result is independent of the particular order $<$ used in the construction, in the sense that it is isomorphic [43] to the universal field of fractions, called also [10] the free skew field.

2.2 Coxeter relations and multidimensional consistency

Consider the group given by generators $\langle r_1, r_2, \dots, r_{N-1} \rangle$ subject to the following relations (of type A_{N-1}), see [33] for details

$$r_i \circ r_i = \text{id} \quad (\text{involutivity}), \quad (2.2)$$

$$r_i \circ r_{i+1} \circ r_i = r_{i+1} \circ r_i \circ r_{i+1}, \quad (\text{braid relations}), \quad (2.3)$$

$$r_i \circ r_j = r_j \circ r_i \quad |i - j| > 1. \quad (2.4)$$

It is well known that such a group is isomorphic to the symmetric group S_N , and in this interpretation r_i can be identified with the transposition $(i, i + 1)$. The standard geometric realization of the group is the symmetry group of the regular N -simplex, where r_i can be identified with the reflection exchanging vertices i with $i + 1$.

Fig. 1 An edge-map and its consistency around the cube

Let us present another geometric realization of the symmetric group, which will be used in the sequel. Consider the N -dimensional cube graph Q_N , whose vertices can be identified with binary sequences of length N , and two vertices belong to an edge if their sequences differ at one place only. The edges can be given an orientation to the vertex with bigger number of ones, and the corresponding index $i = 1, \dots, N$ indicating at which position of the sequence the transition $0 \rightarrow 1$ occurs. This way any oriented path from the initial vertex $(0, 0, \dots, 0)$ to the terminal vertex $(1, 1, \dots, 1)$ of the graph Q_N gives rise to a sequence $(\sigma(1), \sigma(2), \dots, \sigma(N))$ of indices of subsequent edges, and thus to the permutation $\sigma \in S_N$. The left regular action of S_N on itself gives the action on the paths. In what follows, we attach to each edge a weight, and we extend the action of the generators r_i of the symmetric group to the corresponding action on weighted paths.

Let $\mathcal{X}$ be any set, consider map $f: \mathcal{X} \times \mathcal{X} \ni (x_1, x_2) \mapsto (f_1(x_1, x_2), f_2(x_1, x_2)) \in \mathcal{X} \times \mathcal{X}$. We would like to interpret the map as providing shifts in discrete variables for fields attached to the edges of a square lattice, so that $x_{1(2)} = f_1(x_1, x_2)$, $x_{2(1)} = f_2(x_1, x_2)$ as graphically depicted in Fig. 1. In order to keep such an interpretation for multidimensional lattices, the map should satisfy compatibility conditions equivalent to the commutation of shifts in different directions.

The map f is called consistent around the cube [1, 49] if for arbitrary values of the fields (x_1, x_2, x_3) attached to initial edges (containing the vertex $(0, 0, 0)$) of the cube Q_3

$$f_1(f_1(x_1, x_3), f_1(x_2, x_3)) = f_1(f_1(x_1, x_2), f_2(x_2, x_3)) \stackrel{\text{def}}{=} x_{1(23)}, \quad (2.5)$$

$$f_1(f_2(x_1, x_2), f_2(x_1, x_3)) = f_2(f_1(x_1, x_3), f_1(x_2, x_3)) \stackrel{\text{def}}{=} x_{2(13)}, \quad (2.6)$$

$$f_2(f_2(x_1, x_2), f_2(x_1, x_3)) = f_2(f_1(x_1, x_2), f_2(x_2, x_3)) \stackrel{\text{def}}{=} x_{3(12)}, \quad (2.7)$$

i.e., for any final edge (containing the vertex $(1, 1, 1)$) of the cube two possible values of the corresponding field calculated using the map coincide, see Fig. 1. This concept is the crucial one in the theory of discrete integrable systems. In particular, it implies

Fig. 2 The companion map and braid relation $r_2 \circ r_1 \circ r_2 = r_1 \circ r_2 \circ r_1$ in the edge interpretation

possibility of attaching in a consistent way values of the field on all edges of the multidimensional cube Q_N once its values on initial edges have been given.

