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EDITORIAL

Special Issue on Spherical Mathematics and Statistics

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Spherical mathematics and spherical statistics, respectively, often came in second with major achievements; there are numerous examples ranging from geometry and probability to approximation and computer aided geometric design. Spherical probability and statistics came into existence only in the fifties of the last century with the most influential paper by Ronald Aylmer Fisher, published in the Proceedings of the Royal Society in 1953, which was triggered by problems analyzing paleomagnetic data related to Wegener's hypothesis of continental drift and eventually became instrumental to establishing the theory of plate tectonics. About fifty years later, spherical statistics was apparently completed by spherical regression and came to a preliminary standstill as it seems. As of today, there are three major textbooks on spherical statistics, and there are two better-known ones on circular statistics.

There are few instances wherein spherical reasoning took the lead. The Funk transform (introduced by Paul Funk in Mathematische Annalen in 1913 and 1916, respectively), as compared to the Radon transform (presented by Johann Radon to the Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig in 1917), is a famous example. However, nowadays the term "Radon transform" is often used in a generic way to assign the mean values along the geodesics of the domain of definition to a function regardless of what this domain is. Many generalizations and specifications exist, most notably its discrete version known as the Hough transform (U.S. Patent 3069654, 1962).

Often, the sphere in mind is a mathematical model of the Earth's shape, but in crystallography and mineralogy, for instance, it is the sphere of poles, where pole points represent crystallographic directions or axes, that is, pairs of antipodally sym-

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metric directions. In both cases, it is the two-dimensional sphere in three-dimensional space.

The sphere allows only for orthogonal or special orthogonal transformations, that is, rotations and inversion. At first glance, it seems to be a rather poor structure. Things turn more interesting when rotations are considered in terms of unit quaternions (as introduced by Benjamin Olinde Rodrigues in Journal of Mathématiques in 1840, and William Rowan Hamilton in his Lectures on Quaternions in 1853) living on the unit three-dimensional sphere in four-dimensional space. For instance, despite the difference in dimensions, statistics of rotations and axes (which may be subject to rotations) are basically the same. Only in practical applications does the dimension matter numerically when it comes to computations involving normalizing constants.

Other geosciences which largely motivated the development of spherical mathematics are of course modern geodesy, geophysics, and remote imagery, which make excessive use of spherical harmonics, splines, and wavelets. Historically, some extensions or generalizations from an Euclidean to a spherical setting were almost straightforward. Others were involved and cumbersome at times, for example, fast Fourier methods on spheres and the rotation group, while others are still the subject of ongoing research like spherical wavelets.

For an association and its flagship journal devoted to promoting the advancement of mathematics, statistics, and informatics in the Geosciences, there are several good reasons for publishing a special spherical issue. This issue contains seven selected papers: initial versions of some of them have been orally presented on the occasion of the workshop "Transformationen auf der Sphäre und Tomographie" in Hasenwinkel, Germany, September 3 to 6, 2007, organized by Jürgen Prestin, Lübeck, and myself in the frame of a joint research project with financial support from the German Science Foundation (DFG).

The first paper is on Marcinkiewicz–Zygmund inequalities on the rotation group as related to polynomial approximation to scattered data. Marcinkiewicz–Zygmund inequalities relate L_p norms of polynomials on various manifolds to weighted L_p norms of selected samples and have proven instrumental in scattered data approximation theory. In the deterministic setting, there is a direct reciprocal connection between the dimension of the polynomial space and the separation parameter associated with the underlying scattered data set. Manuel Graef and Dominik Schmid show that while keeping the data site fixed, the degree and hence the dimension of the spaces of polynomials to which the Marcinkiewicz–Zygmund inequalities apply can largely be increased by changing the associated deterministic to probabilistic inequalities.

