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A special family of non-symmetric semi-classical forms of class one

Mohamed Zaatra[1]

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Abstract

A form (linear functional) v is called regular if there exists a sequence of polynomials $\{S_n\}_{n\geq 0}$, $\deg(S_n) = n$, which is orthogonal with respect to v. $\{S_n\}_{n\geq 0}$ is fully characterized by the following recurrence relation: $S_{n+2}(x) = (x - \beta_{n+1})S_{n+1}(x) - \gamma_{n+1}S_n(x)$, $n \geq 0$, with $S_0(x) = 1$, $S_1(x) = x - \beta_0$ and $\gamma_{n+1} \neq 0$, $n \geq 0$. Such a form v is said be semi-classical if there exist polynomials $\Psi(x)$ and $\Phi(x)$ with $\deg(\Psi) \geq 1$ such that $(\Phi v)' + \Psi v = 0$. When v is semi-classical and regular, its corresponding polynomial sequences $\{S_n\}_{n\geq 0}$ are called semi-classical. In this work, we solve the system of Laguerre–Freud equations for the recurrence coefficients β_n, γ_{n+1}, $n \geq 0$ of the semi-classical orthogonal polynomials sequences of class one when $\beta_n = t_{n-1} - t_n$ and $\gamma_{n+1} = -t_n(c + t_n)$ with $t_n(c + t_n) \neq 0$ $n \geq 0$, $t_{-1} = 0$ and $c \in \mathbb{C} - \{0\}$. There are essentially five canonical cases.

Keywords Orthogonal polynomials · Semi-classical forms

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1 Introduction and preliminary results

Semi-classical orthogonal polynomials were introduced in [11]. They are a natural generalization of the classical polynomials. Maroni [7, 8] has worked on the linear form of moments and has given a unified theory of this kind of polynomials. A semi-classical form u satisfies the distributional equation $(\Phi u)' + \Psi u = 0$ where $\Phi(x)$ is a monic polynomial and $\Psi(x)$ is a polynomial with $\deg(\Psi) \geq 1$. Since the system of Laguerre– Freud equations corresponding to the problem of determining all the semi-classical forms of class $s \geq 1$ becomes non-linear, the problem was only solved when $s = 1$ and for the symmetric case [1].

So, the aim of this paper is to solve the system in a special nonsymmetric case. Indeed, we exhaustively describe the family of semi-classical sequences $\{S_n\}_{n\geq 0}$ of class $s = 1$,

✉ Mohamed Zaatra
medzaatra@yahoo.fr

1 Institut Supérieur des Sciences et Techniques des Eaux de Gabès, Campus universitaire, 6072 Gabès, Tunisia

verifying the following three-term recurrence relation:

$$\begin{aligned} S_{n+2}(x) &= \big(x-(t_n-t_{n+1})\big)S_{n+1}(x)+t_n(c+t_n)S_n(x), \quad n\geq 0, \\ S_1(x) &= x+t_0, \quad S_0(x)=1, \end{aligned}$$

with $t_n(c+t_n)\neq 0$, $n\geq 0$. This family has been the subject of some works: for instance, Maroni [6, 9] characterized such sequences by a particular quadratic decomposition and by a perturbation of a symmetric form.

The structure of the manuscript is as follows. The first section is devoted to the preliminary results and notations used in the sequel. In the second section, first we give some properties of the regular form associated with the sequence $\{S_n\}_{n\geq 0}$. Specially, we focus our attention in the case where it is semi-classical of class one. Second, using these properties and the Laguerre–Freud equations for the recurrence coefficients β_n, γ_{n+1}, $n\geq 0$, of orthogonal polynomials with respect to a semi-classical form of class one, we obtain all the sequences which we look for. Finally, we show that there are essentially five canonical cases.

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients and let $\mathcal{P}'$ be its dual. The elements of $\mathcal{P}'$ will be called either linear forms or linear functionals. We denote by $\langle v, f\rangle$ the action of $v\in\mathcal{P}'$ on $f\in\mathcal{P}$. For $n\geq 0$, $(v)_n=\langle v, x^n\rangle$ are the moments of v. In particular a linear form v is called symmetric if all of its moments of odd order are zero [3].

For any linear form v, any $a\in\mathbb{C}-\{0\}$ and any polynomial h let $Dv=v'$, hv, $h_a v$, δ_c, and $(x-c)^{-1}v$ be the linear forms defined by $\langle v', f\rangle := -\langle v, f'\rangle$, $\langle hv, f\rangle := \langle v, hf\rangle$, $\langle h_a v, f\rangle := \langle v, h_a f\rangle = \langle v, f(ax)\rangle$, $\langle\delta_c, f\rangle := f(c)$, and $\langle (x-c)^{-1}v, f\rangle := \langle v, \theta_c f\rangle$, where $\big(\theta_c f\big)(x)=\dfrac{f(x)-f(c)}{x-c}$, $c\in\mathbb{C}$, $f\in\mathcal{P}$.

It is straightforward to prove that for $v\in\mathcal{P}'$, $f\in\mathcal{P}$ and $c\in\mathbb{C}$, we have [7]

$$(fv)' = f'v+fv', \tag{1.1}$$

$$(x-c)^{-1}\big((x-c)v\big) = v-(v)_0\delta_c, \tag{1.2}$$

$$(x-c)\big((x-c)^{-1}v\big) = v. \tag{1.3}$$

Let us recall that a form v is said to be regular (quasi-definite) if there exists a sequence $\{S_n\}_{n\geq 0}$ of polynomials with $\deg S_n=n$, $n\geq 0$, such that

$$\langle v, S_n S_m\rangle = r_n\delta_{n,m} \quad , \quad r_n\neq 0, \quad n\geq 0. \tag{1.4}$$

We can always assume that each S_n is monic, i.e. $S_n(x)=x^n+$ lower degree terms. Then the sequence $\{S_n\}_{n\geq 0}$ is said to be orthogonal with respect to v (monic orthogonal polynomial sequence (MOPS) in short).

Moreover, a form v is regular if and only if $\Delta_n(v)=\det\big((v)_{\nu+\mu}\big)_{\nu,\mu=0}^{n}\neq 0$, $n\geq 0$ (Hankel determinants) [7].

It is a well-known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies a three-term recurrence relation (TTRR) (see, for instance, the monograph [3] by Chihara)

$$\begin{aligned} S_{n+2}(x) &= (x-\beta_{n+1})S_{n+1}(x)-\gamma_{n+1}S_n(x), \quad n\geq 0, \\ S_1(x) &= x-\beta_0, \quad S_0(x)=1, \end{aligned} \tag{1.5}$$

with $\big(\beta_n, \gamma_{n+1}\big)\in\mathbb{C}\times(\mathbb{C}-\{0\})$, $n\geq 0$. By convention we set $\gamma_0=(v)_0$.

The form v is said to be normalized if $(v)_0=1$. In this paper, we suppose that any form is normalized.

Now, let us recall some features about the semi-classical character [7].

Definition 1.1 A linear form v is said to be semi-classical when it is regular and there exist two polynomials, Φ, a monic polynomial, and Ψ, $\deg(\Phi) = t \geq 0$, $\deg(\Psi) = p \geq 1$, such that

$$\big(\Phi v\big)' + \Psi v = 0. \tag{1.6}$$

The corresponding MOPS $\{S_n\}_{n\geq 0}$ is said to be semi-classical.

Proposition 1.2 *The semi-classical linear form v satisfying (1.6) is said to be of class $s = \max(t-2, p-1)$ if and only if the following condition is satisfied*

$$\prod_{c\in\mathcal{Z}_\Phi}\Big(|\Phi'(c)+\Psi(c)| + |\langle v, \theta_c^2\Phi + \theta_c\Psi\rangle|\Big) \neq 0, \tag{1.7}$$

where $\mathcal{Z}_\Phi$ is the set of zeros of Φ.

The semi-classical character of a linear form is kept by shifting. Indeed, the shifted form $\tilde{v} = h_{a^{-1}}v,\ a \neq 0$, is also semi-classical having the same class as that v and fulfilling the equation

$$\big(\tilde{\Phi}\tilde{v}\big)' + \tilde{\Psi}\tilde{v} = 0,$$

where

$$\tilde{\Phi}(x) = a^{-t}\Phi(ax),\ \tilde{\Psi}(x) = a^{1-t}\Psi(ax).$$

The sequence $\{\tilde{S}_n\}_{n\geq 0}$, where $\tilde{S}_n = a^{-n}S_n(ax),\ n \geq 0$, is orthogonal with respect to $\tilde{v}$. The recurrence coefficients are given by

$$\tilde{\beta}_n = \frac{\beta_n}{a},\quad \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2},\ n \geq 0.$$

The next result [1] characterizes the elements of the functional equation satisfied by any symmetric semi-classical linear form.

Proposition 1.3 *Let v be a symmetric semi-classical linear form of class s satisfying (1.6). The following statements hold:*

(i) When s is odd then Φ is odd and Ψ is even.
(ii) When s is even then Φ is even and Ψ is odd.

In the sequel, we assume that $\{S_n\}_{n\geq 0}$ is a semi-classical MOPS of class one. This means that

$$\begin{aligned} &\Phi(x) = c_3x^3 + c_2x^2 + c_1x + c_0,\quad \Psi(x) = a_2x^2 + a_1x + a_0,\\ &|c_3| + |a_2| \neq 0, |a_2| + |a_1| \neq 0, |c_3| + |c_2| + |c_1| + |c_0| \neq 0. \end{aligned} \tag{1.8}$$

Furthermore, the Laguerre–Freud system of equations giving recursively the recurrence coefficients for semi-classical orthogonal polynomials of class one are [2].

$$a_2\gamma_1 = -\Psi(\beta_0), \tag{1.9}$$

$$\begin{aligned} &(a_2 - 2nc_3)(\gamma_n + \gamma_{n+1}) - 4c_3\sum_{\nu=0}^{n-2}\gamma_{\nu+1}\\ &\quad = 2\sum_{\nu=0}^{n-1}(\theta_{\beta_n}\Phi)(\beta_\nu) - \Psi(\beta_n),\ n \geq 1,\ \sum_0^{-1} = 0, \end{aligned} \tag{1.10}$$

$$\Xi_n \gamma_{n+1} - 2c_2 \sum_{\nu=0}^{n-1} \gamma_{\nu+1} - 3c_3 \sum_{\nu=0}^{n-1} \gamma_{\nu+1}(\beta_\nu + \beta_{\nu+1}) = \sum_{\nu=0}^{n} \Phi(\beta_\nu), \quad n \geq 0, \tag{1.11}$$

with

$$\Xi_n = a_1 + (\beta_n + \beta_{n+1})(a_2 - 2nc_3) - \beta_{n+1}c_3 - (2n+1)c_2 - 2c_3 \sum_{\nu=0}^{n} \beta_\nu \tag{1.12}$$

2 Semi-classical forms of class one: special case

From now on, let v be a semi-classical form of class $s_v = 1$ satisfying (1.6) and its corresponding MOPS $\{S_n\}_{n\geq 0}$ fulfills

$$\begin{aligned} S_{n+2}(x) &= \Big(x - (t_n - t_{n+1})\Big) S_{n+1}(x) + t_n(c + t_n) S_n(x), \quad n \geq 0, \\ S_1(x) &= x + t_0, \quad S_0(x) = 1, \end{aligned} \tag{2.1}$$

with $t_n(c + t_n) \neq 0$, $n \geq 0$ and $c \in \mathbb{C}$. The next Lemma will play an important role in the sequel.

