Energy methods for Dirac-type equations in two-dimensional Minkowski space

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Abstract In this article, we develop energy methods for a large class of linear and nonlinear Dirac-type equations in two-dimensional Minkowski space. We will derive existence results for several Dirac-type equations originating in quantum field theory, in particular for Dirac-wave maps to compact Riemannian manifolds.

Keywords Nonlinear Dirac equations · Two-dimensional Minkowski space · Energy methods · Dirac-wave maps

Mathematics Subject Classification 35L02 · 35L60 · 58J45 · 53C27

1 Introduction and results

In quantum field theory, spinors are used to describe fermions, which are elementary particles of half-integer spin. The equations that govern their behavior are both linear and nonlinear Dirac equations. Linear Dirac equations are employed to model free fermions. However, to model the interaction of fermions, one has to take into account nonlinearities.

In mathematical terms, spinors are sections in the spinor bundle, which is a vector bundle defined on the underlying manifold. Its existence requires the vanishing of the second Stiefel–Whitney class, which is a topological condition. The natural operator acting on spinors is the Dirac operator, which is a first-order differential operator. If the underlying manifold is Riemannian, the Dirac operator is elliptic; if the manifold is Lorentzian, the Dirac operator is hyperbolic. The fact that the Dirac operator is of first

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order usually leads to technical problems since there are less tools available compared to second-order operators such as the Laplacian.

In the case of a Riemannian manifold many results on the qualitative behavior of nonlinear Dirac equations have been obtained recently (see [15, 17, 22]). However, obtaining an existence result for nonlinear Dirac equations in the Riemannian case is rather complicated.

This article is supposed to be the first step to develop energy methods for linear and nonlinear Dirac equations on Lorentzian manifolds in a geometric framework. For an introduction to linear geometric wave equations on globally hyperbolic manifolds, we refer to [6].

As a starting point, we will stick to the case of two-dimensional Minkowski space, and the generalization to higher-dimensional globally hyperbolic manifolds will be treated in a subsequent work. Most of the analytic results on nonlinear Dirac equation in Minkowski space make use of a global trivialization of the spinor bundle and investigate the resulting coupled system of partial differential equations of complex-valued functions. In our approach, we do not make use of a global trivialization of the spinor bundle, but derive energy estimates for the spinor itself. This approach seems to be the natural one from a geometric point of view.

This article is organized as follows: After presenting the necessary background on spin geometry in two-dimensional Minkowski space in the next subsection, we will focus on the analysis of linear Dirac equations in Sect. 2. Afterward, in Sect. 3 we will consider several models from quantum field theory that involve nonlinear Dirac equations. Making use of the energy methods developed before, we derive existence results for some of these models. The last section is devoted to the study of Dirac-wave maps from two-dimensional Minkowski space taking values in a compact Riemannian manifold. Again, by application of suitable energy methods, we are able to derive an existence result for the latter.

1.1 Spin geometry in two-dimensional Minkowski space

First, let us fix the notations that will be used throughout this article. In addition, we want to recall several facts on spin geometry in the setting of a Lorentzian manifold. In this article, we will make use of the Einstein summation convention, that is, we sum over repeated indices.

In the following, we will consider two-dimensional Minkowski space $\mathbb{R}^{1,1}$ with metric $(+, -)$ and global coordinates (t, x) . The tangent vectors of $\mathbb{R}^{1,1}$ will be denoted by ∂_t, ∂_x .

The spinor bundle over $\mathbb{R}^{1,1}$ will be denoted by $\Sigma\mathbb{R}^{1,1}$, and sections in this bundle will be called *spinors*. Note that $\Sigma\mathbb{R}^{1,1}$ can be globally trivialized; however, we will not often make use of this fact. On $\Sigma\mathbb{R}^{1,1}$ there exists a metric connection and we have a Hermitian, but indefinite, scalar product denoted by $\langle \cdot, \cdot \rangle$. We will use the convention that this scalar product is linear in the first and antilinear in the second slot.

We denote Clifford multiplication of a spinor ψ with a tangent vector X by $X \cdot \psi$. Note that in contrast to the Riemannian case, Clifford multiplication is symmetric, that is

$$\langle X \cdot \xi, \psi \rangle_{\Sigma \mathbb{R}^{1,1}} = \langle \xi, X \cdot \psi \rangle_{\Sigma \mathbb{R}^{1,1}}$$

for all $\psi, \xi \in \Gamma(\Sigma \mathbb{R}^{1,1})$ and all $X \in T\mathbb{R}^{1,1}$. In addition, the Clifford relations

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi$$

hold for all $X, Y \in T\mathbb{R}^{1,1}$, where g is the metric of $\mathbb{R}^{1,1}$.

The Dirac operator on two-dimensional Minkowski is defined as (with $\varepsilon_j = g(e_j, e_j) = \pm 1$)

$$D := \sum_{j=1}^2 \varepsilon_j e_j \cdot \nabla_{e_j} = \partial_t \cdot \nabla_{\partial_t} - \partial_x \cdot \nabla_{\partial_x}.$$

Here, ∇ denotes the connection on $\Sigma \mathbb{R}^{1,1}$ and $e_i, i = 1, 2$ is a pseudo-orthonormal basis of $T\mathbb{R}^{1,1}$.

Note that, in contrast to the Riemannian case, the Dirac operator defined above is not formally self-adjoint with respect to the L^2 -norm but satisfies

$$\int_{\mathbb{R}^{1,1}} \langle \xi, D\psi \rangle d\mu = - \int_{\mathbb{R}^{1,1}} \langle D\xi, \psi \rangle d\mu$$

for all $\psi, \xi \in \Sigma \mathbb{R}^{1,1}$. For this reason, we will mostly consider the operator iD , since this combination is formally self-adjoint with respect to the L^2 -norm.

For many of the analytic questions discussed in this article, it will be necessary to have a positive definite scalar product on the spinor bundle in order to establish energy estimates.

For this reason, we consider the positive definite scalar product

$$\langle \partial_t \cdot, \cdot \rangle,$$

where ∂_t denotes the globally defined timelike vector field. The resulting norm will be denoted by $||_{\beta}$, that is

$$0 \leq |\psi|_{\beta}^2 := \langle \partial_t \cdot \psi, \psi \rangle$$

for $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1})$.

For more details on spin geometry on Lorentzian manifolds, we refer to [8] and also [4, 5, 7].

Remark 1.1 Note that we have two kinds of natural scalar products on the spinor bundle in the semi-Riemannian case. On the one hand, we have the (geometric) scalar product that is invariant under the spin group, but indefinite. On the other hand, we have the (analytic) scalar product, which is positive definite but breaks the geometric invariance.

2 Linear Dirac equations in two-dimensional Minkowski space

In this section, we derive conserved energies for solutions of linear Dirac-type equations. Later on, we will generalize these methods to the nonlinear case.

We start by analyzing solutions of

$$D\psi = 0. \quad (2.1)$$

Note that, for a solution of (2.1), we have the following identities

$$\frac{\partial}{\partial t} \langle \partial_t \cdot \psi, \psi \rangle - \frac{\partial}{\partial x} \langle \partial_x \cdot \psi, \psi \rangle = \langle D\psi, \psi \rangle + \langle \psi, D\psi \rangle = 0, \quad (2.2)$$

$$\begin{aligned} \frac{\partial}{\partial t} \langle \partial_x \cdot \psi, \psi \rangle - \frac{\partial}{\partial x} \langle \partial_t \cdot \psi, \psi \rangle &= \langle (\partial_t \cdot \nabla_{\partial_x} - \partial_x \cdot \nabla_{\partial_t}) \psi, \psi \rangle \\ &\quad - \langle \psi, (\partial_t \cdot \nabla_{\partial_x} + \partial_x \cdot \nabla_{\partial_t}) \psi \rangle = 0. \end{aligned} \quad (2.3)$$

We will use these identities to derive several conservation laws.

Lemma 2.1 *Let $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1})$ be a solution of (2.1). Then, the energies*

$$\begin{aligned} E_1(t) &= \frac{1}{2} \int_{\mathbb{R}} |\psi|_{\beta}^2 dx, \\ E_2(t) &= \frac{1}{2} \int_{\mathbb{R}} \left(\left| \frac{\partial}{\partial x} |\psi|_{\beta}^2 \right|^2 + \left| \frac{\partial}{\partial t} |\psi|_{\beta}^2 \right|^2 \right) dx, \\ E_3(t) &= \frac{1}{2} \int_{\mathbb{R}} (|\nabla_{\partial_t} \psi|_{\beta}^2 + |\nabla_{\partial_x} \psi|_{\beta}^2) dx \end{aligned}$$

are conserved.

Proof The first claim follows from integrating (2.2). Differentiating (2.2) with respect to t and (2.3) with respect to x , we find

$$\frac{\partial^2}{\partial t^2} \langle \partial_t \cdot \psi, \psi \rangle = \frac{\partial^2}{\partial x \partial t} \langle \partial_x \cdot \psi, \psi \rangle = \frac{\partial^2}{\partial x^2} \langle \partial_t \cdot \psi, \psi \rangle.$$

Hence, $|\psi|_{\beta}^2$ solves the one-dimensional wave equation, which yields the second statement. The third assertion follows since ψ solves $\nabla_{\partial_t}^2 \psi = \nabla_{\partial_x}^2 \psi$. $\square$

We can use (2.2) and (2.3) to find conserved energies that involve higher L^p norms of ψ .

Lemma 2.2 *Let $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1})$ be a solution of (2.1). Then, the energy*

$$E_4(t) = \int_{\mathbb{R}} (|\psi|_{\beta}^4 + |\langle \partial_x \cdot \psi, \psi \rangle|^2) dx$$

is conserved.

Proof Making use of (2.2) and (2.3), we calculate

$$\begin{aligned}
 \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} |\psi|_{\beta}^4 dx &= \int_{\mathbb{R}} |\psi|_{\beta}^2 \frac{\partial}{\partial t} \langle \partial_t \cdot \psi, \psi \rangle dx \\
 &= \int_{\mathbb{R}} |\psi|_{\beta}^2 \frac{\partial}{\partial x} \langle \partial_x \cdot \psi, \psi \rangle dx \\
 &= - \int_{\mathbb{R}} \langle \partial_x \cdot \psi, \psi \rangle \frac{\partial}{\partial x} \langle \partial_t \cdot \psi, \psi \rangle dx \\
 &= - \int_{\mathbb{R}} \langle \partial_x \cdot \psi, \psi \rangle \frac{\partial}{\partial t} \langle \partial_x \cdot \psi, \psi \rangle dx \\
 &= - \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} |\langle \partial_x \cdot \psi, \psi \rangle|_{\beta}^2 dx,
 \end{aligned}$$

which proves the claim. $\square$

Remark 2.3 It is straightforward to generalize the previous conservation law to any L^p norm of ψ making use of (2.2) and (2.3).

Proposition 2.4 Let $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1})$ be a solution of (2.1). Then, the following energy

$$E_5(t) = \frac{1}{2} \int_{\mathbb{R}} \left(\left| \frac{\partial}{\partial x} e(\psi) \right|^2 + \left| \frac{\partial}{\partial t} e(\psi) \right|^2 \right) dx$$

is conserved, where

$$e(\psi) := \frac{1}{2} (|\nabla_{\partial_t} \psi|_{\beta}^2 + |\nabla_{\partial_x} \psi|_{\beta}^2).$$

Proof By a direct calculation, we find that $e(\psi)$ solves the one-dimensional wave equation, which yields the statement. $\square$

Remark 2.5 By the Sobolev embedding $H^1 \hookrightarrow L^\infty$, this yields a pointwise bound on $e(\psi)$. It is straightforward to also bound higher derivatives of ψ .

As a next step, we investigate if the same conservation laws still hold if ψ solves a linear Dirac equation with a right-hand side. To this end, we consider the linear Dirac equation

$$iD\psi = \lambda\psi, \quad \lambda \in \mathbb{R}. \quad (2.4)$$

Note that (2.4) arises as critical point of the functional

$$S(\psi) = \int_{\mathbb{R}^{1,1}} (\langle \psi, iD\psi \rangle - \lambda |\psi|^2) d\mu,$$

which leads to the prefactor of i in front of the Dirac operator. Moreover, note that we do not use the definite $|\psi|_{\beta}^2$ -norm.

