

Feynman-Diagrammatic Description of the Asymptotics of the Time Evolution Operator in Quantum Mechanics

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Received: 18 April 2010 / Revised: 2 September 2010 / Accepted: 4 September 2010

Published online: 30 September 2010 – © The Author(s) 2010. This article is published with open access at Springerlink.com

Abstract. We describe the “Feynman diagram” approach to nonrelativistic quantum mechanics on \mathbb{R}^n , with magnetic and potential terms. In particular, for each classical path γ connecting points q_0 and q_1 in time t , we define a formal power series $V_\gamma(t, q_0, q_1)$ in \hbar , given combinatorially by a sum of diagrams that each represent finite-dimensional convergent integrals. We prove that $\exp(V_\gamma)$ satisfies Schrödinger’s equation, and explain in what sense the $t \rightarrow 0$ limit approaches the δ distribution. As such, our construction gives explicitly the full $\hbar \rightarrow 0$ asymptotics of the fundamental solution to Schrödinger’s equation in terms of solutions to the corresponding classical system. These results justify the heuristic expansion of Feynman’s path integral in diagrams.

Mathematics Subject Classification (2010). 81T18, 81S40, 81Q15.

Keywords. quantum mechanics, Feynman diagrams, formal integrals, path integrals, semiclassical asymptotics.

1. Introduction

The goal of this paper is to show that the semiclassical description of the evolution operator in quantum mechanics in terms of Feynman diagrams satisfies all the necessary requirements: it satisfies the Schrödinger equation and has the correct initial conditions. This description is rooted in Feynman’s path integral interpretation of quantum mechanics, first published in [5] (see also [6]), and formal semiclassical arguments. In certain cases, non-perturbative path integrals can be interpreted analytically as Weiner integrals, e.g. [1, 7, 10]. However, for the last half century, physicists have used “formal” perturbative path integrals and Feynman diagrams extensively to make perturbative empirical predictions in theories that do not have adequate mathematical foundations, and there have been similar mathematical applications to the subject of topological quantum field theory [15]. Surprisingly, the diagrammatic methods have not previously been shown to give the correct answers when applied to the one mathematically rigorous nonperturbative theory available, namely nonrelativistic quantum mechanics. We fill that gap.

In this paper we consider quantum mechanics in \mathbb{R}^n with the Hamiltonian being a second-order differential operator. (In [8] we will discuss perturbative quantum mechanics on general Riemannian manifolds.) Recall some basic definitions. Let $B_1, \dots, B_n, C: \mathbb{R}^n \rightarrow \mathbb{R}$ be smooth functions.¹ The nonrelativistic Schrödinger operator with electric potential C and magnetic potential $B = (B_1, \dots, B_n)$ is the following linear second-order differential operator on \mathbb{R}^n :

$$\hat{H} = \sum_{j=1}^n \left(i\hbar \frac{\partial}{\partial q^j} + B_j(q) \right)^2 + C(q) \quad (1.0.1)$$

Solutions to the partial differential equation $\frac{\partial}{\partial t} \psi(t, q) = \frac{1}{i\hbar} \hat{H} \psi(t, q)$ with fixed initial data define a one-parameter family of unitary “time evolution” operators $U(t)$ on $L^2(\mathbb{R}^n)$. The kernel of the time evolution operator is a distribution $U(t, q_0, q_1)$, satisfying $(U(t)\psi)(q_1) = \int_{\mathbb{R}^n} U(t, q_0, q_1) \psi(q_0) dq_0$. It is determined by the following initial value problem:

$$i\hbar \frac{\partial}{\partial t} U(t, q_0, q_1) = \hat{H}_{q_1} [U(t, q_0, q_1)] \quad U(0, q_0, q) = \delta(q - q_0) \quad (1.0.2)$$

The operator \hat{H}_{q_1} acts on the q_1 variable, leaving (t, q_0) fixed. In this paper we describe explicitly the asymptotics of $U(t, q_0, q_1)$ as $\hbar \rightarrow 0$, at least for (t, q_0, q_1) in a dense open subset of $\mathbb{R}_{>0} \times \mathbb{R}^{2n}$.

The asymptotics of $U(t, q_0, q_1)$ are known to be given by the Hamilton-Jacobi action and solutions to the transport equation [16]. Feynman’s path integral also predicts that the asymptotics are given by a power series where coefficients are parameterized by “Feynman diagrams”: each diagram represents a finite-dimensional integral, and the coefficients of the power series are the values of these integrals. In this paper we will define this diagrammatic expansion; the integrands depend on the solutions to the corresponding classical mechanical system. We will prove that the Feynman-diagrammatic power series satisfies formal versions of equation (1.0.2): it satisfies the Schrödinger equation as a function valued in formal power series, and as a distribution valued in formal power series approaches the δ distribution as $t \rightarrow 0$.

In select cases these results are already known. For the quadratic “harmonic oscillator” (the functions B_1, \dots, B_n are linear in position, C is quadratic), the path integral is defined directly from Gauss’s formula, and agrees with the Schrödinger picture [16]. Feynman’s original argument applies when the corresponding classical physics is a perturbation of the free theory, i.e. when B_1, \dots, B_n, C are infinitesimal quantities. Combining these observations, it is not too difficult to prove that the diagrammatic path integral satisfies Schrödinger’s equation

¹Without much difficulty, these functions can be taken to depend explicitly on an external time parameter, but for notational convenience we do not do so. The function $U(t, q_0, q_1)$ that we will discuss should then be a function of two time variables $U(t_0, t_1, q_0, q_1)$, and similar superficial changes must be made throughout this paper.

for infinitesimal perturbations of the harmonic oscillator. More importantly, Duru and Kleinert [3] apply the path integral approach to calculate the energy spectrum of a particle moving in the Coulomb potential, and compare their approach to the Schrödinger methods. Their methods differ from ours: we disallow singular potentials but consider arbitrary smooth ones, so our results do not apply to the Coulomb potential; moreover, they take advantage of numerous changes of variables (without explicitly checking the coordinate invariance of the path integral) and use the “charge” of the potential well as the perturbation parameter, whereas we define the path integral directly and use Planck’s constant \hbar for perturbation. Diagrammatic path integrals do exist in the work of Kleinert and Chervyakov [11] and in the work of DeWitt-Morette [2]; Kleinert and Chervyakov’s methods apply directly only to perturbations of free motion, and neither they nor DeWitt-Morette (who does give an expansion equivalent to ours) check the Schrödinger equation directly, although the result is implicit in their and Duru and Kleinert’s approaches.

We summarize the main definitions and results in Section 2. Section 3 has proofs of two key lemmas. The main theorem is proven in Section 4, and in Section 5 we prove the short-time asymptotics.

2. Statements of the Definitions and Results

2.1. PRELIMINARIES

We begin by establishing some notation. Kronecker’s δ -matrix is $\delta_j^i = 1$ if $i = j$ and 0 otherwise. Dirac’s δ -function is the distribution defined by $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$. Heaviside’s step function is:

$$\Theta(x) = \int_{-\infty}^x \delta(t) dt = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases}$$

We henceforth adopt Einstein’s summation convention: $p_i q^i = q^i p_i = \sum_{i=1}^n p_i q^i$. We raise and lower indices as is convenient: $p^i = p_i$.

DEFINITION 2.1.1. A (*piecewise-smooth parameterized*) *path* is a continuous function $\gamma : [0, t] \rightarrow \mathbb{R}^n$ such that there exists a decomposition $0 = t_0 < t_1 < \dots < t_n = t$ with $\gamma|_{[t_j, t_{j+1}]}$ smooth for each $j = 0, \dots, n-1$. The *duration* of a path $\gamma : [0, t] \rightarrow \mathbb{R}^n$ is the real number t . A *based loop* is a path $\gamma : [0, t] \rightarrow \mathbb{R}^n$ with $\gamma(0) = 0 = \gamma(t)$. A path γ is *classical* if it satisfies the following nonlinear second-order differential equation (we write $\dot{\gamma}(\tau) = \frac{d}{d\tau}\gamma(\tau)$, etc.):

$$0 = \ddot{\gamma}^i(\tau) + \left(\frac{\partial B_i}{\partial q^j}(\gamma(\tau)) - \frac{\partial B_j}{\partial q^i}(\gamma(\tau)) \right) \dot{\gamma}^j(\tau) + \frac{\partial C}{\partial q^i}(\gamma(\tau)) \quad (2.1.2)$$

Let γ, ξ be paths of duration t , and define the differential operator \mathcal{D}_γ given by:

$$\begin{aligned} \mathcal{D}_\gamma[\xi]^i(\tau) = & \ddot{\xi}^i(\tau) + \left(\frac{\partial B_i}{\partial q^j}(\gamma(\tau)) - \frac{\partial B_j}{\partial q^i}(\gamma(\tau)) \right) \dot{\xi}^j(\tau) \\ & + \left(\frac{\partial^2 B_i}{\partial q^j \partial q^k}(\gamma(\tau)) \dot{\gamma}^k(\tau) - \frac{\partial^2 B_j}{\partial q^i \partial q^k}(\gamma(\tau)) \dot{\gamma}^k(\tau) + \frac{\partial^2 C}{\partial q^i \partial q^j}(\gamma(\tau)) \right) \xi^j(\tau) \end{aligned} \tag{2.1.3}$$

A classical path γ of duration t is *nondegenerate* if the kernel of \mathcal{D}_γ among based loops of duration t is trivial.

As equation (2.1.2) is nondegenerate second-order, a classical path γ is determined by its duration and its initial conditions $(\dot{\gamma}(0), \gamma(0))$. Thus the space of classical paths is naturally an open subset of \mathbb{R}^{2n+1} . Nondegeneracy is clearly an open condition among classical paths. A standard result holds that for given initial conditions, the set of durations for which a classical path fails to be nondegenerate is discrete; see e.g. [13]. In particular, the nondegenerate classical paths are a dense open subset of all classical paths. The following lemma follows from e.g. [9]; in Section 3.1 we include the proof rewritten in the language of classical paths.