Lemma 2.1 [9] *The following statements are equivalent:*

(a) The MOPS $\{S_n\}_{n\geq 0}$ satisfies (2.1).
(b) $(v)_{2n+2} = c(v)_{2n+1}, \ n \geq 0$.
(c) There exists a regular symmetric form u such that

$$v = \big((v)_1 - c\big)(x - c)^{-1} u + \delta_c.$$

Remark 2.2 The form v is c-quasi-antisymmetric (i.e $(v)_{2n+2} = c(v)_{2n+1}, n \geq 0$). For more information about these forms, see [9].

2.1 Class and functional equation of the form *u*

In the sequel our aim is to characterize the structure of the polynomial elements of the functional Eq. (1.6) satisfied by the form v which its corresponding MOPS $\{S_n\}_{n\geq 0}$ fulfills (2.1). This is possible through the study of the semi-classical character of the symmetric form u defined by

$$\lambda u = (x - c)v, \ \lambda = \big((v)_1 - c\big) \neq 0. \tag{2.2}$$

The case $c = 0$ is treated in [12] and also in [6], so henceforward we assume $c \neq 0$. Using a dilation in the variable c, we can assume

$$\gamma_{n+1} = -t_n(1 + t_n), \ n \geq 0. \tag{2.3}$$

Then,

$$\lambda u = (x - 1)v, \ \lambda = \big((v)_1 - 1\big) \neq 0. \tag{2.4}$$

Consequently, according to [4], the form u is regular if and only if

$$S_n(1) \neq 0, \ n \geq 0.$$

The form u defined by (2.4) when it is regular, is also semi-classical of class s_u such that $s_u \leq s_v + 1$ and satisfying the functional equation [4]

$$(Eu)' + Fu = 0, \tag{2.5}$$

with

$$E(x) = (x-1)\Phi(x), \quad F(x) = (x-1)\Psi(x) - 2\Phi(x). \tag{2.6}$$

Theorem 2.3 *Let v be a semi-classical form of class one and satisfies (1.6). Then, the form u defined by (2.4) is semi-classical of class s_u satisfying*

$$(\tilde{E}u)' + \tilde{F}u = 0. \tag{2.7}$$

Moreover,

(a) If $\Phi(1) \neq 0$, then

$$\tilde{E}(x) = E(x), \quad \tilde{F}(x) = F(x),$$

and $s_u = 2$.

(b) If $\Phi(1) = 0$ and $\Psi(1) \neq 0$, then

$$\tilde{E}(x) = \Phi(x), \quad \tilde{F}(x) = \Psi(x) - (\theta_1\Phi)(x),$$

and $s_u = 1$.

(c) If $\Phi(1) = 0$ and $\Psi(1) = 0$, then

$$\tilde{E}(x) = (\theta_1\Phi)(x), \quad \tilde{F}(x) = (\theta_1\Psi)(x),$$

and $s_u = 0$.

For the proof, we need the following lemma.

Lemma 2.4 *(i) For all root c of Φ, we have*

$$\langle u, \theta_c^2 E + \theta_c F\rangle = \frac{(c-1)^2}{\lambda}\langle v, \theta_c^2\Phi + \theta_c\Psi\rangle - \frac{c-1}{\lambda}(\Phi' + \Psi)(c), \tag{2.8}$$

$$E'(c) + F(c) = (c-1)(\Phi' + \Psi)(c). \tag{2.9}$$

(ii) The class of the form u depends only the zero $x = 1$.

Proof (i) Let c be a root of Φ. Then we can write

$$E(x) = (x-1)(x-c)\Phi_c(x), \quad \Phi_c(x) = (\theta_c\Phi)(x). \tag{2.10}$$

Using the definition of the operator θ_c, it is easy to prove that, for $f, g \in \mathcal{P}$, we have

$$\Big(\theta_c(fg)\Big)(x) = \Big(\theta_c f\Big)(x)g(x) + f(c)\Big(\theta_c g\Big)(x). \tag{2.11}$$

On account of (2.4) and (2.10), we have

$$\langle u, \theta_c^2 E\rangle = \frac{1}{\lambda}\langle (x-1)v, \theta_c\big((\xi-1)\Phi_c\big)(x)\rangle.$$

Taking $f(x) = x - 1$ and $g(x) = \Phi_c(x)$ in (2.11), we obtain

$$\langle (x-1)v, \theta_c\big((\xi-1)\Phi_c\big)(x)\rangle = (c-1)^2\langle v, \theta_c^2\Phi\rangle$$

$$+2(c-1)\langle v, \Phi_c\rangle + \langle v, \Phi\rangle - (c-1)\Phi'(c),$$

since $x-1=(x-c)+c-1$. Therefore,

$$\begin{aligned}\langle u, \theta_c^2 E\rangle &= \frac{(c-1)^2}{\lambda}\langle v, \theta_c^2\Phi\rangle \\ &+\frac{2(c-1)}{\lambda}\langle v, \Phi_c\rangle + \frac{1}{\lambda}\langle v, \Phi\rangle - \frac{c-1}{\lambda}\Phi'(c).\end{aligned} \tag{2.12}$$

Proceeding as in (2.12), we can easily prove that

$$\begin{aligned}\langle u, \theta_c F\rangle &= \frac{(c-1)^2}{\lambda}\langle v, \theta_c\Psi\rangle + \frac{1}{\lambda}\langle v, (x+c-2)\Psi\rangle \\ &-\frac{2(c-1)}{\lambda}\langle v, \Phi_c\rangle - \frac{2}{\lambda}\langle v, \Phi\rangle - \frac{c-1}{\lambda}\Psi(c).\end{aligned} \tag{2.13}$$

Adding (2.12) and (2.13), we get

$$\begin{aligned}\langle u, \theta_c^2 E + \theta_c F\rangle &= \frac{(c-1)^2}{\lambda}\langle v, \theta_c^2\Phi + \theta_c\Psi\rangle - \frac{c-1}{\lambda}(\Phi'+\Psi)(c) \\ &+\frac{1}{\lambda}\langle (v\Phi)' + \Psi v, x+c-2\rangle.\end{aligned}$$

This yields (2.8), since $\langle (\Phi v)' + \Psi v, x+c-2\rangle = 0$, from (1.6). Next, it is easy to find (2.9) from (2.6).

(ii) Let c be a root of E such that $c \neq 1$. According to (2.6) we get $\Phi(c) = 0$. We suppose $|E'(c) + F(c)| = 0$. By virtue of (2.8) and (2.9), we obtain

$$|\Phi'(c) + \Psi(c)| = 0,$$

and

$$\langle u, \theta_c^2 E + \theta_c F\rangle = \frac{(c-1)^2}{\lambda}\langle v, \theta_c^2\Phi + \theta_c\Psi\rangle \neq 0,$$

since v is semi-classical and so satisfies (1.7). Hence, (2.6) cannot be simplified by $x-c$ for $c \neq 1$. □

Proof of Theorem 2.3 We may write $E'(1) + F(1) = -\Phi(1)$.

(a) If $\Phi(1) \neq 0$, then $|E'(1) + F(1)| \neq 0$. Thus, (14) cannot be simplified and so the form u is of class

$$s_u = \max\big(\deg(E) - 2, \deg(F) - 1\big) = \max\big(\deg(\Phi) - 1, \deg(\Psi)\big).$$

Hence, $s_u = 2$.

(b) If $\Phi(1) = 0$, then

$$|E'(1) + F(1)| = 0 \ \text{ and } \ \langle u, \theta_1^2 E + \theta_1 F\rangle = 0,$$

according to (2.8) and (2.9). So, (2.5) can be simplified by the polynomial $x-1$ and becomes

$$(\tilde{E}u)' + \tilde{F}u = 0, \tag{2.14}$$

where

$$\tilde{E}(x) = \Phi(x), \quad \tilde{F}(x) = \Psi(x) - (\theta_1\Phi)(x). \tag{2.15}$$

It is easy to see that (2.14) cannot be simplified, since

$$\tilde{E}'(1) + \tilde{F}(1) = \Psi(1) \neq 0.$$

Therefore, $s_u = 1$.

(c) If $\Phi(1) = 0$ and $\Psi(1) = 0$, here

$$\tilde{E}'(1) + \tilde{F}(1) = \Psi(1) = 0.$$

A simple calculation gives $\langle u, \theta_1^2 \tilde{E} + \theta_1 \tilde{F} \rangle = \frac{1}{\lambda} \langle v, \Psi \rangle = 0$. So, (2.14) is simplified by the polynomial $x - 1$ and it becomes

$$(\hat{E}u)' + \hat{F}u = 0, \tag{2.16}$$

where

$$\hat{E}(x) = (\theta_1 \Phi)(x), \quad \hat{F}(x) = (\theta_1 \Psi)(x). \tag{2.17}$$

If 1 is a root of $\theta_1 \Phi$, then $\Phi'(1) + \Psi(1) = 0$. Assuming that $|\hat{E}'(1) + \hat{F}(1)| = 0$. A simple calculations gives

$$\langle u, \theta_1^2 \hat{E} + \theta_1 \hat{F} \rangle = \frac{1}{\lambda} \langle v, \theta_1^2 \Phi + \theta_1 \Psi \rangle \neq 0,$$

because v is a semi-classical and it satisfies (1.7). Thus, (2.16) cannot be simplified, and so $s_u = 0$. □

2.2 Structure of the polynomials Φ and Ψ

Let us split up each polynomial from Φ, Ψ, $\theta_1 \Phi$ and $\theta_1 \Psi$ according to its odd and even parts, that is, write

$$\begin{aligned} \Phi(x) &= \Phi^e(x^2) + x\Phi^o(x^2), \ \Psi(x) = \Psi^e(x^2) + x\Psi^o(x^2), \\ (\theta_1 \Phi)(x) &= \Phi_1^e(x^2) + x\Phi_1^o(x^2), \ (\theta_1 \Psi)(x) = \Psi_1^e(x^2) + x\Psi_1^o(x^2). \end{aligned} \tag{2.18}$$

Proposition 2.5 *Let v be a semi-classical form of class one satisfying (1.6) and its corresponding MOPS $\{S_n\}_{n \geq 0}$ fulfills (1.10). The following statements hold:*

(a) If $\Phi(1) \neq 0$, then $\Phi^e = \Phi^o$ and $x\Psi^o(x) - \Psi^e(x) - 2\Phi^e(x) = 0$.
(b) If $\Phi(1) = 0$ and $\Psi(1) \neq 0$, then $\Phi^e = 0$ and $(x-1)\Psi^o(x) - \Phi^o(x) = 0$.
(c) If $\Phi(1) = 0$ and $\Psi(1) = 0$, then $\Phi^e(x) + \Phi^o(x) = 0$ and $x\Psi^o(x) + \Psi^e(x) = 0$.

Proof Write

$$\tilde{E}(x) = \tilde{E}^e(x) + x\tilde{E}^o(x), \ \tilde{F}(x) = \tilde{F}^e(x) + x\tilde{F}^o(x). \tag{2.19}$$

We need to discuss the following situations:

(a) $\Phi(1) \neq 0$. According to (2.18) and from the expression of polynomials $\tilde{E}$ and $\tilde{F}$ given in Theorem 2.3, we infer

$$\begin{aligned} &\tilde{E}^e(x) = x\Phi^o(x) - \Phi^e(x), \quad \tilde{E}^o(x) = \Phi^e(x) - \Phi^o(x), \\ &\tilde{F}^e(x) = x\Psi^o(x) - \Psi^e(x) - 2\Phi^e(x), \quad \tilde{F}^o(x) = \Psi^e(x) - \Psi^o(x) - 2\Phi^o(x). \end{aligned}$$

Then, $\tilde{E}^o = \tilde{F}^e = 0$, from Proposition 1.3, since $s_u = 2$. This gives (a). In the other cases, we are going to proceed with the same technique.