For a solution of (2.4), we have the following identities

$$\frac{\partial}{\partial t} \langle \partial_t \cdot \psi, \psi \rangle - \frac{\partial}{\partial x} \langle \partial_x \cdot \psi, \psi \rangle = -\lambda (\langle i\psi, \psi \rangle + \langle \psi, i\psi \rangle) = 0, \quad (2.5)$$

$$\begin{aligned} \frac{\partial}{\partial t} \langle \partial_x \cdot \psi, \psi \rangle - \frac{\partial}{\partial x} \langle \partial_t \cdot \psi, \psi \rangle &= \langle (\partial_t \cdot \nabla_{\partial_x} - \partial_x \cdot \nabla_{\partial_t}) \psi, \psi \rangle \\ &\quad - \langle \psi, (\partial_t \cdot \nabla_{\partial_x} + \partial_x \cdot \nabla_{\partial_t}) \psi \rangle \\ &= 2\lambda \langle i \partial_x \cdot \partial_t \cdot \psi, \psi \rangle. \end{aligned} \quad (2.6)$$

Lemma 2.6 *Let $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1})$ be a solution of (2.4). Then, the energies*

$$\begin{aligned} E_1(t) &= \frac{1}{2} \int_{\mathbb{R}} |\psi|_{\beta}^2 dx, \\ E_6(t) &= \frac{1}{2} \int_{\mathbb{R}} (|\nabla_{\partial_t} \psi|_{\beta}^2 + |\nabla_{\partial_x} \psi|_{\beta}^2 - \lambda^2 |\psi|_{\beta}^2) dx \end{aligned}$$

are conserved.

Proof The first statement follows from integrating (2.5). Regarding the second claim, we note that due to the prefactor of i we obtain the following wave-type equation when squaring the Dirac operator

$$\nabla_{\partial_t}^2 \psi - \nabla_{\partial_x}^2 \psi = \lambda^2 \psi,$$

which yields the second statement. $\square$

Proposition 2.7 *For given initial data $\psi(0, x) = \psi_0(x) \in H^1(\mathbb{R}, \Sigma \mathbb{R}^{1,1})$, the solution of (2.4) exists globally in that space.*

Proof Making use of the conserved energy $E_6(t)$, we obtain the following energy inequality

$$\int_{\mathbb{R}} |\partial_x \psi|_{\beta}^2 dx \leq \int_{\mathbb{R}} |\nabla_{\partial_x} \psi|_{\beta}^2 dx \leq \lambda^2 \int_{\mathbb{R}} |\psi|_{\beta}^2 dx + E_6(t) \leq C$$

for a uniform constant C . Moreover, by the Sobolev embedding $H^1 \hookrightarrow L^\infty$ this yields a pointwise bound on $|\psi|_{\beta}$. Consequently, the solution of (2.4) exists globally. $\square$

2.1 Twisted spinors

In this subsection, we want to discuss if the previous results still hold when we consider twisted spinors, which are sections in the spinor bundle that is twisted by some additional vector bundle F .

To this end, let F be a Hermitian vector bundle with a metric connection. Moreover, we will assume that we have a positive definite scalar product on F . On the twisted

bundle $\Sigma\mathbb{R}^{1,1} \otimes F$, we obtain a metric connection induced from the connections on $\Sigma\mathbb{R}^{1,1}$ and F , which we will denote by $\tilde{\nabla}$, via setting

$$\tilde{\nabla} := \nabla^{\Sigma\mathbb{R}^{1,1}} \otimes \mathbb{1}^F + \mathbb{1}^{\Sigma\mathbb{R}^{1,1}} \otimes \nabla^F.$$

The twisted Dirac operator $D^F : \Gamma(\Sigma\mathbb{R}^{1,1} \otimes F) \rightarrow \Gamma(\Sigma\mathbb{R}^{1,1} \otimes F)$ is defined by

$$D^F := \varepsilon_j e_j \cdot \tilde{\nabla}_{e_j} = \partial_t \cdot \tilde{\nabla}_{\partial_t} - \partial_x \cdot \tilde{\nabla}_{\partial_x}.$$

In contrast to the spinor bundle $\Sigma\mathbb{R}^{1,1}$ over Minkowski space, the vector bundle F is not supposed to be flat such that it may have non-vanishing curvature. We will denote its curvature endomorphism by $R^F(\cdot, \cdot)$.

Lemma 2.8 *The square of the twisted Dirac operator D^F satisfies the following Weitzenboeck formula*

$$(D^F)^2 = -\tilde{\nabla}_{\partial_t}^2 + \tilde{\nabla}_{\partial_x}^2 - \partial_t \cdot \partial_x \cdot R^F(\partial_t, \partial_x), \quad (2.7)$$

where R^F denotes the curvature of the vector bundle F .

Proof We calculate

$$\begin{aligned} (D^F)^2 &= (\partial_t \cdot \tilde{\nabla}_{\partial_t} - \partial_x \cdot \tilde{\nabla}_{\partial_x})(\partial_t \cdot \tilde{\nabla}_{\partial_t} - \partial_x \cdot \tilde{\nabla}_{\partial_x}) \\ &= -\tilde{\nabla}_{\partial_t}^2 + \tilde{\nabla}_{\partial_x}^2 - \partial_t \cdot \partial_x \cdot \tilde{\nabla}_{\partial_t} \tilde{\nabla}_{\partial_x} - \partial_x \cdot \partial_t \cdot \tilde{\nabla}_{\partial_x} \tilde{\nabla}_{\partial_t} \\ &= -\tilde{\nabla}_{\partial_t}^2 + \tilde{\nabla}_{\partial_x}^2 - \partial_t \cdot \partial_x \cdot R^F(\partial_t, \partial_x), \end{aligned}$$

which completes the proof. $\square$

Note that, compared to the Riemannian case, we have a different sign in front of the curvature term of the vector bundle F . In addition, we do not get a scalar curvature contribution in (2.7) since we are restricting ourselves to two-dimensional Minkowski space.

Remark 2.9 Most of the Dirac-type equations studied in quantum field theory involve twisted Dirac operators [31]. In particular, the spinors that are considered in the standard model of elementary particle physics are sections in the spinor bundle twisted by some vector bundle.

Again, we start by deriving several energy estimates for solutions of

$$D^F \psi = 0. \quad (2.8)$$

For solutions of (2.8), we obtain the following identities

$$\frac{\partial}{\partial t} \langle \partial_t \cdot \psi, \psi \rangle - \frac{\partial}{\partial x} \langle \partial_x \cdot \psi, \psi \rangle = \langle D^F \psi, \psi \rangle + \langle \psi, D^F \psi \rangle = 0, \quad (2.9)$$

$$\begin{aligned} \frac{\partial}{\partial t} \langle \partial_x \cdot \psi, \psi \rangle - \frac{\partial}{\partial x} \langle \partial_t \cdot \psi, \psi \rangle &= \langle (\partial_t \cdot \tilde{\nabla}_{\partial_x} - \partial_x \cdot \tilde{\nabla}_{\partial_t}) \psi, \psi \rangle \\ &\quad - \langle \psi, (\partial_t \cdot \tilde{\nabla}_{\partial_x} + \partial_x \cdot \tilde{\nabla}_{\partial_t}) \psi \rangle = 0. \end{aligned} \quad (2.10)$$

Proposition 2.10 *Let $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1} \otimes F)$ be a solution of (2.8). Then, the energies*

$$\begin{aligned} \tilde{E}_1(t) &= \frac{1}{2} \int_R |\psi|_\beta^2 dx, \\ \tilde{E}_2(t) &= \frac{1}{2} \int_{\mathbb{R}} \left(\left| \frac{\partial}{\partial x} |\psi|_\beta^2 \right|^2 + \left| \frac{\partial}{\partial t} |\psi|_\beta^2 \right|^2 \right) dx \end{aligned}$$

are conserved.

Proof This follows as in the proof of Lemma 2.1. $\square$

Lemma 2.11 *Let $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1} \otimes F)$ be a solution of (2.8). Then, the energy*

$$\tilde{E}_4(t) = \int_{\mathbb{R}} (|\psi|_\beta^4 + |\langle \partial_x \cdot \psi, \psi \rangle|^2) dx$$

is conserved.

Proof This follows as in the proof of Lemma 2.2 making use of (2.9) and (2.10). $\square$

Remark 2.12 Again, it is straightforward to also find conserved energies involving higher L^p norms of ψ for solutions of (2.8).

We set

$$\tilde{E}_3(t) := \frac{1}{2} \int_{\mathbb{R}} (|\tilde{\nabla}_{\partial_t} \psi|_\beta^2 + |\tilde{\nabla}_{\partial_x} \psi|_\beta^2) dx.$$

When we try to control derivatives of solutions of (2.8), it will be necessary to control the curvature of the vector bundle F .

Lemma 2.13 *Let $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1} \otimes F)$ be a solution of (2.8). Then, $\tilde{E}_3(t)$ satisfies*

$$\frac{d}{dt} \tilde{E}_3(t) \leq \tilde{E}_3(t) + \frac{\|\psi\|_{L^\infty}^2}{2} \int_{\mathbb{R}} |R^F(\partial_t, \partial_x)|^2 dx.$$

Proof By assumption, we have $D^F \psi = 0$ and consequently also $(D^F)^2 \psi = 0$. Now, we calculate

$$\begin{aligned} \frac{d}{dt} \tilde{E}_3(t) &= \int_{\mathbb{R}} (\langle \partial_t \cdot \tilde{\nabla}_{\partial_t}^2 \psi, \tilde{\nabla}_{\partial_t} \psi \rangle + \langle \partial_t \cdot \tilde{\nabla}_{\partial_t} \tilde{\nabla}_{\partial_x} \psi, \tilde{\nabla}_{\partial_x} \psi \rangle) dx \\ &= \int_{\mathbb{R}} (\langle \partial_t \cdot (\tilde{\nabla}_{\partial_t}^2 - \tilde{\nabla}_{\partial_x}^2) \psi, \tilde{\nabla}_{\partial_t} \psi \rangle + \langle \partial_t \cdot R^F(\partial_t, \partial_x) \psi, \tilde{\nabla}_{\partial_x} \psi \rangle) dx \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}} (\langle \partial_x \cdot R^F(\partial_t, \partial_x) \psi, \tilde{\nabla}_{\partial_t} \psi \rangle + \langle \partial_t \cdot R^F(\partial_t, \partial_x) \psi, \tilde{\nabla}_{\partial_x} \psi \rangle) dx \\
&\leq \tilde{E}_3(t) + \frac{\|\psi\|_{\beta}^2}{2} \int_{\mathbb{R}} |R^F(\partial_t, \partial_x)|^2 dx
\end{aligned}$$

yielding the result. $\square$

As a next step, we discuss if the previous methods can still be employed if ψ solves a linear Dirac equation with non-trivial right-hand side, that is,

$$i D^F \psi = \lambda \psi, \quad \lambda \in \mathbb{R}. \quad (2.11)$$

For a solution of (2.11), we have the following identities

$$\frac{\partial}{\partial t} \langle \partial_t \cdot \psi, \psi \rangle - \frac{\partial}{\partial x} \langle \partial_x \cdot \psi, \psi \rangle = 0, \quad (2.12)$$

$$\frac{\partial}{\partial t} \langle \partial_x \cdot \psi, \psi \rangle - \frac{\partial}{\partial x} \langle \partial_t \cdot \psi, \psi \rangle = 2\lambda \langle i \partial_x \cdot \partial_t \cdot \psi, \psi \rangle. \quad (2.13)$$

Lemma 2.14 *Let $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1} \otimes F)$ be a solution of (2.11). Then, the energy*

$$\tilde{E}_1(t) = \frac{1}{2} \int_{\mathbb{R}} |\psi|_{\beta}^2 dx$$

is conserved.

Proof This follows by integrating (2.12). $\square$

Lemma 2.15 *Let $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1} \otimes F)$ be a solution of (2.11). Then, the following inequality holds*

$$\int_{\mathbb{R}} (|\psi|_{\beta}^4 + |\langle \partial_x \cdot \psi, \psi \rangle|^2) dx \leq C e^{|\lambda|t},$$

where the positive constant C depends on ψ_0 .

