Contractions of group representations via geometric quantization

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Abstract

We propose a general framework to contract unitary dual of Lie groups via holomorphic quantization of their coadjoint orbits, using geometric quantization. The sufficient condition for the contractibility of a representation is expressed via cocycles on coadjoint orbits. This condition is verified explicitly for the contraction of SU_2 into $\mathbb{H}$. We construct two types of contractions that can be implemented on every matrix Lie group with diagonal contraction matrix.

1 Introduction

Physicists study certain degenerations of group laws which are called contractions. In this paper, we study Lie group contractions in the context of geometric quantization. The degeneration is understood in the sense that group law becomes "more abelian", i.e. more structure constants of Lie algebra become zero. The pioneering work of Inönü and Wigner [14] is a classical original source. Their motivation was simply to relate two different physics: Galilean and Lorentzian. Contractions of representation of Lie groups and Lie algebras and their unitary duals have attracted the attention of

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many authors [5,7,12,13,19,22,23] and continue to be a field of active investigations; see, for example, [3,4,9,10].

The first general global definition of Lie group contraction has been given in [19] where the authors study the contraction of the principal continuous series of the de Sitter groups SO(n, 1) to positive mass representations of both the Euclidean group $E(n)$ and the Poincaré group $E_0(n-1, 1)$. In [22], a contraction of SU(2) to the Heisenberg group $\mathbb{H}$ was studied in detail. The authors showed that the group action on the manifold depends smoothly on the contraction parameter $\varepsilon > 0$. They also established transference results linking Fourier analysis on these two groups. Applying [22], Ricci and Rubin proved in [23] an analogue of de Leeuw's theorem relating L^p-norms of Fourier multipliers on $SU(2)$ and $\mathbb{H}$. A short and elegant proof of the main result of [22] was given in [5] employing Berezin quantization. Furthermore, it was shown by [8] that every semisimple Lie group can be contracted to its Cartan motion group.

Generally, one cannot expect that there always exists a contraction $\Phi_\varepsilon \colon G_0 \to G_1$ for any Lie group G_0 and G_1. Indeed, let $\mathfrak{g}_1$ and $\mathfrak{g}_0$ be the Lie algebras of G_1 and G_0 and $U_\varepsilon \colon \mathfrak{g}_1 \to \mathfrak{g}_0$ be differential of Φ_ε. Then, the existence of contraction U_ε is equivalent to the fact that $\mathfrak{g}_0$ is the limit in the Zariski topology on the affine variety of Lie algebras. It can be seen that not every two points in the affine variety of Lie algebras of dimension less than 6 are path-connected in the Zariski topology [18, Theorem 1]. In other words, the existence of Lie algebra contraction $U_\varepsilon \colon \mathfrak{g}_0 \to \mathfrak{g}_1$ is equivalent to saying that $\mathfrak{g}_0$ is the limit of Lie algebras $U_\varepsilon \mathfrak{g}_1$ in the Zariski topology. Hence, we shall assume that there exists a Lie algebra contraction U_ε of $\mathfrak{g}_1$ into $\mathfrak{g}_0$.

The aim of the paper is to show that the existence of Lie group contraction combined with the geometric quantization yields a systematic approach to link harmonic analysis on two groups. The novelty of the paper is that we do not impose any restriction on the group nor on the contraction. In addition, we tried to answer the following question.

Question 1.1 *Let G_0 and G_1 be two Lie groups and let $\Phi_\varepsilon \colon G_1 \to G_0$ be a Lie group contraction. Does there exist a contraction $\widehat{\Phi}_\varepsilon$ of $\widehat{G}_1$ into $\widehat{G}_0$?. In other words, does the following diagram commute*

$$
\begin{array}{ccc}
G_1 & \xrightarrow{GQ} & \widehat{G}_1 \\
\downarrow{\scriptstyle\Phi_\varepsilon} & & \downarrow{\scriptstyle\widehat{\Phi}_\varepsilon} \\
G_0 & \xrightarrow{GQ} & \widehat{G}_0
\end{array}
\tag{1.1}
$$

where GQ stands for the geometric quantization. We shall give meaning to $\widehat{\Phi}_\varepsilon$ as an MN-contraction of $\widehat{G}_1$ into $\widehat{G}_0$.

A similar problem has been treated in [4] in the context of the algebraic Harish-Chandra modules.

2 Geometric quantization

2.1 General theory

We first recall a few basic elements of the geometric quantization needed here. The procedure of quantization consists of assigning to a smooth function on a symplectic manifold (M, ω), an operator on a Hilbert space $\mathcal{H}$ with the following three properties. For each $f : M \to \mathbb{R}$, there should be an operator $\widehat{f}$ acting on $\mathcal{H}$ such that

(1) the map $f \mapsto \widehat{f}$ is linear;
(2) if f is constant, $\widehat{f}$ is a multiplication operator
(3) if $[f_1, f_2] = f_3$, then $[\widehat{f_1}, \widehat{f_2}] = -i\hbar\widehat{f_3}$,

where $\hbar$ is the Planck constant.