LEMMA 2.1.4. *Let $\gamma : [0, t] \rightarrow \mathbb{R}^n$ be classical and nondegenerate. Then there exists a neighborhood \mathcal{O} of $(\gamma(0), \gamma(t)) \in \mathbb{R}^{2n}$ and a map $\hat{\gamma} : [0, t] \times \mathcal{O} \rightarrow \mathbb{R}^n$ such that (i) $\hat{\gamma}(-, q_0, q_1)$ is classical and nondegenerate for each $(q_0, q_1) \in \mathcal{O}$, (ii) $\hat{\gamma}(0, q_0, q_1) = q_0$ and $\hat{\gamma}(t, q_0, q_1) = q_1$, and (iii) $\hat{\gamma}(-, \gamma(0), \gamma(t)) = \gamma$.*

Let γ be a classical path of duration t . Then for sufficiently small ϵ , γ determines classical paths γ_s for $s \in [t - \epsilon, t + \epsilon]$, either by restriction or by extension via equation (2.1.2), and if γ is nondegenerate, for sufficiently small ϵ so are the γ_s . Thus, to a nondegenerate classical path γ we can associate a family $\hat{\gamma}_t(q_0, q_1, -)$ of nondegenerate classical paths, varying smoothly in the parameters. We will abuse notation and write γ for this whole family. So henceforth γ is a smooth family of solutions to the boundary value problem given by equation (2.1.2) along with Dirichlet boundary values $\gamma(0) = q_0, \gamma(t) = q_1$, where (t, q_0, q_1) ranges over an open subset \mathcal{O} of \mathbb{R}^{2n+1} .

DEFINITION 2.1.5. Let γ be a classical nondegenerate path, extended to a family with boundary values varying in $\mathcal{O} \subseteq \mathbb{R}^{2n+1}$ as above. The corresponding *Hamilton function* $S_\gamma : \mathcal{O} \rightarrow \mathbb{R}$ is:

$$S_\gamma(t, q_0, q_1) = \int_0^t \left(\frac{1}{2} |\dot{\gamma}(\tau)|^2 + B_i(\gamma(\tau)) \dot{\gamma}^i(\tau) - C(\gamma(\tau)) \right) d\tau \tag{2.1.6}$$

Then $S_\gamma(t, q_0, q_1)$ satisfies the following well-known equations:

$$\frac{\partial[-S_\gamma]}{\partial t} = \frac{1}{2} |\nabla_{q_0}[-S_\gamma] - B(q_0)|^2 + C(q_0) = \frac{1}{2} |\nabla_{q_1}[-S_\gamma] + B(q_1)|^2 + C(q_1) \quad (2.1.7)$$

$$\frac{\partial[-S_\gamma]}{\partial q_0^i} = \dot{\gamma}^i(0) + B_i(q_0) \quad \frac{\partial S_\gamma}{\partial q_1^i} = \dot{\gamma}^i(t) + B_i(q_1) \quad (2.1.8)$$

For a few different constructions, we will need the matrix $\frac{\partial^2}{\partial q_0 \partial q_1}[-S(t, q_0, q_1)]$ and its inverse; we define the indices on the inverse matrix by: $\left(\left(\frac{\partial^2[-S]}{\partial q_1 \partial q_0} \right)^{-1} \right)^{kl}$ $\frac{\partial^2[-S]}{\partial q_0^l \partial q_1^k} = \delta_m^k$, and similarly for $q_1 \leftrightarrow q_0$.

LEMMA 2.1.9. *Let γ be a nondegenerate path of duration t , extended to a family as above. Let $G_\gamma: [0, t]^{\times 2} \rightarrow (\mathbb{R}^n)^{\otimes 2}$ be defined by:*

$$\begin{aligned} G_\gamma^{ij}(\varsigma, \tau) = & \Theta(\varsigma - \tau) \frac{\partial \gamma^i}{\partial q_0^k}(\varsigma) \left(\left(\frac{\partial^2(-S_\gamma)}{\partial q_0 \partial q_1} \right)^{-1} \right)^{kl} \frac{\partial \gamma^j}{\partial q_1^l}(\tau) \\ & + \Theta(\tau - \varsigma) \frac{\partial \gamma^i}{\partial q_1^k}(\varsigma) \left(\left(\frac{\partial^2(-S_\gamma)}{\partial q_1 \partial q_0} \right)^{-1} \right)^{kl} \frac{\partial \gamma^j}{\partial q_0^l}(\tau) \end{aligned} \quad (2.1.10)$$

Then $-G_\gamma$ is a Green's function for the operator \mathcal{D}_γ given in equation (2.1.3); i.e. G_γ satisfies $\mathcal{D}_\gamma[G_\gamma]_k^i(\varsigma, \tau) = -\delta_k^i \delta(\tau - \varsigma)$, where \mathcal{D}_γ acts on the j, τ parts of equation (2.1.10), and G_γ vanishes along the boundary of the square $[0, t]^{\times 2}$.

For the proof, see Section 3.2. Statements similar to Lemmas 2.1.4 and 2.1.9 appear in e.g. [2], where degenerate paths are also briefly addressed.

2.2. FEYNMAN DIAGRAMS AND THE MAIN RESULTS

DEFINITION 2.2.1. A *partition* of a set X is a collection of nonempty subsets of X , called the *blocks* of the partition, that are pairwise disjoint and whose union is all of X .

A *Feynman diagram* is a finite combinatorial graph with all vertices trivalent and higher. (Self-loops, parallel edges, and the empty diagram are allowed.) More precisely, a Feynman diagram is a finite set H of “half-edges,” along with two partitions E (the “edges”) and V (the “vertices”) of H , with the requirements $|e| = 2$ for each block $e \in E$, and $|v| \geq 3$ for each block $v \in V$. An *isomorphism* of Feynman diagrams $\Gamma_1 = (H_1, E_1, V_1)$ and $\Gamma_2 = (H_2, E_2, V_2)$ is a bijection $\phi: H_1 \xrightarrow{\sim} H_2$ such that $\phi(E_1) = E_2$ and $\phi(V_1) = V_2$, where $\phi(E_1), \phi(V_1)$ are the obvious partitions of H_2 induced by (ϕ, E_1, V_1) . We will denote Feynman diagrams with the

obvious pictures:



Let $\Gamma = (H, E, V)$ be a Feynman diagram. The *Euler characteristic* of Γ is $\chi(\Gamma) = |V| - |E|$. The *connected components* of Γ are the equivalence classes of H under the equivalence relation generated by $E \cup V$. Let $\pi_0(\Gamma)$ be the set of connected components of Γ ; then $\lambda(\Gamma) = |\pi_0(\Gamma)| - \chi(\Gamma)$ is the *number of loops* of Γ , also called its *first Betti number*. A diagram Γ is *connected* if it has precisely one connected component. For example, the above three pictures comprise all (up to isomorphism) diagrams Γ with $\chi(\Gamma) = -1$, and each is connected.

A *marked Feynman diagram* is a Feynman diagram $\Gamma = (H, E, V)$ along with a subset $M \subseteq H$ of “marked half-edges” such at each vertex $v \in V$, at most one half-edge $\eta \in v$ is marked, i.e. $|v \cap M| \leq 1$. An isomorphism of marked Feynman diagrams is required to respect the marking. We indicate the markings on a marked Feynman diagram with small circles, as in equation (2.2.4).

DEFINITION 2.2.2. Pick a duration $t > 0$. A *totally labeled Feynman diagram* (with duration t) is a Feynman diagram $\Gamma = (H, E, V)$ along with maps $\vec{\tau} : V \rightarrow [0, t]$ and $\vec{j} : H \rightarrow \{1, \dots, n\}$. Let γ be a classical nondegenerate path of duration t and $(\Gamma, M, \vec{\tau}, \vec{j})$ a totally labeled marked Feynman diagram. Then its *value* is (no summation convention):

$$\begin{aligned}
 \mathcal{F}_\gamma(\Gamma, M, \vec{\tau}, \vec{j}) = & \prod_{v \in V, v \cap M = \emptyset} \left(\left(\prod_{\eta \in V} \frac{\partial}{\partial q^{\vec{j}(\eta)}} \right) [C(q)] \Big|_{q = \gamma(\vec{\tau}(v))} \right) \\
 & + \left(\prod_{\eta \in V} \frac{\partial}{\partial q^{\vec{j}(\eta)}} \right) [B_k(q)] \Big|_{q = \gamma(\vec{\tau}(v))} \dot{\gamma}^k(\tau) \\
 & \times \prod_{v \in V, v \cap M = \{\zeta\}} \left(\left(\prod_{\eta \in v \setminus \{\zeta\}} \frac{\partial}{\partial q^{\vec{j}(\eta)}} \right) [-B_{\vec{j}(\zeta)}(q)] \Big|_{q = \gamma(\vec{\tau}(v))} \right) \\
 & \times \prod_{\substack{e = \{\eta, \zeta\} \in E, \\ e \cap M = \emptyset}} G^{\vec{j}(\eta), \vec{j}(\zeta)}(\vec{\tau}(\eta), \vec{\tau}(\zeta)) \\
 & \times \prod_{\substack{e = \{\eta, \zeta\} \in E, \\ e \cap M = \{\eta\}}} \frac{\partial G^{\vec{j}(\eta), \vec{j}(\zeta)}(\varsigma_1, \varsigma_2)}{\partial \varsigma_1} \Big|_{\substack{\varsigma_1 = \vec{\tau}(\eta) \\ \varsigma_2 = \vec{\tau}(\zeta)}} \\
 & \times \prod_{\substack{e = \{\eta, \zeta\} \in E, \\ e \subseteq M}} \frac{\partial^2 G^{\vec{j}(\eta), \vec{j}(\zeta)}(\varsigma_1, \varsigma_2)}{\partial \varsigma_1 \partial \varsigma_2} \Big|_{\substack{\varsigma_1 = \vec{\tau}(\eta) \\ \varsigma_2 = \vec{\tau}(\zeta)}}
 \end{aligned}$$

If (Γ, M) is an (unlabeled) marked Feynman diagram, its *value* is an integral over all total labelings:

$$\mathcal{F}_\gamma(\Gamma, M) = \sum_{\vec{j} \in \{1, \dots, n\}^H} \int_{\vec{\tau} \in [0, t]^V} \mathcal{F}_\gamma(\Gamma, M, \vec{\tau}, \vec{j}) d\vec{\tau}$$

More compactly, Feynman diagrams are evaluated via the following *Feynman rules* and the obvious compositions and contractions (including the Einstein summation convention), where ξ_1, \dots, ξ_k are test-functions ranging over the space of paths of duration t :

$$\mathcal{F}_\gamma \left(\begin{array}{c} \xi_1 \quad \xi_2 \quad \dots \quad \xi_n \\ \diagdown \quad \diagup \quad \dots \quad \diagdown \quad \diagup \\ \bullet \end{array} \right) = \int_{\tau=0}^t \left(\frac{\partial^n C(q)}{\partial q^{i_1} \dots \partial q^{i_n}} + \frac{\partial^n B_k(q)}{\partial q^{i_1} \dots \partial q^{i_n}} \dot{\gamma}^k(\tau) \right) \Big|_{q=\gamma(\tau)} \xi_1^{i_1}(\tau) \dots \xi_n^{i_n}(\tau) d\tau \quad (2.2.3)$$

$$\mathcal{F}_\gamma \left(\begin{array}{c} \xi_1 \quad \xi_2 \quad \dots \quad \xi_n \\ \diagdown \quad \diagup \quad \dots \quad \diagdown \quad \diagup \\ \circ \end{array} \right) = \int_{\tau=0}^t \frac{\partial^{n-1}[-B_{i_1}]}{\partial q^{i_2} \dots \partial q^{i_n}} \Big|_{q=\gamma(\tau)} \xi_1^{i_1}(\tau) \xi_2^{i_2}(\tau) \dots \xi_n^{i_n}(\tau) d\tau \quad (2.2.4)$$

$$\mathcal{F}_\gamma \left(\begin{array}{c} \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \end{array} \right) = G_\gamma^{ij}(\zeta, \tau) \quad (2.2.5)$$

We must check that the value of a marked Feynman diagram converges: the Green's function $G_\gamma(\zeta, \tau)$ is non-smooth like $|\zeta - \tau|$ near the diagonal, so any diagram in which both ends of the same edge are marked includes a Dirac δ -function distribution. But no vertex can have more than one marking, so such δ -functions do not lead to divergence integrals. (We have insisted that B, C be smooth. If they are allowed to be singular, divergences can occur, requiring some renormalization procedure, c.f. [12].) We prove the following in Section 4.