Proof The proof follows by a direct calculation making use of (2.12) and (2.13). $\square$

Lemma 2.16 *Let $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1} \otimes F)$ be a solution of (2.11). Then, the following inequality holds*

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} (|\tilde{\nabla}_{\partial_t} \psi|_{\beta}^2 + |\tilde{\nabla}_{\partial_x} \psi|_{\beta}^2 - \lambda^2 |\psi|_{\beta}^2) dx &\leq \int_{\mathbb{R}} (|\tilde{\nabla}_{\partial_t} \psi|_{\beta}^2 + |\tilde{\nabla}_{\partial_x} \psi|_{\beta}^2) dx \\
&\quad + \int_{\mathbb{R}} |\psi|_{\beta}^2 |R^F(\partial_t, \partial_x)|^2 dx.
\end{aligned}$$

Proof By the Weitzenboeck formula (2.7), we find

$$\tilde{\nabla}_{\partial_t}^2 \psi - \tilde{\nabla}_{\partial_x}^2 \psi = \lambda^2 \psi - \partial_t \cdot \partial_x \cdot R^F(\partial_t, \partial_x) \psi.$$

In addition, we calculate

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} (|\tilde{\nabla}_{\partial_t} \psi|_{\beta}^2 + |\tilde{\nabla}_{\partial_x} \psi|_{\beta}^2 - \lambda^2 |\psi|_{\beta}^2) dx \\ &= \int_{\mathbb{R}} (\langle \partial_x \cdot R^F(\partial_t, \partial_x) \psi, \tilde{\nabla}_{\partial_t} \psi \rangle + \langle \partial_t \cdot R^F(\partial_t, \partial_x) \psi, \tilde{\nabla}_{\partial_x} \psi \rangle) dx \end{aligned}$$

yielding the result. $\square$

Remark 2.17 We have to impose a bound of the form $\int_{\mathbb{R}} |\psi|_{\beta}^2 |R^F(\partial_t, \partial_x)|^2 dx \leq C$ if we want to deduce an energy estimate for solutions of (2.11).

3 Nonlinear Dirac equations in two-dimensional Minkowski space

In this section, we want to investigate if the energy methods developed for linear Dirac equations in the previous section can also be applied to the nonlinear case. The equations we will study mostly arise in quantum field theory; however, some of them also are connected to problems in differential geometry (see, for example, [29]).

Up to now, there exist many analytic results on nonlinear Dirac equations in two-dimensional Minkowski space. A general framework for semilinear hyperbolic systems was developed in [3]; for a recent survey on nonlinear Dirac equations, see [27] and references therein.

3.1 The Thirring Model

First, we will focus on a famous model from quantum field theory, the *Thirring model*. This model was introduced in [33] to describe the self-interaction of a Dirac field in two-dimensional Minkowski space. In the physics literature, there exist a huge number of results on the Thirring model, and also in the mathematical literature, many results, including existence results, have been established. Most of the methods employed so far make use of a global trivialization of the spinor bundle over two-dimensional Minkowski space yielding existence results for the Thirring model (see, for example, [23, 24, 26, 30, 36]). Making use of our energy methods, we will also provide an existence result for the Thirring model.

The action functional for the Thirring model is given by

$$S(\psi) = \int_{\mathbb{R}^{1,1}} \left(\langle \psi, iD\psi \rangle - \lambda |\psi|^2 - \frac{\kappa}{2} \varepsilon_j \langle \psi, e_j \cdot \psi \rangle \langle \psi, e_j \cdot \psi \rangle \right) d\mu \quad (3.1)$$

with real parameters κ and λ . In physics, λ is usually interpreted as mass, whereas κ describes the strength of interaction. The critical points of (3.1) are given by

$$iD\psi = \lambda\psi + \kappa\varepsilon_j\langle\psi, e_j \cdot \psi\rangle e_j \cdot \psi. \quad (3.2)$$

Remark 3.1 Note that for $\lambda = 0$ solutions of (3.2) are invariant under scaling, that is, if ψ is a solution of (3.2), then

$$\psi(t, x) \rightarrow r\psi(r^2t, r^2x),$$

where r is a positive number, is also a solution.

For a solution of (3.2), the following identity holds

$$\begin{aligned} \partial_x \cdot \nabla_{\partial_t} \psi - \partial_t \cdot \nabla_{\partial_x} \psi &= i\lambda \partial_x \cdot \partial_t \cdot \psi + i\varepsilon_j \kappa \langle \psi, e_j \cdot \psi \rangle \partial_x \cdot \partial_t \cdot e_j \cdot \psi \\ &= i\lambda \partial_x \cdot \partial_t \cdot \psi - i\kappa \langle \psi, \partial_t \cdot \psi \rangle \partial_x \cdot \psi + i\kappa \langle \psi, \partial_x \cdot \psi \rangle \partial_t \cdot \psi \end{aligned}$$

leading to the two equations

$$\begin{aligned} \frac{\partial}{\partial t} \langle \partial_t \cdot \psi, \psi \rangle - \frac{\partial}{\partial x} \langle \partial_x \cdot \psi, \psi \rangle &= \lambda (\langle i\psi, \psi \rangle + \langle \psi, i\psi \rangle) - \kappa \langle \psi, e_j \cdot \psi \rangle \langle \langle ie_j \cdot \psi, \psi \rangle \\ &\quad + \langle \psi, ie_j \cdot \psi \rangle \rangle = 0, \end{aligned} \quad (3.3)$$

$$\frac{\partial}{\partial t} \langle \partial_x \cdot \psi, \psi \rangle - \frac{\partial}{\partial x} \langle \partial_t \cdot \psi, \psi \rangle = 2\lambda \langle i\partial_x \cdot \partial_t \cdot \psi, \psi \rangle. \quad (3.4)$$

Hence, for a solution of (3.2) the energy

$$E_1(t) = \frac{1}{2} \int_{\mathbb{R}} |\psi|_{\beta}^2 dx \quad (3.5)$$

is conserved again. By combining (3.3) and (3.4), we find

$$\square |\psi|_{\beta}^2 = 2\lambda \frac{\partial}{\partial x} \langle i\partial_x \cdot \partial_t \cdot \psi, \psi \rangle = 4\lambda (\langle i\nabla_{\partial_t} \psi, \psi \rangle + 2|\psi|_{\beta}^2 (\lambda - \kappa |\psi|^2)).$$

This identity turns out to be very useful in the case of the massless Thirring model, that is for solutions of (3.2) with $\lambda = 0$. More precisely, we find

Proposition 3.2 *Let $\psi \in \Gamma(\Sigma\mathbb{R}^{1,1})$ be a solution of (3.2) with $\lambda = 0$. Then*

$$|\psi|_{\beta}^2 \leq C \quad (3.6)$$

for a positive constant C .

Proof Since $|\psi|_{\beta}^2$ solves the one-dimensional wave equation, we again get a conserved energy and the result follows from the Sobolev embedding $H^1 \hookrightarrow L^{\infty}$. $\square$

In order to treat the massive Thirring model, we need several auxiliary lemmata.

Lemma 3.3 *Let $\psi \in \Gamma(\Sigma\mathbb{R}^{1,1})$ be a solution of (3.2). Then, the following wave-type equation holds*

$$\begin{aligned}\nabla_{\partial_t}^2 \psi - \nabla_{\partial_x}^2 \psi &= \lambda^2 \psi + \varepsilon_j \lambda \kappa \langle \psi, e_j \cdot \psi \rangle e_j \cdot \psi + 2\kappa \lambda \langle i \partial_x \cdot \partial_t \cdot \psi, \psi \rangle i \partial_x \cdot \partial_t \cdot \psi \\ &\quad + \kappa \langle \psi, \partial_t \cdot \psi \rangle (-2i \nabla_{\partial_t} \psi - \lambda \partial_t \cdot \psi) \\ &\quad + \kappa \langle \psi, \partial_x \cdot \psi \rangle (-2i \partial_t \cdot \partial_x \cdot \nabla_{\partial_t} \psi - \lambda \partial_x \cdot \psi) \\ &\quad + \kappa^2 (|\langle \psi, \partial_t \cdot \psi \rangle|^2 \psi + |\langle \psi, \partial_x \cdot \psi \rangle|^2 \psi \\ &\quad - 2 \langle \psi, \partial_x \cdot \psi \rangle \langle \psi, \partial_t \cdot \psi \rangle \partial_x \cdot \partial_t \cdot \psi).\end{aligned}\quad (3.7)$$

Proof Applying iD to (3.2), we find

$$\nabla_{\partial_t}^2 \psi - \nabla_{\partial_x}^2 \psi = \lambda^2 \psi + \varepsilon_j \lambda \kappa \langle \psi, e_j \cdot \psi \rangle e_j \cdot \psi + i\kappa D(\varepsilon_j \langle \psi, e_j \cdot \psi \rangle e_j \cdot \psi).$$

In order to manipulate the last contribution on the right-hand side, we calculate

$$\begin{aligned}(\partial_t \cdot \nabla_{\partial_t} - \partial_x \cdot \nabla_{\partial_x})(\langle \psi, \partial_t \cdot \psi \rangle \partial_t \cdot \psi - \langle \psi, \partial_x \cdot \psi \rangle \partial_x \cdot \psi) \\ = 2\lambda \langle i \partial_x \cdot \partial_t \cdot \psi, \psi \rangle \partial_x \cdot \partial_t \cdot \psi - \langle \psi, \partial_t \cdot \psi \rangle (\nabla_{\partial_t} \psi + \partial_x \cdot \partial_t \cdot \nabla_{\partial_x} \psi) \\ + \langle \psi, \partial_x \cdot \psi \rangle (\nabla_{\partial_x} \psi + \partial_x \cdot \partial_t \cdot \nabla_{\partial_t} \psi),\end{aligned}$$

where we made use of (3.3) and (3.4). Rewriting (3.2) as

$$\nabla_{\partial_x} \psi = \partial_x \cdot \partial_t \cdot \nabla_{\partial_t} \psi + i\lambda \partial_x \cdot \psi + i\kappa \varepsilon_j \langle \psi, e_j \cdot \psi \rangle \partial_x \cdot e_j \cdot \psi$$

and using the identity

$$\begin{aligned}\varepsilon_j \langle \psi, \partial_t \cdot \psi \rangle \langle \psi, e_j \cdot \psi \rangle \partial_t \cdot e_j \cdot \psi + \varepsilon_j \langle \psi, \partial_x \cdot \psi \rangle \langle \psi, e_j \cdot \psi \rangle \partial_x \cdot e_j \cdot \psi \\ = -|\langle \psi, \partial_t \cdot \psi \rangle|^2 \psi - |\langle \psi, \partial_x \cdot \psi \rangle|^2 \psi + 2 \langle \psi, \partial_t \cdot \psi \rangle \langle \psi, \partial_x \cdot \psi \rangle \partial_x \cdot \partial_t \cdot \psi\end{aligned}$$

yields the claim. $\square$

Lemma 3.4 *Let $\psi \in \Gamma(\Sigma\mathbb{R}^{1,1})$ be a solution of (3.2). Then, the following equality holds*

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}} \left(\frac{1}{3} |\psi|_{\beta}^6 + |\langle \partial_x \cdot \psi, \psi \rangle|^2 |\psi|_{\beta}^2 \right) dx \\ = 4\lambda \int_{\mathbb{R}} \langle \partial_t \cdot \psi, \psi \rangle \langle \partial_x \cdot \psi, \psi \rangle \langle i \partial_x \cdot \partial_t \cdot \psi, \psi \rangle dx.\end{aligned}\quad (3.8)$$

Proof This follows by a direct calculation using (3.3) and (3.4). $\square$

Proposition 3.5 *Let $\psi \in \Gamma(\Sigma\mathbb{R}^{1,1})$ be a solution of (3.2) with $\lambda \neq 0$. Then, the following inequality holds*

$$\int_{\mathbb{R}} (|\nabla_{\partial_t} \psi|_{\beta}^2 + |\nabla_{\partial_x} \psi|_{\beta}^2) dx \leq C e^{Ct}, \quad (3.9)$$

where the constant C depends on λ, κ and the initial data.