(b) $\Phi(1)=0$ and $\Psi(1)\neq 0$. Similarly as above, we have

$$\tilde{E}^e(x)=\Phi^e(x),\quad \tilde{E}^o(x)=\Phi^o(x),$$
$$\tilde{F}^e(x)=\Psi^e(x)-\Phi_1^e(x),\quad \tilde{F}^o(x)=\Psi^o(x)-\Phi_1^o(x).$$

If $s_u=1$, then $\tilde{E}^e=\tilde{F}^o=0$. This leads to result (b) because

$$\Phi(x)=(x-1)(\theta_1\Phi)(x).$$

(c) $\Phi(1)=0$ and $\Psi(1)=0$. In this case, we obtain

$$\tilde{E}^e(x)=\Phi_1^e(x),\quad \tilde{E}^o(x)=\Phi_1^o(x),$$
$$\tilde{F}^e(x)=\Psi_1^e,\quad \tilde{F}^o(x)=\Psi_1^o(x).$$

Since u is of even class, $\tilde{E}^o=\tilde{F}^e=0$. Therefore, $\Phi_1^o=0$ and $\Psi_1^e=0$. This gives the desired result since $\Psi(x)=(x-1)(\theta_1\Psi)(x)$. □

Theorem 2.6 *Let v be a semi-classical linear form of class one satisfying (1.6) and $\{S_n\}_{n\geq 0}$ be its corresponding MOPS fulfilling (2.1). The solutions of the functional Eq. (1.6) are*

$$\begin{aligned}\Phi(x)=(x+1)(c_3x^2+c_1),\ \Psi(x)=a_2x^2+(a_2+2c_3)x-2c_1,\\ (c_3+c_1)c_1\neq 0,\quad (a_2+c_3)(u)_1-c_1=0,\end{aligned}\tag{2.20}$$

or

$$\begin{aligned}\Phi(x)=x(x^2-1),\quad \Psi(x)=a_2x^2+x+a_0,\\ a_2+a_0+1\neq 0,\quad (a_2+1)(u)_1+a_0=0,\end{aligned}\tag{2.21}$$

or

$$\Phi(x)=(x-1)(c_3x^2+c_1),\quad \Psi(x)=a_2x(x-1),\quad c_1\neq 0.\tag{2.22}$$

Proof We consider the following four cases.

- If $\deg(\Phi)=0$, then $\Phi(1)\neq 0$. So, we obtain $\Phi^e(x)=1$ and $\Phi^o(x)=0$. From Proposition 2.5, we get $\Phi^e(x)=0$ which yields a contradiction.
- If $\deg(\Phi)=1$, then $c_1=1$, $c_2=c_3=0$ and $a_2\neq 0$. So, we obtain $\Phi^e(x)=c_0$, $\Phi^o(x)=1$, $\Psi^e(x)=a_2x+a_0$ and $\Psi^o(x)=a_1$. Following Proposition 2.5, we can distinguish the following three situations. If $\Phi(1)\neq 0$, then $c_0=1$, $a_1=a_2$ and $a_0=-2$. This leads to result (2.20). If $\Phi(1)=0$ and $\Psi(1)\neq 0$, here $c_0=0$, which yields a contradiction. If $\Phi(1)=0$ and $\Psi(1)=0$, then $c_0=-1$, $a_1=-a_2$ and $a_0=0$. This gives (2.22).
- If $\deg(\Phi)=2$, then $c_2=1$, $c_3=0$ and $a_2\neq 0$. So, we obtain $\Phi^e(x)=x+c_0$ and $\Phi^o(x)=c_1$. By virtue of Proposition 2.5, such assumptions lead us to a contradiction.
- If $\deg(\Phi)=3$, then $1\leq\deg(\Psi)\leq 2$. In this case, we get $\Phi^e(x)=c_2x+c_0$, $\Phi^o(x)=x+c_1$, $\Psi^e(x)=a_2x+a_0$ and $\Psi^o(x)=a_1$. We have to examine three subcases. If $\Phi(1)\neq 0$, then $c_2=1$, $c_0=c_1$, $a_1=a_2+2$ and $a_0=-2c_1$. On the other hand, from (1.6) we have

$$\langle(\Phi v)'+\Psi v, x^{2n+2}\rangle=0,\ n\geq 0.\tag{2.23}$$

If $c_1=0$, then from (2.23) we can deduce

$$(a_2-2n-2)(v)_{2n+3}=0,\ n\geq 0.$$

This leads to result (2.20). If $\Phi(1)=0$ and $\Psi(1)\neq 0$, here $c_2=c_0=0$, $c_1=-1$ and $a_1=1$. This gives (2.21). If $\Phi(1)=0$ and $\Psi(1)=0$, thus $c_2=-1, c_0=-c_1, a_1=-a_2$ and $a_0=0$. If $c_1=0$, then from (2.23) we have $(a_2-2n-1)\big((v)_{2n+3}-(v)_{2n+2}\big)=0,\ n\geq 0$. So, after a certain rang the Hankel determinants [9] associated with v are equal to zero, by following v is not regular. This leads to result (2.22). □

2.3 Recurrence coefficients of $\{S_n\}_{n\geq 0}$

In the sequel we assume that $\{S_n\}_{n\geq 0}$ is 1-quasi-antisymmetric semi-classical orthogonal sequence of class one satisfying

$$\begin{aligned} &S_{n+2}(x)=\Big(x-(t_n-t_{n+1})\Big)S_{n+1}(x)+t_n(1+t_n)S_n(x), \quad n\geq 0,\\ &S_1(x)=x+t_0, \quad S_0(x)=1, \end{aligned} \tag{2.24}$$

By virtue of the Theorem 2.6, it follows that

$$(\Phi v)'+\Psi v=0, \tag{2.25}$$

with

$$\Phi(x)=c_3x^3+c_2x^2+c_1x+c_0,\ \ \Psi(x)=a_2x^2+a_1x+a_0. \tag{2.26}$$

In this case, the system (1.9)–(1.11) becomes (for $n\geq 0$)

$$\begin{aligned} &4c_3\sum_{\nu=0}^{n-1}t_\nu(1+t_{\nu+1})+2[(2n-3)c_3-a_2]t_{n-1}t_n+[2n(c_3+c_2)-a_2-a_1]t_n\\ &\quad+[2(n-1)(c_3-c_2)+a_1-a_2-2c_3]t_{n-1}+a_0-2nc_1=0, \end{aligned} \tag{2.27}$$

$$\begin{aligned} &l2c_2\sum_{\nu=0}^{n-1}t_\nu(1+t_{\nu+1})+t_n(1+t_n)\{(2nc_3-a_2)(t_{n-1}-t_{n+1})-c_3(t_{n+1}+3t_{n-1})\\ &\quad+(2n+1)c_2-a_1\}+c_1t_n-(c_3+c_2)t_n^2-(n+1)c_0=0. \end{aligned} \tag{2.28}$$

According to Theorem 2.6, we are going to consider the following three cases.

2.3.1 The case $\Phi(1)\neq 0$

In this case, taking into account (2.20), the system (2.27)–(2.28) can be written as

$$\begin{aligned} &2c_3\sum_{\nu=0}^{n-1}t_\nu(1+t_{\nu+1})\\ &\quad+\{[(2n-3)c_3-a_2]t_{n-1}+(2n-1)c_3-a_2\}t_n-(n+1)c_1=0,\ \ n\geq 0, \end{aligned} \tag{2.29}$$

$$\begin{aligned} &2c_3\sum_{\nu=0}^{n-1}t_\nu(1+t_{\nu+1})+\{[(2n-3)c_3-a_2]t_{n-1}+[(2n-3)c_3-a_2]t_n(1+t_{n-1})\\ &\quad-[(2n+1)c_3-a_2]t_{n+1}(1+t_n)+(2n-1)c_3-a_2+c_1\}t_n-(n+1)c_1=0,\ \ n\geq 0. \end{aligned} \tag{2.30}$$

Subtracting identities (2.29) and (2.30), we obtain for $n\geq 0$ that

$$[(2n+1)c_3-a_2]t_{n+1}(1+t_n)-[(2n-3)c_3-a_2]t_n(1+t_{n-1})-c_1=0. \tag{2.31}$$

Proposition 2.7 *For $n \geq 0$ we have*

$$t_n(1+t_{n-1}) = \frac{(n+1)[(n-3)c_3 - a_2]c_1}{[(2n-1)c_3 - a_2][(2n-3)c_3 - a_2]}. \tag{2.32}$$

Proof From (2.31), we have for $n \geq 0$

$$[(2n+1)c_3 - a_2][(2n-1)c_3 - a_2]t_{n+1}(1+t_n) \\ -[(2n-1)c_3 - a_2][(2n-3)c_3 - a_2]t_n(1+t_{n-1}) = [(2n-1)c_3 - a_2]c_1.$$

So, for $n \geq 0$ we obtain

$$[(2n+1)c_3 - a_2][(2n-1)c_3 - a_2]t_{n+1}(1+t_n) \\ = (3c_3 + a_2)(c_3 + a_2)t_0 + c_1 \sum_{\nu=0}^{n} [(2\nu - 1)c_3 - a_2].$$

This yields (2.32). □

Corollary 2.8 *The sequences $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ are defined by*

$$\beta_0 = -t_0, \;\; \beta_{n+1} = t_n - t_{n+1}, \;\; n \geq 0, \\ \gamma_{n+1} = -t_n(1+t_n), \;\; n \geq 0, \tag{2.33}$$

where for $n \geq 0$

$$t_{2n} = \begin{cases} -\dfrac{1}{a_2} \dfrac{\sum_{k=0}^{n} \frac{(-1)^k}{(2a_2)^k \Gamma(k+1)\Gamma(2n+1-2k)}}{\sum_{k=0}^{n} \frac{(-1)^k}{(2a_2)^k \Gamma(k+1)\Gamma(2n+2-2k)}}, & \textit{if } (c_3, c_1) = (0,1), \\ \dfrac{(2n-a_2-3)c_1}{2(4n-a_2-1)} \dfrac{\sum_{k=0}^{n} \frac{\Gamma(2n-k-\frac{a_2}{2}-\frac{3}{2})c_1^k}{2^{2k}\Gamma(2n+1-2k)\Gamma(k+1)}}{\sum_{k=0}^{n} \frac{\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})c_1^k}{2^{2k}\Gamma(2n+2-2k)\Gamma(k+1)}}, & \textit{if } c_3 = 1 \textit{ and } c_1 \neq 0, \end{cases} \tag{2.34}$$

$$t_{2n+1} = \begin{cases} -\dfrac{1}{a_2} \dfrac{\sum_{k=0}^{n} \frac{(-1)^k}{(2a_2)^k \Gamma(k+1)\Gamma(2n+2-2k)}}{\sum_{k=0}^{n+1} \frac{(-1)^k}{(2a_2)^k \Gamma(k+1)\Gamma(2n+3-2k)}}, & \textit{if } (c_3, c_1) = (0,1), \\ \dfrac{(2n-a_2-2)c_1}{2(4n+1-a_2)} \dfrac{\sum_{k=0}^{n} \frac{\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})c_1^k}{2^{2k}\Gamma(2n+2-2k)\Gamma(k+1)}}{\sum_{k=0}^{n+1} \frac{\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})c_1^k}{2^{2k}\Gamma(2n+3-2k)\Gamma(k+1)}}, & \textit{if } c_3 = 1 \textit{ and } c_1 \neq 0. \end{cases} \tag{2.35}$$

Proof Firstly, we prove (2.34) by induction on n.

From (2.32) we get $t_0 = -\frac{c_1}{c_3+a_2}$. Hence, (2.34) is true for $n = 0$.

We suppose that (2.34) is true until n and prove that it is true for $n+1$.

From (2.32), we have for $n \geq 0$

$$t_{2n+2} = \\ \frac{(2n+3)[(2n-1)c_3 - a_2][(4n-1)c_3 - a_2](1+t_{2n})c_1}{[(4n+3)c_3 - a_2]\{[(4n-1)c_3 - a_2][(4n+1)c_3 - a_2](1+t_{2n}) + (2n+2)[(2n-2)c_3 - a_2]c_1\}}. \tag{2.36}$$

Two cases arise as follows.

