Periodica Mathematica Hungaria (2023) 87:508–519
https://doi.org/10.1007/s10998-023-00531-8

Regularity and symbolic defect of points on rational normal curves

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Accepted: 11 August 2022 / Published online: 5 June 2023

Abstract

In this paper we study ideals of points lying on rational normal curves defined in projective plane and projective 3-space. We give an explicit formula for the value of Castelnuovo–Mumford regularity for their ordinary powers. Moreover, we compare the m-th symbolic and ordinary powers for such ideals in order to show whenever the m-th symbolic defect is non-zero.

Keywords Rational normal curves · Regularity · Symbolic defect

1 Introduction

Studying Castelnuovo–Mumford regularity $\operatorname{reg}(I)$ of a homogeneous ideal $I \subseteq \mathbb{K}[x_0, \ldots, x_n]$ has a long story starting from the paper of Mumford [15], who introduced the concept of m-regularity for an ideal I, i.e. the number m for which all i-th syzygies of I are generated in degrees not greater than $m + i$, for all i. Bayer and Stillman in [1]went on with Mumford's ideas by showing an explicit criterion for m-regularity. They also proved an equality between $\operatorname{reg}(I)$ and the regularity of initial ideal of I with respect to the reverse lexicographic order in any characteristic of $\mathbb{K}$. A connection between Castelnuovo–Mumford regularity and syzygies of given ideal I justifies why $\operatorname{reg}(I)$ can be viewed as a measure of complexity of I and also explain unflagging interests in this subject.

Iman Bahmani Jafarloo was partially supported by MIUR Grant Dipartimento di Eccellenza 2018-2022 (E11G18000350001). Grzegorz Malara was partially supported by National Science Centre, Poland, Sonata Grant 2018/31/D/ST1/00177.

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Swanson in [16] analysed r-th ordinary powers I^r of homogeneous ideals I, showing that these powers can be expressed in terms of primary decomposition of I^r. As an additional result, it has been proved that $\operatorname{reg}(I^r)$ is bounded above by some linear functions which depend on r. As a consequence, a new way of investigation of $\operatorname{reg}(I^r)$ has begun. In [9] Cutkosky, Herzog, and Trung, building upon papers of Swanson and the paper of Bertram, Ein and Lazarsfeld [2], introduced a new asymptotic invariant, the so-called asymptotic regularity $\operatorname{areg}(I)$ of a homogeneous ideal I. Later, the work on regularity of homogeneous ideals and their powers was significantly improved in [7, 11], for instance for the case of Gorenstein and zero dimensional ideals.

One of the best known classes of curves in projective spaces $\mathbb{P}^n$ are rational normal curves and they have been studied widely, see [4–6, 8]. Studying schemes of fat points lying on rational normal curves has its own long history. In [6] Catalisano and Gimigliano gave an algorithm for computing the Hilbert function for fat point schemes lying on a twisted cubic curve and they extended the work for rational normal curves in $\mathbb{P}^n$ together with Ellia [5]. At the same time, Conca in [8] described the Hilbert function and resolution of symbolic and ordinary powers of ideals of rational normal curves.

Our motivation for this work is computing the regularity of powers of ideals of points on two types of rational normal curves, conic and twisted cubic curve. The main results of this paper concerning the regularity of powers of such ideals are Theorems 3.4 and 4.5 which can be summed up as follows:

Theorem *Let $n \in \{2, 3\}$ and let $C \subset \mathbb{P}^n$ be a rational normal curve. Denote by I_{D_j} the ideal defining a set of s general points on C. Let $0 \le j < n$ be such that $s = nd - j$, then*

$$\operatorname{reg}(I^r_{D_j}) = \begin{cases} rd + 1 & \text{if } j = 0 \\ rd & \text{if } 0 < j < n. \end{cases}$$

The paper is organized as follows. In Sect. 2 we recall all needed definitions and prove basic facts that are used through the paper. The first non-trivial case of a rational normal curve is a conic in $\mathbb{P}^2$. We dedicated Sect. 3 to this case. It culminates with the proof of Theorem 3.4. Section 4 is devoted to the study of twisted cubic curves and the culmination of this section is Theorem 4.5. The last section is a small step towards understanding the structure of symbolic powers of ideals I_{D_j}. We prove that for all integers $m \ge 3$ there is $I^{(m)}_{D_j} \not\subseteq I^m_{D_j}$, and state a conjecture about the relation between symbolic and ordinary powers of ideals I_{D_j}.