Proof From (3.7) and a direct calculation, we obtain

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} (|\nabla_{\partial_t} \psi|_{\beta}^2 + |\nabla_{\partial_x} \psi|_{\beta}^2) dx &\leq \frac{d}{dt} \kappa^2 \int_{\mathbb{R}} \left(\frac{1}{3} |\psi|_{\beta}^6 + |\langle \partial_x \cdot \psi, \psi \rangle|^2 |\psi|_{\beta}^2 \right) dx \\ &\quad + C \int_{\mathbb{R}} |\psi|_{\beta}^3 |\nabla \psi|_{\beta} dx \\ &\leq C \int_{\mathbb{R}} |\psi|_{\beta}^6 dx + C \int_{\mathbb{R}} |\nabla \psi|_{\beta}^2 dx, \end{aligned}$$

where we used (3.8) in the last step. In order to estimate the L^6 -norm of ψ , we make use of the Sobolev embedding theorem in one dimension

$$\begin{aligned} \left(\int_{\mathbb{R}} |\psi|_{\beta}^6 dx \right)^{\frac{1}{2}} &\leq C \left(\int_{\mathbb{R}} |\partial_x \psi|_{\beta}^3 dx \right)^{\frac{3}{2}} \\ &\leq C \left(\int_{\mathbb{R}} |\nabla_{\partial_x} \psi|_{\beta}^{\frac{2}{3}} |\psi|_{\beta}^{\frac{4}{3}} dx \right)^{\frac{3}{2}} \\ &\leq C \left(\int_{\mathbb{R}} |\nabla_{\partial_x} \psi|_{\beta}^2 dx \right)^{\frac{1}{2}} \int_{\mathbb{R}} |\psi|_{\beta}^2 dx. \end{aligned}$$

Since the L^2 -norm of ψ is conserved for a solution of (3.2), we obtain

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} (|\nabla_{\partial_t} \psi|_{\beta}^2 + |\nabla_{\partial_x} \psi|_{\beta}^2) dx \leq C \int_{\mathbb{R}} (|\nabla_{\partial_t} \psi|_{\beta}^2 + |\nabla_{\partial_x} \psi|_{\beta}^2) dx$$

and the result follows by integration of the differential inequality. $\square$

Corollary 3.6 *Let $\psi \in \Gamma(\Sigma\mathbb{R}^{1,1})$ be a solution of (3.2) with $\lambda \neq 0$. Then, the following estimate holds*

$$|\psi|_{\beta} \leq C e^{Ct}, \quad (3.10)$$

where the positive constant C depends on λ, κ and the initial data.

Theorem 3.7 (Existence of a global solution) *For any given initial data of the regularity*

$$\psi(0, x) = \psi_0(x) \in H^1(\mathbb{R}, \Sigma\mathbb{R}^{1,1}),$$

Eq. (3.2) admits a global weak solution in $H^1(\mathbb{R}^{1,1}, \Sigma\mathbb{R}^{1,1})$, which is uniquely determined by the initial data.

Proof The existence of a global solution follows directly since we have a uniform bound on ψ for the massless case $\lambda = 0$. Moreover, in the massive case $\lambda \neq 0$ we have an exponential bound on ψ . These bounds ensure that the solution cannot blow up and has to exist globally.

To achieve uniqueness, let us consider two solutions ψ, ξ of (3.2) that coincide at $t = 0$. Set $\eta := \psi - \xi$. Then, η satisfies

$$\begin{aligned} \partial_t \cdot \nabla_{\partial_t} \eta &= \partial_x \cdot \nabla_{\partial_x} \eta - i\lambda \eta - i\kappa \varepsilon_j (\langle \eta, e_j \cdot \psi \rangle e_j \cdot \psi + \langle \xi, e_j \cdot \eta \rangle e_j \cdot \psi \\ &\quad - \langle \xi, e_j \cdot \xi \rangle e_j \cdot \eta). \end{aligned}$$

Thus, we find

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} |\eta|_{\beta}^2 dx &= 2\kappa \int_{\mathbb{R}} (\varepsilon_j \operatorname{Re} \langle \eta, e_j \cdot \psi \rangle \langle i e_j \cdot \psi, \eta \rangle) + \varepsilon_j \operatorname{Re} \langle \xi, e_j \cdot \eta \rangle \langle i e_j \cdot \psi, \eta \rangle) dx \\ &\leq C \int_{\mathbb{R}} |\eta|_{\beta}^2 (|\psi|_{\beta}^2 + |\chi|_{\beta} |\psi|_{\beta}) dx \\ &\leq C \int_{\mathbb{R}} |\eta|_{\beta}^2 dx, \end{aligned}$$

where we used the pointwise bound on ψ, ξ in the last step. Consequently, we find

$$\int_{\mathbb{R}} |\eta|_{\beta}^2 dx \leq e^{Ct} \int_{\mathbb{R}} |\eta|_{\beta}^2|_{t=0} dx$$

such that if $\psi = \xi$ at $t = 0$, then $\psi = \xi$ for all times. $\square$

Remark 3.8 The existence of a global solution for the Thirring model is due to the algebraic structure of the right-hand side of (3.2). More generally, we could consider

$$iD\psi = \kappa V(\psi) \varepsilon_j \langle e_j \cdot \psi, \psi \rangle e_j \cdot \psi,$$

where $V(\psi)$ is supposed to be a real-valued potential. It can again be checked that

$$\square |\psi|_{\beta}^2 = 0$$

such that we get a global bound on ψ .

Remark 3.9 In the physics literature, the Thirring model is usually formulated as

$$S(\psi, \bar{\psi}) = \int_{\mathbb{R}^{1,1}} (\langle \bar{\psi}, iD\psi \rangle - \lambda |\psi|_{\beta}^2 - \frac{\kappa}{2} \varepsilon_j \langle \bar{\psi}, e_j \cdot \psi \rangle \langle \bar{\psi}, e_j \cdot \psi \rangle) d\mu,$$

where it is assumed that ψ and $\bar{\psi}$ are independent. Hence, the critical points consist of two equations

$$iD\psi = \lambda\psi + \kappa\langle\bar{\psi}, e_j \cdot \psi\rangle e_j \cdot \psi, \quad iD\bar{\psi} = \lambda\bar{\psi} + \kappa\langle\psi, e_j \cdot \bar{\psi}\rangle e_j \cdot \bar{\psi}.$$

These equations have to be considered as independent as can be checked by a direct calculation.

4 Dirac-wave maps from two-dimensional Minkowski space

Dirac-wave maps arise as a mathematical version of the supersymmetric nonlinear sigma model studied in quantum field theory (see, for example, [1] for the physics background). The central object of the supersymmetric nonlinear sigma model is an energy functional that consists of a map between two manifolds and so-called vector spinors. We want to analyze this model with the methods from geometric analysis; hence, in contrast to the physics literature, we will consider standard instead of Grassmann-valued spinors.

In order to define Dirac-wave maps from two-dimensional Minkowski space, we choose the following setup. Let (N, h) be a compact Riemannian manifold and let $\phi: \mathbb{R}^{1,1} \rightarrow N$ be a map. We consider the pullback of the tangent bundle from the target, which will be denoted by ϕ^*TN . As discussed in Sect. 2.1, we form the twisted spinor bundle $\Sigma M \otimes \phi^*TN$. Sections in $\Sigma M \otimes \phi^*TN$ will be called *vector spinors*.

Most of the results that have been obtained in the mathematical literature on the supersymmetric nonlinear sigma model consider the case where both domain and target manifolds are Riemannian. This study was initiated in [20], where the notion of *Dirac-harmonic maps* was introduced. Dirac-harmonic maps form a semilinear elliptic system for a map between two Riemannian manifolds and a spinor along that map. For a given Dirac-harmonic map, many analytic and geometric results have been established, as, for example, the regularity of weak solutions [34]. Motivated from the physics literature, there exist several extensions of the Dirac-harmonic map system such as Dirac-harmonic maps with curvature term [13, 19] and Dirac-harmonic maps with torsion [14].

Making use of the Atiyah–Singer index theorem, uncoupled solutions to the equations for Dirac-harmonic maps have been constructed in [2]. Here, uncoupled refers to the fact that the map part is harmonic. In addition, several approaches to the existence problem that make use of the heat-flow method have been studied in [11, 16, 21, 28, 35] and [10].

Up to now, there is only one reference investigating Dirac-wave maps [25], that is, critical points of the supersymmetric nonlinear sigma model with the domain being two-dimensional Minkowski space. Expressing the Dirac-wave map system in characteristic coordinates, an existence result for smooth initial data could be obtained. In this section, we will extend the analysis of Dirac-wave maps and derive an existence result that also includes distributional initial data. The methods we use here are partly inspired from the analysis of wave maps (see [32] for an introduction to the latter).

A problem similar to the one studied in this section, namely the full bosonic string from two-dimensional Minkowski space to Riemannian manifolds, was treated in [18].

The energy functional for Dirac-wave maps is given by

$$S(\phi, \psi) = \frac{1}{2} \int_{\mathbb{R}^{1,1}} (|d\phi|^2 + \langle \psi, iD^{\phi*TN} \psi \rangle) d\mu. \quad (4.1)$$

Here, $D^{\phi*TN}$ is the twisted Dirac operator acting on vector spinors. Note that $iD^{\phi*TN}$ is self-adjoint with respect to the L^2 -norm such that the energy functional is real-valued.

Whenever choosing local coordinates, we will use Latin indices to denote coordinates on two-dimensional Minkowski space and Greek indices to denote coordinates on the target manifold.

For the sake of completeness, we will give a short derivation of the critical points of (4.1).

Proposition 4.1 *The Euler–Lagrange equations of (4.1) read*

$$\tau(\phi) = \frac{1}{2} R^N(\psi, i e_j \cdot \psi) \varepsilon_j d\phi(e_j), \quad (4.2)$$

$$D^{\phi*TN} \psi = 0. \quad (4.3)$$

Here, R^N denotes the curvature tensor of the target N and e_j , $j = 1, 2$ is an pseudo-orthonormal basis of $T\mathbb{R}^{1,1}$.

Proof First, we consider a variation of ψ , while keeping ϕ fixed, satisfying $\frac{\bar{\nabla} \psi}{\partial s} \big|_{s=0} = \xi$. We calculate

$$\begin{aligned} \frac{d}{ds} \big|_{s=0} S(\phi, \psi) &= \frac{1}{2} \int_{\mathbb{R}^{1,1}} (\langle \xi, iD^{\phi*TN} \psi \rangle + \langle \psi, iD^{\phi*TN} \xi \rangle) d\mu \\ &= \int_{\mathbb{R}^{1,1}} \operatorname{Re} \langle \xi, iD^{\phi*TN} \psi \rangle d\mu \end{aligned}$$

yielding the equation for the vector spinor ψ . To obtain the equation for the map ϕ , we consider a variation of ϕ , while keeping ψ fixed, that is, $\frac{\partial \phi}{\partial s} \big|_{s=0} = \eta$. It is well known that

$$\frac{d}{ds} \big|_{s=0} \frac{1}{2} \int_{\mathbb{R}^{1,1}} |d\phi|^2 d\mu = - \int_{\mathbb{R}^{1,1}} \langle \tau(\phi), \eta \rangle d\mu,$$

where $\tau(\phi) := \nabla_{e_j}^{\phi*TN} d\phi(e_j)$ is the wave map operator. In addition, we find

$$\begin{aligned} \frac{d}{ds} \big|_{s=0} \frac{1}{2} \int_{\mathbb{R}^{1,1}} \langle \psi, iD^{\phi*TN} \psi \rangle d\mu &= \frac{1}{2} \int_{\mathbb{R}^{1,1}} \langle \psi, R^N(d\phi(\partial_s), d\phi(e_j)) i \varepsilon_j e_j \cdot \psi \rangle \big|_{s=0} d\mu \\ &= \frac{1}{2} \int_{\mathbb{R}^{1,1}} \langle \psi, R^N(\psi, i \varepsilon_j e_j \cdot \psi) d\phi(e_j), \eta \rangle d\mu, \end{aligned}$$

completing the proof. $\square$

Solutions of system (4.2), (4.3) are called *Dirac-wave maps* from $\mathbb{R}^{1,1} \rightarrow N$.