The main ingredient in the program of geometric quantization for a symplectic manifold (M, ω) of dimension $2n$ with symplectic (exact) 2-form ω is to construct a Hermitian line bundle $\pi : B \to M$ with Hermitian connection ∇ for which the curvature 2-form is equal to $\hbar^{-1}\omega$. Locally, the connection ∇, or covariant derivative ∇_X for a vector field X on M, is given by

$$\nabla_X(f) = X(f) - \frac{i}{\hbar}\theta(X)f, \tag{2.1}$$

where f is a function on M where the connection 1-form θ can be shown to be locally related [11, Proposition 23.5] to ω as $d\theta = \omega$. It is possible to construct such a line bundle on (M, ω) only if the symplectic manifold satisfies the integrality condition or is quantizable. Such a result is, for example, proved in [11, Theorem 23.9, p.489] where the integrality condition is

$$\frac{1}{2\pi\hbar}\int_{\Sigma}\omega \in \mathbb{Z}, \tag{2.2}$$

where Σ is an arbitrary closed 2-surface in M. This condition is purely geometric and involves the computation of the holonomy of the line bundle for closed loop, which when contracted must be equal to 1. The condition (2.2) is obtained after using Stokes's theorem.

This line bundle has a natural Hilbert space $\mathcal{H}$ consisting of square-integrable global sections with respect to the natural Liouville measure of M and a Hermitian structure on the line bundle B. It is an inner product in the fibres denoted by $(\cdot, \cdot)$ with the property to be smooth in the sense that for a smooth section $v : M \to B$, (v, v) is a smooth function of M. We now have the Hilbert space and we need to find the map $f \mapsto \widehat{f}$ satisfying the three conditions. It is given, for a section $s : M \to B$, by

$$\widehat{f}(s) = -i\hbar\nabla_{X_f}s + fs, \tag{2.3}$$

where X_f is the Hamiltonian vector field associated with the function f, i.e. $\omega(X_f, Y) = df(Y)$ for any vector field Y, and fs is the multiplication operation.

This operator obviously satisfies (1) and (2); the third condition is satisfied by using the properties of the covariant derivative ∇_{X_f}; see [11,28] for more details.

We will now follow [28] to expose a dynamical interpretation of $\widehat{f}$, which will give us a useful formula for later. Let V_f be the vector field on B given in a local trivialization by

$$V_f = X_f + \hbar^{-1} L \frac{\partial}{\partial \phi} , \tag{2.4}$$

where $z = re^{i\phi}$ is the coordinate on the fibre of L and

$$L = \theta(X_f) - f \tag{2.5}$$

is the Lagrangian associated with the function f, or equivalently the Legendre transform of the Hamiltonian f. This vector field is constructed so that it is invariant under gauge transformations and $f \mapsto V_f$ is a Lie algebra isomorphism from $C^\infty(M)$ to a subalgebra of vector fields on B. Let ξ_t denote the flow of V_f which projects to ρ_t, the flow of X_f. From these two flows, we can define the action of the flow ρ_t on the sections $s : M \to B$ by mapping it to $\widehat{\rho}_t$ such that

$$\xi_t(\widehat{\rho}_t s(m)) = s(\rho_t m) . \tag{2.6}$$

This formula shows that $\widehat{\rho}_t$ is the correct pull-back of the sections such that the flow ξ_t corresponds to only moving the base point with ρ_t. In a local trivialization, the sections s and $\widehat{\rho}_t s$ are represented by the complex functions ψ and $\widehat{\rho}_t \psi$ with the following relation

$$\widehat{\rho}_t \psi(m) = \psi(\rho_t m) \exp\left(-\frac{i}{\hbar} \int_0^t L(\rho_{t'} m) \mathrm{d}t' \right) . \tag{2.7}$$

It might happen that the quantum Hilbert space is reducible. A notion of polarization imposes further restrictions.

Definition 2.1 A polarization $\mathcal{P}$ of a symplectic manifold M is a choice at each point $z \in M$ of a Lagrangian subspace $P(z) \subset T_z^{\mathbb{C}} M$ (dimension is n) satisfying the following conditions

(1) If two complex vector fields X and Y lie in $P(z)$ at each point z, then the commutator $[X, Y] \in P(z)$.
(2) The dimension of $P(z) \cap \overline{P(z)}$ is constant.

Definition 2.2 (*The quantum Hilbert space*). Let $\mathcal{P}$ be a polarization of M. A smooth section s of $L \to M$ is said to be polarized if

$$\nabla_X s = 0, \quad \forall X \in \overline{P} . \tag{2.8}$$

The quantum Hilbert space $\mathcal{H}_{\mathcal{P}}$ associated with $\mathcal{P}$ is the closure in the prequantum Hilbert space of the space of smooth, square-integrable, polarized sections $s\colon L \to M$.

Definition 2.3 A polarization $\mathcal{P}$ of a symplectic manifold M is called purely complex if

$$P(z) \cap \overline{P(z)} = \varnothing$$

for every $z \in M$.

A purely complex polarization $\mathcal{P}$ yields existence of a unique integrable almost complex structure J on M such that $P(z)$ is $(1, 0)$-tangent space of M for all $z \in M$.

Definition 2.4 Let $\mathcal{P}$ be a purely complex polarization of M and let J be the integrable almost complex structure associated with $\mathcal{P}$ such that $J(z) = i \operatorname{Id}$ on $P(z)$ and $J(z) = -i \operatorname{Id}$ on $\overline{P(z)}$. We say that $\mathcal{P}$ is a Kähler polarization if the bilinear form

$$g(X, Y) := \omega(X, J(z)Y), \quad X, Y \in T_z M,$$

is positive definite for each $z \in M$.

For a thorough exposition of the geometric quantization, we refer [11,15,16,28].