2 Preliminary

Let $S = K[x_0, \ldots, x_n]$ be the graded ring of polynomials over an algebraically closed field K. Let

$$M = \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}.$$

Denote by $I = I_2(M)$ be the ideal generated by the 2-minors of M (known as the Hankel matrix). It is known that the ideal I defines the rational normal curve (RNC for short) in $\mathbb{P}^n$, which we denoted by C, the Veronese embedding of

$$\nu_n : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n, \quad [s : t] \mapsto [s^n : s^{n-1}t : \cdots : st^{n-1} : t^n]. \tag{2.1}$$

Recall that for any homogeneous ideal J the Hilbert function $\operatorname{HF}(S/J, t)$ of S/J, for $t \in \mathbb{N} \cup \{0\}$, is the dimension over K of degree t homogeneous part of S/J.

Remark 2.1 For the ideal $I = I_2(M)$ the Hilbert function of S/I is known to be

$$\mathrm{HF}(S/I, t) = n(t+1) - (n-1), \quad \text{for } t \geq 0.$$

Let $J \subset S$ be any homogeneous ideal. We denote by $\beta_{ij}(J)$ the (i, j)-th *Betti number* of J, i.e. the dimension of $\mathrm{Tor}_i^S(J, K)$ in degree j. By definition, the Castelnuovo–Mumford regularity $\mathrm{reg}(J)$ of J is

$$\mathrm{reg}(J) = \max\left\{j - i : \beta_{ij}(J) \neq 0\right\}.$$

It is convenient to write $\beta(J)$ and $\alpha(J)$ for the maximum and the minimum degree of the minimal set of generators of J, respectively. In general, we have $\mathrm{reg}(J) \geq \beta(J)$ and $\mathrm{reg}(S/J) = \mathrm{reg}(J) - 1$.

Remark 2.2 It is known that for a zero-dimensional ideal J, if $t \geq 0$ is the least value such that $\Delta\,\mathrm{HF}(S/J, t) = \mathrm{HF}(S/J, t) - \mathrm{HF}(S/J, t-1) = 0$, then $\mathrm{reg}(J) = t$.

Definition 2.3 Let $J \subset S$ be a homogeneous ideal. Then the asymptotic regularity of J is the real number

$$\mathrm{areg}(J) = \lim_{r\to\infty} \frac{\mathrm{reg}(J^r)}{r}.$$

At it was shown in [9, Theorem 1.1], we always have $\mathrm{areg}(J) = \frac{\beta(J^r)}{r}$, since it is known that $\beta(J^r)$ is linear function which depends on r for all $r \gg 0$.

Let $D_j \subset C$ be a set of $nd - j$ general points on the rational normal curve $C \subset \mathbb{P}^n$ for integers $d \geq 2$ and $0 \leq j \leq n-1$. Denote by I_{D_j} the ideal defining the set D_j. In the following we study the ideal I_{D_j} and the next lemma is an observation that we need in order to prove that the forms of order rd does not vanish in $I_{D_j}^r$.

Lemma 2.4 *Let D_j be a set of $nd - j$ points on rational normal curve C. Then, $\beta(I_{D_j}^r) = r\beta(I_{D_j}) = rd$.*

Proof The proof directly follows from [10, Exercise A2.21, d]. More precisely, I_{D_j} is an ideal in the symmetric algebra S/I (the coordinate ring of C) generated at most in degree d. □

Proposition 2.5 *Let D_j be as in Lemma 2.4. If $r \geq 2$ and $d \geq 2$, then*

$$rd \leq \mathrm{reg}(I_{D_j}^r) \leq \mathrm{reg}(I_{D_j}) + (r-1)d.$$

Proof On the one hand from Lemma 2.4 and the fact that $\beta(I_{D_j}^r) \leq \mathrm{reg}(I_{D_j}^r)$, we have $rd \leq \mathrm{reg}(I_{D_j}^r)$. On the other hand since I_{D_j} is a zero-dimension ideal generated at most in degree d, therefore from [11, Corollary 7.9] we have that $\mathrm{reg}(I_{D_j}^r) \leq \mathrm{reg}(I_{D_j}) + (r-1)d$. Hence,

$$rd \leq \mathrm{reg}(I_{D_j}^r) \leq \mathrm{reg}(I_{D_j}) + (r-1)d.$$

□

Lemma 2.6 *The set $\{x_0^{d-1} - x_n^{d-1} = 0\}$ and C meet each other exactly at $n(d-1)$ distinct points.*