System (4.2), (4.3) can be expanded as

$$\frac{\nabla}{\partial t} d\phi(\partial_t) - \frac{\nabla}{\partial x} d\phi(\partial_x) = \frac{1}{2} R^N(\psi, i\partial_t \cdot \psi) d\phi(\partial_t) - \frac{1}{2} R^N(\psi, i\partial_x \cdot \psi) d\phi(\partial_x), \quad (4.4)$$

$$\partial_t \cdot \tilde{\nabla}_{\partial_t} \psi = \partial_x \cdot \tilde{\nabla}_{\partial_x} \psi. \quad (4.5)$$

Choosing local coordinates on the target N , the Euler–Lagrange equations acquire the form ($\alpha = 1, \dots, \dim N$)

$$\begin{aligned} & \frac{\partial^2 \phi^\alpha}{\partial t^2} - \frac{\partial^2 \phi^\alpha}{\partial x^2} + \Gamma_{\beta\gamma}^\alpha \left(\frac{\partial \phi^\beta}{\partial t} \frac{\partial \phi^\gamma}{\partial t} - \frac{\partial \phi^\beta}{\partial x} \frac{\partial \phi^\gamma}{\partial x} \right) \\ &= R_{\beta\gamma\delta}^\alpha \left(\langle \psi^\gamma, i\partial_t \cdot \psi^\delta \rangle \frac{\partial \phi^\beta}{\partial t} - \langle \psi^\gamma, i\partial_x \cdot \psi^\delta \rangle \frac{\partial \phi^\beta}{\partial x} \right), \\ D\psi^\alpha &= -\Gamma_{\beta\gamma}^\alpha \frac{\partial \phi^\beta}{\partial x_j} \varepsilon_j e_j \cdot \psi^\gamma, \end{aligned}$$

where $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols and $R_{\beta\gamma\delta}^\alpha$ the components of the curvature tensor on the target N .

In order to treat a weak version of system (4.2), (4.3), it will be necessary to embed N isometrically into some $\mathbb{R}^q$ making use of the Nash embedding theorem. Then, we have $\phi: \mathbb{R}^{1,1} \rightarrow \mathbb{R}^q$ and $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1} \otimes \mathbb{R}^q)$. In this case, the equations for Dirac-wave maps acquire the form

$$\begin{aligned} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} &= \mathbb{I}(d\phi, d\phi) + P(\mathbb{I}(\psi, d\phi(\partial_t)), i\partial_t \cdot \psi) \\ &\quad - P(\mathbb{I}(\psi, d\phi(\partial_x)), i\partial_x \cdot \psi), \end{aligned} \quad (4.6)$$

$$\partial_t \cdot \nabla_{\partial_t} \psi - \partial_x \cdot \nabla_{\partial_x} \psi = \mathbb{I}\left(\frac{\partial \phi}{\partial t}, \partial_t \cdot \psi\right) - \mathbb{I}\left(\frac{\partial \phi}{\partial x}, \partial_x \cdot \psi\right), \quad (4.7)$$

where $\mathbb{I}$ denotes the second fundamental form of the embedding and P the shape operator defined by

$$\langle P(\xi, X), Y \rangle = \langle \mathbb{I}(X, Y), \xi \rangle$$

for $X, Y \in \Gamma(TN)$ and $\xi \in T^\perp N$. For solutions of (4.3), we obtain the following identities

$$\frac{\partial}{\partial t} \langle \partial_t \cdot \psi, \psi \rangle - \frac{\partial}{\partial x} \langle \partial_x \cdot \psi, \psi \rangle = \langle D^{\phi^*TN} \psi, \psi \rangle + \langle \psi, D^{\phi^*TN} \psi \rangle = 0, \quad (4.8)$$

$$\begin{aligned} \frac{\partial}{\partial t} \langle \partial_x \cdot \psi, \psi \rangle - \frac{\partial}{\partial x} \langle \partial_t \cdot \psi, \psi \rangle &= \langle (\partial_t \cdot \tilde{\nabla}_{\partial_x} - \partial_x \cdot \tilde{\nabla}_{\partial_t}) \psi, \psi \rangle \\ &\quad - \langle \psi, (\partial_t \cdot \tilde{\nabla}_{\partial_x} + \partial_x \cdot \tilde{\nabla}_{\partial_t}) \psi \rangle = 0. \end{aligned} \quad (4.9)$$

Moreover, for a solution of (4.3) the following wave- type equation holds

$$\tilde{\nabla}_{\partial_t}^2 \psi - \tilde{\nabla}_{\partial_x}^2 \psi = -\partial_t \cdot \partial_x \cdot R^N(d\phi(\partial_t), d\phi(\partial_x))\psi$$

due to the Weitzenboeck formula (2.7) with $F = \phi^*TN$.

Remark 4.2 In the physics literature [1], the energy functional for the supersymmetric nonlinear sigma model is defined as follows

$$S(\phi, \psi, \bar{\psi}) = \frac{1}{2} \int_{\mathbb{R}^{1,1}} (|d\phi|^2 + \langle \bar{\psi}, iD^{\phi^*TN}\psi \rangle) d\mu,$$

where $\bar{\psi} := \partial_t \cdot \psi$. Treating $\psi, \bar{\psi}$ as independent fields, we obtain the Euler–Lagrange equations

$$\tau(\phi) = \frac{1}{2} R^N(\bar{\psi}, i e_j \cdot \psi) \varepsilon_j d\phi(e_j), \quad D^{\phi^*TN}\psi = 0, \quad D^{\phi^*TN}\bar{\psi} = 0.$$

Note that the spinors ψ and $\bar{\psi}$ have to be considered as independent since the two equations $D^{\phi^*TN}\psi = 0, D^{\phi^*TN}(\partial_t \cdot \psi) = 0$ are not compatible. In addition, the equation for the map ϕ would acquire the form

$$\tau(\phi) = - \underbrace{\frac{1}{2} R^N(\psi, i \partial_t \cdot \partial_x \cdot \psi) d\phi(\partial_x)}_{=0},$$

when inserting $\bar{\psi} := \partial_t \cdot \psi$. It turns out that the curvature term on the right-hand side vanishes due to symmetry reasons. In the physics literature, one usually considers anticommuting spinors such that this term does not vanish.

In the following, we will only analyze system (4.2), (4.3).

Let us demonstrate how to construct an explicit solution to the Euler–Lagrange equations (4.2), (4.3), where we follow the ideas used for the construction of an explicit solution to the Dirac-harmonic map system in [20, Proposition 2.2]. To this end, we recall the following

Definition 4.3 Let M be a n -dimensional Lorentzian spin manifold. A spinor $\psi \in \Gamma(\Sigma M)$ is called *twistor spinor* if it satisfies

$$P_X \psi := \nabla_X^{\Sigma M} \psi + \frac{1}{n} X \cdot D\psi = 0 \quad (4.10)$$

for all vector fields X .

In two-dimensional Minkowski space, twistor spinors are of the form

$$\psi(x) = \psi_1 + x \cdot \psi_2,$$

where ψ_1, ψ_2 are constant spinors [9].

Proposition 4.4 *Let $\phi: \mathbb{R}^{1,1} \rightarrow N$ be a wave map, that is, a solution of $\tau(\phi) = 0$. We set*

$$\psi := \varepsilon_j e_j \cdot \chi \otimes d\phi(e_j),$$

where χ is a twistor spinor. Then the pair (ϕ, ψ) is a Dirac-wave map, that is uncoupled:

$$\tau(\phi) = 0 = \frac{1}{2} \varepsilon_j R^N(\psi, i e_j \cdot \psi) d\phi(e_j), \quad D^{\phi^*TN} \psi = 0.$$

Proof First, we check that the equation for the vector spinor ψ is satisfied. To this end, we calculate

$$\begin{aligned} D^{\phi^*TN} \psi &= (\partial_t \cdot \tilde{\nabla}_{\partial_t} - \partial_x \cdot \tilde{\nabla}_{\partial_x}) (\partial_t \cdot \chi \otimes d\phi(\partial_t) - \partial_x \cdot \chi \otimes d\phi(\partial_x)) \\ &= -(\nabla_{\partial_t}^{\Sigma M} \chi + \partial_x \cdot \partial_t \cdot \nabla_{\partial_x}^{\Sigma M} \chi) \otimes d\phi(\partial_t) \\ &\quad + (\nabla_{\partial_x}^{\Sigma M} \chi - \partial_t \cdot \partial_x \cdot \nabla_{\partial_t}^{\Sigma M} \chi) \otimes d\phi(\partial_x) \\ &\quad - \chi \otimes \tau(\phi) - \partial_t \cdot \partial_x \cdot \chi \otimes \left(\frac{\nabla}{\partial t} d\phi(\partial_x) - \frac{\nabla}{\partial x} d\phi(\partial_t) \right) \\ &= 0, \end{aligned}$$

the first two terms vanish since χ is a twistor spinor by assumption. As a second step, we check that the curvature term on the right-hand side of (4.2) vanishes. Using the local expression of (4.2)

$$\varepsilon_j R^N(\psi, i e_j \cdot \psi) d\phi(e_j) = R_{\beta\gamma\delta}^{\alpha} \frac{\partial}{\partial y^{\alpha}} \left(\langle \psi^{\gamma}, i \partial_t \cdot \psi^{\delta} \rangle \frac{\partial \phi^{\beta}}{\partial t} - \langle \psi^{\gamma}, i \partial_x \cdot \psi^{\delta} \rangle \frac{\partial \phi^{\beta}}{\partial x} \right),$$

we find (the second term vanishes for the same reason)

$$\begin{aligned} R_{\beta\gamma\delta}^{\alpha} \langle \psi^{\gamma}, i \partial_t \cdot \psi^{\delta} \rangle &= -R_{\beta\gamma\delta}^{\alpha} \langle \chi, i \partial_t \cdot \chi \rangle \left(\frac{\partial \phi^{\gamma}}{\partial t} \frac{\partial \phi^{\delta}}{\partial t} + \frac{\partial \phi^{\gamma}}{\partial x} \frac{\partial \phi^{\delta}}{\partial x} \right) \\ &\quad + R_{\beta\gamma\delta}^{\alpha} \langle \chi, i \partial_x \cdot \chi \rangle \left(\frac{\partial \phi^{\gamma}}{\partial t} \frac{\partial \phi^{\delta}}{\partial x} + \frac{\partial \phi^{\gamma}}{\partial x} \frac{\partial \phi^{\delta}}{\partial t} \right) = 0 \end{aligned}$$

due to the symmetries of the curvature tensor on N . $\square$

4.1 Conserved energies

In this subsection, we give several conserved energies for solutions of system (4.2), (4.3). By integrating (4.8), we directly get that

$$E_1(t) = \int_{\mathbb{R}} |\psi|_{\beta}^2 dx$$

is conserved for a solution of (4.3). Again, it is straightforward to also control higher L^p norms of ψ as discussed in Sect. 2.1.

Moreover, for a solution of (4.3), we have

$$|\tilde{\nabla}_{\partial_t} \psi|_{\beta}^2 = |\tilde{\nabla}_{\partial_x} \psi|_{\beta}^2.$$

Proposition 4.5 *Let $(\phi, \psi): \mathbb{R}^{1,1} \rightarrow N$ be a Dirac-wave map. Then, the energy*

$$E(t) := \frac{1}{2} \int_{\mathbb{R}} (|d\phi(\partial_t)|^2 + |d\phi(\partial_x)|^2 - \langle \psi, i\partial_t \cdot \tilde{\nabla}_{\partial_t} \psi \rangle) dx \quad (4.11)$$

is conserved.