2.2 Lie group representation

We now specify this construction to Lie group representations. Let G be a locally compact Lie group with Lie algebra $\mathfrak{g}$. Each $\xi \in \mathfrak{g}$ generates a right-invariant vector field R_ξ. Let $\mu \in \mathfrak{g}^*$ and $\mathcal{O}_\mu$ be the coadjoint orbit through μ. A coadjoint orbit is a symplectic manifold with the symplectic form

$$\omega_\mu(\xi, \eta) = \mu([\xi, \eta]), \quad \forall \xi, \eta \in \mathfrak{g}, \tag{2.9}$$

corresponding to the reduction of the canonical symplectic form on T^*G using the momentum map $J : T^*G \to \mathfrak{g}^*$. This form is exact and thus $\omega_\mu = \mathrm{d}\theta_\mu$, where

$$\theta_\mu(\xi) = \mu(\xi). \tag{2.10}$$

$(\mathcal{O}_\mu, \mathrm{d}\theta_\mu)$ is a presymplectic manifold, the symplectic manifold of right cosets G/G_μ, where G_μ is the stabilizer of μ, i.e.

$$G_\mu := \{g \in G \colon \operatorname{Ad}^*_g \mu = \mu\}, \tag{2.11}$$

and its Lie algebra

$$\mathfrak{g}_\mu = \{\xi \in \mathfrak{g} \colon \operatorname{ad}^*_\xi \mu = 0\}. \tag{2.12}$$

A coadjoint orbit $\mathcal{O}_\mu$ is integral if the symplectic manifold $(\mathcal{O}_\mu, \omega_\mu)$ is integral in the sense of (2.2). Let $\mathcal{O}_\mu$ be an integral coadjoint orbit, then there exists [11, Theorem 23.9, p.489] a holomorphic line bundle B over $\mathcal{O}_\mu$. From (2.10) the Lagrangian L_μ associated with μ is given by $L_\mu(\xi) = \frac{1}{2}\mu(\xi)$.

Let $\pi \in \widehat{G}$ and let $(U_\alpha, \varphi_\alpha)$ be a local trivialization of $\mathcal{O}_\mu$. Then, it follows from (2.7) that the flow ρ_t acts locally on the space of global holomorphic sections of $B \to \mathcal{O}_\mu$ via the formula

$$\pi(g)s(m) = s(\mathrm{Ad}^*_{g^{-1}}m)e^{\frac{i}{\hbar}\mu\int_m^{m_1} L_\mu \, ds}, \ m \in \mathcal{O}_\mu. \tag{2.13}$$

where $m_1 = \mathrm{Ad}^*_{g^{-1}}m$. We used the fact that the Lagrangian L_μ is invariant under the coadjoint flow and the flow on the base is given by the coadjoint action; see [20, Chapter 13].

The second element of the construction is a choice of polarization. A remarkable result [25,26] yields the existence of polarizing subalgebras for solvable Lie algebras. These subalgebras can be constructed algorithmically [21]. It is unknown to the authors whether there exist complex polarizations for general connected Lie groups. Therefore, we make the following assumption.

Assumption 2.5 *Let G be a locally compact Lie group such that for every coadjoint orbit $\mathcal{O}$ there exists invariant Kähler polarization $\mathcal{P} \subset T\mathcal{O}$.*

3 Contractions

3.1 Lie group contractions

Before using geometric quantization to contract unitary duals, let us recall classic results on Lie group contractions.

Let G_1 and G_0 be two locally compact connected Lie groups of the same dimension.

Definition 3.1 We say that the family $\{\Phi_\varepsilon\}_{\varepsilon \geq 0}$ of differential maps

$$\Phi_\varepsilon \colon G_0 \to G_1, \tag{3.1}$$

mapping the identity e_{G_0} of G_0 to the identity e_{G_1} of G_1, defines a *contraction* of G_1 to G_0, if given any relatively compact open neighbourhood V of e_{G_0}

(1) there is $\varepsilon_V > 0$ such that for $\varepsilon < \varepsilon_V$ the map $\Phi_\varepsilon\big|_V$ is a diffeomorphism;
(2) If W is such that $W^2 \subset V$ and $\varepsilon < \varepsilon_V$, then $\Phi_\varepsilon(W)^2 \subset \Phi_\varepsilon(V)$;
(3) for $x, y \in V$

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon^{-1}(\Phi_\varepsilon(x)\Phi_\varepsilon(y)^{-1}) = xy^{-1}. \tag{3.2}$$

The first two conditions are necessary for the limit of the third condition to be well defined.

The Lie algebra can be contracted by the differential $U_\varepsilon = D_e \Phi_\varepsilon$ of Φ_ε. That is, at the identity e_{G_0}

$$U_\varepsilon := D_e \Phi_\varepsilon : \mathfrak{g}_0 \to \mathfrak{g}_1 . \tag{3.3}$$

This map is then used to contract the Lie algebra $U_\varepsilon : \mathfrak{g}_0 \to \mathfrak{g}_1$. Now, we fix a contraction $\Phi_\varepsilon : G_0 \to G_1$ of G_1 to G_0 as in Definition 3.1.

3.2 Contraction of the unitary dual

Assume that there exists a contraction $\Phi_\varepsilon : G_0 \to G_1$. Then, the differential U_ε of Φ_ε allows us to deform the coadjoint orbits of G_1 into the coadjoint orbits of G_0. We express the adjoint action Ad_0 of G_0 on $\mathfrak{g}_0$ as follows:

$$\mathrm{Ad}_0(e^X) = \lim_{\varepsilon \to 0} U_\varepsilon^{-1} \circ \mathrm{Ad}(e^{U_\varepsilon(X)}) \circ U_\varepsilon, \quad X \in \mathfrak{g}_0 . \tag{3.4}$$

