

Cremmer–Gervais r -Matrices and the Cherednik Algebras of Type GL_2

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Abstract. We give an interpretation of the Cremmer–Gervais r -matrices for \mathfrak{sl}_n in terms of actions of elements in the rational and trigonometric Cherednik algebras of type GL_2 on certain subspaces of their polynomial representations. This is used to compute the nilpotency index of the Jordanian r -matrices, thus answering a question of Gerstenhaber and Giaquinto. We also give an interpretation of the Cremmer–Gervais quantization in terms of the corresponding double affine Hecke algebra.

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1. Introduction

Let \mathfrak{sl}_n denote the Lie algebra of traceless $n \times n$ matrices having entries in a field k . Let V denote the vector representation of \mathfrak{sl}_n and let $r \in \mathfrak{sl}_n \wedge \mathfrak{sl}_n \subset \text{End}(V \otimes V)$ be a skew-symmetric linear operator. Define $r_{12} := r \otimes 1$, $r_{23} := 1 \otimes r$, and $r_{13} := P_{23}r_{12}P_{23}$ where P_{23} is the permutation operator on $V^{\otimes 3}$: $P_{23}(u \otimes v \otimes w) = u \otimes w \otimes v$. An important class of operators which arise in studying Lie bialgebras and Poisson–Lie groups are those satisfying the modified classical Yang–Baxter equation (MCYBE)

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] - \lambda(P_{123} - P_{213}) = 0$$

for some $\lambda \in k$ (see [3, 8] for more details). The solutions to the MCYBE are called *classical r -matrices* and fall into two classes: those satisfying the MCYBE for λ nonzero (resp. zero) are called *quasitriangular* (resp. *triangular*).

In the early 1980s, Belavin and Drinfel'd successfully classified all quasitriangular r -matrices in the case when k is the field of complex numbers [1]. This classification gives us a solution space which we view as a disjoint union of quasi-projective subvarieties of $\mathbb{P}(\mathfrak{sl}_n \wedge \mathfrak{sl}_n)$. In contrast, the triangular r -matrices are

more mysterious as there is not a constructive classification of them (except in the smaller cases of \mathfrak{sl}_2 and \mathfrak{sl}_3 , see [3, Chapter 3]), only a homological interpretation exists (due to the work of Stolin [15]: for details see [3, Section 3.1.D.] and [8, Section 3.5]), and there are currently few known examples.

In the paper [9], Gerstenhaber and Giaquinto investigate the behavior along the boundaries of the aforementioned quasi-projective varieties and show that the boundary points are all triangular r -matrices. In the same paper, they construct the most general class of known examples of triangular r -matrices, the so-called *generalized Jordanian r -matrices* $r_{J,n}$ (see [7,9]). They prove that the Jordanian r -matrices lie on the boundary of the component corresponding to the quasitriangular Cremmer–Gervais r -matrices (discussed in [6,7,9,11–13]) and conjecture that $r_{J,n}^3 = 0$.

In Theorem 4.3, we prove that the nilpotency index of $r_{J,n}$ is quite different than conjectured. We do this by first interpreting the quantum Cremmer–Gervais R -matrix R in terms of the double affine Hecke algebra (DAHA) $\mathcal{H}^{q,t}$ of type GL_2 (Theorem 2.2) and the classical Cremmer–Gervais r -matrix in terms of the degenerate DAHA \mathcal{H}'_c (Theorem 3.4). Using Suzuki’s embedding [16] of the rational Cherednik algebra \mathcal{H}''_{-c} into \mathcal{H}'_c , we give a simple interpretation of both the Cremmer–Gervais and Jordanian r -matrices as operators on the polynomial representation of \mathcal{H}''_{-c} . Using the relations in \mathcal{H}''_{-c} , we find that the nilpotency index of $r_{J,n}$ is n when n is odd, and $2n - 1$ when n is even. The conceptual difference between the two cases has a representation theoretic origin: the polynomial representation of the rational Cherednik algebra \mathcal{H}''_c of type A_1 is reducible if and only if the deformation parameter c has the form $n/2$ for an odd positive integer n (a special case of Dunkl’s theorem [2]).