Proof We calculate

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} (|d\phi(\partial_t)|^2 + |d\phi(\partial_x)|^2) dx &= \int_{\mathbb{R}} \langle d\phi(\partial_t), \tau(\phi) \rangle dx \\ &= -\frac{1}{2} \int_{\mathbb{R}} \langle d\phi(\partial_t), R^N(\psi, i\partial_x \cdot \psi) d\phi(\partial_x) \rangle dx. \end{aligned}$$

Differentiating (4.3) with respect to t , we obtain the following identity

$$\partial_t \cdot \tilde{\nabla}_{\partial_t}^2 \psi = \partial_x \cdot \tilde{\nabla}_{\partial_t} \tilde{\nabla}_{\partial_x} \psi = \partial_x \cdot R^N(d\phi(\partial_t), d\phi(\partial_x))\psi + \partial_x \cdot \tilde{\nabla}_{\partial_x} \tilde{\nabla}_{\partial_t} \psi.$$

Thus, we find

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} \langle \psi, i\partial_t \cdot \tilde{\nabla}_{\partial_t} \psi \rangle dx &= \frac{1}{2} \int_{\mathbb{R}} \underbrace{((\partial_t \cdot \tilde{\nabla}_{\partial_t} - \partial_x \cdot \tilde{\nabla}_{\partial_x})\psi, i\tilde{\nabla}_{\partial_t} \psi)}_{=0} \\ &\quad + \langle \psi, i\partial_x \cdot R^N(d\phi(\partial_t), d\phi(\partial_x))\psi \rangle dx, \end{aligned}$$

where we used integration by parts in the last step. The assertion then follows by adding up both contributions. $\square$

Having gained control over ψ , we now establish a bound on the derivatives of ϕ . To this end, we set

$$e(\phi) := \frac{1}{2} (|d\phi(\partial_t)|^2 + |d\phi(\partial_x)|^2).$$

Then, we obtain the following

Proposition 4.6 *Let $(\phi, \psi): \mathbb{R}^{1,1} \rightarrow N$ be a Dirac-wave map. Then, the following formula holds*

$$\square e(\phi) = \frac{\partial^2}{\partial t^2} \langle \tilde{\nabla}_{\partial_x} \psi, i\partial_x \cdot \psi \rangle - \frac{\partial^2}{\partial x^2} \langle \tilde{\nabla}_{\partial_t} \psi, i\partial_t \cdot \psi \rangle, \quad (4.12)$$

where $\square := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$.

Proof Testing (4.4) with $d\phi(\partial_t)$ and $d\phi(\partial_x)$, we obtain the two equations

$$\begin{aligned}\frac{\partial}{\partial t}e(\phi) - \frac{\partial}{\partial x}\langle d\phi(\partial_t), d\phi(\partial_x) \rangle &= -\langle R^N(\psi, i\partial_x \cdot \psi)d\phi(\partial_x), d\phi(\partial_t) \rangle, \\ -\frac{\partial}{\partial x}e(\phi) + \frac{\partial}{\partial t}\langle d\phi(\partial_t), d\phi(\partial_x) \rangle &= \langle R^N(\psi, i\partial_t \cdot \psi)d\phi(\partial_t), d\phi(\partial_x) \rangle.\end{aligned}$$

Differentiating the first equation with respect to t , the second one with respect to x and adding up both contributions, we find

$$\square e(\phi) = \frac{\partial}{\partial x}\langle R^N(\psi, i\partial_t \cdot \psi)d\phi(\partial_t), d\phi(\partial_x) \rangle - \frac{\partial}{\partial t}\langle R^N(\psi, i\partial_x \cdot \psi)d\phi(\partial_x), d\phi(\partial_t) \rangle.$$

We may rewrite the right-hand side as follows

$$\begin{aligned}& \frac{\partial}{\partial x}\langle R^N(\psi, i\partial_t \cdot \psi)d\phi(\partial_t), d\phi(\partial_x) \rangle - \frac{\partial}{\partial t}\langle R^N(\psi, i\partial_x \cdot \psi)d\phi(\partial_x), d\phi(\partial_t) \rangle \\&= \frac{\partial}{\partial x}\langle \tilde{\nabla}_{\partial_t}\tilde{\nabla}_{\partial_x}\psi, i\partial_t \cdot \psi \rangle - \frac{\partial}{\partial x}\langle \tilde{\nabla}_{\partial_x}\tilde{\nabla}_{\partial_t}\psi, i\partial_t \cdot \psi \rangle - \frac{\partial}{\partial t}\langle \tilde{\nabla}_{\partial_x}\tilde{\nabla}_{\partial_t}\psi, i\partial_x \cdot \psi \rangle \\&\quad + \frac{\partial}{\partial t}\langle \tilde{\nabla}_{\partial_t}\tilde{\nabla}_{\partial_x}\psi, i\partial_x \cdot \psi \rangle \\&= \frac{\partial^2}{\partial x\partial t}\langle \tilde{\nabla}_{\partial_x}\psi, i\partial_t \cdot \psi \rangle - \frac{\partial}{\partial x}\langle \tilde{\nabla}_{\partial_x}\psi, i\partial_t \cdot \tilde{\nabla}_{\partial_t}\psi \rangle - \frac{\partial^2}{\partial x^2}\langle \tilde{\nabla}_{\partial_t}\psi, i\partial_t \cdot \psi \rangle \\&\quad + \frac{\partial}{\partial x}\langle \tilde{\nabla}_{\partial_t}\psi, i\partial_t \cdot \tilde{\nabla}_{\partial_x}\psi \rangle \\&\quad - \frac{\partial^2}{\partial x\partial t}\langle \tilde{\nabla}_{\partial_t}\psi, i\partial_x \cdot \psi \rangle + \frac{\partial}{\partial t}\langle \tilde{\nabla}_{\partial_t}\psi, i\partial_x \cdot \tilde{\nabla}_{\partial_x}\psi \rangle + \frac{\partial^2}{\partial t^2}\langle \tilde{\nabla}_{\partial_x}\psi, i\partial_x \cdot \psi \rangle \\&\quad - \frac{\partial}{\partial t}\langle \tilde{\nabla}_{\partial_x}\psi, i\partial_x \cdot \tilde{\nabla}_{\partial_t}\psi \rangle \\&= \frac{\partial^2}{\partial x\partial t}\underbrace{\langle (\partial_t \cdot \tilde{\nabla}_{\partial_x} - \partial_x \cdot \tilde{\nabla}_{\partial_t})\psi, i\psi \rangle}_{=0} \\&\quad + \frac{\partial}{\partial x}\underbrace{(\langle \tilde{\nabla}_{\partial_t}\psi, i\partial_t \cdot \tilde{\nabla}_{\partial_x}\psi \rangle - \langle \tilde{\nabla}_{\partial_x}\psi, i\partial_t \cdot \tilde{\nabla}_{\partial_t}\psi \rangle)}_{=0} \\&\quad + \frac{\partial}{\partial t}\underbrace{(\langle \tilde{\nabla}_{\partial_t}\psi, i\partial_x \cdot \tilde{\nabla}_{\partial_x}\psi \rangle - \langle \tilde{\nabla}_{\partial_x}\psi, i\partial_x \cdot \tilde{\nabla}_{\partial_t}\psi \rangle)}_{=0} + \frac{\partial^2}{\partial t^2}\langle \tilde{\nabla}_{\partial_x}\psi, i\partial_x \cdot \psi \rangle \\&\quad - \frac{\partial^2}{\partial x^2}\langle \tilde{\nabla}_{\partial_t}\psi, i\partial_t \cdot \psi \rangle \\&= \frac{\partial^2}{\partial t^2}\langle \tilde{\nabla}_{\partial_x}\psi, i\partial_x \cdot \psi \rangle - \frac{\partial^2}{\partial x^2}\langle \tilde{\nabla}_{\partial_t}\psi, i\partial_t \cdot \psi \rangle,\end{aligned}$$

where we used that ψ is a solution of (4.5) several times completing the proof. $\square$

Remark 4.7 The conserved energies that we have presented so far all reflect the hyperbolic nature of the Dirac-wave map system (4.2), (4.3). In addition, as in the Riemannian case, we also have the energy-momentum tensor, which is conserved for a Dirac-wave map. More precisely, the symmetric 2-tensor T_{ij} defined by

$$T_{ij} := 2\langle d\phi(e_i), d\phi(e_j) \rangle - g_{ij}|d\phi|^2 - \frac{1}{2}\langle \psi, (e_i \cdot \tilde{\nabla}_{e_j} + e_j \cdot \tilde{\nabla}_{e_i})\psi \rangle$$

is divergence free for a solution of (4.2), (4.3).

4.2 An existence result via energy methods

In this subsection, we will derive an existence result for the Cauchy problem associated with (4.2), (4.3). Since we want to be able to treat initial data of low regularity, we have to use the extrinsic version of the Dirac-wave map system (4.6), (4.7).

Making use of the conserved energies from the last subsection, we find

Proposition 4.8 *Let $(\phi, \psi): \mathbb{R}^{1,1} \rightarrow \mathbb{R}^q$ be a Dirac-wave map. Then, the following energy is conserved*

$$E_{DW}(t) := \frac{1}{2} \int_{\mathbb{R}} \left(\left| \frac{\partial \phi}{\partial t} \right|^2 + \left| \frac{\partial \phi}{\partial x} \right|^2 - \langle \psi, i \partial_t \cdot \nabla_{\partial_t} \psi \rangle \right) dx. \quad (4.13)$$

Proof By (4.7), we have a conserved energy for the intrinsic version of the Dirac-wave map system. To obtain the conserved energy for the extrinsic version, we consider the isometric embedding $\iota: N \rightarrow \mathbb{R}^q$. Since ι is an isometry, we may apply its differential to all terms in (4.13), which gives the statement. $\square$

Corollary 4.9 *Let $(\phi, \psi): \mathbb{R}^{1,1} \rightarrow \mathbb{R}^q$ be a Dirac-wave map. Then the energy*

$$E(t) = \int_{\mathbb{R}} \left(\left| \frac{\partial}{\partial x} (e(\phi) - \langle i \partial_t \cdot \nabla_{\partial_t} \psi, \psi \rangle) \right|^2 + \left| \frac{\partial}{\partial t} (e(\phi) - \langle i \partial_t \cdot \nabla_{\partial_t} \psi, \psi \rangle) \right|^2 \right) dx \quad (4.14)$$

is conserved.

Proof We obtain a conserved energy from (4.12). Applying the isometric embedding ι again yields the claim. $\square$

Proposition 4.10 *Let ψ be a solution of (4.3). Then, the following pointwise bound holds*

$$|\psi|_{\beta} \leq C, \quad (4.15)$$

where the positive constant C depends on ψ_0 .

Proof This follows from Proposition 2.10 with $F = \phi^*TN$ and the Sobolev embedding $H^1 \hookrightarrow L^\infty$. $\square$

Lemma 4.11 *Let $(\phi, \psi): \mathbb{R}^{1,1} \rightarrow \mathbb{R}^q$ be a Dirac-wave map. Then the following estimate holds*

$$\left| \frac{\partial \phi}{\partial t} \right|^2 + \left| \frac{\partial \phi}{\partial x} \right|^2 \leq C(1 + |\nabla \psi|_\beta), \quad (4.16)$$

where the positive constant C depends on $E_{DW}(0)$ and ψ_0 .