- If $(c_3, c_1) = (0, 1)$, then (2.36) reduces to

$$t_{2n+2} = \frac{(2n+3)(1+t_{2n})}{2n+2-a_2(1+t_{2n})}. \tag{2.37}$$

Substitution of t_{2n} from (2.34) into (2.37) gives

$$\begin{aligned} t_{2n+2} &= -\frac{1}{a_2}\frac{(2n+3)\Big(\sum_{k=0}^{n}\frac{(-1)^k}{(2a_2)^k\Gamma(2n+2-2k)\Gamma(k+1)}+\sum_{k=0}^{n}\frac{2(-1)^{k+1}}{(2a_2)^{k+1}\Gamma(2n+1-2k)\Gamma(k+1)}\Big)}{\sum_{k=0}^{n}\frac{(-1)^k}{(2a_2)^k\Gamma(2n+2-2k)\Gamma(k+1)}+\sum_{k=0}^{n}\frac{2(4n+3-2k)(-1)^{k+1}}{(2a_2)^{k+1}\Gamma(2n+2-2k)\Gamma(k+1)}} \\ &= -\frac{1}{a_2}\frac{(2n+3)\Big(\sum_{k=0}^{n}\frac{(-1)^k(2n+2-2k)}{(2a_2)^k\Gamma(2n+3-2k)\Gamma(k+1)}+\sum_{k=0}^{n+1}\frac{2k(-1)^{k}}{(2a_2)^{k}\Gamma(2n+3-2k)\Gamma(k+1)}\Big)}{\sum_{k=0}^{n}\frac{(-1)^k(2n+2-2k)(2n+3-2k)}{(2a_2)^k\Gamma(2n+4-2k)\Gamma(k+1)}+\sum_{k=0}^{n+1}\frac{2k(4n+5-2k)(-1)^{k}}{(2a_2)^{k}\Gamma(2n+4-2k)\Gamma(k+1)}}. \end{aligned}$$

Here, we obtain (2.34) when $(c_3, c_1) = (0, 1)$ since $(2n+2-2k)(2n+3-2k)+2k(4n+5-2k) = (2n+2)(2n+3)$.

- If $c_3 = 1$ with $c_1 \neq 0$, then (2.36) becomes

$$t_{2n+2} = \frac{(2n+3)(2n-1-a_2)(4n-1-a_2)(1+t_{2n})c_1}{(4n+3-a_2)\{(4n-1-a_2)(4n+1-a_2)(1+t_{2n})+(2n+2)(2n-2-a_2)c_1\}}. \tag{2.38}$$

Substitution of t_{2n} from (2.34) into (2.38) gives

$$\begin{aligned} t_{2n+2} &= \frac{(2n-1-a_2)(2n+3)c_1}{2(4n+3-a_2)}\Bigg((2n-\frac{a_2}{2}-\frac{1}{2})\sum_{k=0}^{n}\frac{\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})c_1^k}{2^{2k}\Gamma(2n+2-2k)\Gamma(k+1)} \\ &\quad+(2n-a_2-3)\sum_{k=0}^{n}\frac{\Gamma(2n-k-\frac{a_2}{2}-\frac{3}{2})c_1^{k+1}}{2^{2k+2}\Gamma(2n+1-2k)\Gamma(k+1)}\Bigg) \\ &\quad\times\Bigg((2n-\frac{a_2}{2}-\frac{1}{2})(2n-\frac{a_2}{2}+\frac{1}{2})\sum_{k=0}^{n}\frac{\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})c_1^k}{2^{2k}\Gamma(2n+2-2k)\Gamma(k+1)} \\ &\quad+(2n+2)(2n-a_2-2)\sum_{k=0}^{n}\frac{\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})c_1^{k+1}}{2^{2k+2}\Gamma(2n+2-2k)\Gamma(k+1)} \\ &\quad+(2n-\frac{a_2}{2}+\frac{1}{2})(2n-a_2-3)\sum_{k=0}^{n}\frac{\Gamma(2n-k-\frac{a_2}{2}-\frac{3}{2})c_1^{k+1}}{2^{2k+2}\Gamma(2n+1-2k)\Gamma(k+1)}\Bigg)^{-1} \\ &= \frac{(2n-1-a_2)(2n+3)c_1}{2(4n+3-a_2)} \\ &\quad\Bigg((2n-\frac{a_2}{2}-\frac{1}{2})\sum_{k=0}^{n}\frac{(2n+2-2k)\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})c_1^k}{2^{2k}\Gamma(2n+3-2k)\Gamma(k+1)} \\ &\quad+(2n-a_2-3)\sum_{k=0}^{n+1}\frac{k\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})c_1^{k}}{2^{2k}\Gamma(2n+3-2k)\Gamma(k+1)}\Bigg) \\ &\quad\times\Bigg((2n+2)(2n-a_2-2)\sum_{k=0}^{n+1}\frac{k\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})c_1^{k}}{2^{2k}\Gamma(2n+4-2k)\Gamma(k+1)} \end{aligned}$$

$$+ (2n - \frac{a_2}{2} + \frac{1}{2})(2n - a_2 - 3) \sum_{k=0}^{n+1} \frac{(2n+3-2k)k\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})c_1^k}{2^{2k}\Gamma(2n+4-2k)\Gamma(k+1)}$$

$$+ (2n - \frac{a_2}{2} - \frac{1}{2})(2n - \frac{a_2}{2} + \frac{1}{2})$$

$$\sum_{k=0}^{n} \frac{(2n+2-2k)(2n+3-2k)\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})c_1^k}{2^{2k}\Gamma(2n+4-2k)\Gamma(k+1)}\Bigg)^{-1}$$

$$= \frac{(2n-1-a_2)(2n+3)c_1}{2(4n+3-a_2)}$$

$$\left(\sum_{k=0}^{n} \frac{\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})c_1^k}{2^{2k}\Gamma(2n+3-2k)\Gamma(k+1)} + \frac{\Gamma(n-\frac{a_2}{2}-\frac{1}{2})c_1^{n+1}}{2^{2n+2}\Gamma(n+2)}\right)$$

$$\times \Bigg((2n-a_2-2)\sum_{k=0}^{n+1} \frac{k\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})c_1^k}{2^{2k}\Gamma(2n+4-2k)\Gamma(k+1)}$$

$$+ (2n - \frac{a_2}{2} + \frac{1}{2})\sum_{k=0}^{n+1} \frac{(2n+3-2k)k\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})c_1^k}{2^{2k}\Gamma(2n+4-2k)\Gamma(k+1)}\Bigg)^{-1}.$$

Hence, the result (2.34) because

$$(2n+3-2k)(2n - \frac{a_2}{2} + \frac{1}{2}) + k(2n - a_2 - 2) = (2n+3)(2n - k - \frac{a_2}{2} + \frac{1}{2}).$$

Finally, from (2.34) and (2.32), we can deduce (2.35) after some calculations. □

2.3.2 The case $\Phi(1) = 0$ and $\Psi(1) \neq 0$

In this case, taking into account the relation (2.21), the system (2.27)–(2.28) reads as for $n \geq 0$

$$4\sum_{\nu=0}^{n-1} t_\nu(1+t_{\nu+1}) + 2(2n-3-a_2)t_{n-1}t_n + (2n-3-a_2)t_{n-1}$$
$$+(2n-1-a_2)t_n + a_0 + 2n = 0, \quad (2.39)$$
$$(2n+1-a_2)t_{n+1} - (2n-3-a_2)t_{n-1} + 2 = 0. \quad (2.40)$$

Proposition 2.9 *The sequences* $\{\beta_n\}_{n\geq 0}$ *and* $\{\gamma_{n+1}\}_{n\geq 0}$ *are defined by*

$$\beta_0 = -t_0, \ \beta_{2n+2} = t_{2n+1} - t_{2n+2}, \ \beta_{2n+1} = t_{2n} - t_{2n+1}, \ n \geq 0,$$
$$\gamma_{2n+1} = -t_{2n}(1+t_{2n}), \ \gamma_{2n+2} = -t_{2n+1}(1+t_{2n+1}), \ n \geq 0, \quad (2.41)$$

where for $n \geq 0$

$$t_{2n} = -\frac{2n+(1+a_2)t_0}{4n-1-a_2}, \ t_{2n+1} = -2\frac{n+1}{4n+1-a_2}. \quad (2.42)$$

Proof Using (2.40), we get

$$(4n+1-a_2)t_{2n+1} - (4n-3-a_2)t_{2n-1} = -2, \ n \geq 0,$$
$$(4n+3-a_2)t_{2n+2} - (4n-1-a_2)t_{2n} = -2, \ n \geq 0.$$

So, we obtain (2.41) and (2.42). □

2.3.3 The case $\Phi(1) = 0$ and $\Psi(1) = 0$

In this case, thanks to Eq. (2.22), the system (2.27)–(2.28) becomes for $n \geq 0$

$$2c_3 \sum_{\nu=0}^{n-1} t_\nu(1 + t_{\nu+1}) + [(2n - 3)c_3 - a_2]t_{n-1}(1 + t_n) - nc_1 = 0, \tag{2.43}$$

$$\begin{aligned}&-2c_3 \sum_{\nu=0}^{n-1} t_\nu(1 + t_{\nu+1}) - t_n(1 + t_n)\{[(2n + 1)c_3 - a_2](1 + t_{n+1}) - [(2n - 3)c_3 \\ &-a_2]t_{n-1}\} + c_1(1 + t_n) + nc_1 = 0.\end{aligned} \tag{2.44}$$

Adding identities (2.43) and (2.44), we obtain for $n \geq 0$

$$[(2n + 1)c_3 - a_2]t_n(1 + t_{n+1}) - [(2n - 3)c_3 - a_2]t_{n-1}(1 + t_n) = c_1. \tag{2.45}$$

Proceeding as in (2.32), we can easily prove the next result.

Proposition 2.10 *We have*

$$t_n(1 + t_{n+1}) = \frac{(n + 1)[(n - 1)c_3 - a_2]c_1}{[(2n + 1)c_3 - a_2][(2n - 1)c_3 - a_2]}, \; n \geq 0. \tag{2.46}$$