Proof One can see that

$$x_0^{d-1} - x_n^{d-1} = \prod_{i=1}^{d-1}(x_0 - \xi_i x_n),$$

where ξ_i is the i-th primitive root of unity for $i = 1, \ldots, d-1$. By (2.1) we have that

$$\prod_{i=1}^{d-1}(x_0 - \xi_i x_n) = \prod_{i=1}^{d-1}(s^n - \xi_i t^n) = \prod_{i=1}^{d-1}(\zeta^n - \xi_i).$$

It follows that $\{x_0^{d-1} - x_n^{d-1} = 0\}$ intersects C at $n(d-1)$ distinct points, therefore the desired result follows. Moreover, we conclude that no two hyperplanes $\{x_0 - \xi_\alpha x_n = 0\}$ and $\{x_0 - \xi_\beta x_n = 0\}$, with $\alpha \neq \beta$, intersect C at the same point for all $\alpha, \beta \in \{1, 2, \ldots, d-1\}$. □

In the following sections, we study the regularity of $I_{D_j}^r$ where D_j lies on a conic in $\mathbb{P}^2$, or on a twisted cubic curve (TCC) in $\mathbb{P}^3$. Since we are considering these points in two separate sections, we agree to use the same notation of C for both, conic and TCC.

3 Regularity of points on a conic

This section is devoted to study the regularity of $I_{D_j}^r$ where $D_j \subset C \subset \mathbb{P}^2$. By the definition of ideal I, we have that

$$I = \det\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix} = \langle x_1^2 - x_0 x_2 \rangle = \langle Q \rangle .$$

Lemma 3.1 *Let D_j be a set of $2d - j$ distinct points in $\mathbb{P}^2$ lie on C for $d \geq 2$ and $j \in \{0, 1\}$. Then its defining ideal can be represented as:*

$$I_{D_j} = \begin{cases} I + \left\langle x_1(x_0^{d-1} - x_2^{d-1}) \right\rangle, & \text{if } j = 0 \\ I + \left\langle x_1(x_0^{d-1} - x_2^{d-1}), x_0(x_0^{d-1} - x_2^{d-1}) \right\rangle, & \text{if } j = 1. \end{cases}$$

Proof We proceed as follows:

- Let $j = 0$. By Lemma 2.6 one can see that $\{x_1 = 0\} \cap \{x_0 - \xi_\alpha x_2 = 0\} \cap C = \emptyset$ for $\alpha = 1, 2, \ldots, d-1$. Since the line $\{x_1 = 0\}$ does not contain any tangent line to C, therefore the intersection of $\{x_1(x_0^{d-1} - x_2^{d-1}) = 0\}$ and C is a set $2(d-1) + 2 = 2d$ distinct points.
- Let $j = 1$. Since the point $\{\langle x_1, x_0 \rangle\} \notin \{x_0^{d-1} - x_2^{d-1} = 0\}$, the desired result follows from Lemma 2.6.

□

Proposition 3.2 *Let D_j be as in Lemma 3.1. Then*

$$\operatorname{reg}(I_{D_j}) = \begin{cases} d+1, & \text{if } j = 0, \\ d, & \text{if } j = 1. \end{cases}$$

Proof Let $j=0$. Then the syzygy matrices of S/I_{D_0} are as follows

$$A_1=\left(Q\ x_1(x_0^{d-1}-x_2^{d-1})\right),\ A_2=\begin{pmatrix} x_1(x_0^{d-1}-x_2^{d-1}) \\ -Q \end{pmatrix}.$$

Therefore, we have its minimal free resolution

$$0\to S(-d-2)\xrightarrow{A_2} S(-2)\oplus S(-d)\xrightarrow{A_1} S\to S/I_{D_0}\to 0,$$

and from that $\operatorname{reg}(S/I_{D_0})=d$. Accordingly, $\operatorname{reg}(I_{D_0})=d+1$

Similarly, for $j=1$, we compute the syzygy matrices for S/I_{D_1},

$$A_1=\left(Q\ x_1(x_0^{d-1}-x_2^{d-1})\ x_0(x_0^{d-1}-x_2^{d-1})\right),\ A_2=\begin{pmatrix} 0 & x_0^{d-1}-x_2^{d-1} \\ x_0 & -x_1 \\ -x_1 & x_2 \end{pmatrix}.$$

Hence,

$$0\to S^2(-d-1)\xrightarrow{A_2} S(-2)\oplus S^2(-d)\xrightarrow{A_1} S\to S/I_{D_1}\to 0.$$

We see that $\operatorname{reg}(S/I_{D_1})=d-1$ and, consequently, $\operatorname{reg}(I_{D_1})=d$. □