Proof From the conserved energy (4.14) and the Sobolev embedding $H^1 \hookrightarrow L^\infty$, we obtain the following bound

$$|e(\phi) - \langle i \partial_t \cdot \nabla_{\partial_t} \psi, \psi \rangle| \leq E_{DW}(0).$$

Using the pointwise bound on ψ , we obtain the claim. $\square$

In order to obtain control over the derivatives of ψ , we turn (4.7) into a wave-type equation. Applying the Dirac operator D on both sides of (4.7), we obtain the wave-type equation

$$\begin{aligned} \nabla_{\partial_t}^2 \psi - \nabla_{\partial_x}^2 \psi &= \mathbb{I}(\square \phi, \psi) + \mathbb{I}\left(\frac{\partial \phi}{\partial t}, \nabla_{\partial_t} \psi + \partial_x \cdot \partial_t \cdot \nabla_{\partial_x} \psi\right) \\ &\quad + \mathbb{I}\left(\frac{\partial \phi}{\partial x}, -\nabla_{\partial_x} \psi + \partial_t \cdot \partial_x \cdot \nabla_{\partial_t} \psi\right) \\ &\quad + (\nabla_{d\phi(\partial_t)} \mathbb{I})\left(\frac{\partial \phi}{\partial t}, \psi\right) + (\nabla_{d\phi(\partial_t)} \mathbb{I})\left(\frac{\partial \phi}{\partial x}, \partial_t \cdot \partial_x \cdot \psi\right) \\ &\quad - (\nabla_{d\phi(\partial_x)} \mathbb{I})\left(\frac{\partial \phi}{\partial x}, \partial_t \cdot \partial_x \cdot \psi\right) + (\nabla_{d\phi(\partial_x)} \mathbb{I})\left(\frac{\partial \phi}{\partial t}, \psi\right). \end{aligned} \quad (4.17)$$

We set

$$E_{(\psi,1,2)}(t) := \frac{1}{2} \int_{\mathbb{R}} (|\nabla_{\partial_t} \psi|_\beta^2 + |\nabla_{\partial_x} \psi|_\beta^2) dx.$$

The energy $E_{(\psi,1,2)}(t)$ satisfies the following differential inequality

Lemma 4.12 *Let $(\phi, \psi): \mathbb{R}^{1,1} \rightarrow \mathbb{R}^q$ be a Dirac-wave map. Then the following inequality holds*

$$\frac{d}{dt} E_{(\psi,1,2)}(t) \leq C(E_{(\psi,1,2)}(t) + (E_{(\psi,1,2)}(t))^{\frac{1}{2}}), \quad (4.18)$$

where the positive constant C depends on N and the initial data.

Proof Making use of (4.17) a direct calculation yields

$$\begin{aligned} \frac{d}{dt} E_{(\psi,1,2)}(t) &= \int_{\mathbb{R}} (\langle \partial_t \cdot \nabla_{\partial_t} \psi, \mathbb{I}(\square \phi, \psi) \rangle \\ &\quad + \langle \partial_t \cdot \nabla_{\partial_t} \psi, \mathbb{I}\left(\frac{\partial \phi}{\partial t}, \nabla_{\partial_t} \psi + \partial_x \cdot \partial_t \cdot \nabla_{\partial_x} \psi\right) \rangle) \end{aligned}$$

$$\begin{aligned}
& + \left\langle \partial_t \cdot \nabla \partial_t \psi, \mathbb{I} \left(\frac{\partial \phi}{\partial x}, -\nabla \partial_x \psi + \partial_t \cdot \partial_x \cdot \nabla \partial_t \psi \right) \right\rangle \\
& + \left\langle \partial_t \cdot \nabla \partial_t \psi, (\nabla_{d\phi(\partial_t)} \mathbb{I}) \left(\frac{\partial \phi}{\partial t}, \psi \right) \right\rangle \\
& + \left\langle \partial_t \cdot \nabla \partial_t \psi, (\nabla_{d\phi(\partial_t)} \mathbb{I}) \left(\frac{\partial \phi}{\partial x}, \partial_t \cdot \partial_x \cdot \psi \right) \right\rangle \\
& + \left\langle \partial_t \cdot \nabla \partial_t \psi, (\nabla_{d\phi(\partial_x)} \mathbb{I}) \left(\frac{\partial \phi}{\partial t}, \psi \right) \right\rangle \\
& - \left\langle \partial_t \cdot \nabla \partial_t \psi, (\nabla_{d\phi(\partial_x)} \mathbb{I}) \left(\frac{\partial \phi}{\partial x}, \partial_t \cdot \partial_x \cdot \psi \right) \right\rangle dx.
\end{aligned}$$

Note that the second and the third term on the right-hand side can be rewritten as

$$\begin{aligned}
& \left\langle \partial_t \cdot \nabla \partial_t \psi, \mathbb{I} \left(\frac{\partial \phi}{\partial t}, \nabla \partial_t \psi + \partial_x \cdot \partial_t \cdot \nabla \partial_x \psi \right) \right\rangle \\
& = - \left\langle \partial_t \cdot \psi, (\nabla_{d\phi(\partial_t)} \mathbb{I}) \left(\frac{\partial \phi}{\partial t}, \nabla \partial_t \psi + \partial_x \cdot \partial_t \cdot \nabla \partial_x \psi \right) \right\rangle
\end{aligned}$$

since $\psi \perp \mathbb{I}$. Consequently, we get the following inequality

$$\begin{aligned}
\frac{d}{dt} E_{(\psi, 1, 2)}(t) & \leq C \int_{\mathbb{R}} (|\psi|_{\beta} |\nabla \psi|_{\beta} |\square \phi| + |\psi|_{\beta} |\nabla \psi|_{\beta} |d\phi|^2) dx \\
& \leq C \int_{\mathbb{R}} (|\psi|_{\beta} |\nabla \psi|_{\beta} |d\phi|^2 + |\psi|_{\beta}^5 |\nabla \psi|_{\beta}) dx \\
& \leq C \left(\int_{\mathbb{R}} |\nabla \psi|_{\beta}^2 dx + \left(\int_{\mathbb{R}} |\nabla \psi|_{\beta}^2 dx \right)^{\frac{1}{2}} \right)
\end{aligned}$$

where we used (4.16) in the last step, completing the proof. $\square$

Corollary 4.13 *Let $(\phi, \psi): \mathbb{R}^{1,1} \rightarrow \mathbb{R}^q$ be a Dirac-wave map. Then the following estimate holds*

$$E_{(\psi, 1, 2)}(t) \leq C e^{Ct},$$

where the positive constant C depends on N and the initial data.

Proof This follows from the last lemma and the Gronwall inequality. $\square$

Corollary 4.14 *Let $(\phi, \psi): \mathbb{R}^{1,1} \rightarrow \mathbb{R}^q$ be a Dirac-wave map. Then the following estimate holds*

$$\int_{\mathbb{R}} \left(\left| \frac{\partial \phi}{\partial t} \right|^2 + \left| \frac{\partial \phi}{\partial x} \right|^2 \right) dx \leq C(1 + e^{Ct}),$$

where the positive constant C depends on N and the initial data.

Proof From the conserved energy (4.13), we obtain

$$\begin{aligned} \int_{\mathbb{R}} (|\frac{\partial \phi}{\partial t}|^2 + |\frac{\partial \phi}{\partial x}|^2) dx &\leq E_{DW}(0) + C \left(\int_{\mathbb{R}} |\psi|_{\beta}^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\nabla \psi|_{\beta}^2 dx \right)^{\frac{1}{2}} \\ &\leq E_{DW}(0) + C e^{Ct}, \end{aligned}$$

yielding the result. $\square$

It turns out that we need to gain control over the L^4 -norm of $\nabla \psi$. To this end, let us recall the following fact.

Remark 4.15 Suppose that $f: \mathbb{R}^{1,1} \rightarrow \mathbb{R}$ is a solution of the scalar wave equation, that is, $\square f = 0$. Then, we have the following conservation law

$$\frac{d}{dt} \int_{\mathbb{R}} (|\frac{\partial f}{\partial t}|^4 + |\frac{\partial f}{\partial x}|^4 + 6|\frac{\partial f}{\partial t}|^2 |\frac{\partial f}{\partial x}|^2) dx = 0.$$

Motivated from the conserved energy for solutions of the scalar wave equation, we define

$$\begin{aligned} E_{(\psi, 1, 4)}(t) &:= \int_{\mathbb{R}} (|\nabla_{\partial_t} \psi|_{\beta}^4 + |\nabla_{\partial_x} \psi|_{\beta}^4 + 2|\nabla_{\partial_x} \psi|_{\beta}^2 |\nabla_{\partial_t} \psi|_{\beta}^2 \\ &\quad + 4|\langle \partial_t \cdot \nabla_{\partial_x} \psi, \nabla_{\partial_t} \psi \rangle|^2) dx. \end{aligned}$$

It can be checked by a direct calculation that $E_{(\psi, 1, 4)}(t)$ is conserved when $\psi \in \Gamma(\Sigma \mathbb{R}^{1,1})$ is a solution of $\nabla_{\partial_t}^2 \psi = \nabla_{\partial_x}^2 \psi$.

Lemma 4.16 Let $(\phi, \psi): \mathbb{R}^{1,1} \rightarrow \mathbb{R}^q$ be a Dirac-wave map. Then the following inequality holds

$$\frac{d}{dt} E_{(\psi, 1, 4)}(t) \leq C (E_{(\psi, 1, 4)}(t) + (E_{(\psi, 1, 4)}(t))^{\frac{1}{2}}), \quad (4.19)$$

where the positive constant C depends on N and the initial data.

Proof Let ψ be a solution of $\nabla_{\partial_t}^2 \psi - \nabla_{\partial_x}^2 \psi = f$. Then, a direct, but lengthy calculation yields

$$\begin{aligned} \frac{d}{dt} E_{(\psi, 1, 4)}(t) &= \int_{\mathbb{R}} (4|\nabla_{\partial_t} \psi|_{\beta}^2 \langle \partial_t \cdot \nabla_{\partial_t} \psi, f \rangle - 4|\nabla_{\partial_x} \psi|_{\beta}^2 \langle \partial_t \cdot \nabla_{\partial_t} \psi, f \rangle \\ &\quad + 8\langle \partial_t \cdot \nabla_{\partial_x} \psi, f \rangle \langle \partial_t \cdot \nabla_{\partial_t} \psi, \nabla_{\partial_x} \psi \rangle) dx. \end{aligned}$$

At this point, we choose

$$\begin{aligned}
f &= \mathbb{I}(\square\phi, \psi) + \mathbb{I}\left(\frac{\partial\phi}{\partial t}, \nabla_{\partial_t}\psi + \partial_x \cdot \partial_t \cdot \nabla_{\partial_x}\psi\right) \\
&\quad + \mathbb{I}\left(\frac{\partial\phi}{\partial x}, -\nabla_{\partial_x}\psi + \partial_t \cdot \partial_x \cdot \nabla_{\partial_t}\psi\right) \\
&\quad + (\nabla_{d\phi(\partial_t)}\mathbb{I})\left(\frac{\partial\phi}{\partial t}, \psi\right) + (\nabla_{d\phi(\partial_t)}\mathbb{I})\left(\frac{\partial\phi}{\partial x}, \partial_t \cdot \partial_x \cdot \psi\right) \\
&\quad - (\nabla_{d\phi(\partial_x)}\mathbb{I})\left(\frac{\partial\phi}{\partial x}, \partial_t \cdot \partial_x \cdot \psi\right) + (\nabla_{d\phi(\partial_x)}\mathbb{I})\left(\frac{\partial\phi}{\partial t}, \psi\right).
\end{aligned}$$

By the same reasoning as in the proof of Lemma 4.12, we find

$$\begin{aligned}
\frac{d}{dt}E_{(\psi,1,4)}(t) &\leq C \int_{\mathbb{R}} (|\psi|_{\beta} |\nabla\psi|_{\beta}^3 |\square\phi| + |\psi|_{\beta} |\nabla\psi|_{\beta}^3 |d\phi|^2) dx \\
&\leq C \int_{\mathbb{R}} |\nabla\psi|_{\beta}^4 dx + C \left(\int_{\mathbb{R}} |\nabla\psi|_{\beta}^4 dx \right)^{\frac{1}{2}},
\end{aligned}$$

which completes the proof. $\square$

Corollary 4.17 *Let $(\phi, \psi): \mathbb{R}^{1,1} \rightarrow \mathbb{R}^q$ be a Dirac-wave map. Then the following estimate holds*

$$E_{(\psi,1,4)}(t) \leq Ce^{Ct}, \quad (4.20)$$

where the positive constant C depends on N and the initial data.

Proof This follows from the last lemma and the Gronwall inequality. $\square$

After having gained control over the derivatives of ψ , we now control the second derivative of ϕ . To this end, we set

$$E_{(\phi,2,2)}(t) := \frac{1}{2} \int_{\mathbb{R}} \left(\left| \frac{\partial^2\phi}{\partial x^2} \right|^2 + \left| \frac{\partial^2\phi}{\partial x \partial t} \right|^2 \right) dx.$$

Proposition 4.18 *Let $(\phi, \psi): \mathbb{R}^{1,1} \rightarrow \mathbb{R}^q$ be a Dirac-wave map. Then the following inequality holds*

$$\frac{d}{dt}E_{(\phi,2,2)}(t) \leq C(E_{(\phi,2,2)}(t) \int_{\mathbb{R}} e(\phi) dx + E_{(\phi,2,2)}(t) + \int_{\mathbb{R}} (|\psi|_{\beta}^2 + |\nabla\psi|_{\beta}^4) dx), \quad (4.21)$$

where the positive constant C depends on N and the initial data.