Corollary 2.11 *The sequences* $\{\beta_n\}_{n\geq 0}$ *and* $\{\gamma_{n+1}\}_{n\geq 0}$ *are defined by*

$$\begin{aligned}\beta_0 = -\, t_0, \; \beta_{n+1} = t_n - t_{n+1}, \; n \geq 0, \\ \gamma_{n+1} = -t_n(1 + t_n), \; n \geq 0,\end{aligned} \tag{2.47}$$

where for $n \geq 0$

$$t_{2n} = \begin{cases} -(2n+1)\dfrac{\sum_{k=0}^{n} \frac{(-1)^k(1-(t_0+1)\Lambda_{n-k})}{(2a_2)^k\Gamma(k+1)\Gamma(2n+2-2k)}}{\sum_{k=0}^{n} \frac{(-1)^k(1-(t_0+1)\Lambda_{n-1-k})}{(2a_2)^k\Gamma(k+1)\Gamma(2n+1-2k)}}, & \textit{if}(c_3, c_1) = (0, 1), \\ -\dfrac{(4n+2)}{4n-a_2-1}\dfrac{\sum_{k=0}^{n} \frac{\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})(1-(1+t_0)\Omega_{n-k})c_1^k}{2^{2k}\Gamma(k+1)\Gamma(2n+2-2k)}}{\sum_{k=0}^{n} \frac{\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})(1-(1+t_0)\Omega_{n-1-k})c_1^k}{2^{2k}\Gamma(k+1)\Gamma(2n+1-2k)}}, & \textit{if}\, c_3 = 1 \textit{ and } c_1 \neq 0, \end{cases} \tag{2.48}$$

$$t_{2n+1} = \begin{cases} -(2n+2)\dfrac{\sum_{k=0}^{n+1} \frac{(-1)^k(1-(t_0+1)\Lambda_{n-k})}{(2a_2)^k\Gamma(k+1)\Gamma(2n+3-2k)}}{\sum_{k=0}^{n} \frac{(-1)^k(1-(t_0+1)\Lambda_{n-k})}{(2a_2)^k\Gamma(k+1)\Gamma(2n+2-2k)}}, & \textit{if}(c_3, c_1) = (0, 1), \\ -\dfrac{(4n+4)}{4n+1-a_2}\dfrac{\sum_{k=0}^{n+1} \frac{\Gamma(2n-k-\frac{a_2}{2}+\frac{3}{2})(1-(1+t_0)\Omega_{n-k})c_1^k}{2^{2k}\Gamma(k+1)\Gamma(2n+3-2k)}}{\sum_{k=0}^{n} \frac{\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})(1-(1+t_0)\Omega_{n-k})c_1^k}{2^{2k}\Gamma(k+1)\Gamma(2n+2-2k)}}, & \textit{if}\, c_3 = 1 \textit{ and } c_1 \neq 0, \end{cases} \tag{2.49}$$

with

$$\Lambda_n = \sum_{\nu=0}^{n} \frac{\Gamma(2\nu + 1)}{(2a_2)^\nu\Gamma(\nu + 1)}, \; n \geq 0, \; \Lambda_{-1} = 0, \tag{2.50}$$

$$\Omega_n = \frac{\Gamma(\frac{1}{2} - \frac{a_2}{2})}{\sqrt{\pi}} \sum_{\nu=0}^{n} (-1)^\nu \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu - \frac{a_2}{2} + \frac{1}{2})} c_1^\nu, \; n \geq 0, \; \Omega_{-1} = 0. \tag{2.51}$$

For the proof, we need the following lemma.

Lemma 2.12 *For $n \geq 1$ and $a \in \mathbb{C} - \mathbb{Z}_{-}^{*}$, we have*

$$\sum_{k=0}^{n}(-1)^k \frac{\Gamma(a+n+k)}{\Gamma(k+1)\Gamma(n-k+1)\Gamma(a+k+1)} = 0.$$

Proof Writing $1^{(n)} = (x^{n+a}x^{-a-n})^{(n)}$ and applying the Leibniz formula, we get

$$\begin{aligned} 0 &= \sum_{k=0}^{n} \frac{\Gamma(n+1)(x^{a+n})^{(n-k)}(x^{-a-n})^{(k)}}{\Gamma(k+1)\Gamma(n-k+1)} \\ &= \Gamma(n+1)\sum_{k=0}^{n}(-1)^k \frac{\Gamma(a+n+k)}{\Gamma(k+1)\Gamma(n-k+1)\Gamma(a+k+1)} x^{-n}. \end{aligned}$$

Putting $x = 1$, we obtain the desired result. □

Proof of Corollary 2.11 Firstly, we prove (2.48) by induction on n.

It's clear that (2.48) is true for $n = 0$.

We suppose that it is true until n and prove that it is true for $n+1$. From (2.46), we have for $n \geq 0$

$$\begin{aligned} t_{2n+2} = &\Big\{[(4n-1)c_3 - a_2]\Big(2(n+1)(2nc_3 - a_2)c_1 + [(4n+3)c_3 - a_2][(4n+1)c_3 - a_2]\Big)t_{2n} \\ &- (2n+1)[(2n-1)c_3 - a_2][(4n+3)c_3 - a_2]c_1\Big\} \\ &\Big\{[(4n+3)c_3 - a_2]\Big((2n+1)[(2n-1)c_3 - a_2]c_1 \\ &- [(4n-1)c_3 - a_2][(4n+1)c_3 - a_2]t_{2n}\Big)\Big\}^{-1}. \end{aligned} \tag{2.52}$$

The following two cases arise:

- If $(c_3, c_1) = (0, 1)$, then (2.52) reduce to

$$t_{2n+2} = -\frac{2n+1-(2n+2-a_2)t_{2n}}{2n+1+a_2 t_{2n}}. \tag{2.53}$$

Substitution of t_{2n} from (2.48) into (2.53) gives

$$\begin{aligned} t_{2n+2} \; &= -\Bigg(\sum_{k=0}^{n} \frac{(-1)^k\big(1-(t_0+1)\Lambda_{n-1-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+1-2k)} \\ &\quad + (2n+2-a_2)\sum_{k=0}^{n} \frac{(-1)^k\big(1-(t_0+1)\Lambda_{n-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+2-2k)}\Bigg) \\ &\quad \times \Bigg(\sum_{k=0}^{n} \frac{(-1)^k\big(1-(t_0+1)\Lambda_{n-1-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+1-2k)} \\ &\quad - a_2\sum_{k=0}^{n} \frac{(-1)^k\big(1-(t_0+1)\Lambda_{n-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+2-2k)}\Bigg)^{-1} \\ &= -\Bigg(2\sum_{k=0}^{n+1} \frac{(-1)^k k(2n+3-2k)\big(1-(t_0+1)\Lambda_{n-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+4-2k)} \end{aligned}$$

$$+ 4(n+1)\sum_{k=0}^{n+1}\frac{k(-1)^k\big(1-(t_0+1)\Lambda_{n+1-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+4-2k)}$$
$$+\sum_{k=0}^{n}\frac{(-1)^k(2n+2-2k)(2n+3-2k)\big(1-(t_0+1)\Lambda_{n-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+4-2k)}\Bigg)$$
$$\times\Bigg(2\sum_{k=0}^{n+1}\frac{k(-1)^k\big(1-(t_0+1)\Lambda_{n-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+3-2k)}$$
$$+\sum_{k=0}^{n}\frac{(-1)^k(2n+2-2k)\big(1-(t_0+1)\Lambda_{n-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+3-2k)}\Bigg)^{-1}.$$

But, from (2.50), we have

$$\Lambda_{n+1-k}=\Lambda_{n-k}+\frac{\Gamma(2n+3-2k)}{(2a_2)^{n+1-k}\Gamma(n+2-k)}.$$

Therefore,

$$t_{2n+2} = -\Bigg(2\sum_{k=0}^{n+1}\frac{(-1)^k k(2n+3-2k)\big(1-(t_0+1)\Lambda_{n+1-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+4-2k)}$$
$$+4(n+1)\sum_{k=0}^{n+1}\frac{k(-1)^k\big(1-(t_0+1)\Lambda_{n+1-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+4-2k)}$$
$$+\sum_{k=0}^{n}\frac{(-1)^k(2n+2-2k)(2n+3-2k)\big(1-(t_0+1)\Lambda_{n+1-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+4-2k)}$$
$$+\frac{2(t_0+1)}{(2a_2)^{n+1}}\Bigg(\sum_{k=0}^{n+1}\frac{k(-1)^k}{\Gamma(k+1)\Gamma(n+2-k)}+\sum_{k=0}^{n}\frac{(-1)^k}{\Gamma(k+1)\Gamma(n+1-k)}\Bigg)\Bigg)$$
$$\times\Bigg(2\sum_{k=0}^{n+1}\frac{k(-1)^k\big(1-(t_0+1)\Lambda_{n-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+3-2k)}$$
$$+\sum_{k=0}^{n}\frac{(-1)^k(2n+2-2k)\big(1-(t_0+1)\Lambda_{n-k}\big)}{(2a_2)^k\Gamma(k+1)\Gamma(2n+3-2k)}\Bigg)^{-1}.$$

Then, we can deduce the result (2.48) when $(c_3,c_1)=(0,1)$ since

$$2k(2n+3-2k)+4k(n+1)+(2n+2-2k)(2n+3-2k)=(2n+2)(2n+3),$$
$$\sum_{k=0}^{n+1}\frac{k(-1)^k}{\Gamma(k+1)\Gamma(n+2-k)}+\sum_{k=0}^{n}\frac{(-1)^k}{\Gamma(k+1)\Gamma(n+1-k)}=0.$$

- If $c_3=1$ with $c_1\neq 0$, then (2.52) becomes

$$t_{2n+2} = \Big\{(4n-1-a_2)\Big(2(n+1)(2n-a_2)c_1+(4n+3-a_2)(4n+1-a_2)\Big)t_{2n}$$
$$-(2n+1)(2n-1-a_2)(4n+3-a_2)c_1\Big\}$$
$$\Big\{(4n+3-a_2)\Big((2n+1)(2n-1-a_2)c_1$$

$$- (4n-1-a_2)(4n+1-a_2)t_{2n}\Big)\Big\}^{-1}. \tag{2.54}$$

Substitution of t_{2n} from (2.48) into (2.54) gives

$$\begin{aligned}
t_{2n+2} &= -\frac{1}{4n+3-a_2}\Bigg((2n-1-a_2)(4n+3-a_2)\\
&\sum_{k=0}^{n}\frac{\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})\big(1-(1+t_0)\Omega_{n-1-k}\big)c_1^{k+1}}{2^{2k}\Gamma(k+1)\Gamma(2n+1-2k)}\\
&\quad+4(n+1)(2n-a_2)\sum_{k=0}^{n}\frac{\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\big(1-(1+t_0)\Omega_{n-k}\big)c_1^{k+1}}{2^{2k}\Gamma(k+1)\Gamma(2n+2-2k)}\\
&\quad+2(4n+1-a_2)(4n+3-a_2)\sum_{k=0}^{n}\frac{\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\big(1-(1+t_0)\Omega_{n-k}\big)c_1^{k}}{2^{2k}\Gamma(k+1)\Gamma(2n+2-2k)}\Bigg)\\
&\quad\times\Bigg((2n-1-a_2)\sum_{k=0}^{n}\frac{\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})\big(1-(1+t_0)\Omega_{n-1-k}\big)c_1^{k+1}}{2^{2k}\Gamma(k+1)\Gamma(2n+1-2k)}\\
&\quad+2(4n+1-a_2)\sum_{k=0}^{n}\frac{\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\big(1-(1+t_0)\Omega_{n-k}\big)c_1^{k}}{2^{2k}\Gamma(k+1)\Gamma(2n+2-2k)}\Bigg)^{-1}\\
&= -\frac{1}{4n+3-a_2}\Bigg((2n-1-a_2)(4n+3-a_2)\\
&\sum_{k=0}^{n+1}\frac{4k(2n+3-2k)\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\big(1-(1+t_0)\Omega_{n-k}\big)c_1^{k}}{2^{2k}\Gamma(k+1)\Gamma(2n+4-2k)}\\
&\quad+2(4n+1-a_2)(4n+3-a_2)\\
&\sum_{k=0}^{n}\frac{(2n+2-2k)(2n+3-2k)\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\big(1-(1+t_0)\Omega_{n-k}\big)c_1^{k}}{2^{2k}\Gamma(k+1)\Gamma(2n+4-2k)}\\
&\quad+4(n+1)(2n-a_2)\sum_{k=0}^{n+1}\frac{4k\Gamma(2n-k-\frac{a_2}{2}+\frac{3}{2})\big(1-(1+t_0)\Omega_{n+1-k}\big)c_1^{k}}{2^{2k}\Gamma(k+1)\Gamma(2n+4-2k)}\Bigg)\\
&\quad\times\Bigg((2n-1-a_2)\sum_{k=0}^{n+1}\frac{4k\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\big(1-(1+t_0)\Omega_{n-k}\big)c_1^{k}}{2^{2k}\Gamma(k+1)\Gamma(2n+3-2k)}\\
&\quad+2(4n+1-a_2)\sum_{k=0}^{n}\frac{(2n+2-2k)\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\big(1-(1+t_0)\Omega_{n-k}\big)c_1^{k}}{2^{2k}\Gamma(k+1)\Gamma(2n+3-2k)}\Bigg)^{-1}.
\end{aligned}$$