THEOREM 2.2.6 (Schrödinger's Equation). *Let γ be a classical nondegenerate path, extended to a family with boundary values (t, q_0, q_1) ranging over some open set $\mathcal{O} \in \mathbb{R}^{2n+1}$. Define on \mathcal{O} the following $\mathbb{R}[[\hbar]]$ -valued function:*

$$\begin{aligned} V_\gamma(t, q_0, q_1) = & -S_\gamma(t, q_0, q_1) + i\hbar \frac{1}{2} \log \left| \det \frac{\partial^2[-S_\gamma]}{\partial q_0 \partial q_1} \right| \\ & + \sum_{\substack{\text{equivalence classes } (\Gamma, M) \\ \text{of connected marked} \\ \text{Feynman diagrams}}} \frac{(i\hbar)^{\lambda(\Gamma)} \mathcal{F}_\gamma(\Gamma, M)}{|\text{Aut}(\Gamma, M)|} \end{aligned} \quad (2.2.7)$$

Then $U_\gamma = \exp((i\hbar)^{-1} V_\gamma)$ is a formal solution to Schrödinger's equation: $i\hbar \frac{\partial U_\gamma}{\partial t} = \hat{H}[U_\gamma]$ as formal power series in \hbar .

We remark that for a connected diagram $\Gamma = (H, E, V)$, we have $\lambda(\Gamma) = |E| - |V| + 1 \leq |E| = 2|H|$, and so the sum in equation (2.2.7) is finite at each order in \hbar .

In general, the series has zero radius of convergence in \hbar . Note that U_γ can be described as:

$$U_\gamma(t, q_0, q_1) = e^{-(i\hbar)^{-1}S_\gamma} \sqrt{\left| \det \frac{\partial[-S_\gamma]}{\partial q_0 \partial q_1} \right|} \sum_{\substack{\text{equivalence classes of} \\ \text{marked Feynman} \\ \text{diagrams } (\Gamma, M)}} \frac{(i\hbar)^{-\chi(\Gamma)} \mathcal{F}_\gamma(\Gamma, M)}{|\text{Aut}(\Gamma, M)|}$$

2.3. SUMS AND LIMITS OF U_γ

Temporarily, let \hbar range over “small” non-zero real numbers. Pick N a positive integer and suppose that $\mathcal{O} \subseteq \mathbb{R}^n$ is open, and $f_1, f_2 : \mathcal{O} \rightarrow \mathbb{R}[i\hbar]$ satisfy $f_1 = f_2 + O(\hbar^{N+1})$. Let $g : \mathcal{O} \rightarrow \mathbb{R}^n$ have compact support. Then:

$$\int e^{(i\hbar)^{-1}f_1(q)} g(q) dq = \int e^{(i\hbar)^{-1}f_2(q)} g(q) dq + O(\hbar^{N+n/2}) \tag{2.3.1}$$

For details, see [4, Chapter 3]. Thus, if $f, g : \mathcal{O} \rightarrow \mathbb{R}[[i\hbar]]$ are smooth term-by-term and g has compact support, then we are justified in defining $\int \exp((i\hbar)^{-1}f(q)) g(q) dq$ as a formal expression in \hbar , by successively approximating f by polynomials in \hbar . We will write \int^{formal} for this formal integral.

We will not totally define what we mean by a “formal expression in \hbar ,” although it is not hard to do so. Essentially, a formal expression in \hbar is a $\mathbb{C}[[\sqrt{\hbar}]]$ -linear combination of terms of the form $\exp((i\hbar)^{-1}f)$, where $f \in \mathbb{R}[[i\hbar]]$, modulo the obvious equivalences and with the obvious arithmetic. If $f \pmod{\hbar}$ has finitely many critical points within the support of g , and if at each critical point the Hessian of f is nondegenerate, then $\int^{\text{formal}} \exp((i\hbar)^{-1}f) g$ is a formal expression in \hbar in this sense.

DEFINITION 2.3.2. Let $f_1, f_2, \dots, f_\infty$ be distributions on $\mathcal{O} \subseteq \mathbb{R}^n$, valued in formal expressions in \hbar . Then $\lim f_N = f_\infty$ if and only if for every $g : \mathcal{O} \rightarrow \mathbb{R}$ with compact support, $\lim_{N \rightarrow \infty} \int_{\mathcal{O}}^{\text{formal}} f_N(q) g(q) dq = \int_{\mathcal{O}}^{\text{formal}} f_\infty(x) g(x) dx$. This defines the *pointwise convergence* topology on the space of distributions. The topology on $\mathbb{R}[[i\hbar]]$ is given coefficient-by-coefficient.

In Section 5 we will prove:

THEOREM 2.3.3 (The initial value problem). *Pick $\mathcal{O} \subseteq \mathbb{R}^{2n+1}$ open, and let A be the set of all families of nondegenerate classical paths with boundary values varying over \mathcal{O} . Let $\{g_\gamma\}_{\gamma \in A}$ be a collection of smooth functions $\mathcal{O} \rightarrow \mathbb{R}[[i\hbar]]$. Then for fixed t, q_1 , the possibly-infinite sum $\sum_{\gamma \in A} g_\gamma(t, q_0, q_1) \exp((i\hbar)^{-1}V_\gamma(t, q_0, q_1))$ converges pointwise as a distribution in q_0 valued in formal expressions in \hbar .*

Moreover, suppose that $\mathcal{O} = (0, \epsilon) \times \mathcal{O}_0 \times \mathcal{O}_0$, where $\mathcal{O}_0 \subseteq \mathbb{R}^n$ is open with compact closure and $\epsilon > 0$ is sufficiently small depending on \mathcal{O}_0 . Then for fixed q_1 , as a

distribution valued in formal expressions in \hbar we have:

$$\lim_{t \rightarrow 0} \sum_{\gamma \in A} \exp\left((i\hbar)^{-1} V_\gamma(t, q_0, q_1)\right) = (2\pi\hbar)^{n/2} e^{in\pi/4} \delta(q_0 - q_1) \tag{2.3.4}$$

The reader should be mildly disappointed in Theorem 2.3.3: we have first set \hbar to a formal parameter, and then, in equation (2.3.4), taken a limit as $t \rightarrow 0$. Much better would be to approximate V_γ by a polynomial in \hbar , interpret \hbar as a positive real parameter, take any necessary limits, and then take the asymptotics as $\hbar \rightarrow 0$. But when the limits are performed in this order, we do not believe that $\sum_{\gamma \in A} \exp((i\hbar)^{-1} V_\gamma)$ always converges. Moreover, even if it does, some extra assumption is needed in order to take the $t \rightarrow 0$ limit for non-formal \hbar in equation (2.3.4). For example, it is enough if the paths $\gamma \in A$ are of bounded length, although we will not prove this. However, in as natural a system as the one-dimensional mechanics with electric potential $C(q) = q^4$, the classical paths can be arbitrarily long, and as $t \rightarrow 0$ with fixed endpoints the classical paths run away with length growing as $t^{-1/2}$.

One final remark is in order. It is tempting to conclude from Theorems 2.2.6 and 2.3.3 that the $\hbar \rightarrow 0$ asymptotics of the full propagator $U(t, q_0, q_1)$ are given by:

$$U(t, q_0, q_1) \approx \sum_{\substack{\gamma \text{ classical,} \\ \gamma(0)=q_0, \gamma(t)=q_1}} (2\pi\hbar)^{-n/2} e^{-in\pi/4} \exp\left((i\hbar)^{-1} V_\gamma(t, q_0, q_1)\right)$$

But a modification must be made. The formal power series V_γ are defined only for γ nondegenerate, and Theorem 2.2.6 only applies in this case. Thus the above formal sum solves equation (1.0.2) only for short times in a neighborhood of the diagonal. In general, if we know the asymptotics of U near short nondegenerate paths, we can determine the full asymptotics by means of the composition law: $U(t_0 + t_1, q_0, q_1) = \int U(t_0, q_0, q) U(t_1, q, q_1) dq$. In [8] we prove the following:

THEOREM 2.3.5 (Semigroup law). *Let γ, V_γ be as in Theorem 2.2.6. Suppose that $t = t_0 + t_1$ and the restrictions $\gamma_0 = \gamma|_{[0, t_0]}$ and $\gamma_1 = \gamma|_{[t_0, t]}$ are nondegenerate, and define the corresponding formal power series V_{γ_0} and V_{γ_1} . Let $\eta(\gamma)$ be the Morse index of γ with respect to the Morse function $\gamma \mapsto \int_0^t (\frac{1}{2}|\dot{\gamma}|^2 + B(\gamma) \cdot \dot{\gamma} - C(\gamma)) d\tau$ on the space of paths with the same boundary values, and similarly define Morse indexes $\eta(\gamma_0)$ and $\eta(\gamma_1)$ (see e.g. [13]). Set $U_\gamma = (-i)^{\eta(\gamma)} (2\pi\hbar)^{-n/2} e^{-in\pi/4} \exp((i\hbar)^{-1} V_\gamma(t, q_0, q_1))$, and similarly define U_{γ_0} and U_{γ_1} . Then for a sufficiently small neighborhood \mathcal{Q} of $\gamma(t_0)$ we have:*

$$U_\gamma(t_0 + t_1, q_0, q_1) = \int_{\mathcal{Q}}^{\text{formal}} U_{\gamma_0}(t_0, q_0, q) U_{\gamma_1}(t_1, q, q_1) dq$$

Thus, the correct asymptotics of the full propagator $U(t, q_0, q_1)$ are essentially given by:

$$U(t, q_0, q_1) \approx (2\pi\hbar)^{-n/2} e^{in\pi/4} \sum_{\substack{\gamma \text{ classical,} \\ \gamma(0)=q_0, \gamma(t)=q_1}} (-i)^{n(\gamma)} \exp\left((i\hbar)^{-1} V_\gamma(t, q_0, q_1)\right)$$

Here the sum makes sense provided that there is no degenerate classical path of duration t connecting q_0 to q_1 , and converges pointwise as a distribution. This justifies the method of formal Feynman path integration.