Assume that multidimensionally consistent map f is invertible and admits existence of its unique companion map r defined by

$$r(x, y) = (x', y'), \quad \text{where} \quad (y', y) = f(x, x'). \quad (2.8)$$

It is known [2] that the multidimensional consistency in that case implies braid relations for the companion map in the form

$$r_2 \circ r_1 \circ r_2 = r_1 \circ r_2 \circ r_1, \quad \text{in} \quad \mathcal{X} \times \mathcal{X} \times \mathcal{X}, \quad (2.9)$$

where $r_1 = r \times \text{id}_{\mathcal{X}}$ and $r_2 = \text{id}_{\mathcal{X}} \times r$.

Actually, the connection has been formulated in [2] in terms of the Yang–Baxter relations

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12} \quad \text{in} \quad \mathcal{X} \times \mathcal{X} \times \mathcal{X}, \quad (2.10)$$

where $R = \pi \circ r$, π is the transposition (i.e., $(R(x, y) = (y', x'))$), and subscripts denote the arguments on which R acts. We remark that eventual involutivity of r means the so-called reversibility condition

$$\pi \circ R \circ \pi \circ R = \text{id}_{\mathcal{X} \times \mathcal{X}}, \quad (2.11)$$

on the level of the corresponding Yang–Baxter map.

2.3 The non-commutative KP hierarchy and the KP map

The linear problem for the non-commutative difference KP hierarchy [17] (see also [35] for the commutative case) reads as follows

$$\phi_{k+1} - \phi_{k(i)} = \phi_k u_{i,k}, \quad (2.12)$$

here $\phi_k(n_1, n_2, \dots)$, is k th component, $k \in \mathbb{Z}$, of the wave function with values in a division ring, $n_i \in \mathbb{Z}$ are independent discrete variables, and index in brackets denotes

the forward shift in the corresponding variable, i.e., $\phi_{k(i)}(n_1, n_2, \dots, n_i, \dots) = \phi_k(n_1, n_2, \dots, n_i + 1, \dots)$. The compatibility condition of Eq. (2.12) gives the so-called non-commutative difference KP hierarchy

$$u_{j,k}u_{i,k(j)} = u_{i,k}u_{j,k(i)}, \quad i \neq j, \quad (2.13)$$

$$u_{i,k(j)} + u_{j,k+1} = u_{j,k(i)} + u_{i,k+1}. \quad (2.14)$$

It is not difficult to find explicit form of the transformation rule

$$u_{i,k(j)} = (u_{i,k} - u_{j,k})^{-1}u_{i,k}(u_{i,k+1} - u_{j,k+1}), \quad i \neq j, \quad (2.15)$$

called in [17] the non-commutative KP map. Let us attach the vector $\mathbf{u}_i = (u_{i,k})_{k \in \mathbb{Z}}$ to the corresponding i th edge of multidimensional cube (see Fig. 1). Then, the map involves edges of a single quadrilateral of the cube. Our paper is based on the following result [16] equivalent to integrability of the non-commutative KP hierarchy

Proposition 2.1 *The KP map is multidimensionally consistent.*

Remark Equations of the non-commutative discrete KP hierarchy can be obtained from the non-Abelian Hirota–Miwa system [52], see also [14] for geometric meaning of the corresponding linear problem and of the nonlinear equations in terms of the so-called Desargues maps. Notice that the KP map can be considered as two-dimensional only on the formal level of the infinite-component vector $\mathbf{u}_i$. On the local level of the three-dimensional non-Abelian Hirota–Miwa system, its four dimensional consistency was established in [14] in connection to the Desargues theorem of projective geometry. Then, instead of the Yang–Baxter equation, one has the so-called pentagon equation [21].

Corollary 2.2 *The KP map is birational with the inverse given by*

$$u_{i,k} = (u_{i,k-1(j)} - u_{j,k-1(i)})u_{i,k(j)}(u_{i,k(j)} - u_{j,k(i)})^{-1}. \quad (2.16)$$

By direct calculations, along lines described in [16] for the KP map, one can show the following result.

Corollary 2.3 *The inverse of the KP map is three-dimensionally consistent as well, i.e., given values $(\mathbf{u}_{i(jl)}, \mathbf{u}_{j(il)}, \mathbf{u}_{l(ij)})$ on three final edges of the cube Q_3 , the various ways to obtain values $(\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_l)$ on the initial edges give the same result.*

3 The companion of the KP map and its properties

Define the companion map to the KP map $r: (\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}', \mathbf{y}')$ by solution of the equations

$$x_k y_k = x'_k y'_k, \quad y_k + x_{k+1} = y'_k + x'_{k+1}, \quad k \in \mathbb{Z}, \quad (3.1)$$

which is just system (2.13) and (2.14) written in new variables

$$x_k = u_{i,k}, \quad y_k = u_{j,k(i)}, \quad x'_k = u_{j,k}, \quad y'_k = u_{i,k(j)} \quad (3.2)$$

for fixed pair of distinct indices i, j . We assume that the arguments $(\mathbf{x}, \mathbf{y}) = (x_k, y_k)_{k \in \mathbb{Z}}$ of the map are free variables.