2. The Yang–Baxter Equations and the Double Affine Hecke Algebra

Let k be a field of characteristic 0 and let $K = k(q, t^{1/2})$. We begin with a

DEFINITION 2.1. (see [4,5]) The double affine Hecke algebra $\mathcal{H}^{q,t}$ of type GL_2 is the associative K -algebra with generators $X_1^{\pm 1}, X_2^{\pm 1}, Y_1^{\pm 1}, Y_2^{\pm 1}, T$ and relations

$$\begin{aligned} X_j X_j^{-1} &= X_j^{-1} X_j = Y_j Y_j^{-1} = Y_j^{-1} Y_j = 1, \\ (T - t^{1/2})(T + t^{-1/2}) &= 0, \quad T X_1 T = X_2, \quad T Y_2 T = Y_1, \\ Y_2^{-1} X_1 Y_2 X_1^{-1} &= T^2, \quad Y_1 Y_2 X_j = q X_j Y_1 Y_2, \\ Y_j X_1 X_2 &= q X_1 X_2 Y_j, \quad [Y_1, Y_2] = 0, \quad [X_1, X_2] = 0, \end{aligned}$$

for $j = 1, 2$.

The K -vector space $K[X_1^{\pm 1}, X_2^{\pm 1}]$ can be made into a $\mathcal{H}^{q,t}$ -module, called the *polynomial representation*, defined as follows. Let (12) act on $K[X_1^{\pm 1}, X_2^{\pm 1}]$ by swapping variables and let $S = \frac{1-(12)}{X_1 - X_2}$. For integers a, b define $\Gamma_{a,b}.f(X_1, X_2) :=$

$f(q^a X_1, q^b X_2)$. The double affine Hecke algebra $\mathcal{H}^{q,t}$ acts faithfully on $K[X_1^{\pm 1}, X_2^{\pm 1}]$ via

$$\begin{aligned} T &\mapsto t^{1/2}(12) - (t^{1/2} - t^{-1/2})X_2S, \\ Y_1 &\mapsto T\Gamma_{0,1}(12), \quad Y_2 \mapsto \Gamma_{0,1}(12)T^{-1}, \end{aligned}$$

and the X 's act via multiplication. For an operator $R \in \text{End}_K(K[X_1^{\pm 1}, X_2^{\pm 1}])$, let R_{12}, R_{13}, R_{23} denote the corresponding operators on $K[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$. In [10], Gerstenhaber and Giaquinto introduced the *modified quantum Yang–Baxter equation* (MQYBE). An operator is called a *modified quantum R -matrix* if it satisfies the MQYBE

$$R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} - \lambda(P_{123}R_{12} - P_{213}R_{23}) = 0$$

for some scalar λ . Here, P_{ijk} denotes the permutation on the variables $X_i \mapsto X_j \mapsto X_k \mapsto X_i$. Furthermore, R is called *unitary* if $R(12)R(12) = 1$.

The classical analogue of the MQYBE is called the *modified classical Yang–Baxter equation* (MCYBE). An operator r is called a *classical r -matrix* if it satisfies the MCYBE

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] - \lambda(P_{123} - P_{213}) = 0$$

for some scalar λ (for more details, see [3]). Furthermore, r is called *skew-symmetric* if $(12)r(12) = -r$. For shorthand, we will denote the left hand sides of the above equations by $\text{MQYBE}_\lambda(R)$ and $\text{MCYBE}_\lambda(r)$, respectively.