By the duality, we obtain the coadjoint action

$$\mathrm{Ad}_0^*(e^X) = \lim_{\varepsilon \to 0} [U_\varepsilon]^* \circ \mathrm{Ad}^*(e^{U_\varepsilon(X)}) \circ [U_\varepsilon^{-1}]^*, \quad X \in \mathfrak{g}_0 , \tag{3.5}$$

where

$$U_\varepsilon^*(X)(Y) = Y(U_\varepsilon(X)), \quad X, Y \in \mathfrak{g}_0^* . \tag{3.6}$$

The corresponding character of the contracted group G_0 is given by

$$\chi_\mu(e^X) = \lim_{\varepsilon \to 0} e^{U_\varepsilon^*(f)(X)}, \quad \mu \in \mathfrak{g}_0^*, \quad X \in \mathfrak{g}_0 . \tag{3.7}$$

Following from example [2], the coadjoint orbits of the contracted group G_0 can be obtained via the following procedure

(1) Fix an element $\mu \in \mathfrak{g}_0^*$ and fix a coadjoint orbit $\mathcal{O}_\mu$.
(2) Transport the point μ to $\mathfrak{g}_1^*$ by using the map $(U_\varepsilon^*)^{-1}$.
(3) The action of G_1 on $\mu_\varepsilon = (U_\varepsilon^*)^{-1}(\mu)$ originates a G_1-coadjoint orbit $\tilde{\mathcal{O}}_\varepsilon \subset \mathfrak{g}_1^*$.
(4) The orbit $\tilde{\mathcal{O}}_\varepsilon$ is translated to $\mathfrak{g}_0^*$ by using the map U_ε^*, i.e.

$$\mathcal{O}_\mu^\varepsilon = U_\varepsilon^*(\tilde{\mathcal{O}}_\varepsilon) . \tag{3.8}$$

The orbit method of Kirillov is a method in geometric representation theory which identifies the unitary representations of Lie groups with the canonical G-action on spaces of sections of specific line bundles over the coadjoint orbits. The general case of a Lie group is not yet fully understood. Vogan rigorously justified [27] the orbit method for reductive Lie groups. However, we shall restrict ourselves to solvable type I Lie groups to use results of Kostant [15] and Auslander [1].

Coadjoint orbits can be described by Ad_G-invariant functions $f : \mathfrak{g}^* \to \mathbb{C}$. However, it is not clear whether the holomorphic line bundles over the coadjoint orbits $\mathcal{O}_\varepsilon$ are contracted properly. Therefore, we introduce.

Definition 3.2 Let $(U_\alpha, \varphi_\alpha)$ be a local trivialization of the line bundle $\mathcal{O}_\mu \to B$. We shall say that a family $\{\mathcal{O}_\varepsilon\}_{\varepsilon > 0}$ of integral polarized coadjoint orbits $\mathcal{O}_\varepsilon$ is an admissible contraction of $\mathcal{O}_\mu$ if there is a family of local trivializations $(U_\alpha, c_\alpha^\varepsilon)$ of the line bundles $\mathcal{O}_\varepsilon \to B_\varepsilon$ such that

$$\lim_{\varepsilon \to 0} c_{\alpha\beta}^\varepsilon = c_{\alpha\beta}, \quad z \in V, \tag{3.9}$$

uniformly on compact subsets $V \subset \mathbb{C}^r$, where $c_{\alpha\beta}^\varepsilon$ and $c_{\alpha\beta}$ are the transition functions associated with the trivializations $(U_\alpha, \varphi_\alpha)$ and $(U_\alpha, \varphi_\alpha^\varepsilon)$.

Definition 3.3 Let $B \to M$ be a holomorphic line bundle. We say a sequence of global sections $\{s_n : M \to B\}$ converges to the global section $s : M \to B$ if

$$\lim_{n \to \infty} c_{\alpha\beta}^n(x) s_{n\beta}(x) = s_\alpha(x), \quad x \in U_{\alpha\beta}, \tag{3.10}$$

where $s_{n\beta}$ and s_α are the localization of s_n and s on U_β and U_α, respectively.

A rigorous notion of contraction of representations was first introduced in [19].

Definition 3.4 [19] We say that a representation σ of G_0 in $\mathcal{H}$ is MN-contraction of the sequence $\{\pi^n\}$ of representations π^n of G_1 in $\mathcal{H}^n$ if there exists a sequence ε_n with limit 0, a sequence of unitary operators $A_n : \mathcal{H}^n \to \mathcal{H}$ and a dense subspace D of $\mathcal{H}$ such that

(1) For each $v \in D$ there is an integer $n(v)$ such that $n \geq n(v)$ implies $v \in A_n(\mathcal{H}^n)$;
(2) For each $v \in D$ and $g \in G_0$

$$\lim_{n \to 0} \|A_n \pi^n(\Phi_{\varepsilon_n}(g)) A_n^{-1} v - \sigma(g) v\|_{\mathcal{H}} = 0. \tag{3.11}$$

Question 3.5 [6] *Let G_1 be a semisimple compact Lie group. Assume that there is a group contraction of G_1 to G_0. What unitary representations π of G_0 are MN-contractions of the unitary irreducible representations of G_1?*

In Theorem 3.6 we find sufficient condition on elements $\pi \in \widehat{G}$ to be contraction of a sequence $\{\pi_\varepsilon\} \subset \widehat{G_1}$. The condition is expressed in terms of the corresponding coadjoint orbits.