3. Proofs of the Lemmas

3.1. PROOF OF LEMMA 2.1.4

Let \mathcal{P} be the space of paths of duration t and Λ the space of based loops of duration t . Each is an infinite-dimensional vector space; we think of \mathcal{P} as an infinite-dimensional smooth manifold with tangent bundle $T\mathcal{P} = \mathcal{P} \times \mathcal{P}$. Recall that the derivative df of a function $f : \mathcal{P} \rightarrow \mathbb{R}$ is defined by $df_\gamma \cdot \xi = \frac{\partial}{\partial \epsilon} |_{\epsilon=0} (f(\gamma + \epsilon\xi))$; f is differentiable if such a derivative exists for all $(\xi, \gamma) \in T\mathcal{P}$. Let $\pi : \mathcal{P} \rightarrow \mathbb{R}^{2n}$ be the map $\gamma \mapsto (\gamma(0), \gamma(t))$; it is a trivial bundle with fiber Λ . As Λ is a vector space, we can identify the fiber tangent spaces: if $\gamma \in \mathcal{P}$, then $T_\gamma(\pi^{-1}(\pi(\gamma))) = \Lambda$.

Consider the function $f : \mathcal{P} \rightarrow \mathbb{R}$ given by:

$$f(\gamma) = \int_0^t \left(\frac{1}{2} |\dot{\gamma}(\tau)|^2 + B_i(\gamma(\tau)) \dot{\gamma}^i(\tau) - C(\gamma(\tau)) \right) d\tau$$

As is well-known, a path γ is classical if and only if $\Lambda \subseteq \ker df_\gamma$. The Hessian of f at γ is the symmetric bilinear form:

$$h_\gamma : (\xi_1, \xi_2) \mapsto \int_0^t \left(\xi_1^i(\tau) \xi_2^j(\tau) + \frac{\partial B_i(q)}{\partial q^j} \left(\xi_1^i(\tau) \xi_2^j(\tau) + \xi_1^j(\tau) \xi_2^i(\tau) \right) + \left(\frac{\partial^2 B_k(q)}{\partial q^i \partial q^j} \dot{\gamma}^k(\tau) - \frac{\partial^2 C(q)}{\partial q^i \partial q^j} \right) \xi_1^i(\tau) \xi_2^j(\tau) \right) \Big|_{q=\gamma(\tau)} d\tau$$

When ξ_1, ξ_2 are based loops, integration by parts gives $h_\gamma(\xi_1, \xi_2) = \int_0^t \xi_1(\tau) \cdot \mathcal{D}_\gamma[\xi_2](\tau) d\tau$. Thus γ is nondegenerate if and only if $\Lambda \cap \ker h_\gamma = 0$.

Let \mathcal{C} be the space of classical paths. By considering the derivatives of df along paths in \mathcal{C} , one easily checks that $T_\gamma \mathcal{C} \subseteq \ker h_\gamma$. Thus if γ is classical and nondegenerate, then $d\pi : T_\gamma \mathcal{C} \rightarrow T_{\pi(\gamma)} \mathbb{R}^{2n}$ is an injection. But the space of classical paths of a given duration has dimension $2n$, and so $d\pi$ is a bijection. This completes the proof of Lemma 2.1.4.

3.2. PROOF OF LEMMA 2.1.9

We begin with the observation that for each $a = 0, 1$ and each $j = \{1, \dots, n\}$, the path $\phi_{a,j}(\tau) = \frac{\partial \gamma}{\partial q_a^j}(\tau)$ satisfies $\mathcal{D}_\gamma[\phi] = 0$; this follows by differentiating equation (2.1.2). In other words, the $2n$ paths $\phi_{a,j}$ are *Jacobi fields* along γ . The boundary values are: $\phi_{0,j}^i(0) = \delta_j^i = \phi_{1,j}^i(t)$ and $\phi_{0,j}^i(t) = \delta_j^i = \phi_{1,j}^i(0)$. Let $\phi_0(\tau)$, $\phi_1(\tau)$ be the $n \times n$ matrices $(\phi_a(\tau))_j^i = \phi_{a,j}^i(\tau)$. Each solution to the initial value problem $\{\mathcal{D}_\gamma[\varphi], \varphi(0) = 0\}$ is determined by $\dot{\varphi}(0)$. Since $\phi_1(t) = \delta$ is full-rank, $\dot{\phi}_1(0)$ must be also.

Define the following function on $[0, t]$ valued in $2n \times 2n$ matrices:

$$M(\tau) = \begin{pmatrix} \phi_0(\tau) & \phi_1(\tau) \\ \dot{\phi}_0(\tau) & \dot{\phi}_1(\tau) \end{pmatrix}$$

Then $M(0)$ is lower-triangular with full-rank blocks on the diagonal, and hence invertible. Since the coefficient on $\dot{\xi}$ in $\mathcal{D}_\gamma[\xi]$ is traceless, Liouville's formula gives $\det M(\tau) = \det M(0)$, and so $M(\tau)$ is invertible for each $\tau \in [0, t]$. Let ψ_0, ψ_1 be the $(n \times n)$ -matrix valued functions on $[0, t]$ satisfying:

$$\begin{pmatrix} \phi_{0,j}^i(\tau) & \phi_{1,j}^i(\tau) \\ \dot{\phi}_{0,j}^i(\tau) & \dot{\phi}_{1,j}^i(\tau) \end{pmatrix} \begin{pmatrix} \psi_{0,k}^j(\tau) \\ \psi_{1,k}^j(\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ \delta_k^i \end{pmatrix} \quad (3.2.1)$$

Then it is straightforward to check that the Green's function for $-\mathcal{D}_\gamma$ is given by:

$$G_k^i(\zeta, \tau) = -\Theta(\zeta - \tau) \phi_{0,j}^i(\tau) \psi_{0,k}^j(\zeta) + \Theta(\tau - \zeta) \phi_{1,j}^{i,i}(\tau) \psi_{1,k}^j(\zeta)$$

But the differential operator $-\mathcal{D}_\gamma$ is self-adjoint for the inner product $\langle \xi_1, \xi_2 \rangle = \int_0^t \xi_1 \cdot \xi_2$ on the space of based loops. Thus if $G_k^i(\zeta, \tau)$ is its Green's function, then $G_k^i(\zeta, \tau) = G_i^k(\tau, \zeta)$. In particular, each ψ_a satisfies $\mathcal{D}_\gamma[\psi_a] = 0$, and so the ψ_a s are linear combinations of the ϕ_b s. To find the coefficients, we need only check the boundary values. We have $\psi_{0,k}^j(0) = 0 = \psi_{1,k}^j(t)$, by evaluating the top row of equation (3.2.1) at $\tau = 0, t$. On the other hand, the bottom row then gives:

$$(\psi_0(t))^{-1} = \dot{\phi}_0(t) = \frac{\partial^2 \gamma(\tau)}{\partial \tau \partial q_0} \Big|_{\tau=t} = \frac{\partial}{\partial q_0} (\dot{\gamma}(t)) = \frac{\partial}{\partial q_0} \left(\frac{\partial S_\gamma}{\partial q_1} - B(q_1) \right) = \frac{\partial^2 S_\gamma}{\partial q_0 \partial q_1} \quad (3.2.2)$$

We have used equation (2.1.8) to do the necessary simplifications. Thus $\psi_{0,k}^j(\zeta) = \phi_{1,l}^j(\zeta) \left(\left(\frac{\partial^2 S_\gamma}{\partial q_0 \partial q_1} \right)^{-1} \right)^{lk}$. A similar argument shows that $\psi_{1,k}^j(\zeta) = -\phi_{0,l}^j(\zeta) \left(\left(\frac{\partial^2 S_\gamma}{\partial q_1 \partial q_0} \right)^{-1} \right)^{lk}$. This completes the proof of Lemma 2.1.9.

Here the test functions ξ, ζ are paths of duration t , and in the second line on the right-hand side the bivalent vertices have the natural meanings extending equations (2.2.3) and (2.2.4) to $n=2$. In particular, $\blacksquare^\xi = 0$ if ξ is a based loop. By integrating by parts, $\blacksquare^{\xi, \zeta} = \int_0^t \xi(\tau) \cdot \mathcal{D}_\gamma[\zeta](\tau) d\tau$ whenever ξ is a based loop, where \mathcal{D}_γ is the differential operator given in equation (2.1.3).

We represent differentiation with respect to t, q_0, q_1 with dotted circles:

$$\begin{aligned} \text{Diagram with } \Gamma \text{ in a dotted circle} &= \frac{\partial}{\partial t} [\mathcal{F}(\Gamma)] & \text{Diagram with } q_a \text{ in a dotted circle} &= \frac{\partial}{\partial q_a^i} [\mathcal{F}(\Gamma)] \end{aligned}$$

Then the product rule can be written graphically as:

$$\text{Diagram with } \Gamma_1, \Gamma_2 \text{ in a dotted circle} = \text{Diagram with } \Gamma_1 \text{ in a dotted circle} + \text{Diagram with } \Gamma_2 \text{ in a dotted circle}$$

Suppose then that Γ_1 is a subdiagram of Γ whose images $\Gamma_1, \dots, \Gamma_n$ under the group of automorphisms of Γ do not intersect, so that $\Gamma = \bar{\Gamma} \cup \Gamma_1 \cup \dots \cup \Gamma_n$ (we do allow automorphism of Γ to induce nontrivial automorphisms of Γ_1). Then:

$$\begin{aligned} \text{Diagram with } \Gamma \text{ in a dotted circle} &= \text{Diagram with } \bar{\Gamma} \Gamma_1 \Gamma_2 \dots \Gamma_n \text{ in a dotted circle} \\ &= \text{Diagram with } \bar{\Gamma} \Gamma_1 \Gamma_2 \dots \Gamma_n \text{ in a dotted circle} + \bar{\Gamma} \text{ (Diagram with } \Gamma_1 \text{ in a dotted circle)} \Gamma_2 \dots \Gamma_n + \dots + \bar{\Gamma} \Gamma_1 \Gamma_2 \dots \text{ (Diagram with } \Gamma_n \text{ in a dotted circle)} \\ &= \text{Diagram with } \bar{\Gamma} \Gamma_1 \Gamma_2 \dots \Gamma_n \text{ in a dotted circle} + n \bar{\Gamma} \text{ (Diagram with } \Gamma_1 \text{ in a dotted circle)} \Gamma_2 \dots \Gamma_n \end{aligned}$$

It is an elementary counting lemma that $|\text{Aut}(\bar{\Gamma} \text{ (Diagram with } \Gamma_1 \text{ in a dotted circle)} \Gamma_2 \dots \Gamma_n)| = \frac{1}{n} |\text{Aut}(\bar{\Gamma} \Gamma_1 \Gamma_2 \dots \Gamma_n)|$. From this observation, we derive the fundamental result that for $z = q_1^i$ or t :

$$\frac{\partial}{\partial z} \sum_{\Gamma \text{ connected}} \frac{(i\hbar)^{\lambda(\Gamma)} \mathcal{F}_\gamma(\Gamma)}{|\text{Aut } \Gamma|} = \sum_{\substack{\Gamma \text{ connected with} \\ \text{one } \blacksquare^\xi \text{ or } \text{Diagram with } \Gamma_1 \text{ in a dotted circle}}} \frac{(i\hbar)^{\lambda(\Gamma)} \mathcal{F}_\gamma(\Gamma)}{|\text{Aut } \Gamma|} \tag{4.1.1}$$

The sum on the right-hand side ranges over connected components with precisely one differentiated basic subgraph—either a single differentiated vertex (of valence three or more) or a single differentiated edge.