THEOREM 2.2. *The operator $R = (12)Y_2(12)Y_2^{-1}$ is unitary and satisfies the modified quantum Yang–Baxter equation*

$$R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} = (1 - t^{-1})^2 (P_{123}R_{12} - P_{213}R_{23}).$$

Proof. It is obvious that R is unitary. To show R satisfies the MQYBE, we first set $v = (X_1 + X_2)S$. This operator satisfies $\text{MQYBE}_1(v) = 0$, $\text{MCYBE}_1(v) = 0$, and $v^2 = 0$. Therefore, it follows that $\text{MQYBE}_{\lambda^2}(\exp(\lambda v)) = 0$ for all scalars λ . Setting $F = \Gamma_{0,-1}$ and $\hat{R} = \exp((1 - t^{-1})v)$, we have $R = F_{21}^{-1} \hat{R} F_{12}$. Furthermore

- (i) $F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}$,
- (ii) $\hat{R}_{12}F_{23}F_{13} = F_{13}F_{23}\hat{R}_{12}$,
- (iii) $\hat{R}_{23}F_{12}F_{13} = F_{13}F_{12}\hat{R}_{23}$.

So R is obtained by twisting \hat{R} by F and hence satisfies the MQYBE (cf. [14, Thm. 1]). □

3. Semiclassical Limit

Setting $q = e^h$ and $t = e^{ch}$, we view K as a subfield of $k[[c, h]]$ and the above formulae for the Y 's in the polynomial representation become

$$Y_1 \mapsto 1 + h \left(X_1 \frac{\partial}{\partial X_1} + cX_2S + \frac{c}{2} \right) + O(h^2)$$

$$Y_2 \mapsto 1 + h \left(X_2 \frac{\partial}{\partial X_2} - cX_2S - \frac{c}{2} \right) + O(h^2)$$

Define y_i as the coefficient of h in Y_i in the above expressions. The operators y_1, y_2 obey commutation relations which motivate the following

DEFINITION 3.1. (see Cherednik [5]) The degenerate (or trigonometric) double affine Hecke algebra \mathcal{H}'_c of type GL_2 is the $k(c)$ -algebra having generators $y_1, y_2, X_1^{\pm 1}, X_2^{\pm 1}$, and (12) and relations

$$(12)^2 = 1, \quad [X_1, X_2] = 0, \quad (12)X_1(12) = X_2,$$

$$(12)y_1 - y_2(12) = c, \quad [y_1, y_2] = 0,$$

$$[y_i, X_j] = \begin{cases} X_i + c(12)X_1 & \text{if } i = j \\ -c(12)X_1 & \text{if } i \neq j \end{cases}$$

Remark 3.2. If $\text{MQYBE}_\lambda(1 + hr + O(h^2)) = 0$, then λ is of the form $\epsilon h^2 + O(h^3)$ for some scalar ϵ and $\text{MCYBE}_\epsilon(r) = 0$. Furthermore, if $1 + hr + O(h^2)$ is unitary, then r is skew-symmetric.

One can readily obtain the following

LEMMA 3.3. $R = 1 + h(y_1 - y_2 - c(12)) + O(h^2)$

Using Remark 3.2 together with Lemma 3.3, we have

COROLLARY 3.4. R, y_1, y_2 as above

- (i) $r := y_1 - y_2 - c(12)$ is skew-symmetric
- (ii) $\text{MCYBE}_{c^2}(r) = 0$.

In this situation, r is called the *semiclassical limit* of R . Since r is homogeneous, it follows that for any natural number n , we can restrict the action of r to the subspace $k(c)^{n,n}$ of $k(c)[X_1^\pm, X_2^\pm]$ spanned by the monomials $X_1^a X_2^b$ with $0 \leq a, b \leq n - 1$. Doing this yields

$$r_n = 2 \sum_{1 \leq k < l \leq n} (k - l - c) e_{kk} \wedge e_{ll} + 2c \sum_{1 \leq k < l \leq n} e_{kl} \wedge e_{lk} + 4c \sum_{1 \leq i < k < j \leq n} e_{i+j-k, j} \wedge e_{ki}.$$