From (2.51) we have

$$\Omega_{n+1-k} = \Omega_{n-k} + \frac{(-1)^{n+1-k}\Gamma(\frac{1}{2}-\frac{a_2}{2})\Gamma(n+\frac{3}{2}-k)c_1^{n+1-k}}{\sqrt{\pi}\,\Gamma(n+\frac{3}{2}-k-\frac{a_2}{2})}.$$

Here

$$\begin{aligned}
t_{2n+2} &= -\frac{1}{4n+3-a_2}\Bigg(4(2n-1-a_2)(4n+3-a_2)\\
&\sum_{k=0}^{n+1}\frac{k(2n+3-2k)\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\big(1-(1+t_0)\Omega_{n+1-k}\big)c_1^{k}}{2^{2k}\Gamma(k+1)\Gamma(2n+4-2k)}
\end{aligned}$$

$$
\begin{aligned}
&+2(4n+1-a_2)(4n+3-a_2)\\
&\sum_{k=0}^{n} \frac{(2n+2-2k)(2n+3-2k)\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\big(1-(1+t_0)\Omega_{n+1-k}\big)c_1^k}{2^{2k}\Gamma(k+1)\Gamma(2n+4-2k)}\\
&+16(n+1)(2n-a_2)\sum_{k=0}^{n+1} \frac{k\Gamma(2n-k-\frac{a_2}{2}+\frac{3}{2})\big(1-(1+t_0)\Omega_{n+1-k}\big)c_1^k}{2^{2k}\Gamma(k+1)\Gamma(2n+4-2k)}\\
&+2(1+t_0)(-c_1)^{n+1}(4n+3-a_2)\frac{\Gamma(\frac{1}{2}-\frac{a_2}{2})}{\sqrt{\pi}}\Big(4(2n-1-a_2)\\
&\sum_{k=0}^{n+1} \frac{k(-1)^k\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\Gamma(n+\frac{5}{2}-k)}{2^{2k}\Gamma(n+\frac{3}{2}-k-\frac{a_2}{2})\Gamma(k+1)\Gamma(2n+4-2k)}\\
&+2(4n+1-a_2)\sum_{k=0}^{n} \frac{(-1)^k(2n+2-2k)\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\Gamma(n+\frac{5}{2}-k)}{2^{2k}\Gamma(n+\frac{3}{2}-k-\frac{a_2}{2})\Gamma(k+1)\Gamma(2n+4-2k)}\Big)\Big)\\
&\times\Big(4(2n-1-a_2)\sum_{k=0}^{n+1} \frac{k\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\big(1-(1+t_0)\Omega_{n-1-k}\big)c_1^k}{2^{2k}\Gamma(k+1)\Gamma(2n+3-2k)}\\
&+2(4n+1-a_2)\sum_{k=0}^{n} \frac{(2n+2-2k)\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\big(1-(1+t_0)\Omega_{n-k}\big)c_1^k}{2^{2k}\Gamma(k+1)\Gamma(2n+3-2k)}\Big)^{-1}.
\end{aligned}
$$

Thus, we can deduce (2.48), because

$$
\begin{aligned}
&k(2n-1-a_2)+2(4n+1-a_2)(2n+2-2k)=8(n+1)(2n-k-\frac{a_2}{2}+\frac{1}{2}),\\
&4k(2n+3-2k)(2n-1-a_2)(4n+3-a_2)+16k(n+1)(2n-a_2)(2n-k-\frac{a_2}{2}+\frac{1}{2})\\
&\quad+2(4n+1-a_2)(4n+3-a_2)(2n+2-2k)(2n+3-2k)\\
&=8(n+1)(4n+6)(2n-k-\frac{a_2}{2}+\frac{1}{2})(2n-k-\frac{a_2}{2}+\frac{3}{2}),
\end{aligned}
$$

and, taking into account Lemma 2.12, we have

$$
\begin{aligned}
&4(2n-1-a_2)\sum_{k=0}^{n+1} \frac{k(-1)^k\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\Gamma(n+\frac{5}{2}-k)}{2^{2k}\Gamma(n+\frac{3}{2}-k-\frac{a_2}{2})\Gamma(k+1)\Gamma(2n+4-2k)}\\
&\quad+2(4n+1-a_2)\sum_{k=0}^{n} \frac{(-1)^k(2n+2-2k)\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})\Gamma(n+\frac{5}{2}-k)}{2^{2k}\Gamma(n+\frac{3}{2}-k-\frac{a_2}{2})\Gamma(k+1)\Gamma(2n+4-2k)}\\
&=-2^{2n+1}\sqrt{\pi}\Big((2n-1-a_2)\sum_{k=0}^{n} \frac{(-1)^k\Gamma(2n-k-\frac{a_2}{2}-\frac{1}{2})}{\Gamma(n+\frac{1}{2}-k-\frac{a_2}{2})\Gamma(k+1)\Gamma(n+1-k)}\\
&\quad-4(4n+1-a_2)\sum_{k=0}^{n} \frac{(-1)^k\Gamma(2n-k-\frac{a_2}{2}+\frac{1}{2})}{\Gamma(n+\frac{3}{2}-k-\frac{a_2}{2})\Gamma(k+1)\Gamma(n+1-k)}\Big)\\
&=-2^{2n+1}\sqrt{\pi}(-1)^n\Big((2n-1-a_2)\sum_{k=0}^{n} \frac{(-1)^k\Gamma(n+k-\frac{a_2}{2}-\frac{1}{2})}{\Gamma(k+\frac{1}{2}-\frac{a_2}{2})\Gamma(k+1)\Gamma(n+1-k)}\\
&\quad-4(4n+1-a_2)\sum_{k=0}^{n} \frac{(-1)^k\Gamma(n+k-\frac{a_2}{2}+\frac{1}{2})}{\Gamma(k+\frac{3}{2}-\frac{a_2}{2})\Gamma(k+1)\Gamma(n+1-k)}\Big)=0.
\end{aligned}
$$

Finally, from (2.48) and (2.46), we can deduce (2.49) after some straightforward calculation. □

2.4 The canonical cases

2.4.1 The case $\Phi(1) \neq 0$

Theorem 2.13 *The following canonical cases arise:*

(a) When $\Phi(x) = x + 1$, *we have*

$$\begin{cases} \beta_0 = -t_0, \ \beta_{n+1} = t_n - t_{n+1}, \ n \geq 0, \\ \gamma_{n+1} = -t_n(1 + t_n), \ n \geq 0, \\ \big((x+1)v\big)' + \big(2\lambda x(x+1) - 2\big)v = 0, \end{cases} \tag{2.55}$$

where

$$t_{2n} = -\frac{a_n}{2\lambda b_n}, \quad t_{2n+1} = -\frac{b_n}{2\lambda a_{n+1}}, \quad n \geq 0, \tag{2.56}$$

with, for $n \geq 0$,

$$\begin{aligned} a_n &= \sum_{k=0}^{n} \frac{(-1)^k}{(4\lambda)^k \Gamma(k+1)\Gamma(2n+1-2k)}, \\ b_n &= \sum_{k=0}^{n} \frac{(-1)^k}{(4\lambda)^k \Gamma(k+1)\Gamma(2n+2-2k)}. \end{aligned} \tag{2.57}$$

(b) When $\Phi(x) = (x+1)(x^2 + c_1)$, $c_1(c_1 + 1) \neq 0$, *we have the following canonical case:*

$$\begin{cases} \beta_0 = -t_0, \ \beta_{n+1} = t_n - t_{n+1}, \ n \geq 0, \\ \gamma_{n+1} = -t_n(1 + t_n), \ n \geq 0, \\ \big((x+1)\big(x^2 - \mu^2\big)v\big)' + \big(-2(\alpha+2)x^2 - 2(\alpha+1)x + 2\mu^2\big)v = 0, \\ (\mu^2 - 1)\{|\alpha\mu| + |\mu + 2\alpha + 1|\}\{|\alpha\mu| + |2\alpha + 1 - \mu|\} \neq 0, \end{cases} \tag{2.58}$$

where

$$t_{2n} = -\frac{(2n+2\alpha+1)\mu^2 c_n(\alpha)}{2(4n+2\alpha+3)d_n(\alpha)}, \quad t_{2n+1} = -\frac{(2n+2\alpha+2)\mu^2 d_n(\alpha)}{2(4n+2\alpha+5)c_{n+1}(\alpha)}, \quad n \geq 0 \tag{2.59}$$

with for $n \geq 0$

$$\begin{cases} c_n(\alpha) = \sum_{k=0}^{n} \frac{(-1)^k \mu^{2k} \Gamma(2n-k+\alpha+\frac{1}{2})}{2^{2k}\Gamma(2n+1-2k)\Gamma(k+1)}, \\ d_n(\alpha) = \sum_{k=0}^{n} \frac{(-1)^k \mu^{2k} \Gamma(2n-k+\alpha+\frac{3}{2})}{2^{2k}\Gamma(2n+2-2k)\Gamma(k+1)}. \end{cases} \tag{2.60}$$

Proof From Theorem 2.6, we have the following:

If $\deg(\Phi) = 1$, denoting $a_2 = 2\lambda$. Then, from (2.20) and (2.34)–(2.35) we get (2.55)–(2.57).

If $\deg(\Phi) = 3$, putting $c_1 = -\mu^2$ and $a_2 = -2(\alpha+2)$. Hence, from (2.20) and (2.34)–(2.35) we obtain (2.58)–(2.60). □

Remark 2.14 (i) Multiplying (2.55) by $(x-1)^2$ and on account of (1), we obtain

$$\Big((x^2-1)(x-1)v\Big)' + 2x\Big(\lambda x^2-(\lambda+\beta+1)x\Big)(x-1)v = 0. \tag{2.61}$$

Then we have

$$v = \frac{1-2\lambda}{2\lambda}(x-1)^{-1}u(\beta,\lambda) + \delta_1, \tag{2.62}$$

where $u(\beta,\lambda)$ is the symmetric semi-classical form of class two satisfying [5]:

$$\begin{cases} \big((x^2-1)u(\beta,\lambda)\big)' + 2x\big(\lambda x^2-(\lambda+\beta+1)x\big)u(\beta,\lambda) = 0, \\ |\beta| + |2\lambda\big(u(\beta,\lambda)\big)_2 - 4\beta - 1| \neq 0. \end{cases}$$

In this case, the form v is called the Geronimus transformation of the form $u(\beta,\lambda)$ (see [6, 13]). The form $u(\beta,\lambda)$ has the following integral representation [5]:

$$\begin{aligned} \langle u(\beta,\lambda), f\rangle \; &= \int_{-\infty}^{+\infty} f(x)|x^2-1|^{\beta} e^{-\lambda x^2} \\ &\Big(A + (B-A)\chi_{[-1,1]}(x)\Big)\,dx,\ \lambda>0,\ \Re(\beta)>-1,\ f\in\mathcal{P}, \end{aligned} \tag{2.63}$$

where $\chi_{[a,b]}(x) = 1$ when $x \in [a,b]$ and zero otherwise (A and B will be chosen in such a way that the form is normalized), or

$$\langle u(\beta,\lambda), f\rangle = A_1 \int_{-1}^{1} (1-x^2)^{\beta} e^{-\lambda x^2} f(x)\,dx,\ \lambda\in\mathbb{R},\ \Re(\beta)>-1,\ f\in\mathcal{P}. \tag{2.64}$$