In this notation, equation (2.1.7) reads:

$$\text{Diagram with } \blacksquare^\xi \text{ in a dotted circle} = \frac{1}{2} \text{Diagram with } \blacksquare^\xi \text{ in a dotted circle} + \text{Diagram with } \blacksquare^\xi \text{ in a dotted circle} + \frac{1}{2} \text{Diagram with } \text{Diagram with } \Gamma_1 \text{ in a dotted circle} + \text{Diagram with } \text{Diagram with } \Gamma_2 \text{ in a dotted circle} + \text{Diagram with } \text{Diagram with } \Gamma_3 \text{ in a dotted circle} \tag{4.1.2}$$

A similar argument gives:

$$\begin{array}{c} j \\ \xi_1 \xi_2 \dots \xi_n \\ q_a \end{array} = \begin{array}{c} j \\ q_a \gamma \xi_1 \xi_2 \dots \xi_n \end{array} + \begin{array}{c} j \\ q_a \gamma \xi_1 \xi_2 \dots \xi_n \end{array}$$

Summing and noting also that the non-diagrammatic part of the bivalent vertex does not depend on the q_a s, we have, for all n :

$$\begin{array}{c} j \\ \xi_1 \xi_2 \dots \xi_n \\ q_a \end{array} = \begin{array}{c} j \\ q_a \gamma \xi_1 \xi_2 \dots \xi_n \end{array} \tag{4.2.1}$$

The derivatives with respect to t are harder because of the boundary terms. At valence $n = 0$, equation (4.1.2) already describes $t \begin{array}{c} \bullet \\ \square \end{array}$. We will not describe the other t -derivatives in full generality, as we will only need them for valence $n \geq 3$ when all incoming edges are based loops. In this case, when $n \geq 3$ at least one incoming edge is not marked, and hence the boundary terms vanish. We have:

$$\begin{array}{c} \xi_1 \xi_2 \dots \xi_n \\ t \end{array} = \begin{array}{c} t \\ \gamma \xi_1 \xi_2 \dots \xi_n \end{array}, \quad n \geq 3, \text{ all } \xi_i\text{'s are based loops} \tag{4.2.2}$$

We conclude this section by addressing the second derivative of a vertex with respect to the q_a . By the product rule:

$$\begin{array}{c} n \\ \dots \\ q_a \end{array} = \begin{array}{c} n \\ \gamma \dots \end{array} = \begin{array}{c} n \\ \gamma \gamma \dots \end{array} + \begin{array}{c} n \\ \gamma \dots \end{array}$$

In particular:

$$\begin{array}{c} \xi \\ \gamma \gamma \dots \end{array} + \begin{array}{c} \xi \\ \gamma \dots \end{array} = \begin{array}{c} \xi \\ \gamma \dots \end{array} = 0 \text{ if } \xi \text{ is a based loop.}$$

The Green's function G_γ , which we denote by an edge, is a based loop in each variable, so:

$$0 = \begin{array}{c} \gamma \\ \dots \end{array} + \begin{array}{c} \gamma \\ \dots \end{array}$$

We have observed already that when contracted with based loops, the bivalent vertex acts as \mathcal{D}_γ . But $\mathcal{D}_\gamma[G_\gamma]^i_j(\zeta, \tau) = -\delta^i_j \delta(\zeta - \tau)$ by definition. In short-hand:

$$\begin{array}{c} \xi \\ | \\ \text{---} \text{---} \\ | \\ \blacksquare \end{array} = - \begin{array}{c} \xi \\ | \\ \text{---} \\ | \\ \blacksquare \end{array}, \text{ provided } \xi \text{ is a based loop.} \tag{4.2.3}$$

And $\frac{\partial^2 \gamma(\tau)}{\partial q_a \partial q_b}$ vanishes at both endpoints $\tau = 0, t$. Therefore:

$$\frac{\partial^2 \gamma^i(\tau)}{\partial q_a^j \partial q_b^k} = \begin{array}{c} j \quad k \\ | \quad | \\ q_a \quad q_b \\ | \quad | \\ \text{---} \text{---} \\ | \\ \blacksquare \\ | \\ \tau, i \end{array} = \begin{array}{c} j \quad k \\ | \quad | \\ q_a \quad q_b \\ | \quad | \\ \text{---} \text{---} \\ | \\ \blacksquare \\ | \\ \tau, i \end{array} \tag{4.2.4}$$

Thus in general to take the second derivative of a vertex one either adds two $\frac{\partial \gamma}{\partial q_a}$ s or an edge connecting to a trivalent vertex with two $\frac{\partial \gamma}{\partial q_a}$ s. The lowest-valence example is:

$$\begin{array}{c} j \quad k \\ | \quad | \\ q_a \quad q_b \\ | \quad | \\ \text{---} \text{---} \\ | \\ \blacksquare \end{array} = \begin{array}{c} j \quad k \\ | \quad | \\ q_a \quad q_b \\ | \quad | \\ \text{---} \text{---} \\ | \\ \blacksquare \end{array} + \begin{array}{c} j \quad k \\ | \quad | \\ q_a \quad q_b \\ | \quad | \\ \text{---} \text{---} \\ | \\ \blacksquare \end{array} = \begin{array}{c} j \quad k \\ | \quad | \\ q_a \quad q_b \\ | \quad | \\ \text{---} \text{---} \\ | \\ \blacksquare \end{array} + 0 \tag{4.2.5}$$

The second summand vanishes because G_γ is a based loop in each variable. As we will need it later, we record one further derivative:

$$\begin{array}{c} j \quad k \quad l \\ | \quad | \quad | \\ q_a \quad q_c \quad q_b \\ | \quad | \quad | \\ \text{---} \text{---} \text{---} \\ | \\ \blacksquare \end{array} = \begin{array}{c} j \quad k \quad l \\ | \quad | \quad | \\ q_a \quad q_c \quad q_b \\ | \quad | \quad | \\ \text{---} \text{---} \text{---} \\ | \\ \blacksquare \end{array} + \begin{array}{c} j \quad k \quad l \\ | \quad | \quad | \\ q_a \quad q_c \quad q_b \\ | \quad | \quad | \\ \text{---} \text{---} \text{---} \\ | \\ \blacksquare \end{array} + \begin{array}{c} j \quad k \quad l \\ | \quad | \quad | \\ q_a \quad q_c \quad q_b \\ | \quad | \quad | \\ \text{---} \text{---} \text{---} \\ | \\ \blacksquare \end{array} \\ = \begin{array}{c} j \quad k \quad l \\ | \quad | \quad | \\ q_a \quad q_c \quad q_b \\ | \quad | \quad | \\ \text{---} \text{---} \text{---} \\ | \\ \blacksquare \end{array} + \begin{array}{c} j \quad k \quad l \\ | \quad | \quad | \\ q_a \quad q_c \quad q_b \\ | \quad | \quad | \\ \text{---} \text{---} \text{---} \\ | \\ \blacksquare \end{array} + \begin{array}{c} j \quad k \quad l \\ | \quad | \quad | \\ q_a \quad q_c \quad q_b \\ | \quad | \quad | \\ \text{---} \text{---} \text{---} \\ | \\ \blacksquare \end{array} = 0 + \begin{array}{c} j \quad k \quad l \\ | \quad | \quad | \\ q_a \quad q_c \quad q_b \\ | \quad | \quad | \\ \text{---} \text{---} \text{---} \\ | \\ \blacksquare \end{array} + 0 \tag{4.2.6}$$

4.3. DERIVATIVES OF THE GREEN'S FUNCTION AND THE DETERMINANT

In this section, we evaluate the derivatives of the Green's function G_γ given by equation (2.1.10) with respect to q_a, t . We record also the derivatives of $\log \left| \det \frac{\partial^2 [-S_\gamma]}{\partial q_0 \partial q_1} \right|$.

It is easy to evaluate $\frac{\partial G_\gamma}{\partial q_a}$. Let ξ be a based loop, and consider the derivative of equation (4.2.3):

$$0 = \begin{array}{c} \xi \\ | \\ \text{---} \text{---} \\ | \\ \blacksquare \end{array} = \begin{array}{c} \xi \\ | \\ \text{---} \text{---} \\ | \\ \blacksquare \end{array} + \begin{array}{c} \xi \\ | \\ \text{---} \text{---} \\ | \\ \blacksquare \end{array} \tag{4.3.1}$$

We contract with G_γ , which is a based loop in each variable, and use equations (4.2.1) and (4.2.3):

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} q_a = - \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_a = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \\ \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_a \tag{4.3.2}$$

The first equality in equation (4.3.2) requires that $\frac{\partial G_\gamma}{\partial q_a}$ is a based loop, which follows from differentiating $G_\gamma(\zeta, 0) = 0 = G_\gamma(\zeta, t)$ with respect to q_a .