Here, $e_{ij} \wedge e_{kl}$ is the operator

$$X_1^a X_2^b \mapsto \frac{1}{2} \left(\delta_{j, a+1} \delta_{l, b+1} X_1^{i-1} X_2^{k-1} - \delta_{l, a+1} \delta_{j, b+1} X_1^{k-1} X_2^{i-1} \right).$$

Remark 3.5. The formula for r_n above suggests that we can view it as being in $\mathfrak{gl}_n \wedge \mathfrak{gl}_n$. Thus, we have a one-parameter family of solutions to the MCYBE over \mathfrak{gl}_n . Setting $c = -\frac{n}{2}$ is the only instance that r_n will be in $\mathfrak{sl}_n \wedge \mathfrak{sl}_n$.

As mentioned in the introduction, the skew-symmetric solutions to the MCYBE

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] - \lambda(P_{123} - P_{213}) = 0$$

fall into two classes: those solutions satisfying the MCYBE with λ nonzero (resp. zero) are called *quasitriangular* (resp. *triangular*) r -matrices. The quasitriangular r -matrices (over \mathbb{C}) were classified in the early 1980's by Belavin and Drinfel'd using combinatorial objects on the Dynkin graph, called BD-triples [1]. The classification tells us that the solution space of quasitriangular r -matrices may be viewed as a disjoint union of quasi-projective subvarieties of $\mathbb{P}(\mathfrak{sl}_n \wedge \mathfrak{sl}_n)$: the aforementioned subvarieties are indexed by the BD-triples. In this case r_n corresponds to the maximal BD-triple obtained by deleting an extremal node. It is the so-called Cremmer–Gervais r -matrix (discussed in [6, 7, 9, 11–13]).

THEOREM 3.6. *When $c = -\frac{n}{2}$, r_n is the Cremmer–Gervais r -matrix.*

Proof. Apply the Lie algebra automorphism $e_{ij} \mapsto -e_{n+1-j, n+1-i}$ of \mathfrak{gl}_n to the formula for r_n above, then multiply the result by $-\frac{1}{n}$, and finally set $c = -\frac{n}{2}$; we obtain the same formula for the Cremmer–Gervais r -matrix r_{CG} as it appears in [9]. \square

4. Connections with the Rational Cherednik Algebra

We begin this section by recalling a

DEFINITION 4.1. (see Cherednik [5]) The rational Cherednik algebra (over $k(c)$) \mathcal{H}''_c of type GL_2 has generators (12), x_1 , x_2 , u_1 , u_2 and relations

$$\begin{aligned} (12)^2 &= 1, & (12)x_1(12) &= x_2, & (12)u_1(12) &= u_2, \\ [x_1, x_2] &= 0, & [u_1, u_2] &= 0, \\ [u_i, x_j] &= \begin{cases} 1 - c(12) & \text{if } i = j \\ c(12) & \text{if } i \neq j \end{cases} \end{aligned}$$

The polynomial representation $k(c)[x_1, x_2]$ of \mathcal{H}''_c is defined where (12) permutes the variables, x_1 and x_2 act via multiplication, and the u_i act by the Dunkl operators

$$u_i \mapsto \frac{\partial}{\partial x_i} + c(-1)^i S.$$

In [16], Suzuki shows that there is an algebra embedding $\psi : \mathcal{H}''_{-c} \rightarrow \mathcal{H}'_c$ defined on generators by

$$\begin{aligned} (12) &\mapsto (12) \\ x_1 &\mapsto X_1 \\ x_2 &\mapsto X_2 \\ u_1 &\mapsto X_1^{-1} \left(y_1 + \frac{c}{2} - c(12) \right) \\ u_2 &\mapsto X_2^{-1} \left(y_2 + \frac{c}{2} \right) \end{aligned}$$

Using this algebra embedding, we see that the Cremmer–Gervais r -matrix has an interpretation in the rational Cherednik algebra. Here, r_n corresponds to $x_1u_1 - x_2u_2 \in \mathcal{H}''_{-c}$.