Lemma 3.3 *Let D_0 be as in Lemma 3.1. Then,* $\operatorname{reg}(I_{D_0}^r)\geq rd+1$ *for $r\geq 2$.*

Proof Set $G=x_1(x_0^{d-1}-x_2^{d-1})$. Directly from the definition of ordinary power $I_{D_0}^r=\langle\{Q^{r-t}G^t\}_{t=0}^r\rangle$. Hence, the first syzygy matrix of $S/I_{D_0}^r$ is

$$A_1=\left[Q^r\ Q^{r-1}G\ Q^{r-2}G^2\cdots Q^2G^{r-2}\ QG^{r-1}\ G^r\right].$$

It is a straightforward computation that the second syzygy matrix can be expressed as follows:

$$A_2=\underbrace{\overset{A_{21}}{\begin{bmatrix} -G & 0 & 0 & 0 & \cdots \\ Q & -G & 0 & 0 & \cdots \\ 0 & Q & -G & 0 & \cdots \\ 0 & 0 & Q & -G & \cdots \\ 0 & 0 & 0 & Q & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}}}_{S(-(2r+1))\cdots}\cdots\underbrace{\overset{A_{22}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ -G \\ Q \\ 0 \end{bmatrix}}}_{S(-(rd+1))}\cdots\underbrace{\overset{A_{23}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -G \\ Q \end{bmatrix}}}_{S(-(rd+2))}.$$

This proves that $\operatorname{reg}(S/I_{D_0}^r)\geq rd$, in consequence $\operatorname{reg}(I_{D_0}^r)\geq rd+1$. □

Theorem 3.4 *Let D_j be as in Lemma 3.1. If $r\geq 2$, then*

(1) $\operatorname{reg}(I_{D_0}^r)=rd+1$,
(2) $\operatorname{reg}(I_{D_1}^r)=rd$.

Proof The case (1) follows from Propositions 2.5, 3.2 and Lemma 3.3. The proof of (2) follows directly from Propositions 2.5 and 3.2. □

4 Regularity of points on a TCC

Let $n = 3$. In this section we study the $\mathrm{reg}(I^r_{D_j})$, where D_j is a set of $3d - j$ points which lie on the twisted cubic curve C defined by the ideal

$$I = \left\langle x_2^2 - x_1x_3, x_1x_2 - x_0x_3, x_1^2 - x_0x_2 \right\rangle = \langle Q_1, Q_2, Q_3 \rangle .$$

Lemma 4.1 *the ideal I_{D_j} defines the set D_j of $3d - j$ distinct points on C for $j = 0, 1, 2$.*

$$I_{D_j} = \begin{cases} I + \left\langle (x_2 - x_1)(x_0^{d-1} - x_3^{d-1}) \right\rangle, & \text{if } j = 0 \\ I + \left\langle x_2(x_0^{d-1} - x_3^{d-1}), x_1(x_0^{d-1} - x_3^{d-1}) \right\rangle, & \text{if } j = 1 \\ I + \left\langle x_2(x_0^{d-1} - x_3^{d-1}), x_1(x_0^{d-1} - x_3^{d-1}), x_0(x_0^{d-1} - x_3^{d-1}) \right\rangle, & \text{if } j = 2. \end{cases}$$

Proof We divide the proof into three cases as follows.

- Let $j = 0$. By Lemma 2.6, an elementary calculation shows that planes $\{x_2 - x_1 = 0\}$ and $\{x_0 - \xi_\alpha x_3 = 0\}$ do not meet C at the same point for $\alpha = 1, 2, \ldots, d-1$. Since the plane $\{x_2 - x_1 = 0\}$ does not contain any tangent line to C, we conclude that $\{(x_2 - x_1)(x_0^{d-1} - x_3^{d-1}) = 0\}$ intersects C at $3(d-1) + 3 = 3d$ points.
- Let $j = 1$. We have

$$\langle x_2(x_0^{d-1} - x_3; d - 1), x_1(x_0^{d-1} - x_3^{d-1}) \rangle = \langle x_2, x_1 \rangle \langle x_0^{d-1} - x_3^{d-1} \rangle.$$

  One can see that the line $\{\langle x_2, x_1 \rangle\}$ is not tangent to C, moreover by Lemma 2.6, we have $\{\langle x_2, x_1 \rangle\} \cap \{x_0^{d-1} - x_3^{d-1} = 0\} \cap C = \emptyset$. Therefore, $\{\langle x_2, x_1 \rangle \langle x_0^{d-1} - x_3^{d-1} \rangle\}$ intersects C at $3(d-1) + 2 = 3d - 1$ points.
- Let $j = 2$. Since the point $\langle x_2, x_1, x_0 \rangle \notin \{x_0^{d-1} - x_3^{d-1} = 0\}$, therefore by Lemma 2.6 the desired result follows.