Theorem 3.6 *Let G_1 be a compact Lie group and G_0 be a locally compact solvable type I Lie group. Let $\pi \in \widehat{G_0}$ be an infinite-dimensional irreducible unitary representation of G_0. Then, π is an MN-contraction of the sequence of unitary irreducible representations if $\mathcal{O}_\varepsilon$ is an admissible contraction of $\{\mathcal{O}_\pi\}$.*

Question 3.7 *Is the converse true? In other words, let π be an MN-contraction of the sequence of unitary representations $\pi_\varepsilon \in \widehat{G_1}$. Then, the coadjoint orbits $\mathcal{O}_\varepsilon$ are an admissible contraction of the coadjoint orbit $\{\mathcal{O}_\pi\}$.*

Sketch of the argument for Question 3.7 Thinking locally, we should have MN-contraction on each U_α, i.e.

$$\pi_\varepsilon(g) s_\alpha$$

on each local section s_α. Each local section gives rise to cocycles $c_{\alpha,\beta}$. Adding a bit more mathematical rigour, one should be able to conclude the statement of Question 3.7.   □

Prior to the proof of Theorem 3.6, we consider the classical contraction $\Phi_\varepsilon \colon \mathbb{H}^1 \to$ SU$_2$.

Example 3.8 Let $G_1 = SU(2)$ and $G_0 = \mathbb{H}$. A Lie group contraction $\Phi_\varepsilon = \exp(U_\varepsilon) \colon G_0 \to G_1$ is given by

$$U_\varepsilon = \begin{pmatrix} \varepsilon^{\frac{1}{2}} & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}. \tag{3.12}$$

The coadjoint orbits $\{\mathcal{O}_s\}$ of SU$_2$ are the two-dimensional spheres with half-integer radius $s \in \frac{1}{2}\mathbb{N}$, i.e.

$$\mathcal{O}_s(\mathrm{SU}_2) = \{(x_1, x_2, x_3) \colon x_1^2 + x_2^2 + x_3^2 = s^2, \quad s \neq 0\}. \tag{3.13}$$

Using that $\mathcal{O}_s$ is diffeomorphic to the complex projective line $\mathbb{CP}^1$, we transport the coordinate atlas $U_1, U_2 \subset \mathbb{C}$ on $\mathcal{O}_s$. There exists a holomorphic line bundle $\pi \colon B \to \mathcal{O}_s$ with the transition function

$$c_s(z) = z^{2s}. \tag{3.14}$$

The existence follows from the fact that $\mathcal{O}_s$ is a quantizable symplectic manifold. The choice (3.14) picks up a specific bundle.

A global holomorphic section s of the bundle $\pi \colon B \to \mathcal{O}_s$ is given by two holomorphic functions $s_1 \colon U_1 \to \mathbb{C}$ and $s_2 \colon U_2 \to \mathbb{C}$ related by

$$s_1(z) = c_s(z) s_2\left(\frac{1}{z}\right), \quad z \in U_1 \cap U_2. \tag{3.15}$$

Expanding holomorphic functions s_1 and s_2 into Taylor series

$$\sum_{n=0} a_m z^m = z^{2s} \sum_{n=0} \tilde{a}_n z^{-n} a \tag{3.16}$$

in respective coordinates and equating coefficients, we find that $\tilde{a}_n = a_n = 0$ for $m > 2s$ and $\tilde{a}_0 = a_{2s}$, $\tilde{a}_1 = a_{2s-1}$, etc. Hence, the section s is given by a polynomial

$$s(z) = \sum_{n=0}^{2s} a_n z^n. \tag{3.17}$$

The coadjoint orbits $\mathcal{O}_\lambda$ of the Heisenberg group $\mathbb{H}$ are given by

$$\mathcal{O}_\lambda = \left\{ z \in \mathbb{C} \colon \frac{z + z^*}{2} = \lambda, \quad \frac{z - z^*}{2} = 0 \right\}. \tag{3.18}$$

In other words, the orbits $\mathcal{O}_\lambda$ are planes through the point $\mathcal{O}_\lambda = \{(\lambda, 0)\}$. There exists a holomorphic line bundle $\pi \colon B \to \mathcal{O}_\lambda$ with the transition function

$$c^\lambda(z) = e^{i\lambda z}, \ z \in \mathbb{C}. \tag{3.19}$$

A global holomorphic section F of $\pi \colon B \to \mathcal{O}_\lambda$ is a holomorphic function. We deform the transition function c_s defined in (3.14) as follows:

$$c_s^\varepsilon(z) = (1 + i\varepsilon z)^{2s_\varepsilon}, \tag{3.20}$$

where

$$s_\varepsilon = \left[\frac{\lambda}{\varepsilon} \right], \quad \lambda \in \mathbb{R}. \tag{3.21}$$

An elementary calculation then yields

$$\lim_{\varepsilon \to 0} (1 + i\varepsilon z)^{2s_\varepsilon} = e^{i\lambda z}. \tag{3.22}$$

This shows the condition (3.9) holds true.

The extension of Example 3.8 to a broader class of Lie groups (e.g. semisimple Lie groups) remains a challenging problem [24] and will be considered elsewhere.

Proof of Theorem 3.6 For every $g \in G_0$ the sections $\pi^\varepsilon(g^\varepsilon)s^\varepsilon$ converge to the section $\pi(g)s_\alpha$ in the sense of Definition 3.3, i.e.

$$\lim_{\varepsilon \to 0} \pi^\varepsilon(g^\varepsilon)s_\alpha^\varepsilon = \pi(g)s_\alpha, \tag{3.23}$$

uniformly for every compact subset $V \subset U_\alpha \cap U_\beta$.