Evaluating $\frac{\partial G}{\partial t}$ is not so easy, because it is not a based loop: from equation (4.3.9) it will follow that $\frac{\partial G^{ij}}{\partial t}(t, t) = \delta^{ij}$. Instead, we will differentiate equation (4.1.4) directly. To begin, we evaluate $\frac{\partial^2 \gamma}{\partial t \partial q_a}$ by differentiating equation (4.1.3) with respect to q_a , using equations (4.2.4) and (4.2.5):

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} t = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_a = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_1 + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} (B) \\
 = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_a q_1 + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_1 q_a + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_1 (B) + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} (B) q_1 \\
 = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_a q_1 q_1 + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_a q_1 (B) + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_1 q_1 q_a + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_1 q_1 (B) \\
 = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_a q_1 t + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_1 q_1 q_a + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_1 (B) q_a \tag{4.3.3}$$

We remark that $\begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_0 = \frac{\partial B(q_1)}{\partial q_0} = 0$.

We turn now to $(\partial^2[-S])^{-1}$. Recall equation (4.1.2). Then:

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} t = \left(\frac{1}{2}\right) \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_0 q_1 + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} (B) q_1 + \left(\frac{1}{2}\right) \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} (B) (B) + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} (C) \\
 = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_0 q_1 q_1 + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_0 q_1 q_1 q_1 + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_0 q_1 (B) + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_0 q_1 (B) q_1 + 0 \\
 = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_0 q_1 t + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_0 q_1 q_1 q_1 + \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \blacksquare \end{array} q_0 q_1 q_1 (B) \tag{4.3.4}$$

In the second line above we used the product rule and dropped any terms involving derivatives with respect to q_0 of functions only of q_1 . In the third line we used equations (4.1.3) (4.2.5) and (4.2.6).

By the quotient rule:

Thus, we can differentiate the Green's function by recalling equation (4.1.4) and using equations(4.3.3) and (4.3.5). Many terms cancel:

$$\begin{aligned}
 &= \Theta(\tau - \varsigma) \left(\begin{aligned} & \text{Diagram 1: } (\partial^2[-S])^{-1} \text{ with vertices } \varsigma, \tau, q_1, q_0, 1, 0, t \end{aligned} \right. \\
 & \quad - \begin{aligned} & \text{Diagram 2: } (\partial^2[-S])^{-1} \text{ with vertices } \varsigma, \tau, q_1, q_0, 1, 1, t \end{aligned} \\
 & \quad + \begin{aligned} & \text{Diagram 3: } (\partial^2[-S])^{-1} \text{ with vertices } \varsigma, \tau, q_1, q_0, 1, 0, t \end{aligned} \left. \right) + (\varsigma \leftrightarrow \tau) + \begin{aligned} & \text{Diagram 4: } (\partial^2[-S])^{-1} \text{ with vertices } \varsigma, \tau, q_1, q_1 \end{aligned} \tag{4.3.8}
 \end{aligned}$$

Finally, consider the integrals implicit in the trivalent vertices in equation (4.3.6). Write ρ for the variable of integration for each, and consider the sum as a single integral $\int_{\rho=0}^t$. When $\rho \leq \varsigma \leq \tau$, the second and third terms exactly cancel; when $\varsigma \leq \rho \leq \tau$, the three terms agree up to sign; and when $\varsigma \leq \tau \leq \rho$, the first two terms cancel. (When $\tau \leq \varsigma$, the contribution to the total expression is from the part we have abbreviate “ $(\varsigma \leftrightarrow \tau)$ ”, and the analysis is similar.) But consider breaking the inte-

gral \int_{ς}^{τ} as a similar sum of contributions from $\rho \leq \varsigma \leq \tau$, etc. Equation (4.1.4) implies that the contributions from each part exactly match the integrals in equation (4.3.6). This proves:

$$\int_{\varsigma}^{\tau} \text{Diagram} = \int_{\varsigma}^{\tau} \text{Diagram 1} + \int_{\varsigma}^{\tau} \text{Diagram 2} \tag{4.3.9}$$

Given equations (4.2.6) and (4.3.4) and the well-known formula for the logarithmic derivative of a determinant, we also have:

$$\frac{\partial}{\partial q_a} \left[\log \left| \det \frac{\partial^2[-S]}{\partial q_0 \partial q_1} \right| \right] = \text{Diagram 1} = \text{Diagram 2} \tag{4.3.10}$$

$$\frac{\partial}{\partial t} \left[\log \left| \det \frac{\partial^2[-S]}{\partial q_1 \partial q_2} \right| \right] = \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} \tag{4.3.11}$$

We conclude this section by simplifying the right-hand side of equation (4.3.10) and the first term on the right-hand side of equation (4.3.11). We first expand the

■-vertex:

$$\begin{array}{c}
 \text{Diagram 1} \\
 \text{Diagram 2} \\
 \text{Diagram 3} \\
 \text{Diagram 4} \\
 \text{Diagram 5}
 \end{array}
 =
 \begin{array}{c}
 \text{Diagram 6} \\
 \text{Diagram 7} \\
 \text{Diagram 8} \\
 \text{Diagram 9} \\
 \text{Diagram 10}
 \end{array}
 +
 \begin{array}{c}
 \text{Diagram 11} \\
 \text{Diagram 12} \\
 \text{Diagram 13} \\
 \text{Diagram 14} \\
 \text{Diagram 15}
 \end{array}
 +
 \begin{array}{c}
 \text{Diagram 16} \\
 \text{Diagram 17} \\
 \text{Diagram 18} \\
 \text{Diagram 19} \\
 \text{Diagram 20}
 \end{array}
 +
 \begin{array}{c}
 \text{Diagram 21} \\
 \text{Diagram 22} \\
 \text{Diagram 23} \\
 \text{Diagram 24} \\
 \text{Diagram 25}
 \end{array}
 +
 \begin{array}{c}
 \text{Diagram 26} \\
 \text{Diagram 27} \\
 \text{Diagram 28} \\
 \text{Diagram 29} \\
 \text{Diagram 30}
 \end{array}$$

But the integrals implicit in the first two summands above are obviously symmetric under switching the two “ γ ” legs. Recalling equation (4.1.4):

$$\begin{array}{c}
 \text{Diagram 31} \\
 \text{Diagram 32}
 \end{array}
 +
 \begin{array}{c}
 \text{Diagram 33} \\
 \text{Diagram 34}
 \end{array}
 =
 \begin{array}{c}
 \text{Diagram 35} \\
 \text{Diagram 36}
 \end{array}
 +
 \begin{array}{c}
 \text{Diagram 37} \\
 \text{Diagram 38}
 \end{array}$$

Similarly:

$$\begin{array}{c}
 \text{Diagram 39} \\
 \text{Diagram 40}
 \end{array}
 +
 \begin{array}{c}
 \text{Diagram 41} \\
 \text{Diagram 42}
 \end{array}
 =
 \begin{array}{c}
 \text{Diagram 43} \\
 \text{Diagram 44}
 \end{array}
 +
 \begin{array}{c}
 \text{Diagram 45} \\
 \text{Diagram 46}
 \end{array}$$

All together, we have the following simplification:

$$\begin{array}{c}
 \text{Diagram 47} \\
 \text{Diagram 48}
 \end{array}
 =
 \begin{array}{c}
 \text{Diagram 49} \\
 \text{Diagram 50}
 \end{array}
 \tag{4.3.12}$$

4.4. THE CANCELLATIONS

In the previous sections, we evaluated the derivatives of every component of V_γ . In this section, we check equation (4.0.1), which in the graphical notation reads:

$$\begin{array}{c}
 \text{Diagram 51} \\
 \text{Diagram 52} \\
 \text{Diagram 53} \\
 \text{Diagram 54} \\
 \text{Diagram 55} \\
 \text{Diagram 56} \\
 \text{Diagram 57} \\
 \text{Diagram 58} \\
 \text{Diagram 59} \\
 \text{Diagram 60}
 \end{array}
 =
 \frac{1}{2}
 \begin{array}{c}
 \text{Diagram 61} \\
 \text{Diagram 62}
 \end{array}
 +
 \frac{1}{2}
 \begin{array}{c}
 \text{Diagram 63} \\
 \text{Diagram 64}
 \end{array}
 +
 \begin{array}{c}
 \text{Diagram 65} \\
 \text{Diagram 66}
 \end{array}
 +
 \frac{1}{2}
 \begin{array}{c}
 \text{Diagram 67} \\
 \text{Diagram 68}
 \end{array}
 +
 \frac{1}{2}
 \begin{array}{c}
 \text{Diagram 69} \\
 \text{Diagram 70}
 \end{array}
 +
 \text{Diagram 71}
 \tag{4.4.1}$$

We have adopted the following convention: the dashed lines (although not the dotted circles) count as edges and the new nodes count as vertices for the purposes of computing Euler characteristic, etc., and in equations with connected diagrams with different numbers of loops, we multiply each diagram Γ by $(i\hbar)^{\lambda(\Gamma)}$.

Recalling that $V_\gamma = \blacksquare + (\frac{i\hbar}{2}) \log |\det \frac{\partial^2[-S_\gamma]}{\partial q_0 \partial q_1}| + \sum \{\text{connected diagrams}\} / \{\text{symmetry}\}$, the calculations from the previous sections give:

$$\overset{t}{V}_\gamma = \overset{t}{\blacksquare} + \frac{1}{2} \overset{q_1}{\gamma} \text{---} \overset{q_1}{\gamma} + \frac{1}{2} \overset{q_1}{B} + \sum_{\substack{\Gamma \text{ connected} \\ \text{with one } \overset{t}{\gamma} \text{ or} \\ \text{one } \overset{q_1}{\gamma} \text{---} \overset{q_1}{\gamma}}} \frac{(i\hbar)^{\lambda(\Gamma)} \mathcal{F}_\gamma(\Gamma)}{|\text{Aut } \Gamma|} \tag{4.4.2}$$

In the sum, it is implied that all other vertices are usual \blacksquare -vertices and are of valence $n \geq 3$. The sum includes the diagram $\overset{t}{\gamma} \text{---} \text{loop}$ coming from $\frac{\partial}{\partial t} \log |\det \frac{\partial^2[-S_\gamma]}{\partial q_0 \partial q_1}|$. Otherwise, each diagram with a $\overset{t}{\gamma}$ comes from the diagram formed by removing it (if the $\overset{t}{\gamma}$ attaches to a trivalent vertex, its removal leaves just a solid edge), and the combinatorics are as in equation (4.1.1): if there are n equivalent ways to attach a $\overset{t}{\gamma}$ to a diagram Γ , then $|\text{Aut } \square \text{---} \overset{t}{\gamma}| = \frac{1}{n} |\text{Aut } \Gamma|$. The second sum occurs similarly, with each $\overset{q_1}{\gamma} \text{---} \overset{q_1}{\gamma}$ replacing a Green's function, as per the second term in equation (4.3.9).