This completes the proof. □

Proposition 4.2 *Let D_j be as in Lemma 4.1. Then,*

$$\mathrm{reg}(I_{D_j}) = \begin{cases} d + 1, & \text{if } j = 0, 1 \\ d, & \text{if } j = 2. \end{cases}$$

Proof We are looking for minimal free resolutions of the form

$$0 \to F_3 \xrightarrow{A_3} F_2 \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0 \to R/I_{D_j} \to 0$$

for any ideal I_{D_j}. Since for any j we know the generators of ideals I_{D_j}, we can write matrices A_i explicitly.

For the sake of the completeness, denote by H the form $x_0^{d-1} - x_3^{d-1}$. With some aid of any algebraic software program, such as `Macaulay2` [14], we compute the syzygy matrices of S/I_{D_j}. In case of $j = 0$ we have

$$A_1 = \left[\, Q_1 \;\; Q_2 \;\; Q_3 \;\; (x_2 - x_1)H \,\right],$$

$$A_2 = \begin{bmatrix} x_1 & x_0 & x_2x_3^{d-1} & -x_2x_3^{d-1} & 0 \\ -x_2 & -x_1 & x_2(H - x_0^{d-2}x_1) & x_1(H - x_0^{d-2}x_2) & 0 \\ x_3 & x_2 & x_0^{d-2}x_2^2 - x_3H & x_0^{d-2}x_2^2 - x_3H & (x_1 - x_2)H \\ 0 & 0 & -Q_1 & -Q_1 - Q_2 & -Q_1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} x_2H & x_0H \\ -x_0^{d-2}x_1x_2 & -x_1H - x_2x_3^{d-1} \\ x_1 + x_2 & x_0 + x_1 \\ -x_2 & -x_1 \\ x_3 & x_2 \end{bmatrix}.$$

Therefore the minimal free resolution of I_{D_0} is

$$0 \to S^2(-d-3) \xrightarrow{A_3} S^2(-3) \oplus S^3(-d-2) \xrightarrow{A_2} S^3(-2) \oplus S(-d) \xrightarrow{A_1} S \to S/I_{D_0} \to 0.$$

While for $j = 1$, there is

$$A_1 = \begin{bmatrix} Q_1 & Q_2 & Q_3 & x_2H & x_1H \end{bmatrix},$$

$$A_2 = \begin{bmatrix} x_1 & 0 & x_0 & -x_3^{d-1} & 0 \\ -x_2 & 0 & -x_1 & x_0^{d-2}x_1 & 0 \\ x_3 & 0 & x_2 & -x_0^{d-2}x_2 & H \\ 0 & x_1 & 0 & -x_2 & x_0 \\ 0 & -x_2 & 0 & x_3 & -x_1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} x_1H \\ -Q_2 \\ -x_0^{d-2}x_1^2 + x_2x_3^{d-1} \\ -Q_3 \\ Q_1 \end{bmatrix}.$$

Thus

$$0 \to S(-d-3) \xrightarrow{A_3} S^2(-3) \oplus S^3(-d-1) \xrightarrow{A_2} S^3(-2) \oplus S^2(-d) \xrightarrow{A_1} S \to S/I_{D_1} \to 0.$$

For the last remaining case, $j = 2$, the matrices are the following

$$A_1 = \begin{bmatrix} Q_1 & Q_2 & Q_3 & x_2H & x_1H & x_0H \end{bmatrix},$$

$$A_2 = \begin{bmatrix} x_1 & 0 & x_0 & 0 & 0 & -x_3^{d-1} & 0 & 0 \\ -x_2 & 0 & -x_1 & 0 & 0 & x_0^{d-2}x_1 & H & 0 \\ x_3 & 0 & x_2 & 0 & 0 & -x_0^{d-2}x_2 & 0 & H \\ 0 & x_1 & 0 & x_0 & 0 & -x_2 & 0 & 0 \\ 0 & -x_2 & 0 & 0 & x_0 & x_3 & -x_2 & -x_1 \\ 0 & 0 & 0 & -x_2 & -x_1 & 0 & x_3 & x_2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & H \\ x_0 & 0 & -x_2 \\ 0 & x_3^{d-1} & -x_0^{d-2}x_1 \\ -x_1 & x_2 & 0 \\ x_2 & -x_3 & 0 \\ 0 & x_0 & -x_1 \\ 0 & -x_1 & x_2 \\ 0 & x_2 & -x_3 \end{bmatrix}.$$