Notice that Eq. (3.1) does not determine the map uniquely. Given one component of the solution, x'_0 say, all other components can be obtained then by the twofold recursion

$$y'_k = (x'_k)^{-1} x_k y_k, \quad x'_{k+1} = y_k + x_{k+1} - y'_k, \quad k \geq 0, \quad (3.3)$$

$$y'_{k-1} = y_{k-1} + x_k - x'_k, \quad x'_{k-1} = x_{k-1} y_{k-1} (y'_{k-1})^{-1}, \quad k \leq 0. \quad (3.4)$$

One can consider then x'_0 as free parameter (notice that the assumption $x'_0 = x_0$ leads to the identity mapping). When we are interested in K -periodic solutions to the companion map equations (3.1), i.e., $k \in \mathbb{Z}_K$, then the freedom vanishes. In this Section, we study properties of the companion map (with free parameter) which hold even without the periodicity restriction.

Define recursively polynomials

$$\mathcal{P}_k^{(0)} = 1, \quad \mathcal{P}_k^{(\ell)} = \mathcal{P}_k^{(\ell-1)} x_{k+\ell} + y_k \dots y_{k+\ell-1} = x_{k+1} \dots x_{k+\ell} + y_k \mathcal{P}_{k+1}^{(\ell-1)}. \quad (3.5)$$

In [17], it was shown that the polynomials are invariants of the companion map. Following [54], we can represent the polynomials as formal sums of all weighted paths between two vertices of the graph as on Fig. 3. This observation can be also given meaning in terms of matrices.

Proposition 3.1 *The polynomial $\mathcal{P}_k^{(\ell)}$ is $(2, 1)$ element of the following product of 2×2 matrices*

$$\begin{pmatrix} x_k & 0 \\ 1 & y_k \end{pmatrix} \begin{pmatrix} x_{k+1} & 0 \\ 1 & y_{k+1} \end{pmatrix} \dots \begin{pmatrix} x_{k+\ell} & 0 \\ 1 & y_{k+\ell} \end{pmatrix} = \begin{pmatrix} x_k \dots x_{k+\ell} & 0 \\ \mathcal{P}_k^{(\ell)} & y_k \dots y_{k+\ell} \end{pmatrix} \quad (3.6)$$

Proof By mathematical induction using the recursive definition (3.5). The combinatorial interpretation of weighted paths in graphs in terms of matrices is standard [58, 64]. $\square$

Corollary 3.2 *Define functions $\alpha_k(\mathbf{x}, \mathbf{y}), \beta_k(\mathbf{x}, \mathbf{y}), k \in \mathbb{Z}$, by*

$$\alpha_k(\mathbf{x}, \mathbf{y}) = x_k y_k, \quad (3.7)$$

$$\beta_k(\mathbf{x}, \mathbf{y}) = y_k + x_{k+1}, \quad (3.8)$$

Fig. 3 The combinatorial interpretation of the polynomial $\mathcal{P}_k^{(\ell)}$ as the sum over all weighted paths between two vertices (marked in black)

or using infinite matrices

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & x_{k-1} & 0 & 0 & \ddots \\ \ddots & 1 & x_k & 0 & \ddots \\ \ddots & 0 & 1 & x_{k+1} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & y_{k-1} & 0 & 0 & \ddots \\ \ddots & 1 & y_k & 0 & \ddots \\ \ddots & 0 & 1 & y_{k+1} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \alpha_{k-1} & 0 & 0 & \ddots \\ \ddots & \beta_{k-1} & \alpha_k & 0 & \ddots \\ \ddots & 1 & \beta_k & \alpha_{k+1} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then, the problem of finding the companion to the KP map (3.1) can be formulated as the problem of finding another solution x'_k, y'_k of the system (3.7) and (3.8) with the data α_k, β_k , once a solution x_k, y_k has been given or, equivalently, as a matrix refactorization problem.