Here, from (2.62) and (2.63)–(2.64), we have for $\Re(\beta) > -1$,

$$\begin{aligned} \langle v, f\rangle \; &= \frac{1-2\lambda}{2\lambda}\int_{-\infty}^{+\infty} sgn(x-1)|x+1||x^2-1|^{\beta-1}e^{-\lambda x^2}\big(f(x) \\ &-f(1)\big)\Big(A+(B-A)\chi_{[-1,1]}(x)\Big)\,dx + f(1),\ \lambda>0, \end{aligned} \tag{2.65}$$

or

$$\begin{aligned} \langle v, f\rangle \; &= \frac{2\lambda-1}{2\lambda}A_1\int_{-1}^{1}(x+1)(1-x^2)^{\beta-1}e^{-\lambda x^2} \\ &f(x)\,dx + f(1)\big(1-\frac{2\lambda-1}{2\lambda}A_1\big),\ \lambda\in\mathbb{R}-\{0\}. \end{aligned} \tag{2.66}$$

(ii) Proceeding as in (2.61), we can deduce from (2.58)

$$\Big((x^2-1)(x^2-\mu^2)(x-1)v\Big)' + 2x\Big(-(\alpha+3)x^2+\alpha+2\mu^2+1\Big)(x-1)v = 0.$$

Hence,

$$v = \frac{\mu^2-2\alpha-3}{2\alpha+3}(x-1)^{-1}u(\alpha,1) + \delta_1, \tag{2.67}$$

where $u(\alpha,\beta)$ is the symmetric semi-classical form of class two satisfying ([5])

$$\begin{cases} \Big((x^2-1)(x^2-\mu^2)u(\alpha,\beta)\Big)' + 2x\Big(-(\alpha+\beta+1)x^2+\alpha+(\beta+1)\mu^2+1\Big)u(\alpha,\beta) = 0, \\ |\beta| + |-(2\alpha+2\beta+3)\big(u(\alpha,\beta)\big)_2 + \mu^2(2\beta+1)| \neq 0, \\ |\alpha| + |-(2\alpha+2\beta+3)\big(u(\alpha,\beta)\big)_2 + 2\alpha(1-\mu^2)+1| \neq 0. \end{cases}$$

From (2.67), we can deduce that the form v is a Geronimus transformation of the form $u(\alpha,1)$. The form $u(\alpha,\beta)$ has the following integral representation (see [5]):

$$\langle u(\alpha,\beta), f\rangle = \int_{-1}^{1}(1-x^2)^{\beta}|x^2-\mu^2|^{\alpha} f(x)\Big(A+(B-A)\chi_{[-\mu,\mu]}(x)\Big)\,dx,\ \ \mu\in]0,1[,\\ \Re(\alpha)>-1,\ \Re(\beta)>-1,\ f\in\mathcal{P}, \tag{2.68}$$

or

$$\langle u(\alpha,\beta), f\rangle = \int_{-\mu}^{\mu}|x^2-1|^{\beta}(\mu^2-x^2)^{\alpha} f(x)\big(A_1+(B_1-A_1)\chi_{[-1,1]}(x)\big)\,dx,\ \ \mu>1,\\ \Re(\alpha)>-1,\ \Re(\beta)>-1,\ f\in\mathcal{P}. \tag{2.69}$$

Therefore, using (2.67) and (2.68)–(2.69), we get for $\Re(\alpha)>-1$, $\Re(\beta)>-1$,

$$\begin{aligned}\langle v, f\rangle &= -\frac{\mu^2-2\alpha-3}{2\alpha+3}\int_{-1}^{1}(x+1)(1-x^2)^{\beta-1}|x^2\\ &\quad -\mu^2|^{\alpha}\big(f(x)-f(1)\big)\big(A+(B-A)\chi_{[-\mu,\mu]}(x)\big)\,dx+f(1),\ \ \mu\in\,]0,1[\,,\end{aligned} \tag{2.70}$$

or

$$\langle v, f\rangle = \tfrac{\mu^2-2\alpha-3}{2\alpha+3}\int_{-\mu}^{\mu}\operatorname{sgn}(x-1)|x+1||x^2-1|^{\beta-1}(\mu^2-x^2)^{\alpha}\big(f(x)-f(1)\big)\\ \times\big(A_1+(B_1-A_1)\chi_{[-1,1]}(x)\big)\,dx+f(1),\ \ \mu>1. \tag{2.71}$$

2.4.2 The case $\Phi(1)=0$ and $\Psi(1)\neq 0$

Theorem 2.15 *The following canonical case arises:*

$$\begin{cases}\beta_0=-t_0,\ \ \beta_{n+1}=t_n-t_{n+1},\ \ n\geq 0,\\ \gamma_{n+1}=-t_n(1+t_n),\ \ n\geq 0,\\ \big(x(x^2-1)v\big)'+\big(-(2\alpha+2\beta+3)x^2+x+2(\beta+1)\big)v=0,\\ \alpha(2\beta+1)\{|\alpha+2\beta+1|+|\alpha-1|\}\neq 0,\end{cases} \tag{2.72}$$

with

$$t_{2n}=-\frac{n+\beta+1}{2n+\alpha+\beta+1},\ \ t_{2n+1}=-\frac{n+1}{2n+\alpha+\beta+2}. \tag{2.73}$$

Proof If we choose $a_2+a_0=-(2\alpha+1)$ and $a_2-a_0=-(2\alpha+4\beta+5)$, we get from the Theorem 2.6 and (2.42) the desired result (2.72) and (2.73). □

Remark 2.16 Multiplying (2.72) by $x-1$ and on account of (1), we obtain

$$\big(x(x^2-1)(x-1)v\big)'+\big(-2(\alpha+\beta+2)x^2+2(\beta+1)\big)(x-1)v=0.$$

Then,

$$v=-\frac{\alpha}{\alpha+\beta+1}(x-1)^{-1}\mathcal{G}.\mathcal{G}(\alpha,\beta)+\delta_1, \tag{2.74}$$

where $\mathcal{G}.\mathcal{G}(\alpha,\beta)$ is the Generalized Gegenbauer form satisfying (see [3])

$$\begin{cases}\big(x(x^2-1)\mathcal{G}.\mathcal{G}(\alpha,\beta)\big)'+\big(-2(\alpha+\beta+2)x^2+2(\beta+1)\big)\mathcal{G}.\mathcal{G}(\alpha,\beta)=0,\\ (2\beta+1)\{|\alpha|+|\alpha+\beta+1|\}\neq 0.\end{cases}$$

In this case, the form v is a Geronimus transformation of the Generalized Gegenbauer form. This form $\mathcal{G}.\mathcal{G}(\alpha, \beta)$ has the following integral representation (see [3]):

$$\langle \mathcal{G}.\mathcal{G}(\alpha, \beta), f \rangle = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} |x|^{2\beta+1}(1-x^2)^{\alpha} f(x)\, dx,\ \Re(\alpha) > -1,\ \Re(\beta) > -1,\ f \in \mathcal{P}. \tag{2.75}$$

Then, from (2.74) and (2.75), we get for $\Re(\alpha) > 0$, $\Re(\beta) > -1$ and $f \in \mathcal{P}$,

$$\langle v, f \rangle = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} \int_{-1}^{1} |x|^{2\beta+1}(1+x)^{\alpha}(1-x)^{\alpha-1}\big(f(x) - f(1)\big)\, dx + f(1). \tag{2.76}$$

2.4.3 The case $(\Phi(1), \Psi(1)) = (0, 0)$

Theorem 2.17 *The following two canonical cases arise:*

(a) When $\Phi(x) = x - 1$, *we have*

$$\begin{cases} \beta_0 = -t_0,\ \beta_{n+1} = t_n - t_{n+1},\ n \geq 0, \\ \gamma_{n+1} = -t_n(1+t_n),\ n \geq 0, \\ \big((x-1)v\big)' + 2\mu^2 x(x-1)v = 0, \end{cases} \tag{2.77}$$

where

$$t_{2n} = -(2n+1)\frac{e_n}{f_n}, \quad t_{2n+1} = -(2n+2)\frac{f_{n+1}}{e_n}, \quad n \geq 0, \tag{2.78}$$

with

$$\begin{cases} e_n = \sum_{k=0}^{n} \frac{(-1)^k\left(1+\lambda\Lambda_{n-k}\right)}{(2\mu)^{2k}\Gamma(k+1)\Gamma(2n+2-2k)}, \quad n \geq 0, \\ f_n = \sum_{k=0}^{n} \frac{(-1)^k\left(1+\lambda\Lambda_{n-1-k}\right)}{(2\mu)^{2k}\Gamma(k+1)\Gamma(2n+1-2k)}, \quad n \geq 0, \\ \Lambda_n = \sum_{\nu=0}^{n} \frac{\Gamma(2\nu+1)}{(2\mu)^{2\nu}\Gamma(\nu+1)},\ n \geq 0,\ \Lambda_{-1} = 0. \end{cases} \tag{2.79}$$

(b) When $\Phi(x) = (x-1)(x^2+c_1)$, $c_1 \neq 0$, *we have the following canonical case:*

$$\begin{cases} \beta_0 = -t_0,\ \beta_{n+1} = t_n - t_{n+1},\ n \geq 0, \\ \gamma_{n+1} = -t_n(1+t_n),\ n \geq 0, \\ \big((x-1)(x^2-\mu^2)v\big)' - 2(\alpha+1)x(x-1)v = 0, \\ \{|\mu^2-1| + |(2\alpha+1)(\lambda+1)-1|\}\{|\alpha(\mu-1)| + |2\alpha(\lambda+\mu)+\lambda|\} \neq 0, \\ \mu\{|\alpha(\mu+1)| + |2\alpha(\lambda-\mu)+\lambda|\} \neq 0, \end{cases} \tag{2.80}$$

where

$$t_{2n} = -\frac{(4n+2)g_n(\alpha)}{(4n+2\alpha+1)h_n(\alpha)}, \quad t_{2n+1} = -\frac{(4n+4)h_{n+1}(\alpha)}{(4n+2\alpha+3)g_n(\alpha)}, \quad n \geq 0, \tag{2.81}$$

with

$$\begin{cases} g_n(\alpha) = \sum_{k=0}^{n} \frac{(-1)^k \mu^{2k} \Gamma(2n-k+\alpha+\frac{3}{2})\left(1+\lambda\Omega_{n-k}\right)}{2^{2k}\Gamma(k+1)\Gamma(2n+2-2k)}, \quad n \geq 0, \\ h_n(\alpha) = \sum_{k=0}^{n} \frac{\Gamma(2n-k+\alpha+\frac{1}{2})\left(1+\lambda\Omega_{n-1-k}\right)(-1)^k \mu^{2k}}{2^{2k}\Gamma(k+1)\Gamma(2n+1-2k)}, \quad n \geq 0, \\ \Omega_n = \frac{\Gamma(\alpha+\frac{3}{2})}{\sqrt{\pi}} \sum_{\nu=0}^{n} \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+\alpha+\frac{3}{2})} \mu^{2\nu}, \ n \geq 0, \ \Omega_{-1} = 0. \end{cases} \tag{2.82}$$

Proof From the Theorem 2.6, we have the following:

If $\deg(\Phi) = 1$, denoting $a_2 = 2\mu^2$ and $t_0 = -\lambda - 1$. Then, from (2.22), (2.48)–(2.49) and (2.50)–(2.51) we get (2.77)–(2.78).