The subject of the second paper is least squares approximation and optimum interpolation of scattered data on the bi-sphere with spherical harmonics applying fast Fourier methods. The results presented by Frank Filbir and Daniel Potts rely on the analysis of Marcinkiewicz–Zygmund inequalities and spherical tensor Fourier transform for fast summation by using the non-equispaced discrete Fourier transform algorithm on the sphere as provided by the free NFFT subroutine library. Special attention is placed on the choice of localized kernels for optimum interpolation. Critical computational aspects concerning the stability and efficiency are addressed in detail, and a short discussion of one relevant application concerning analysis of crystallographic preferred orientation in geo- and material sciences concludes the paper.

The third contribution is devoted to fast summation of kernel functions, as related to scattered data approximation on $SO(3)$. Ralf Hielscher, Jürgen Prestin and Antje Vollrath suggest algorithms for the fast approximate evaluation of sums of radial and other functions based on non-equispaced fast Fourier transforms on $SO(3)$. For special radial functions, kernels of applied harmonic analysis, they estimate the approximation error introduced by truncating the Fourier series of the kernel. For synthetic as well as experimental orientation data arising from electron back scatter diffraction (EBSD), results of numerical tests concerning the error and the computational performance are reported.

A spline method based on reproducing kernels resolving a joint inverse problem is presented in the fourth paper by Paula Berkel and Volker Michel. It is motivated by the general quest for information about properties of the Earth's interior from gravimetric data providing information about the density distribution, and from seismic data related to the compressional velocity, shear velocity, and density, respectively. Since spline methods have been successfully used to resolve the corresponding inverse problems separately, they are generalized to vectorial splines to resolve the joint inverse problem. Existence and uniqueness of an interpolating vectorial spline is proven and computational aspects of its numerical evaluation are considered.

Multiscale modeling of poloidal and toroidal fields by locally supported wavelets is the subject of the fifth contribution by Christian Gerhards and Willi Freeden. Since satellites collecting geomagnetic data are orbiting the ionosphere, that is, a source region of the geomagnetic field, the classic Gauss representation of the magnetic field does not apply. The Mie representation for solenoidal fields can be considered as a canonical generalization splitting the geomagnetic field into its poloidal and toroidal parts and applies to the ionosphere where electric current densities must not be neglected. To describe the geomagnetic field and currents, vector wavelets are constructed in the setting of the Green's function with respect to the Beltrami operator on the unit sphere, and regularizations of the Green's function are used as a tool to introduce locally supported vector wavelets. Finally, a reconstruction scheme for multiscale modeling of the magnetic field is given.

In the sixth paper, Gerlind Plonka and Daniela Rosca develop a spherical Easy Path Wavelet Transform to obtain sparse image representations. The general idea of the easy path wavelet transform is to rearrange the entries of the vector indexing the pixels of an image according to spatial correlation, that is, to draw pathways through the set of entries by linking the index of a given pixel to the index of the adjacent pixel with the closest value. Proceeding in this way results in a complete path covering all pixels corresponding to a new vector generated by permutation. Then a one-dimensional discrete wavelet transform is applied to this vector and, by construction, the choice of the path ensures that most wavelet coefficients remain small. Thus, after thresholding, one gets a sparse representation of the original image. This idea is taken to the sphere by virtue of its triangulation.

The seventh and final paper is on a spectral characterization of the Hilbert transform on the two-dimensional sphere with applications to signal processing. Oliver Fleischmann, Lennart Wietzke, and Gerald Sommer elaborate on an interpretation in terms of signal processing of the Hilbert transform on the two-dimensional sphere expanded into spherical harmonics. The relevance of the Hilbert transform itself is justified by its relationship to the monogenic signal, which is thus generalized for a spherical setting. To accomplish such a spherical generalization, a spherical convolution with the sphere itself as integration domain (instead of $SO(3)$), a canonical spherical Poisson scale space, and conjugate Poisson kernels formulated in Clifford algebraic terms are applied.

Finally, I would like to thank all the authors for their contributions, all reviewers for their constructive criticism, and last but not least, the Editor-in-Chief, Roussos Dimitrakopoulos, for his patience with all of us.

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