Remark It is immediate to see that the functions $\alpha_k(\mathbf{x}, \mathbf{y}), \beta_k(\mathbf{x}, \mathbf{y})$ are invariant with respect to the companion map. This holds true independently of the value of the free parameter which specifies the map.

Remark Notice that $\mathcal{P}_k^{(1)} = \beta_k$, and the recurrence formula (3.5) gives

$$\mathcal{P}_k^{(\ell)} = \mathcal{P}_k^{(\ell-1)} \beta_{k+\ell-1} - \mathcal{P}_k^{(\ell-2)} \alpha_{k+\ell-1}, \quad (3.9)$$

which easily implies invariance of the polynomials $\mathcal{P}_k^{(\ell)}$ with respect to the companion map.

Let us close this Section with similar discussion of braid relations (2.9) in the context of the companion map. Given three sets of indeterminates $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (x_k, y_k, z_k)_{k \in \mathbb{Z}}$, define (using the same symbols as above but with different number of arguments, the meaning will be clear from the context) functions $\alpha_k(\mathbf{x}, \mathbf{y}, \mathbf{z}), \beta_k(\mathbf{x}, \mathbf{y}, \mathbf{z})$, and $\gamma_k(\mathbf{x}, \mathbf{y}, \mathbf{z}), k \in \mathbb{Z}$, by

$$\alpha_k(\mathbf{x}, \mathbf{y}, \mathbf{z}) = x_k y_k z_k, \quad (3.10)$$

$$\beta_k(\mathbf{x}, \mathbf{y}, \mathbf{z}) = y_k z_k + x_{k+1} z_k + x_{k+1} y_{k+1}, \quad (3.11)$$

$$\gamma_k(\mathbf{x}, \mathbf{y}, \mathbf{z}) = z_k + y_{k+1} + x_{k+2}. \quad (3.12)$$

Proposition 3.3 *The above three-argument functions $(\alpha_k, \beta_k, \gamma_k)_{k \in \mathbb{Z}}$ are invariant with respect to the maps $r_1 = r \times \text{id}$ and $r_2 = \text{id} \times r$ whose superpositions give the braid relations (2.9).*

Proof The result follows from the observation that

$$\begin{aligned}\alpha_k(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= \alpha_k(\mathbf{x}, \mathbf{y}) z_k = x_k \alpha_k(\mathbf{y}, \mathbf{z}), \\ \beta_k(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= \beta_k(\mathbf{x}, \mathbf{y}) z_k + \alpha_{k+1}(\mathbf{x}, \mathbf{y}) = \alpha_k(\mathbf{y}, \mathbf{z}) + x_{k+1} \beta_k(\mathbf{y}, \mathbf{z}), \\ \gamma_k(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= z_k + \beta_{k+1}(\mathbf{x}, \mathbf{y}) = \beta_k(\mathbf{y}, \mathbf{z}) + x_{k+2},\end{aligned}$$

and the invariance of the two-argument functions with respect to the companion map. $\square$

4 The companion map in the periodic reduction

4.1 Explicit formulas for periodic companion map

In this Section, we assume that the arguments (x_k, y_k) of the companion map are periodic with period K , i.e., $x_{k+K} = x_k$, $y_{k+K} = y_k$. We will be looking for the explicit form of the map which preserves the periodicity condition, i.e., $x'_{k+K} = x'_k$, $y'_{k+K} = y'_k$ as well. Ruling out the identity mapping, we define h_k , as it was done in the commutative case in [54], by

$$x'_{k+1} = x_{k+1} + h_k^{-1}. \quad (4.1)$$

Then, the second part of the system (3.1) gives

$$y'_k + h_k^{-1} = y_k, \quad (4.2)$$

and the first part leads to equations

$$h_k x_{k+1} + 1 = y_{k+1} h_{k+1}. \quad (4.3)$$

In order to make subsequent formulas more readable, let us introduce the following notation

$$[y^{-1}x]_k^0 = 1, \quad [y^{-1}x]_k^{n+1} = y_{k-1}^{-1} [y^{-1}x]_{k-1}^n x_k \quad \text{where } n \geq 0.$$