Let π be a fixed unitary irreducible representation, and let us denote by $\mathcal{O}_\pi \subset \mathfrak{g}_0$ the coadjoint orbit associated with π. Let us denote by π_ε the representations corresponding to the coadjoint orbits $\mathcal{O}_\pi^\varepsilon$. The latter is a contraction of $\mathcal{O}_\pi$ by the hypothesis. Let $(U_\alpha, \varphi_\alpha)$ be a local trivialization of $\mathcal{O}_\pi \to B$, and let us fix an arbitrary element $s \in \mathcal{H}^\pi$. We denote by s_α its localization to $U_\alpha \subset O_\pi$. Abusing our notation, we shall also write s_α for the coordinate representation of s_α, i.e. $s_\alpha(z) = s_\alpha(\varphi_\alpha^{-1}(z))$, $z \in \mathbb{C}^r$. Let us denote

$$\mathrm{Ad}_g^{*\varepsilon} \colon \mathcal{O}_\pi \to \mathcal{O}_\pi^\varepsilon \tag{3.24}$$

given by

$$\mathrm{Ad}_g^{*\,\varepsilon}(X) = U_\varepsilon^* \mathrm{Ad}_g^{*\,1}(U_\varepsilon^{-1})^*(X), \quad X \in \mathfrak{g}_0. \tag{3.25}$$

Indeed, by (2.13)

$$\pi^\varepsilon(g)s_\alpha(m) = s_\alpha(\mathrm{Ad}_g^{*\,\varepsilon}m)\psi_\mu(m, g), \tag{3.26}$$

where we denote

$$\psi_\mu(m, g) = \exp\left(\frac{i}{2\hbar} \int\limits_{\gamma\,:\,m\to \mathrm{Ad}_g^* m} \mathcal{L}_\mu \, \mathrm{d}s\right). \tag{3.27}$$

Thus, we get

$$\left|\pi(g)s_\alpha(m) - \pi^\varepsilon(\Phi_\varepsilon(g))s_\alpha^\varepsilon(m)\right| \le \left|s_\alpha(\mathrm{Ad}_g^* m) - s_\alpha^\varepsilon(\mathrm{Ad}_g^{*\,\varepsilon}m)\right|$$
$$+ C\left|\psi_\mu(m, g) - \psi_{\mu_\varepsilon}(m, \Phi_{\varepsilon_n}(g))\right|, \tag{3.28}$$

where we used (3.27) and the fact that $\left|\psi_\mu(m, g)\right| \le 1$. This shows that

$$\lim_{\varepsilon\to 0}\left|s_\alpha(\mathrm{Ad}_g^* m) - s_\alpha^\varepsilon(\mathrm{Ad}_g^{*\,\varepsilon}m)\right| \quad \text{uniformly in } U_\alpha. \tag{3.29}$$

The sections s and s^ε are holomorphic. Hence, their localizations s_α and s_α^ε admit Taylor expansions

$$s_\alpha(z) = \sum_{n\in\mathbb{N}^r} c_k^\alpha z^k, \ z \in \varphi(U_\alpha) \subset \mathbb{C}^r, \tag{3.30}$$

$$s^\varepsilon\alpha(z) = \sum_{n\in\mathbb{N}^r} c_k^{\alpha\,\varepsilon} z^k, \ z \in \varphi(U_\alpha) \subset \mathbb{C}^r. \tag{3.31}$$

Since every coadjoint orbit $\mathcal{O}^\pi \varepsilon$ is compact, the Hilbert space $\mathcal{H}^\varepsilon$ of global holomorphic sections s^ε is finite dimensional in view of the Riemann–Roch theorem. The dimension of the representation space is given [17] by

$$\dim(\pi^\varepsilon) = \mathrm{vol}(\mathcal{O}_\pi^\varepsilon). \tag{3.32}$$

Hence, we get

$$s_\alpha^\varepsilon(z) = \sum_{|k|\le N_\varepsilon} c_k^{\alpha\,\varepsilon} z^k, \quad z \in \mathbb{C}^r, \tag{3.33}$$

in where $N_\varepsilon \ge \dim(\pi^\varepsilon)$. Let us choose

$$c_k^{\alpha\,\varepsilon} = c_k^\alpha, \quad |k| \le N_\varepsilon. \tag{3.34}$$

We have

$$
\begin{aligned}
\left| s_\alpha(\mathrm{Ad}_g^* m) - s_\alpha^\varepsilon(\mathrm{Ad}_g^{*\,\varepsilon} m) \right| &= \left| s_\alpha(\mathrm{Ad}_g^* m) - s_\alpha^\varepsilon(\mathrm{Ad}_g^* m) + s_\alpha^\varepsilon(\mathrm{Ad}_g^* m) - s_\alpha^\varepsilon(\mathrm{Ad}_g^{*\,\varepsilon} m) \right| \\
&\le \left| s_\alpha(\mathrm{Ad}_g^* m) - s_\alpha^\varepsilon(\mathrm{Ad}_g^* m) \right| + \left| s_\alpha^\varepsilon(\mathrm{Ad}_g^* m) - s_\alpha^\varepsilon(\mathrm{Ad}_g^{*\,\varepsilon} m) \right|.
\end{aligned}
\tag{3.35}
$$

Since s_α^ε is a continuous function and using (3.5), we get that the second term in the last inequality of (3.35) goes to zero as $\varepsilon \to 0$.

Composing (3.30) and (3.31) and (3.33), we get

$$\left| s_\alpha(\mathrm{Ad}_g^* m) - s_\alpha^\varepsilon(\mathrm{Ad}_g^* m) \right| = \sum_{k \in \mathbb{N}^r \,:\, |k| \ge N_\varepsilon} c_k^\alpha (\mathrm{Ad}_g^* m)^k. \tag{3.36}$$

The last sum converges to zero uniformly by the holomorphicity of s_α. This establishes (3.29). By the existence of contraction, we immediately obtain that

$$\lim_{\varepsilon \to 0} \int_{\gamma\,:\, m \to \mathrm{Ad}_g^* m} \mathcal{L}_\mu \, \mathrm{d}s = \int_{\gamma_\varepsilon\,:\, m \to \mathrm{Ad}_{\Phi_\varepsilon(g)}^{*\,\varepsilon} m} \mathcal{L}_\mu^\varepsilon \, \mathrm{d}s. \tag{3.37}$$

Thus, we have just shown (3.23).