In [9], Gerstenhaber and Giaquinto provide the largest known class of examples of triangular r -matrices, the so-called *generalized Jordanian r -matrices* (also discussed in [7]). They demonstrate that the Jordanian r -matrices lie on the boundary of the orbit $SL_n.r_n$, where SL_n acts via the adjoint action. One can translate this into the setting of the rational Cherednik algebra. Here,

$$e^{\tau \cdot ad(u_1+u_2)}.(x_1u_1 - x_2u_2) = x_1u_1 - x_2u_2 + \tau(u_1 - u_2).$$

Therefore, we have the following

COROLLARY 4.2. $u_1 - u_2 \in \mathcal{H}''_{-c}$ is a boundary solution to the MCYBE (in particular, $MCYBE_0(u_1 - u_2) = 0$). Restricting its action to the linear subspace $k(c)^{n,n}$ and setting $c = -n/2$ corresponds to the Jordanian r -matrix $r_{J,n}$.

As an operator on the polynomial representation of \mathcal{H}''_c ,

$$u_1 - u_2 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - 2cS.$$

This gives us a one-parameter family of triangular r -matrices.

In [9], Gerstenhaber and Giaquinto conjecture that $r_{J,n}^3 = 0$. We use the above interpretation to show that this is not true and to compute the nilpotency index.

THEOREM 4.3. (*Gerstenhaber–Giaquinto Conjecture*) *The nilpotency index of $r_{J,n} \in \mathcal{H}''_{n/2}$ is n when n is odd and $2n - 1$ when n is even.*

Proof. Set $x = \frac{1}{2}(x_1 - x_2)$, $x' = \frac{1}{2}(x_1 + x_2)$. We have $[u_1 - u_2, x'] = 0$ and $(u_1 - u_2)x^m = (m - n[[m]])x^{m-1}$. Here $[[m]] = 1$ if m is odd and $[[m]] = 0$ otherwise. Observe that

$$x_1^i x_2^j = (-1)^j x^{i+j} + (-1)^j (i - j)x^{i+j-1} x' + \dots$$

One computes that in the case when n is odd, we have

$$(u_1 - u_2)^{n-1} x_2^{n-1} = (n - 1)(-2)(n - 3)(-4) \cdots (3 - n)(2)(1 - n) \neq 0$$

and for all $0 \leq i, j \leq n-1$, $(u_1 - u_2)^n (x_1^i x_2^j) = 0$. In the case when n is even, one computes

$$(u_1 - u_2)^{2(n-1)} (x_1^{n-1} x_2^{n-1}) = (2n-2)(n-3)(2n-4)(n-5) \cdots (3-n)(2)(1-n) \neq 0$$

and for all $1 \leq i, j \leq n-1$, $(u_1 - u_2)^{2n-1} (x_1^i x_2^j) = 0$. \square

Remarks. For $n \geq 2$, the Cremmer–Gervais r -matrix r_n is not nilpotent except when $c = -1/2$ and $n = 2$. One can see this by viewing r_n as the operator

$$X_1 \frac{\partial}{\partial X_1} - X_2 \frac{\partial}{\partial X_2} + c(X_1 + X_2) S$$

and verifying $r_n^2(X_1 - X_2) = (1 + 2c)(X_1 - X_2)$ and $r_n^2(X_1^2 - X_2^2) = 4(1 + c)(X_1^2 - X_2^2)$. So in this particular case, nilpotency is only a boundary condition.

The conceptual difference between the even and odd cases in Theorem 4.3 has a representation theoretic origin: the polynomial representation of the rational Cherednik algebra \mathcal{H}_c'' of type A_1 is reducible if and only if the deformation parameter c has the form $n/2$ for an odd positive integer n (a special case of Dunkl's theorem [2]).

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