For example

$$[y^{-1}x]_2^4 = y_1^{-1} y_0^{-1} y_{-1}^{-1} y_{-2}^{-1} x_{-1} x_0 x_1 x_2$$

which in $K = 2$ -periodic case should mean

$$[y^{-1}x]_2^4 = y_1^{-1} y_2^{-1} y_1^{-1} y_2^{-1} x_1 x_2 x_1 x_2,$$

Proposition 4.1 *The formal series solution of the system (4.3) is given by*

$$h_k = y_k^{-1} \sum_{n \geq 0} [y^{-1}x]_k^n = \sum_{n \geq 0} (y_{k-n}y_{k-n+1} \dots y_k)^{-1} x_{k-n+1}x_{k-n+2} \dots x_k, \quad (4.4)$$

i.e., we regard h_k as Mal'cev–Neumann series in the symbols y_k^{-1} , x_k for $k \in \mathbb{Z}_K$.

Proof It is convenient to introduce $Z_k = y_k h_k$, and write down the above system in the form

$$Z_k = 1 + y_{k-1}^{-1} Z_{k-1} x_k. \quad (4.5)$$

The above system can be solved by the successive approximation method [56,58] starting from $Z_k^{(0)} = 0$, and proceeding recursively by

$$Z_k^{(j+1)} = 1 + y_{k-1}^{-1} Z_{k-1}^{(j)} x_k, \quad j \geq 0.$$

The series

$$\lim_{j \rightarrow \infty} Z_k^{(j)} = \sum_{n \geq 0} [y^{-1}x]_k^n$$

is the fixed point of equation (4.5). $\square$

Our next step is finding the series representing inverse of the series (4.4). We introduce the following standard notation [44]. By a composition α of n , denoted $\alpha \models n$, we mean a finite sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ of positive integers with $n = \alpha_1 + \alpha_2 + \dots + \alpha_m$. The number $m = |\alpha|$ is called the length of the composition. Given composition $\alpha \models n$, define

$$[y^{-1}x]_k^\alpha = [y^{-1}x]_k^{\alpha_1} \dots [y^{-1}x]_k^{\alpha_m}.$$

Proposition 4.2 *The inverse of the series*

$$h_k = y_k^{-1} \sum_{n \geq 0} [y^{-1}x]_k^n = y_k^{-1} \left(1 + y_{k-1}^{-1} x_k + y_{k-1}^{-1} y_{k-2}^{-1} x_{k-1} x_k + \dots \right)$$

is given by

$$\begin{aligned} h_k^{-1} &= \left(\sum_{n \geq 0} \sum_{\alpha \models n} (-1)^{|\alpha|} [y^{-1}x]_k^\alpha \right) y_k \\ &= \left(1 - y_{k-1}^{-1} x_k + y_{k-1}^{-1} x_k y_{k-1}^{-1} x_k - y_{k-1}^{-1} y_{k-2}^{-1} x_{k-1} x_k + \dots \right) y_k. \end{aligned}$$

Proof Let us look for the inverse of the series $Z_k = \sum_{n \geq 0} [y^{-1}x]_k^n$ in the form $Z_k^{-1} = \sum_{n \geq 0} c_n$, where c_n is homogeneous polynomial of degree $2n$ in the symbols y_l^{-1} and x_l . Then, equation $Z_k^{-1} Z_k = 1$ implies

$$c_0 = 1, \quad c_0[y^{-1}x]_k^n + c_1[y^{-1}x]_k^{n-1} + \cdots + c_n = 0, \quad n > 0.$$

The identification

$$c_n = \sum_{\alpha \models n} (-1)^{|\alpha|} [y^{-1}x]_k^\alpha$$

follows by separation of the last component α_m from composition α

$$\sum_{\alpha \models n} (-1)^{|\alpha|} [y^{-1}x]_k^\alpha = - \sum_{i=1}^n \left(\sum_{\beta \models n-i} (-1)^{|\beta|} [y^{-1}x]_k^\beta \right) [y^{-1}x]_k^i.$$

□

Remark There is a natural bijection between compositions of n and subsets of $\{1, 2, \dots, n-1\}$, see [44],

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \mapsto \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}\},$$

which gives the number 2^{n-1} of compositions of n . We conclude that each homogenous term c_n has 2^{n-1} summands, $n > 0$.

Remark By separating the first components of compositions, we can check directly that the left inverse of Z_k equals its right inverse.

Finally, we can state the explicit transition formulas.