This completes the proof.   □

4 Two classes of contractions

We exhibit here two types of contractions that can be implemented on any matrix Lie group and which encompass most of the contractions with a diagonal contraction matrix U_ε. First note that in the limit $\varepsilon \to 0$, the important terms in the matrix U_ε will be the lowest powers in ε. Notice also that ε is arbitrary, so only the relative scaling between terms is important. From these considerations, we will study the following two forms of the contraction matrices U_ε.

$$U_{\mathrm{IW}} = \varepsilon \mathrm{Id} + (1 - \varepsilon)u \tag{4.1}$$

$$U_{\mathrm{anti-IW}} = \varepsilon^2 \mathrm{Id} + \varepsilon(1 - \varepsilon)u, \tag{4.2}$$

where u is a matrix to be determined and does not depend on ε. The first contraction is the classical Inönü–Wigner contraction, and the second will be called the anti-Inönü–Wigner contraction for reasons that will become obvious soon. First, we need to define a particular decomposition of the Lie algebra.

Definition 4.1 For a diagonal matrix u of dimension $n = \dim(\mathfrak{g}_1)$, define the two subspaces of $V = \mathbb{R}^n$, the underlying vector space of $\mathfrak{g}_1$ by

$$uV_N = 0 \quad \text{and} \quad uV = V_R. \tag{4.3}$$

Furthermore, these subspaces satisfy

$$\mathfrak{g}_1 = V_N \oplus V_R. \tag{4.4}$$

We now want to derive a geometrical condition on u that will ensure that we obtain a contraction. For this, we will need the following object.

Definition 4.2 (*IW-tensor*). We the following 2-tensor

$$T[u](\xi, \eta) := u^2[\xi, \eta] - u\left([u\xi, \eta] + [\xi, u\eta]\right) + [u\xi, u\eta], \tag{4.5}$$

that we will call the IW-tensor.

In the case when $T[u](\xi, \eta) = 0$, it reduces to the well-known Nijenhuis tensor of complex geometry.

We introduce the following conditions on u in terms of this tensor and the subspaces defined above.

Theorem 4.3 *The following conditions on u give a valid Lie algebra contraction for a choice of splitting $\mathfrak{g}_1 = V_N \oplus V_R$ if*

(1) $T[u](\xi, \eta) \in V_R$, U_{IW} *is an Inönü–Wigner contraction, or*
(2) $T[u](\xi, \eta) \in V_N$, $U_{\text{anti-IW}}$ *is an anti-Inönü–Wigner contraction.*

Proof We begin by proving the IW contraction condition by rewriting

$$U_{\text{IW}} = (1 - \varepsilon)(\lambda \text{Id} + u),$$

where $\lambda = \frac{\varepsilon}{1-\varepsilon}$ converges to 0 when $\varepsilon \to 0$. We first rewrite the Lie bracket using (4.2) to get

$$U^{-1}[U\xi, U\eta] = U^{-1}\left\{\lambda^2[\xi, \eta] + \lambda\left([u\xi, \eta] + [\xi, u\eta]\right) + [u\xi, u\eta]\right\}. \tag{4.6}$$

We then have to ensure that the inner term is in V_R, where U is invertible in the limit $\varepsilon \to 0$. First notice that from the definition of the contraction (4.2) we have the relation $U^{-1}(\lambda\zeta) = \zeta - U^{-1}u\zeta$ for an arbitrary element $\zeta \in \mathfrak{g}$. We can thus rewrite the previous equation as

$$
\begin{aligned}
U^{-1}[U\xi, U\eta] &= \lambda[\xi, \eta] + [u\xi, \eta] + [\xi, u\eta] \\
&\quad + U^{-1}\left\{-\lambda u[\xi, \eta] - u\left([u\xi, \eta] + [\xi, u\eta]\right) + [u\xi, u\eta]\right\} \\
&= \lambda[\xi, \eta] + [u\xi, \eta] + [\xi, u\eta] - u[\xi, \eta] \\
&\quad + U^{-1}\left\{u^2[\xi, \eta] - u\left([u\xi, \eta] + [\xi, u\eta]\right) + [u\xi, u\eta]\right\}.
\end{aligned}
$$

This computation assumes that u is diagonal, and will not apply for more general contractions. Then, as U^{-1} is only defined on V_R in the limit $\varepsilon \to 0$, to the condition for u to be a valid contraction, i.e. for the limit $\varepsilon \to 0$ to exist translates to the condition on the IW-tensor,

$$T[u](\xi, \eta) \in V_R \,. \tag{4.7}$$

The new Lie bracket is

$$[\xi, \eta]_0 = [\xi, \eta]' + u^{-1} T[u](\xi, \eta) \,, \tag{4.8}$$

where we defined

$$[\xi, \eta]' := [u\xi, \eta] + [\xi, u\eta] - u[\xi, \eta] \tag{4.9}$$

$$T[u](\eta, \xi) = [u\xi, u\eta] - u[\xi, \eta]' \,. \tag{4.10}$$

We now turn to the condition for the anti-IW contraction where we can still use the equivalent formulation

$$U_{\text{anti}-\text{IW}} = \varepsilon(\lambda Id + u) \,, \tag{4.11}$$

where $\lambda = \frac{\varepsilon}{1-\varepsilon}$. We then follow the same procedure as before. We first expand the bracket using (4.11) to get

$$U^{-1}[U\xi, U\eta] = U^{-1} \left\{ \varepsilon^{2\alpha} \left(\lambda^2 [\xi, \eta] + \lambda \left([u\xi, \eta] + [\xi, u\eta] \right) + [u\xi, u\eta] \right) \right\} \,. \tag{4.12}$$