Hence we can write

$$0 \to S^3(-d-2) \xrightarrow{A_3} S^2(-3) \oplus S^6(-d-1) \xrightarrow{A_2} S^3(-2) \oplus S^3(-d) \xrightarrow{A_1} S \to S/I_{D_2} \to 0,$$

By a straightforward calculation from the definition of regularity, we get the desired assertion. □

The minimal free resolution of I_{D_1}, calculated in the previous theorem, gives us immediately the following corollary.

Corollary 4.3 *The ideal I_{D_1} is a Gorenstein ideal.*

Lemma 4.4 *Let D_0 be as in Lemma 4.1. Then,* $\operatorname{reg}(I_{D_0}^r) \geq rd + 1$ *for $r \geq 2$.*

Proof Set $G = (x_2 - x_1)(x_0^{d-1} - x_3^{d-1})$. The r-th power of I_{D_0} is as the following

$$I_{D_0}^r = \left\langle Q_1^r, Q_1^{r-1}Q_2, \cdots, Q_1G^{r-1}, \cdots, Q_2G^{r-1}, \cdots, Q_3G^{r-1}, G^r \right\rangle.$$

Consider the 0-*dimensional* ideal $J = \langle Q_1, Q_2, Q_3, G^r\rangle$. Since $I_{D_0}^r \subset J$, therefore we have the following exact sequence:

$$0 \to I_{D_0}^r \to J \to \frac{J}{I_{D_0}^r} \to 0. \tag{4.1}$$

Hence we have

$$\operatorname{reg}(J) \leq \max\left\{\operatorname{reg}\left(\frac{J}{I_{D_0}^r}\right), \operatorname{reg}(I_{D_0}^r)\right\}.$$

We claim that $\operatorname{reg}(J) = rd + 1$. Since $I_C \subset J$ we have $[I_C]_t = [J]_t$ for $t \leq rd - 1$, and it is known that $\operatorname{HF}(S/I_C, t) = 3t + 1$ for $t \geq 0$. Therefore $\operatorname{HF}(S/J, t) = 3t + 1$ for $t \leq rd - 1$. We know that the degree of J is $3rd$, which means that either $\operatorname{HF}(S/J, rd)$ is $3rd - 1$ or $3rd$. To prove by contradiction, assume that $\operatorname{HF}(S/J, rd) = 3rd - 1$. Hence, the first difference of the Hilbert function of S/J is

$$1\ 3\ 3\ 3 \cdots 1\ 1\ 0\,.$$

So, by [13, Proposition 5.2] it follows that $V(J)$ contains a subset of $rd + 2$ collinear points having multiplicities r. It contradicts the fact that $V(J)$ has only subsets of at most $2r$ collinear points. Therefore,

t	0	1	2	3	$\cdots$	$rd-1$	rd	$rd+1$	$\cdots$
$\operatorname{HF}(S/J,t)$	1	4	7	10	$\cdots$	$3rd-2$	$3rd$	$3rd$	$\cdots$

We conclude that $\operatorname{reg}(J) = rd + 1$.

We know from (4.1) that

$$\operatorname{HF}(S/(J/I_{D_0}^r), t) = \operatorname{HF}(S/I_{D_0}^r, t) - \operatorname{HF}(S/J, t), \qquad \forall t \geq 0.$$

Since the set minimal generators of $I_{D_0}^r$ has only one form of degree $\beta(I_{D_0}^r) = rd$, we conclude that $\operatorname{HF}(S/I_{D_0}^r, t) - \operatorname{HF}(S/J, t) = c \in \mathbb{Z}^+$, for all $t \geq rd$. Therefore, the Hilbert function of $S/(J/I_{D_0}^r)$ is partially as follows:

t	0	1	2	3	$\cdots$	rd	$rd+1$	$rd+2$	$\cdots$
$\operatorname{HF}(S/(J/I_{D_0}^r),t)$	0	0	3	10	$\cdots$	c	c	c	$\cdots$