Proposition 4.3 *The formal series solutions of equations (3.1) which define the companion map are given by*

$$x'_k = x_k + \sum_{n \geq 0} \sum_{\alpha \models n} (-1)^{|\alpha|} [y^{-1}x]_{k-1}^\alpha y_{k-1} = x_k + y_{k-1} - y_{k-2}^{-1} x_{k-1} y_{k-1} + \cdots, \quad (4.6)$$

$$\begin{aligned} y'_k &= - \sum_{n > 0} \sum_{\alpha \models n} (-1)^{|\alpha|} [y^{-1}x]_k^\alpha y_k = y_{k-1}^{-1} x_k y_k - y_{k-1}^{-1} x_k y_{k-1}^{-1} x_k y_k \\ &\quad + y_{k-1}^{-1} y_{k-2}^{-1} x_{k-1} x_k y_k + \cdots \end{aligned} \quad (4.7)$$

Remark Equations (4.1)–(4.3) imply another form of the companion map

$$x'_k = h_{k-1}^{-1} y_k h_k, \quad y'_k = h_{k-1} x_k h_k^{-1}. \quad (4.8)$$

Remark Equations (4.3) and (3.5) imply

$$h_k x_{k+1} x_{k+2} \dots x_{k+\ell+1} + \mathcal{P}_{k+1}^{(\ell)} = y_{k+1} y_{k+2} \dots y_{k+\ell+1} h_{k+\ell+1},$$

which for $\ell = K - 1$ under K -periodicity assumption leads to following equation satisfied by h_k

$$y_{k+1} y_{k+2} \dots y_{k+K} h_k = h_k x_{k+1} x_{k+2} \dots x_{k+K} + \mathcal{P}_{k+1}^{(K-1)}. \quad (4.9)$$

By the successive approximation technique, one can obtain the same series solution (4.4). Under additional condition of centrality of the products $x_{k+1} x_{k+2} \dots x_{k+K}$ and $y_{k+1} y_{k+2} \dots y_{k+K}$, the above equation was considered in [17]. Then, its solution reads

$$\begin{aligned} x'_k &= \left[\mathcal{P}_k^{(K-1)} \right]^{-1} y_k \mathcal{P}_{k+1}^{(K-1)}, \\ y'_k &= \mathcal{P}_k^{(K-1)} x_k \left[\mathcal{P}_{k+1}^{(K-1)} \right]^{-1}, \end{aligned} \quad (4.10)$$

and agrees with the corresponding result obtained in the fully commutative case [34, 35].

Remark In the periodic reduction equations (3.7) and (3.8) and the matrix refactorization problem stated in Corollary 3.2 read

$$\begin{aligned} & \begin{pmatrix} x_1 & 0 & \dots & 0 & 1 \\ 1 & x_2 & 0 & \dots & 0 \\ 0 & 1 & x_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & x_K \end{pmatrix} \begin{pmatrix} y_1 & 0 & \dots & 0 & 1 \\ 1 & y_2 & 0 & \dots & 0 \\ 0 & 1 & y_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & y_K \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1 & 0 & \dots & 1 & \beta_K \\ \beta_1 & \alpha_2 & 0 & \dots & 1 \\ 1 & \beta_2 & \alpha_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & \beta_{K-1} & \alpha_K \end{pmatrix}. \end{aligned}$$

Observe that every element x_k or y_k satisfies an equation in a form of a periodic continued fraction. For example, elimination of all variables except from y_1 gives

$$y_1 = \beta_1 - \alpha_2 \left(\beta_2 - \alpha_3 \left(\beta_3 - \dots - \alpha_K \left(\beta_K - \alpha_{K-1} y_1^{-1} \right)^{-1} \dots \right)^{-1} \right)^{-1}$$

As it was observed in [19], solution of the refactorization problem can be interpreted as a non-commutative analog of the Galois theorem for periodic continued fractions.

For more information concerning relation of continued fraction theory to integrable systems see for example [29,31], and especially [53] in the context of Weyl groups. Non-commutative continued fractions and integrable systems were considered also in [12].