We now have the relation $U^{-1}(\varepsilon\lambda\zeta) = \zeta - U^{-1}(\varepsilon u\zeta)$, so we can rewrite

$$
\begin{aligned}
U^{-1}[U\xi, U\eta] &= \varepsilon \left(\lambda[\xi, \eta] + [u\xi, \eta] + [\xi, u\eta] \right) \\
&\quad + U^{-1} \left\{ \varepsilon^2 \left(-\lambda u[\xi, \eta] - u \left([u\xi, \eta] + [\xi, u\eta] \right) + [u\xi, u\eta] \right) \right\} \\
&= \varepsilon \left(\lambda[\xi, \eta] + [u\xi, \eta] + [\xi, u\eta] - \varepsilon u[\xi, \eta] \right) \\
&\quad + U^{-1} \left\{ \varepsilon^2 \left(u^2[\xi, \eta] - u \left([u\xi, \eta] + [\xi, u\eta] \right) + [u\xi, u\eta] \right) \right\} \,.
\end{aligned}
$$

We can go further in the computation to extract non-vanishing terms by using the relation $U^{-1}(\varepsilon\lambda(\varepsilon\lambda^{-1}\zeta)) = \varepsilon\lambda^{-1}\zeta - U^{-1}(\varepsilon u\varepsilon\lambda^{-1}\zeta)$ to get

$$
\begin{aligned}
U^{-1}[U\xi, U\eta] &= \varepsilon \left(\lambda[\xi, \eta] + [u\xi, \eta] + [\xi, u\eta] - \varepsilon u[\xi, \eta] \right) \\
&\quad + \varepsilon\lambda^{-1} \left(u^2[\xi, \eta] - u \left([u\xi, \eta] + [\xi, u\eta] \right) + [u\xi, u\eta] \right) \\
&\quad + U^{-1} \left\{ \varepsilon^2 \lambda^{-1} u \left(u^2[\xi, \eta] - u \left([u\xi, \eta] + [\xi, u\eta] \right) + [u\xi, u\eta] \right) \right\} \,.
\end{aligned}
$$

Note that $\lambda \propto \varepsilon$ when $\varepsilon \to 0$, thus the limit $\varepsilon \to 0$ in the previous equation gives the new Lie bracket only in terms of the IW-tensor

$$[\xi, \eta]_0 = T[u](\xi, \eta) \,. \tag{4.13}$$

In this case, the condition for existence of the contraction is given by

$$T[u](\xi, \eta) \in V_N \,. \tag{4.14}$$

□

We can make a couple of interesting remarks at this point.

(1) If $T[u](\xi, \eta) = 0, \forall \xi, \eta \in \mathfrak{g}$, then the contracted Lie algebra is isomorphic to the original Lie algebra and u is a Nijenhuis tensor. This can be seen by the fact that $T[u](\xi, \eta) = 0$ is equivalent to $[u\xi, u\eta] = u[\xi, \eta]'$, and thus, u defines a homomorphism of Lie algebras between $[\cdot, \cdot]$ and $[\cdot, \cdot]_0 = [\cdot, \cdot]'$.

(2) If $[\xi, \eta]' = 0, \forall \xi, \eta \in \mathfrak{g}$, u is a derivation of the Lie algebra $\mathfrak{g}$ as it will satisfy the Leibniz rule $u[\xi, \eta] = [u\xi, \eta] + [\xi, u\eta]$. The new bracket is then simply $[\xi, \eta]_0 = u^{-1}[u\xi, u\eta]$, but this would also mean that u is invertible everywhere, and thus, the two Lie algebras would be isomorphic.

From these two remarks we get that in order to have a contraction, u must not be a homomorphism of Lie algebras nor a derivation of Lie algebras.

We can even go further and give an extra condition on the splitting, as well the type of Lie algebra that results from the contraction.

Theorem 4.4 *Given a finite-dimensional complex Lie algebra $\mathfrak{g}_1$, for any subspace $\mathfrak{u}$ of a subalgebra of $\mathfrak{g}_1$, let $V_R = \mathfrak{u}$ yield an IWS contraction to a semi-direct algebra and let $V_N = \mathfrak{u}$ yield an anti-IWS contraction to a nilpotent Lie algebra.*

Proof Let us first look at the anti-IWS contractions. Let $V_N \subset \mathfrak{p}$, so that we have $[V_N, V_N] \subset V_N$ and V_N are nilpotent. One can easily check that $T[u](\xi_N, \eta_N) = 0$, and that

$$T[u](\xi_R, \eta_R) = -[\xi_R, \eta_R]_R + [\xi_R, \eta_R] = [\xi_R, \eta_R]_N \subset V_N \,.$$

The last commutation trivially gives $T[u](\xi_N, \eta_R) = [\xi_N, \eta_R]_R - [\xi_N, \eta_R]_R = 0$. We thus have that if $V_N \subset \mathfrak{b}$, then the contraction is anti-IWS.

We now look at the IWS contractions. We first have $T[u](\xi_R, \eta_R) = 0$ and then $T[u](\xi_R, \eta_R) = [\xi_R, \eta_R]_R \subset V_R$ and $T[u](\xi_N, \eta_R) = 0$. This gives the corresponding result. □

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