If $\deg(\Phi) = 3$, putting $c_1 = -\mu^2, a_2 = -2(\alpha+1)$ and $t_0 = -\lambda - 1$. Hence, from (2.22), (2.48)–(2.49) and (2.50)–(2.51) we obtain (2.80)–(2.82). □

Remark 2.18 (i) From (2.77), we obtain

$$v = \lambda(x-1)^{-1}\left(h_{\mu^{-1}}\mathcal{H}\right) + \delta_1, \tag{2.83}$$

where $\mathcal{H}$ is the Hermite form satisfying (see [10]):

$$\left(\mathcal{H}\right)' + 2x\mathcal{H} = 0.$$

From (2.83), we can conclude that the form v is a Geronimus transformation of the Hermite form. This form $\mathcal{H}$ has the following integral representation [10]:

$$\langle \mathcal{H}, f \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} f(x)\, dx. \tag{2.84}$$

Thus, by (2.83) and (2.84), we have

$$\langle v, f \rangle = \frac{\operatorname{sgn}(\mu)\lambda}{\sqrt{\pi}} P \int_{-\infty}^{+\infty} \frac{e^{-\mu^2 x^2}}{x-1} \left(f(x) - f(1)\right) dx + f(1),$$

where

$$P \int_{-\infty}^{+\infty} \frac{V(x)}{x-1} f(x)\, dx = \lim_{\varepsilon \to 0} \left\{ \int_{-\infty}^{1-\varepsilon} \frac{V(x)}{x-1} f(x)\, dx + \int_{1+\varepsilon}^{+\infty} \frac{V(x)}{x-1} f(x)\, dx \right\},$$

and V is a locally integrable function with rapid decay and continuous at the point $x = 1$.

(ii) From (2.80), we get

$$v = \lambda(x-1)^{-1}\left(h_{\mu^{-1}}\mathcal{G}(\alpha)\right) + \delta_1, \tag{2.85}$$

where $\mathcal{G}(\alpha)$ is the Gegenbauer form satisfying [10]:

$$\left((x^2-1)\mathcal{G}(\alpha)\right)' - 2(\alpha+1)x\mathcal{G}(\alpha) = 0.$$

In this case, the linear functional v is a Geronimus transformation of the Gegenbauer form. This form $\mathcal{G}(\alpha)$ has the following integral representation [10]:

$$\langle \mathcal{G}(\alpha), f \rangle = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma^2(\alpha+1)} \int_{-1}^{1} (1-x^2)^{\alpha} f(x)\, dx, \ \Re(\alpha) > 0. \tag{2.86}$$

So, from (2.85) and (2.86), we obtain

$$\langle v, f \rangle = \frac{\lambda\Gamma(2\alpha+2)}{(2\mu)^{2\alpha+1}\Gamma^2(\alpha+1)} P \int_{-\mu}^{\mu} \frac{(\mu^2-x^2)^{\alpha}}{x-1} \left(f(x) - f(1)\right) dx$$

A special family of non-symmetric semi-classical forms of class one

Pearson equation	The coefficients of the TTRR	Integral representation
$\big((x+1)v\big)' + \big(2\lambda x(x+1) - 2\big)v = 0.$	$\beta_0 = -t_0,\ \beta_{n+1} = t_n - t_{n+1},\ n \geq 0,$ $\gamma_{n+1} = -t_n(1+t_n),\ n \geq 0,$ $t_{2n} = -\frac{a_n}{2\lambda b_n},\quad t_{2n+1} = -\frac{b_n}{2\lambda a_{n+1}},\quad n \geq 0,$ $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{(4\lambda)^k \Gamma(k+1)\Gamma(2n+1-2k)},\ n \geq 0,$ $b_n = \sum_{k=0}^{n} \frac{(-1)^k}{(4\lambda)^k \Gamma(k+1)\Gamma(2n+2-2k)},\ n \geq 0.$	$\langle v, f\rangle = \frac{1-2\lambda}{2\lambda} \int_{-\infty}^{+\infty} sgn(x-1)\|x+1\|$ $\|x^2-1\|^{\beta-1} e^{-\lambda x^2}\big(f(x) - f(1)\big)$ $\Big(A + (B-A)\chi_{[-1,1]}(x)\Big)$ $dx + f(1),\ \lambda > 0,\ \Re(\beta) > -1, or$ $\langle v, f\rangle = \frac{2\lambda-1}{2\lambda} A_1 \int_{-1}^{1} (x+1)(1-x^2)^{\beta-1} e^{-\lambda x^2}$ $f(x)\,dx + f(1)\big(1 - \frac{2\lambda-1}{2\lambda} A_1\big), \lambda \in \mathbb{R} - \{0\},\ \Re(\beta) > -1.$
$\big((x-1)v\big)'$ $+2\mu^2 x(x-1)v =$ $0.$	$\beta_0 = -t_0,\ \beta_{n+1} = t_n - t_{n+1},\ n \geq 0,$ $\gamma_{n+1} = -t_n(1+t_n),\ n \geq 0,$ $t_{2n} = -(2n+1)\frac{e_n}{f_n},$ $t_{2n+1} = -(2n+2)\frac{f_{n+1}}{e_n},\quad n \geq 0,$ $e_n = \sum_{k=0}^{n} \frac{(-1)^k\big(1+\lambda\Lambda_{n-k}\big)}{(2\mu)^{2k}\Gamma(k+1)\Gamma(2n+2-2k)},\quad n \geq 0,$ $f_n = \sum_{k=0}^{n} \frac{(-1)^k\big(1+\lambda\Lambda_{n-1-k}\big)}{(2\mu)^{2k}\Gamma(k+1)\Gamma(2n+1-2k)},,\quad n \geq 0,$ $\Lambda_n = \sum_{\nu=0}^{n} \frac{\Gamma(2\nu+1)}{(2\mu)^{2\nu}\Gamma(\nu+1)},\ n \geq 0,\ \Lambda_{-1} = 0.$	$\langle v, f\rangle = \frac{sgn(\mu)\lambda}{\sqrt{\pi}} P \int_{-\infty}^{+\infty} \frac{e^{-\mu^2 x^2}}{x-1}\big(f(x)$ $-f(1)\big)\,dx + f(1),\ \mu \neq 0.$

A special family of non-symmetric semi-classical forms of class one

$\begin{cases} \Big((x+1)\big(x^2-\mu^2\big)v\Big)' + \Big(-2(\alpha+2)x^2 \\ \quad -2(\alpha+1)x + 2\mu^2\Big)v = 0, \\ (\mu^2-1)\{\vert\alpha\mu\vert + \vert\mu+2\alpha+1\vert\}\{\vert\alpha\mu\vert \\ +\vert 2\alpha+1-\mu\vert\} \neq 0. \end{cases}$	$\beta_0 = -t_0,\ \beta_{n+1} = t_n - t_{n+1},\ n \geq 0,$ $\gamma_{n+1} = -t_n(1+t_n),\ n \geq 0,$ $t_{2n} = t_{2n} = -\frac{(2n+2\alpha+1)\mu^2 c_n(\alpha)}{2(4n+2\alpha+3)d_n(\alpha)},$ $t_{2n+1} = -\frac{(2n+2\alpha+2)\mu^2 d_n(\alpha)}{2(4n+2\alpha+5)c_{n+1}(\alpha)},\quad n \geq 0,$ $c_n(\alpha) = \sum_{k=0}^{n} \frac{(-1)^k \mu^{2k}\Gamma(2n-k+\alpha+\frac{1}{2})}{2^{2k}\Gamma(2n+1-2k)\Gamma(k+1)},\ n \geq 0,$ $d_n(\alpha) = \sum_{k=0}^{n} \frac{(-1)^k \mu^{2k}\Gamma(2n-k+\alpha+\frac{3}{2})}{2^{2k}\Gamma(2n+2-2k)\Gamma(k+1)},\ n \geq 0.$	$\langle v, f\rangle = -\frac{\mu^2-2\alpha-3}{2\alpha+3}\int_{-1}^{1}(x+1)(1-x^2)^{\beta-1}\vert x^2$ $-\mu^2\vert^{\alpha}\big(f(x)-f(1)\big)\Big(A+(B-A)\chi_{[-\mu,\mu]}(x)\Big)\,dx$ $+f(1),\ \mu \in]0,1[,\ \Re(\alpha) > -1,\ \Re(\beta) > -1, or$ $\langle v, f\rangle = \frac{\mu^2-2\alpha-3}{2\alpha+3}\int_{-\mu}^{\mu} sgn(x-1)\vert x+1\vert\vert x^2-1\vert^{\beta-1}$ $(\mu^2-x^2)^{\alpha}\big(f(x)-f(1)\big)\Big(A_1+(B_1-A_1)\chi_{[-1,1]}(x)Big\big)$ $dx + f(1),\ \mu > 1,\ \Re(\alpha) > -1,\ \Re(\beta) > -1.$
$\begin{cases} \big(x(x^2-1)v\big)' + \big(-(2\alpha+2\beta+3)x^2 \\ \quad +x+2(\beta+1)\big)v = 0, \\ \alpha(2\beta+1)\{\vert\alpha+2\beta+1\vert + \vert\alpha-1\vert\} \neq 0, \end{cases}$	$\beta_0 = -t_0,\ \beta_{n+1} = t_n - t_{n+1},\ n \geq 0,$ $\gamma_{n+1} = -t_n(1+t_n),\ n \geq 0,$ $t_{2n} = -\frac{n+\beta+1}{2n+\alpha+\beta+1},$ $t_{2n+1} = -\frac{n+1}{2n+\alpha+\beta+2},\quad n \geq 0.$	$\langle v, f\rangle = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\int_{-1}^{1}\vert x\vert^{2\beta+1}(1+x)^{\alpha}(1-x)^{\alpha-1}$ $\big(f(x)-f(1)\big)\,dx + f(1), \Re(\alpha) > -1, \Re(\beta) > -1.$
$\begin{cases} \big((x-1)(x^2-\mu^2)v\big)' \\ \quad -2(\alpha+1)x(x-1)v = 0, \\ \mu\{\vert\mu^2-1\vert + \vert(2\alpha+1)(\lambda+1)-1\vert\} \\ \times\{\vert\alpha(\mu-1)\vert + \vert 2\alpha(\lambda+\mu)+\lambda\vert\} \\ \times\{\vert\alpha(\mu+1)\vert + \vert 2\alpha(\lambda-\mu)+\lambda\vert\} \neq 0, \end{cases}$	$\beta_0 = -t_0,\ \beta_{n+1} = t_n - t_{n+1},\ n \geq 0,$ $\gamma_{n+1} = -t_n(1+t_n),\ n \geq 0,$ $t_{2n} = -\frac{(4n+2)g_n(\alpha)}{(4n+2\alpha+1)h_n(\alpha)},$ $t_{2n+1} = -\frac{(4n+4)h_{n+1}(\alpha)}{(4n+2\alpha+3)g_n(\alpha)},\quad n \geq 0,$ $g_n(\alpha) = \sum_{k=0}^{n} \frac{(-1)^k \mu^{2k}\Gamma(2n-k+\alpha+\frac{3}{2})\big(1+\lambda\Omega_{n-k}\big)}{2^{2k}\Gamma(k+1)\Gamma(2n+2-2k)},\quad n \geq 0,$ $h_n(\alpha) = \sum_{k=0}^{n} \frac{\Gamma(2n-k+\alpha+\frac{1}{2})\big(1+\lambda\Omega_{n-1-k}\big)(-1)^k\mu^{2k}}{2^{2k}\Gamma(k+1)\Gamma(2n+1-2k)},\quad n \geq 0,$ $\Omega_n = \frac{\Gamma(\alpha+\frac{3}{2})}{\sqrt{\pi}}\sum_{\nu=0}^{n}\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+\alpha+\frac{3}{2})}\mu^{2\nu},\ n \geq 0,\ \Omega_{-1} = 0.$	$\langle v, f\rangle = \frac{\lambda\Gamma(2\alpha+2)}{(2\mu)^{2\alpha+1}\Gamma^2(\alpha+1)} P\int_{-\mu}^{\mu}\frac{(\mu^2-x^2)^{\alpha}}{x-1}\big(f(x)-f(1)\big)$ $dx + f(1),\ \Re(\alpha) > 0,\ \mu > 0.$

$$+f(1),\ \Re(\alpha) > 0,\ \mu > 0.$$

Finally, we summarize our results of this section in the following table.

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