4.2 Warning: new “absurd formula”

It turns out that if we rewrite the basic system (4.3) in the form

$$h_k = -x_{k+1}^{-1} + y_{k+1}h_{k+1}x_{k+1}^{-1}, \quad (4.11)$$

then the successive approximation technique gives its unique solution as

$$h_k = -\sum_{n \geq 0} [yx^{-1}]_k^n x_{k+1}^{-1}, \quad (4.12)$$

where $[yx^{-1}]_k^n$ is defined recursively by

$$[yx^{-1}]_k^0 = 1, \quad [yx^{-1}]_k^{n+1} = y_{k+1}[yx^{-1}]_{k+1}^n x_{k+2}^{-1}, \quad n \geq 0.$$

Extending the notation for compositions in the same way as in the previous case, we obtain the solution

$$\begin{aligned} x'_k &= -x_k \sum_{n \geq 0} \sum_{\alpha \models n} (-1)^{|\alpha|} [yx^{-1}]_{k-1}^\alpha \\ &= x_k y_k x_{k+1}^{-1} - x_k y_k x_{k+1}^{-1} y_k x_{k+1}^{-1} + x_k y_k y_{k+1} x_{k+2}^{-1} x_{k+1}^{-1} + \dots, \end{aligned} \quad (4.13)$$

$$y'_k = y_k + x_{k+1} \sum_{n \geq 0} \sum_{\alpha \models n} (-1)^{|\alpha|} [yx^{-1}]_k^\alpha = y_k + x_{k+1} - x_{k+1} y_{k+1} x_{k+2}^{-1} + \dots, \quad (4.14)$$

which seems to contradict uniqueness of the companion map.

This paradox can be explained as follows. Both expressions (4.4) and (4.12) for h_k represent Mal'cev–Neumann series in different spaces and cannot be considered simultaneously. A similar situation happens with the following Euler's “absurd formula”

$$\sum_{n \in \mathbb{Z}} x^n = 0,$$

see detailed discussion in [9]. From various aspects of the above “identity” given there, we can pick up the most relevant one in our context. When considered as rational expressions, the two parts of the series (which, however, again cannot be considered

simultaneously on the series level)

$$\sum_{n \geq 0} x^n = (1 - x)^{-1}, \quad \sum_{n < 0} x^n = \sum_{n > 0} (x^{-1})^n = x^{-1}(1 - x^{-1})^{-1}, \quad (4.15)$$

summed up give the desired “result.”

One can, as a by-product of our considerations, add to the list new “absurd formula”

$$\sum_{n \in \mathbb{Z}} x^n y^n = 0,$$

where x and y are non-commuting symbols (for commuting symbols it reduces to the above one). Notice that two Mal’cev–Neumann series

$$h^{(1)} = \sum_{n \geq 0} x^n y^n, \quad h^{(2)} = - \sum_{n > 0} (x^{-1})^n (y^{-1})^n,$$

$h^{(1)} \in \mathbb{Q}((x, y))$, and $h^{(2)} \in \mathbb{Q}((x^{-1}, y^{-1}))$ satisfy equations

$$h^{(1)} = 1 + x h^{(1)} y, \quad h^{(2)} = -x^{-1} y^{-1} + x^{-1} h^{(2)} y^{-1},$$

which can be easily transformed one into another.

5 The companion map and context-free languages

The present Section arose from (unsuccessful) attempts to rewrite the transformation formulas (4.6) and (4.7) in a more compact form as a rational expression. As we mentioned in Sect. 4, such formulas (4.10) have been previously obtained under the centrality assumption in [17] or in the fully commutative case in [34,35]. The answer came from the theory of formal languages [32], where analogous questions were discussed [8,39,56] earlier.

To make the paper self-contained, let us recall relevant definitions [32]. For a set A , called an *alphabet*, we denote by A^* the set consisting of all words (finite sequences of letters in A including the empty sequence λ). A context-free grammar $G(N, A, R, Y)$ is given by

- (1) an alphabet N whose elements are called variables or nonterminals,
- (2) an alphabet A (disjoint from N) whose elements are called terminals,
- (3) a finite set R of rewriting rules or productions, each rule being a pair $Y_i \rightarrow \gamma$, where $Y_i \in N$ and $\gamma \in (A \cup N)^*$,
- (4) an element $Y \in N$ called the initial variable.

For two words γ_1, γ_2 in $(A \cup N)^*$, we write $\gamma_1 \Rightarrow \gamma_2$ iff $\gamma_1 = \delta_1 Y_i \delta_2$, $\gamma_2 = \delta_1 \gamma \delta_2$ and $Y_i \rightarrow \gamma$ is in R . Let $\overset{*}{\Rightarrow}$ be the reflexive transitive closure of $\Rightarrow$, then the language $L(G)$ generated by G is

$$L(G) = \{w \in A^* | Y \overset{*}{\Rightarrow} w\}.$$

A context-free grammar is termed regular iff the right-hand side of every production belongs to $AN \cup \lambda$. A language L is referred to as context-free or regular iff $L = L(G)$ for some context-free or regular grammar, respectively.