Proof Using the extrinsic version for Dirac-wave maps (4.6), we calculate

$$\begin{aligned}
\frac{d}{dt}E_{(\phi,2,2)}(t) &= \int_{\mathbb{R}} \langle \partial_x(\square\phi), \partial_{x_t}^2\phi \rangle dx \\
&= \int_{\mathbb{R}} \langle \partial_x(\mathbb{I}(d\phi, d\phi)), \partial_{x_t}^2\phi \rangle dx
\end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}} \langle \partial_x (P(\mathbb{I}(\psi, d\phi(\partial_t)), i\partial_t \cdot \psi)), \partial_{xt}^2 \phi \rangle dx \\
& - \int_{\mathbb{R}} \langle \partial_x (P(\mathbb{I}(\psi, d\phi(\partial_x)), i\partial_x \cdot \psi)), \partial_{xt}^2 \phi \rangle dx.
\end{aligned}$$

It is well known that the contribution containing the second fundamental form can be estimated as

$$|\langle \partial_x (\mathbb{I}(d\phi, d\phi)), \partial_{xt}^2 \phi \rangle| \leq C |d\phi|^3 |d^2 \phi|$$

such that

$$\begin{aligned}
\| |d\phi|^3 |d^2 \phi| \|_{L^1(\mathbb{R})} & \leq \| |d\phi|^3 \|_{L^2(\mathbb{R})} \| |d^2 \phi| \|_{L^2(\mathbb{R})} \\
& \leq C \| |d\phi|^3 \|_{L^{\frac{2}{3}}(\mathbb{R})} \| |d^2 \phi| \|_{L^2(\mathbb{R})} \\
& \leq C \| |d\phi|^2 |d^2 \phi| \|_{L^{\frac{2}{3}}(\mathbb{R})} \| |d^2 \phi| \|_{L^2(\mathbb{R})} \\
& \leq C \| |d\phi| \|_{L^2(\mathbb{R})}^2 \| |d^2 \phi| \|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

where we used the Sobolev embedding theorem in one dimension in the second step. Regarding the other two terms, we note

$$\begin{aligned}
|\langle \partial_x (P(\mathbb{I}(\psi, d\phi(\partial_t)), i\partial_t \cdot \psi)), \partial_{xt}^2 \phi \rangle| & \leq C (|\psi|_\beta^2 |d\phi|^2 |d^2 \phi| + |\nabla \psi|_\beta |\psi|_\beta |d\phi| |d^2 \phi| \\
& \quad + |\psi|_\beta^2 |d^2 \phi|^2) \\
& \leq C (|d^2 \phi|^2 + |\psi|_\beta^4 |d\phi|^4 + |\nabla \psi|_\beta^2 |\psi|_\beta^2 |d\phi|^2) \\
& \leq C (|d^2 \phi|^2 + |\psi|_\beta^2 + |\nabla \psi|_\beta^4)
\end{aligned}$$

and the remaining term can be estimated by the same method. By adding up the various contributions, we obtain the claim. $\square$

Corollary 4.19 *Let $(\phi, \psi): \mathbb{R}^{1,1} \rightarrow \mathbb{R}^q$ be a Dirac-wave map. Then the following estimate holds*

$$E_{(\phi, 2, 2)}(t) \leq C f(t), \quad (4.22)$$

where the positive constant C depends on N and the initial data and the function $f(t)$ is finite for finite values of t .

Proof By the last proposition, we obtain an inequality of the form

$$\frac{d}{dt} E_{(\phi, 2, 2)}(t) \leq C(e^{Ct} E_{(\phi, 2, 2)}(t) + E_{(\phi, 2, 2)}(t) + e^{Ct}).$$

The claim follows by the Gronwall lemma. $\square$

At this point, we have obtained sufficient control over the pair (ϕ, ψ) and its derivatives, which is what we need to obtain a global solution of system (4.2), (4.3). In order to obtain a unique solution, we will need the following

Proposition 4.20 (Uniqueness) *Suppose we have two solutions of the Dirac-wave map system (4.2), (4.3). If their initial data coincides, then they coincide for all times.*

Proof We make use of the extrinsic version of the Dirac-wave map system (4.6), (4.7). Suppose that $u, v: \mathbb{R}^{1,1} \rightarrow \mathbb{R}^q$ both solve (4.6) and $\psi, \xi \in \Gamma(\Sigma \mathbb{R}^{1,1} \otimes \mathbb{R}^q)$ both solve (4.7), where ψ is a vector spinor along u and ξ a vector spinor along v . Since we have gained control over the L^2 -norm of the second derivatives of ϕ , we have a pointwise bound on its first derivatives. We set $w := u - v$, $\eta := \psi - \xi$ and calculate

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} |dw|^2 dx &= \int_{\mathbb{R}} \left\langle \frac{\partial w}{\partial t}, \square w \right\rangle dx \\ &= \int_{\mathbb{R}} \left\langle \frac{\partial w}{\partial t}, \mathbb{I}(u)(du, du) - \mathbb{I}(v)(dv, dv) \right\rangle dx \\ &\quad + \int_{\mathbb{R}} \varepsilon_j \left\langle \frac{\partial w}{\partial t}, P(u)(\mathbb{I}(u)(e_j \cdot \psi, du(e_j)), i\psi) \right. \\ &\quad \left. - P(v)(\mathbb{I}(v)(e_j \cdot \xi, dv(e_j)), i\xi) \right\rangle dx. \end{aligned}$$

We rewrite the first term on the right-hand side as follows

$$\mathbb{I}(u)(du, du) - \mathbb{I}(v)(dv, dv) = \mathbb{I}(u)(du, du) - \mathbb{I}(v)(du, du) + \mathbb{I}(v)(du + dv, dw).$$

This yields

$$\left\langle \frac{\partial w}{\partial t}, (\mathbb{I}(u) - \mathbb{I}(v))(du, du) \right\rangle \leq C |du|^2 |w| |dw|.$$

Using the orthogonality $\partial_t \perp \mathbb{I}$, we find

$$\begin{aligned} \left\langle \frac{\partial w}{\partial t}, \mathbb{I}(v)(du, dw) \right\rangle &= \left\langle \frac{\partial u}{\partial t}, \mathbb{I}(v)(du, dw) \right\rangle = \left\langle \frac{\partial u}{\partial t}, (\mathbb{I}(v) - \mathbb{I}(u))(du, dw) \right\rangle \\ &\leq C |du|^2 |w| |dw|. \end{aligned}$$

The same argument also holds for the term involving dv . We rewrite

$$\begin{aligned} &P(u)(\mathbb{I}(u)(e_j \cdot \psi, du(e_j)), i\psi) - P(v)(\mathbb{I}(v)(e_j \cdot \xi, dv(e_j)), i\xi) \\ &= (P(u) - P(v))(\mathbb{I}(u)(e_j \cdot \psi, du(e_j)), i\psi) \\ &\quad + P(v)((\mathbb{I}(u) - \mathbb{I}(v))(e_j \cdot \psi, du(e_j)), i\psi) \\ &\quad + P(v)(\mathbb{I}(v)(e_j \cdot \eta, du(e_j)), i\psi) + P(v)(\mathbb{I}(v)(e_j \cdot \xi, dw(e_j)), i\psi) \\ &\quad + P(v)(\mathbb{I}(v)(e_j \cdot \psi, du(e_j)), i\eta) \end{aligned}$$

such that we may estimate

$$\begin{aligned} & \left| \left\langle \frac{\partial w}{\partial t}, P(u)(\mathbb{I}(u)(e_j \cdot \psi, du(e_j)), i\psi) - P(v)(\mathbb{I}(v)(e_j \cdot \xi, dv(e_j)), i\xi) \right\rangle \right| \\ & \leq C(|w||dw||\psi|_\beta^2|du| + |\eta|_\beta|w||du||\psi|_\beta + |w||dv||\eta|_\beta|\psi|_\beta) \\ & \leq C(|\eta|_\beta^2 + |w|^2 + |dw|^2). \end{aligned}$$

Regarding the spinors ψ, ξ , we calculate

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} \langle \partial_t \cdot \eta, \eta \rangle dx = \int_{\mathbb{R}} \varepsilon_j \langle \mathbb{I}(u)(e_j \cdot \psi, du(e_j)) - \mathbb{I}(v)(e_j \cdot \xi, dv(e_j)), \eta \rangle dx.$$

We may rewrite the right-hand side as follows

$$\begin{aligned} \mathbb{I}(u)(e_j \cdot \psi, du(e_j)) - \mathbb{I}(v)(e_j \cdot \xi, dv(e_j)) &= (\mathbb{I}(u) - \mathbb{I}(v))(e_j \cdot \psi, du(e_j)) \\ &\quad + \mathbb{I}(v)(e_j \cdot \eta, dv(e_j)) \\ &\quad + \mathbb{I}(v)(e_j \cdot \xi, dw(e_j)). \end{aligned}$$

This allows us to obtain the following inequality

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} |\eta|_\beta^2 dx &\leq C \int_{\mathbb{R}} (|\eta|_\beta|w| + |\eta|_\beta^2|dv| + |\eta|_\beta|dw||\xi|) dx \\ &\leq C \int_{\mathbb{R}} (|\eta|_\beta^2 + |w|^2 + |dw|^2) dx. \end{aligned}$$

In total, we find

$$\frac{d}{dt} \int_{\mathbb{R}} (|\eta|_\beta^2 + |w|^2 + |dw|^2) dx \leq C \int_{\mathbb{R}} (|\eta|_\beta^2 + |w|^2 + |dw|^2) dx$$

such that

$$\int_{\mathbb{R}} (|\eta|_\beta^2 + |w|^2 + |dw|^2) dx \leq e^{Ct} \int_{\mathbb{R}} (|\eta_0|_\beta^2 + |w_0|^2 + |dw_0|^2) dx.$$

Consequently, we have $u = v$ and $\psi = \xi$ for all t , whenever the initial data coincides. $\square$

In the end, we obtain the following existence result

Theorem 4.21 *Let $\mathbb{R}^{1,1}$ be two-dimensional Minkowski space and (N, h) be a compact Riemannian manifold. Then, for any given initial data of the regularity*

$$\begin{aligned} \phi(0, x) &= \phi_0(x) \in H^2(\mathbb{R}, N), \\ \frac{\partial \phi}{\partial t}(0, x) &= \phi_1(x) \in H^1(\mathbb{R}, TN), \end{aligned}$$

$$\psi(0, x) = \psi_0(x) \in H^1(\mathbb{R}, \Sigma \mathbb{R}^{1,1} \otimes \phi^*TN) \cap W^{1,4}(\mathbb{R}, \Sigma \mathbb{R}^{1,1} \otimes \phi^*TN)$$

the equations for Dirac-wave maps (4.2), (4.3) admit a global weak solution of the class

$$\phi \in H^2(\mathbb{R}^{1,1}, N), \quad \psi \in H^1(\mathbb{R}^{1,1}, \Sigma \mathbb{R}^{1,1} \otimes \phi^*TN) \cap W^{1,4}(\mathbb{R}, \Sigma \mathbb{R}^{1,1} \otimes \phi^*TN),$$

which is uniquely determined by the initial data.

Proof By the energy estimates derived throughout this section, we have enough control on the right-hand sides of (4.6) and (4.7) to extend the solution of the Dirac-wave map system for all times. The uniqueness follows from Proposition 4.20. $\square$

Remark 4.22 It is very desirable to get rid of the requirement $\psi_0(x) \in W^{1,4}(\mathbb{R}, \Sigma \mathbb{R}^{1,1} \otimes \phi^*TN)$ in Theorem 4.21. However, in order to control the energy $E_{(\phi, 2, 2)}(t)$, it seems to be necessary to control L^p norms of ψ with $p \geq 2$.

Remark 4.23 The statement of Theorem 4.21 still holds if we take into account an additional two-form as was done in [12] for the case of the domain being a closed Riemannian surface.

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