This follows that $\operatorname{reg}\left(\frac{J}{I_{D_0}^r}\right)$ is at most $rd - 1$. From Proposition 2.5, we know that $\operatorname{reg}(I_{D_0}^r) \geq rd$, hence, $\operatorname{reg}\left(\frac{J}{I_{D_0}^r}\right) < \operatorname{reg}(I_{D_0}^r)$. Therefore,

$$rd + 1 = \operatorname{reg}(J) \leq \max\left\{\operatorname{reg}\left(\frac{J}{I_{D_0}^r}\right), \operatorname{reg}(I_{D_0}^r)\right\} = \operatorname{reg}(I_{D_0}^r).$$

The proof is completed. □

Theorem 4.5 *Let D_j be as in Lemma 4.1. If $r \geq 2$ and $d \geq 2$, then*

(1) $\operatorname{reg}(I_{D_0}^r) = rd + 1$,
(2) $\operatorname{reg}(I_{D_1}^r) = \operatorname{reg}(I_{D_2}^r) = rd$,

Proof The proof of (1) is a direct consequence of Propositions 2.5,4.2 and Lemma 4.4. The proof for $j = 1$ follows from Propositions 4.2,2.5 and [7, Proposition 1.12.6]. The last remaining case for $j = 2$ similarly the result follows from Propositions 2.5 and 4.2. The proof is complete. □

Corollary 4.6 *For the ideals I_{D_j} defined in Lemma 4.1, we have*

$$\operatorname{areg}(I_{D_j}) = \lim_{r\to\infty} \frac{\operatorname{reg}(I_{D_j}^r)}{r} = d.$$

Remarks in $\mathbb{P}^n$

It is natural to ask about the regularity of the same type of ideals in higher projective spaces. However, simply calculations can show that the formula for $\operatorname{reg}(I_{D_j}^r)$, with $r > 1$, is much more complicated than for cases of $\mathbb{P}^2$ and $\mathbb{P}^3$, and can not be easily described. Thus, we dedicate this section to be a leading step on further investigations in this subject, by proving the lemma which concerns $\operatorname{reg}(I_{D_j})$.

Definition 4.7 Let $n \geq 4$ and $0 \leq j \leq n-1$. Let I_{D_j} be the ideal of a set $nd - j$ points on C defined by the ideal $I = I_2(M)$ as follows,

$$\begin{cases} I_{D_0} = I + \left\langle (x_{n-1} - x_1)(x_0^{d-1} - x_n^{d-1}) \right\rangle, \\ I_{D_1} = I + \left\langle x_{n-1}(x_0^{d-1} - x_n^{d-1}), x_{n-2}(x_0^{d-1} - x_n^{d-1}) \right\rangle, \\ I_{D_j} = I_{D_{j-1}} + \left\langle x_{n-j-1}(x_0^{d-1} - x_n^{d-1}) \right\rangle, \quad \text{if } 2 \leq j \leq n-1. \end{cases}$$

One can easily observe that the proof of the fact that ideals I_{D_j} indeed describes the set of $nd - j$ distinct points can be mimic from the proof of Lemma 4.1. Also the next remark is similar to the result obtained in Proposition 4.2.

Remark 4.8 For ideals I_{D_j} defined as in Definition 4.7, one can compute the $\operatorname{reg}(I_{D_j})$ as in Proposition 4.2 by writing their free resolutions or directly by computing their Hilbert functions,

$$\operatorname{reg}(I_{D_j}) = \begin{cases} d+1, & \text{if } 0 \leq j < n-1 \\ d, & \text{if } j = n-1. \end{cases}$$

5 Symbolic defect

Comparing symbolic and ordinary powers of ideals of points in $\mathbb{P}^N$ has became very popular in recent years. There are a few different concepts that are concerning "the ideal containment problem". In this section we want to analyse one of them in the case of ideals I_{D_j}. Let us recall first the definition of symbolic power of ideal.

Definition 5.1 Let I be a homogeneous ideal in a polynomial ring R. For $m \geq 1$, the m-th symbolic power of I is the ideal

$$I^{(m)} = R \cap \left(\bigcap_{\mathfrak{p}\in \operatorname{Ass}(I)} (I^m)_{\mathfrak{p}} \right),$$

where the intersection is taken over all associated primes $\mathfrak{p}$ of I.

It is known that for any m the inclusion $I^m \subseteq I^{(m)}$ holds, but the reverse does not hold in general. Therefore it is natural to ask about the number of generators in the module $I^{(m)}/I^m$.

Definition 5.2 We define the m-th symbolic defect of I for any integer $m \geq 2$ to be

$$\mathrm{sdefect}(I, m) = \text{ the number of minimal generators of } I^{(m)}/I^m.$$

We refer the interested readers in this subject to [12].