Similarly, we have:

$$\overset{q_1}{V}_\gamma = \overset{q_1}{\blacksquare} + \sum_{\substack{\Gamma \text{ connected} \\ \text{with one } \overset{q_1}{\gamma}}} \frac{(i\hbar)^{\lambda(\Gamma)} \mathcal{F}_\gamma(\Gamma)}{|\text{Aut } \Gamma|} \tag{4.4.3}$$

Then $\frac{1}{2} \overset{q_1}{V}_\gamma \text{---} \overset{q_1}{V}_\gamma = \frac{1}{2} \overset{q_1}{\gamma} \text{---} \overset{q_1}{\gamma} +$ a sum of diagrams each with one $\overset{q_1}{\gamma} \text{---} \overset{q_1}{\gamma}$, such that its removal yields a diagram that is still connected.

The diagrams in $\frac{1}{2} \overset{q_1}{V}_\gamma \text{---} \overset{q_1}{V}_\gamma$ come in two types. Other than the simply-connected diagram $\frac{1}{2} \overset{q_1}{\blacksquare} \text{---} \overset{q_1}{\blacksquare}$, there are diagrams where the dashed line connects a complicated part to a $\overset{q_1}{\blacksquare}$, and diagrams where the dashed line connects two non-simply-connected parts, which are not connected otherwise. But recall

equation (4.1.3); then:

$$\begin{aligned}
 & \frac{1}{2} \overset{q_1}{\textcircled{V}_\gamma} \overset{q_1}{\textcircled{V}_\gamma} + \overset{q_1}{\textcircled{V}_\gamma} \textcircled{B} + \frac{1}{2} \overset{q_1}{\textcircled{V}_\gamma} \overset{q_1}{\textcircled{V}_\gamma} \\
 &= \frac{1}{2} \overset{q_1}{\blacksquare} \overset{q_1}{\blacksquare} + \overset{q_1}{\blacksquare} \textcircled{B} + \frac{1}{2} \overset{q_1}{\textcircled{\gamma}} \overset{q_1}{\textcircled{\gamma}} + \sum_{\substack{\Gamma \text{ connected} \\ \text{with one } \overset{q_1}{\textcircled{\gamma}} \text{ or} \\ \text{one } \overset{q_1}{\textcircled{\gamma}} - \overset{q_1}{\textcircled{\gamma}}}} \frac{(i\hbar)^{\lambda(\Gamma)} \mathcal{F}_\gamma(\Gamma)}{|\text{Aut } \Gamma|}
 \end{aligned}
 \tag{4.4.4}$$

Comparing equations (4.4.1), (4.4.2) and (4.4.4), we see that the only remaining terms are equal by equation (4.1.2). Thus we have proven Theorem 2.2.6.

5. Proof of Theorem 2.3.3

Recall that the pointwise convergence of distributions is defined by testing against compactly-supported functions. For a chosen open neighborhood $\mathcal{O} \subseteq \mathbb{R}^{2n+1}$, let A be the set of all smooth families of classical paths with boundary values in \mathcal{O} , and fix $(t, q_1) \in \mathbb{R}^{n+1}$. Throughout this section, we will let $g: \mathbb{R}^n \rightarrow \mathbb{R}$ be smooth with compact support K such that $\{t\} \times K \times \{q_1\} \subseteq \mathcal{O}$. We will prove Theorem 2.3.3 in three steps. In Section 5.1, we characterize those families $\gamma \in A$ such that $\int_K^{\text{formal}} \exp(-i\hbar)^{-1} V_\gamma(t, q_0, q_1) g(q_0) dq_0$ is non-zero, and show that there are only finitely many such families. In Section 5.2 we pick $\mathcal{O}_0 \subseteq \mathbb{R}^n$ open with compact closure and show that for $\epsilon > 0$ sufficiently small depending on \mathcal{O}_0 , for $\mathcal{O} = (0, \epsilon) \times \mathcal{O}_0 \times \mathcal{O}_0$ there is a unique family of classical paths γ with $\int_K^{\text{formal}} \exp(-i\hbar)^{-1} V_\gamma(t, q_0, q_1) g(q_0) dq_0 \neq 0$. In Section 5.3 we compute the limit as $t \rightarrow 0$ of this formal integral.

5.1. ON THE SUPPORT OF FORMAL INTEGRALS

We wish to understand the formal integral:

$$\int_K^{\text{formal}} \exp\left((i\hbar)^{-1} V_\gamma(t, q_0, q_1)\right) g(q_0) dq_0
 \tag{5.1.1}$$

We begin by splitting the exponent $V_\gamma = (-S_\gamma) + (V_\gamma + S_\gamma)$. Then $V_\gamma + S_\gamma$ is valued in formal power series that begin in degree \hbar , so $\exp((i\hbar)^{-1}(V_\gamma + S_\gamma))$ is a formal power series. Thus the formal integral in equation (5.1.1) is determined by the $\hbar \rightarrow 0$ asymptotics of integrals of the form $\int_K \exp(-i\hbar)^{-1} S_\gamma(t, q_0, q_1) g(q_0) dq_0$.

Such integrals are conventionally studied by the method of stationary phase. Indeed, by [4, Lemma 3.13], if $\frac{\partial S_\gamma}{\partial q_0} \neq 0$ in K , then:

$$\int_K \exp\left(-(\hbar)^{-1} S_\gamma(t, q_0, q_1)\right) g(q_0) dq_0 = O(\hbar^\infty) \text{ as } \hbar \rightarrow 0 \tag{5.1.2}$$

By equation (2.1.8), $\frac{\partial S_\gamma}{\partial q_0}(t, q_0, q_1) = 0$ if and only if $\dot{\gamma}_{(t, q_0, q_1)}(0) = -B(q_0)$, where $\gamma_{(t, q_0, q_1)}$ is the member of the family of paths γ that connects q_0 to q_1 in time t . If the family γ contains a path with initial values $(\dot{\gamma}(0), \gamma(0)) = (-B(q_0), q_0)$ for some $q_0 \in K$, then it does so in an open neighborhood of q_0 , and γ is determined on the connected component of $\{t\} \times K \times \{q_1\}$ containing (t, q_0, q_1) . Thus, as K is compact, there are only finitely many families γ with $\frac{\partial S_\gamma}{\partial q_0}(t, q_0, q_1) = 0$ for some $q_0 \in K$, and so there are only finitely many paths γ such that the formal integral in equation (5.1.1) is non-zero as a formal expression in \hbar .

5.2. SHORT-DURATION CLASSICAL PATHS CAN BE ALMOST GEODESIC

For $\epsilon \in \mathbb{R}$, consider the non-degenerate second-order ODE given by:

$$0 = \ddot{\gamma}^i + \epsilon \left(\frac{\partial B_i}{\partial q^j}(\gamma) - \frac{\partial B_j}{\partial q^i}(\gamma) \right) \dot{\gamma}^j + \epsilon^2 \frac{\partial C}{\partial q^i}(\gamma) \tag{5.2.1}$$

Let φ_ϵ be the ‘‘flow for duration 1’’ map, i.e. $\varphi_\epsilon(v, q) = (\dot{\gamma}_\epsilon(1), \gamma_\epsilon(1))$ for γ_ϵ the unique solution to equation (5.2.1) with $(\dot{\gamma}_\epsilon(0), \gamma_\epsilon(0)) = (v, q)$. It is defined on an open neighborhood of the 0 section in $\mathbb{T}\mathbb{R}^n = \mathbb{R}^{2n}$, it is smooth when it is defined, and it depends smoothly on ϵ . Let $\phi_\epsilon(v, q) = (q, \pi \circ \varphi_\epsilon(v, q))$, where $\pi : \mathbb{T}\mathbb{R}^n \rightarrow \mathbb{R}^n$ is the natural projection. Then ϕ_0 is the isomorphism $(v, q) \mapsto (q, q + v)$. So the partial function $\phi_{[-1]} : \mathbb{T}\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{2n}$ is defined on an open neighborhood of $\mathbb{T}\mathbb{R}^n \times \{0\}$.

Fix $\mathcal{O}_0 \subseteq \mathbb{R}^n$ with compact closure $\overline{\mathcal{O}_0}$, and let $\mathcal{O}_1 \supseteq \overline{\mathcal{O}_0}$ be open with compact closure $\overline{\mathcal{O}_1}$. Let $\mathcal{P} = \phi_0^{-1}(\overline{\mathcal{O}_1} \times \overline{\mathcal{O}_1})$. It is a compact subset of $\mathbb{T}\mathbb{R}^n$ containing $\{0\} \times \overline{\mathcal{O}_1}$. Then we can find $\epsilon_0 > 0$ such that $\mathcal{P} \times (-\epsilon_0, \epsilon_0)$ is contained in the domain of $\phi_{[-1]}$. Let $(v, q) \in \mathcal{P}$ such that $q \in \mathcal{O}_1$. For $\epsilon \in (-\epsilon_0, \epsilon_0)$, let γ_ϵ be the duration-1 solution to equation (5.2.1) with initial conditions $(\dot{\gamma}_\epsilon(0), \gamma_\epsilon(0)) = (v, q)$. Then γ_0 is nondegenerate, and the nondegeneracy condition depends smoothly on ϵ . Thus for each (v, q) there is some number $0 < \epsilon_1(v, q) < \epsilon_0$ so that for $\epsilon \in (-\epsilon_1, \epsilon_1)$, γ_ϵ is nondegenerate. By lemma 2.1.4, ϵ_1 can be taken to depend lower-semicontinuously on (v, q) . Thus it has a minimum value ϵ_2 on the compact set $\{(v, q) \in \mathcal{P} \text{ s.t. } q \in \overline{\mathcal{O}_0}\}$.

Then $\epsilon_2 > 0$ satisfies the following: for each $\epsilon \in (-\epsilon_2, \epsilon_2)$ and for each $(q_0, q_1) \in \mathcal{O}_0 \times \mathcal{O}_0$, we have chosen a nondegenerate duration-1 solution to equation (5.2.1) connecting q_0 to q_1 , and the set of all these chosen paths for a given ϵ is precisely the set of solutions γ_ϵ to equation (5.2.1) with $(\dot{\gamma}_\epsilon(0), \gamma_\epsilon(0)) \in \mathcal{P}$ and $\gamma(0), \gamma(1) \in \mathcal{O}_0$. Thus this family $\gamma_{[-1]}$ is smooth.

However, if $\epsilon > 0$ and γ_ϵ is a duration-1 solution to equation (5.2.1), then $\tau \mapsto \gamma_\epsilon(\epsilon^{-1}\tau)$ is a duration- ϵ solution to equation (2.1.2) (equation (5.2.1) with $\epsilon = 1$).

In particular, we have constructed a smooth family γ of classical paths with boundary values ranging in $\mathcal{O}_0 \times \mathcal{O}_0 \times (0, \epsilon_2)$.

