Correction to: Matrix versions of the Hellinger distance

Rajendra Bhatia¹ · Stephane Gaubert² · Tanvi Jain³

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Theorem 9 in our paper [1] is wrong. The statement should be replaced by the following:

Theorem *When $d = d_3$, the minimum in (13) is attained at a unique point X which is the solution of the matrix equation*

$$X^2 = \frac{2}{\pi} \sum_{j=1}^m w_j \int_0^{\infty} \left(\lambda X^{-1} + A_j^{-1} \right)^{-2} \sqrt{\lambda} d\lambda. \quad (1)$$

This minimiser is the 1/2-power mean $Q_{1/2}$ given by (14) if $Q_{1/2}$ commutes with every A_j . In particular, the minimiser is $Q_{1/2}$ if

- (i) *All A_j 's commute, or*
- (ii) *$Q_{1/2} = I$.*

An outline of Proof Let

$$f(X) = \sum_{j=1}^n w_j \Phi_3(X, A_j)$$

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✉ Tanvi Jain
tanvi@isid.ac.in

Rajendra Bhatia
rajendra.bhatia@ashoka.edu.in

Stephane Gaubert
Stephane.Gaubert@inria.fr

¹ Ashoka University, Sonapat, Haryana 131029, India

² INRIA and CMAP, Ecole Polytechnique, CNRS, 91128 Palaiseau, France

³ Indian Statistical Institute, New Delhi 110016, India

be the objective function (13) of [1]. Using Proposition 1 we see that

$$\begin{aligned} Df(X)(Y) &= \operatorname{tr} \left(Y - 2 \sum_{j=1}^m w_j \int_0^\infty (\lambda + X A_j^{-1})^{-1} Y (\lambda + A_j^{-1} X)^{-1} d\nu(\lambda) \right) \\ &= \operatorname{tr} \left(\left(I - 2 \sum_{j=1}^m w_j \int_0^\infty ((\lambda + X A_j^{-1})(\lambda + A_j^{-1} X))^{-1} d\nu(\lambda) \right) Y \right). \end{aligned}$$

Hence X_0 is a critical point of f if and only if X_0 satisfies the matrix equation

$$I = 2 \sum_{j=1}^m w_j \int_0^\infty ((\lambda + X A_j^{-1})(\lambda + A_j^{-1} X))^{-1} d\nu(\lambda). \quad (2)$$

Taking congruence with X on both sides, we see that this equation is equivalent to (1).

The function in (13) is strictly convex. So it has at most one critical point. This entails the uniqueness of the solution to (1). The existence of a solution follows from the Brouwer fixed point theorem. The argument is similar to that in our proof of Theorem 10.

On differentiating (25) of [1], we get

$$\frac{1}{2} x^{-1/2} = \int_0^\infty \frac{1}{(\lambda + x)^2} d\nu(\lambda). \quad (3)$$

So if $Q_{1/2} A_j^{-1} = A_j^{-1} Q_{1/2}$ for all $1 \leq j \leq m$, from (2) and (3) we get

$$\begin{aligned} I &= Q_{1/2}^{1/2} Q_{1/2}^{-1/2} = \sum_{j=1}^m w_j (A_j^{1/2} Q_{1/2}^{-1/2}) \\ &= \sum_{j=1}^m w_j (A_j Q_{1/2}^{-1})^{1/2} \\ &= 2 \sum_{j=1}^m w_j \int_0^\infty (\lambda + A_j^{-1} Q_{1/2})^{-2} d\nu(\lambda) \\ &= 2 \sum_{j=1}^m w_j \int_0^\infty ((\lambda + Q_{1/2} A_j^{-1})(\lambda + A_j^{-1} Q_{1/2}))^{-1} d\nu(\lambda). \end{aligned}$$

This proves the second statement of the theorem. $\square$

We became aware of the mistake in Theorem 9 of [1] from the arxiv preprint [2]. The authors there give a correct version (in a different notation) and also give an example to show that the barycentre obtained in (1) does not reduce to formula (17) in our paper [1].

References

1. Bhatia, R., Gaubert, S., Jain, T.: Matrix versions of the Hellinger distance. *Lett. Math. Phys.* **109**, 1777–1804 (2019)
2. Pitrik, J., Virosztek, D.: Quantum Hellinger distances revisited. [arXiv:1903.10455v3](https://arxiv.org/abs/1903.10455v3)

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