Motivated by the result of relation between symbolic and ordinary powers obtained for ideal of s general points on smooth conic [3], we take another step towards description of this behaviour for ideals of s general points on a TCC, by analysing the symbolic defect of I_{D_j}. What we can prove for those ideals is the following:

Theorem 5.3 *Let I_{D_j} be the ideals of points defined in Lemma 4.1. Then*

(1) $\mathrm{sdefect}(I_{D_1}, m) > 0$, *if* $m \geq 3$.
(2) $\mathrm{sdefect}(I_{D_j}, m) > 0$ *for* $j = 0, 2$.

Proof Our proof is based on simply observation that a particular element, different for each case, belongs to $I_{D_j}^{(m)} \setminus I_{D_j}^m$.

For the case (1) consider the polynomial

$$f_1 = Q_1 Q_3 (x_0^{d-1} - x_3^{d-1}).$$

We prove by induction on $k \geq 1$ that

$$f_1^k \in I_{D_1}^{(3k)}, \quad Q_2 f_1^k \in I_{D_1}^{(3k+1)}, \quad Q_2 Q_3 f_1^k \in I_{D_1}^{(3k+2)},$$

while

$$f_1^k \notin I_{D_1}^{3k}, \quad Q_2 f_1^k \notin I_{D_1}^{3k+1}, \quad Q_2 Q_3 f_1^k \notin I_{D_1}^{3k+2}. \tag{5.1}$$

First observe that $f_1 \in I_{D_1}^{(3)}$, which is a straightforward consequence of the Zariski–Nagata theorem (see [10, Theorem 3.14]). Assume for the induction hypothesis that we have $f_1^k \in I_{D_1}^{(3k)}$, for some $k > 1$. Then one can easily check that there are $Q_2 f_1^k \in I_{D_1}^{(3k+1)}$, $Q_2 Q_3 f_1^k \in I_{D_1}^{(3k+2)}$, once again by the Zariski–Nagata theorem. The fact that $f_1^{k+1} \in I_{D_1}^{(3k+3)}$ follows from induction hypothesis together with

$$f_1^k f_1 \in I_{D_1}^{(3k)} I_{D_1}^{(3)} \subset I_{D_1}^{(3k+3)},$$

since symbolic powers of any homogeneous ideal I form graded sequence of ideals.

Now we turn to the second part of the proof of (1). It can be checked by any symbolic algebra program, or check by hand, that $f_1 \notin I_{D_1}^3$. Therefore directly from the definition of ordinary power we get

$$f_1^k \notin I_{D_1}^{3k}.$$

Multiplying element f_1^k by appropriate $Q_i \in I_{D_1}$ gives the desired assertion (5.1).

The proof of the case (2) is identical as the case (1), if we instead of taking f_1 consider the polynomials

$$f_0 = Q_3(x_2 - x_3)(x_0^{d-1} - x_3^{d-1}), \quad f_2 = Q_3(x_0^{d-1} - x_3^{d-1}),$$

and proceed by induction on $k \geq 1$ in order to show that

$$f_{0,2}^k \in I_{D_{0,2}}^{(2k)}, \quad Q_1 f_{0,2}^k \in I_{D_{0,2}}^{(2k+1)},$$

and

$$f_{0,2}^k \notin I_{D_{0,2}}^{2k}, \quad Q_1 f_{0,2}^k \notin I_{D_{0,2}}^{2k+1}.$$

□

Remark 5.4 There is one missing case of sdefect$(I_{D_1}, 2)$ in the statement of Theorem 5.3. We expect that sdefect$(I_{D_1}, 2) = 0$, however we do not have a theoretical proof of this hypothesis.

Motivated by numerous tests and observations that we made, we want to finish this section with a conjecture that we was not able to prove, but we believe to be true.

Conjecture 5.5 *Let D_j be a set of $3d - j$ general points on a TCC, where $0 \leq j \leq 2$. Then*

1) $I_{D_j}^{(m)} \subseteq I_{D_j}^r$ *if and only if $m \geq r + 1$ for any integer $r \geq 2$, in the case $j = 0, 2$.*
2) $I_{D_1}^{(m)} \subseteq I_{D_1}^r$ *if and only if $m \geq r + 1$ for $r \geq 3$, and moreover, $I_{D_1}^{(m)} \subseteq I_{D_1}^2$ if and only if $m \geq 2$.*

Acknowledgements We want to warmly thank Enrico Carlini and Tomasz Szemberg for all inspiring discussions through the whole process of writing this article. Bahmani Jafarloo thanks Navid Nemati and Aldo Conca for useful conversations.

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