Moreover, suppose that γ is a duration- ϵ solution to equation (2.1.2) with initial conditions $\gamma(0) \in \mathcal{O}_0$ and $\dot{\gamma}(0) = -B(\gamma(0))$. Then if ϵ is sufficiently small depending on $\gamma(0)$, we have $(\epsilon \dot{\gamma}(0), \gamma(0)) \in \mathcal{P}$, and so γ is a member of our family. In particular, our family contains all paths with $\dot{\gamma}(0) = -B(\gamma(0))$ that start and end in \mathcal{O}_0 with sufficiently small duration depending on the endpoints.

5.3. THE LIMIT AS $t \rightarrow 0$

In Section 5.1, we showed that the only contributions to an integral of the form

$$\begin{aligned} & \int_K^{\text{formal}} \exp\left((i\hbar)^{-1} V_\gamma(t, q_0, q_1)\right) g(q_0) dq_0 \\ &= \int_K^{\text{formal}} e^{-(i\hbar)^{-1} S_\gamma(t, q_0, q_1)} \sqrt{\left| \det \frac{\partial^2 [-S_\gamma]}{\partial q_0 \partial q_1} \right|} \exp(\sum \text{diagrams}) g(q_0) dq_0 \end{aligned} \tag{5.3.1}$$

come from families γ that include paths with initial conditions $\dot{\gamma}(0) = -B(\gamma(0))$. Indeed, for fixed t, q_1 , equation (5.3.1) is supported only at the points $q_0 \in K$ so that the duration- t path with initial conditions $(\dot{\gamma}(0), \gamma(0)) = (-B(q_0), q_0)$ ends at $\gamma(t) = q_1$. In Section 5.2 we showed that for $\mathcal{O} = (0, \epsilon_2) \times \mathcal{O}_0 \times \mathcal{O}_0$ there is a unique such family, and that for $q_1 \in \mathcal{O}_0$ and sufficiently small t depending on q_1 , there is a unique such point q_0 . Indeed, we have $q_1 - q_0 = -t B(q_0) + O(t^2)$. We remark also that $\exp(\sum \text{diagrams}) = 1 + O(\hbar)$.

We claim that for these triples (t, q_0, q_1) , we have:

$$\left(\left(\frac{\partial^2 [-S_\gamma]}{\partial q_1 \partial q_0} \right)^{-1} \right)^{kl} = t \delta^{kl} + O(t^2) \tag{5.3.2}$$

Indeed, consider integrating equation (2.1.2) twice, with initial conditions $(\dot{\gamma}(0), \gamma(0)) = (v_0, q_0)$:

$$\gamma^i(t) = q_0^i + t v_0^i + \int_0^t \int_0^\tau \left(\left(\frac{\partial B_k}{\partial q^i} - \frac{\partial B_i}{\partial q^k} \right) \dot{\gamma}^k(\varsigma) - \frac{\partial C}{\partial q^i} \right) \Big|_{q=\gamma(\varsigma)} d\varsigma d\tau$$

Then holding q_0 fixed we have:

$$\begin{aligned} \frac{\partial \gamma^i(t)}{\partial v_0^j} &= t \delta_j^i + \int_0^t \int_0^\tau \left(\frac{\partial}{\partial q^l} \left(\frac{\partial B_k}{\partial q^i} - \frac{\partial B_i}{\partial q^k} \right) \frac{\partial \gamma^l(\varsigma)}{\partial v_0^j} \dot{\gamma}^k(\varsigma) \right. \\ &\quad \left. + \left(\frac{\partial B_k}{\partial q^i} - \frac{\partial B_i}{\partial q^k} \right) \frac{\partial \dot{\gamma}^k(\varsigma)}{\partial v_0^j} - \frac{\partial}{\partial q^l} \left(\frac{\partial C}{\partial q^i} \right) \right) \Big|_{q=\gamma(\varsigma)} d\varsigma d\tau \end{aligned} \tag{5.3.3}$$

We now take t to be sufficiently small and v_0 to vary only in a compact neighborhood of $-B(q_0)$. Then $\frac{\partial \gamma(\varsigma)}{\partial v_0}, \frac{\partial \dot{\gamma}(\varsigma)}{\partial v_0} = O(1)$ as $t \rightarrow 0$, but then equation (5.3.3) gives $\frac{\partial \gamma^i(t)}{\partial v_0^j} = t \delta_j^i + O(t^2)$. On the other hand, as we observed in equation (3.2.2), it follows from equation (2.1.8) that $(\frac{\partial^2[-S_\gamma]}{\partial q_1 \partial q_0})^{-1} = (\frac{\partial v_0}{\partial q_1})^{-1} = \frac{\partial q_1}{\partial v_0}$. This gives equation (5.3.2).

We make a similar argument, now holding q_1 fixed:

$$q_0^i = q_1^i - t v_0^i - \int_0^t \int_0^\tau \left(\left(\frac{\partial B_k}{\partial q^i} - \frac{\partial B_i}{\partial q^k} \right) \dot{\gamma}^k(\varsigma) - \frac{\partial C}{\partial q^i} \right) \Big|_{q=\gamma(\varsigma)} d\varsigma d\tau \quad (5.3.4)$$

$$\frac{\partial q_0^i}{\partial v_0^j} = -t \delta_j^i + O(t^2) \quad (5.3.5)$$

$$\begin{aligned} \left(\left(\frac{\partial^2[S_\gamma]}{(\partial q_0)^2} \right)^{-1} \right)^{kl} &= \left(\left(-\frac{\partial}{\partial q_0} (v_0 + B(q_0)) \right)^{-1} \right)^{kl} \\ &= \left((t^{-1} \delta + O(1))^{-1} \right)^{kl} = t \delta^{kl} + O(t^2) \end{aligned} \quad (5.3.6)$$

Then, by [4, Theorem 3.15(ii)], we have:

$$\begin{aligned} &\int^{\text{formal}} \exp \left((i\hbar)^{-1} V_\gamma(t, q_0, q_1) \right) g(q_0) dq_0 \\ &= (2\pi\hbar)^{n/2} \times e^{i\pi/4} \times \det \left(t \delta_j^i + O(t^2) \right)^{-1/2} \times e^{-(i\hbar)^{-1} S_\gamma(t, q_1 + B(q_1) + O(t^2), q_1)} \\ &\quad \times \det \left(t \delta_j^i + O(t^2) \right)^{1/2} \times g \left(q_1 + B(q_1) + O(t^2) \right) \times (1 + O(\hbar)) \end{aligned}$$

But $S_\gamma(t, q_1 + B(q_1) + O(t^2), q_1) = \int_0^t (\frac{1}{2} |\dot{\gamma}|^2 + B(\gamma) \cdot \dot{\gamma} - C(\gamma)) d\tau = \int_0^t (\frac{1}{2} (\dot{\gamma} + B)^2 + O(1)) d\tau = \int_0^t (O(t) + O(1)) d\tau = O(t)$. Thus:

$$\lim_{t \rightarrow 0} \int^{\text{formal}} \exp \left((i\hbar)^{-1} V_\gamma(t, q_0, q_1) \right) g(q_0) dq_0 = (2\pi\hbar)^{n/2} e^{i\pi/4} (g(q_1) + O(\hbar)) \quad (5.3.7)$$

What about the order- \hbar terms? They can be given explicitly as a sum along the lines of our equation (2.2.7)—c.f. [14, 15, 8]—or implicitly as in [4]. All we need is the following fact: each degree in \hbar is given by a finite sum in which for each summand there is some M so that the summand scales as $(\frac{\partial^2[-S_\gamma]}{(\partial q_0)^2})^{-M}$ times a product of strictly fewer than M terms given by derivatives of g and V_γ at $q_0 = q_1 + O(t^2)$.

The calculations in Section 4 in fact allow us to write down all derivatives of V_γ with respect to q_0 . In addition to the derivatives computed in Section 4.4, we

have:

$$\frac{\partial^k V_\gamma}{(\partial q_0)^k} = \sum_{\substack{\text{connected marked} \\ \text{Feynman diagrams } (\Gamma, M) \\ \text{with } n \text{ ordered } \overset{q_0}{\vdots} \underset{\vdots}{s}}} \frac{(i\hbar)^{\lambda(\Gamma)} \mathcal{F}_\gamma(\Gamma, M)}{|\text{Aut}(\Gamma, M)|}, \quad k \geq 3 \tag{5.3.8}$$

Consider expanding each edge as in equation (4.1.4). Then in each diagram a marked half edge converts some $\frac{\partial \gamma}{\partial q_a}$ into $\frac{\partial \dot{\gamma}}{\partial q_a} = -B(q_0) + O(t)$. On the other hand, each vertex corresponds to an integration \int_0^t of bounded functions, and so the vertices are $O(t)$. Thus, by equation (5.3.2), $\mathcal{F}_\gamma(\Gamma, M) = O(t^{|V|+|E|})$ for each diagram Γ in equation (5.3.8). As the number of diagrams at each order is finite, this proves that for $k \geq 3$, $\frac{\partial^k V_\gamma}{\partial q_0^k} = O(t)$.

But the derivatives of g do not depend on t ; $\frac{\partial V_\gamma}{\partial q_0} = -\frac{\partial S_\gamma}{\partial q_0} + \sum\{\text{diagrams}\} = 0 + \sum\{\text{diagrams}\} = O(t)$; and $\frac{\partial^2 V_\gamma}{(\partial q_0)^2} = -\frac{\partial^2 S_\gamma}{(\partial q_0)^2} + \sum\{\text{diagrams}\} = O(t^{-1})$. As each term after the first one in the asymptotic expansion of equation (5.3.7) has strictly more occurrences of $\left(\frac{\partial^2[-S_\gamma]}{(\partial q_0)^2}\right)^{-1}$ than of derivatives of g and V_γ , we see that the $O(\hbar)$ part of equation (5.3.7) is also $O(t)$. Thus, as formal expressions in \hbar :

$$\lim_{t \rightarrow 0} \int^{\text{formal}} \exp\left((i\hbar)^{-1} V_\gamma(t, q_0, q_1)\right) g(q_0) dq_0 = (2\pi\hbar)^{n/2} e^{in\pi/4} g(q_1) \tag{5.3.9}$$

Theorem 2.3.3 is proved.

Acknowledgements

This project was suggested by N. Reshetikhin, who provided support and suggestions throughout all stages of it. K. Datchev, C. Schommer-Pries, G. Thompson, and I. Ventura provided valuable discussions. I would like to also thank the anonymous referee for alerting me to the work of H. Kleinert and collaborators. I am grateful to Aarhus University for the hospitality. This work is supported by NSF grant DMS-0901431.

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