To simplify formulas, in this Section we use the notation $\bar{x}_k = y_{k-1}^{-1}$. Then, the system (4.5) reads

$$Z_k = 1 + \bar{x}_k Z_{k-1} x_k, \quad k = 1, \dots, K. \quad (5.1)$$

Let us consider the support L_k of the series Z_k as a subset of the set $\mathcal{X}_K^*$ consisting of all words over the alphabet $\mathcal{X}_K = \{x_1, \dots, x_K, \bar{x}_1, \dots, \bar{x}_K\}$.

Proposition 5.1 *The language L_k , $k = 1, \dots, K$, is generated by the context-free grammar with variables $\{Y_1, Y_2, \dots, Y_K\}$, terminals $\{x_1, \dots, x_K, \bar{x}_1, \dots, \bar{x}_K\}$, the production rules*

$$Y_i \rightarrow \lambda, \quad Y_i \rightarrow \bar{x}_i Y_{i-1} x_i, \quad (5.2)$$

and the initial variable Y_k .

Example 5.1 In the simplest case $K = 1$, the unique solution of the equation (we skip the subscripts)

$$Z = 1 + \bar{x} Z x, \quad (5.3)$$

is the series

$$Z = \sum_{n=0}^{\infty} \bar{x}^n x^n = 1 + \bar{x} x + \bar{x} \bar{x} x x + \bar{x} \bar{x} \bar{x} x x x + \dots, \quad (5.4)$$

with the support $L = \{\bar{x}^n x^n\}_{n \geq 0} = \{\lambda, \bar{x} x, \bar{x} \bar{x} x x, \bar{x} \bar{x} \bar{x} x x x, \dots\}$. The language L is generated by the grammar with variable Y , which is also the initial symbol, terminals $\{\bar{x}, x\}$ and the productions

$$Y \rightarrow \lambda, \quad Y \rightarrow \bar{x} Y x. \quad (5.5)$$

The above language is the standard and the simplest example of a context-free language, which is not regular. In the theory of computing machines, this means that it can be accepted by a push-down automaton but cannot be accepted by a finite-state automaton (see [32] or any textbook of the theory of formal languages). This can be shown by the so-called pumping lemma, and the arguments go over to the case of languages L_k . By the known relation between regular languages and rational series [8, 39, 56], the corresponding solutions Z_k to system (5.1) cannot be written as rational expressions.

Recall that to find transformation formulas we had to calculate inverses of the series Z_k , $k = 1, \dots, K$. We remark that the inverse Z^{-1} of the series (5.4) was calculated

in [7]. By Proposition 4.2, the support of the series Z_k^{-1} is the language

$$L_k^* = \bigcup_{n \geq 0} \{[\bar{x}x]_k^\alpha \mid \alpha \models n\}, \quad (5.6)$$

whose elements are all possible concatenations of words of L_k . Notice that elements of languages L_k^* are in bijection with compositions. From that point of view, the elements of the language L of the above example can be put in bijection with generators of the algebra of the non-commutative symmetric functions [25,44]. This Hopf algebra is a generalization of the algebra of symmetric functions [45], which plays pivotal role in the theory of KP hierarchy [46]. The corresponding extension for $K > 1$ of the Hopf algebra of the non-commutative symmetric functions together with its graded dual (quasi-symmetric functions) was the subject of [20].

6 Conclusion and open problems

Starting from periodic reduction in the KP-map for non-commuting symbols, we have constructed its companion map in terms of Mal'cev–Neumann series which cannot be reduced to rational expressions. By multidimensional consistency of the KP map, our formulas give a realization of the A -type Coxeter relations. We studied the action of the finite Coxeter group only, leaving for further research the case of affine Coxeter groups and their eventual connection [53] with non-commutative discrete Painlevé equations [18,28,42].

An interesting output of our research is the relation of the companion to KP map in the periodic reduction to theory of formal languages. The type of series in non-commuting variables, we have encountered in the present research, corresponds to the so-called context-free languages and push-down automata. This aspect certainly deserves deeper studies, especially in the spirit of current applications of integrable systems to combinatorics.

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Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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