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On separation by function of bounded variation

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Abstract
In the present paper we are concerned with the problem of separation of two given functions by an increasing (decreasing) function or by a function of bounded variation. In the context of separation by a function of bounded variation we give two approaches. The first one concerns the separation with respect to the classical order in the set of real numbers, and this allow us to the new concept of variation involving two functions. This approach in fact leads us to the unusual phenomenon, namely the bounded joint variation of two functions forces these functions to be essentially the same. The second approach concerns the separation with respect to the partial order generated by the Lorentz cone.

Keywords Monotonic function · Function of bounded variation · Separation theorem

Mathematics Subject Classification 26A45 · 26A48 · 39B62 · 39B72

1 Introduction

The purpose of this paper is to find (if it is possible) the necessary and sufficient conditions under which two given functions can be separated by an increasing function as well as a decreasing function or by a function of bounded variation. A problem of finding for a given pair of real valued functions a function belonging to a given class which lies between them has a long story. Probably the first result of this type was obtained by Hahn in 1917 (see [5]) who proved that if a function $g\colon X \to \mathbb{R}$ is upper semicontinuous and $f\colon X \to \mathbb{R}$ is lower semicontinuous (where X stands for a metrizable topological space) and $g \le f$ then there exists a continuous function $h\colon X \to \mathbb{R}$ such that $g \le h \le f$ on X. The another very important separation type result is a consequence of a geometrical version of Hahn–Banach theorem proved by Kakutani [7] and says that if $f, -g$ are convex functions and $g \le f$ then there always exists an affine function h which separates f and g. The corresponding result for separating a superadditive function g and a subadditive f by an additive one was proved by Kranz [9] (see also [8]). These results give only the necessary conditions for separation.

We already know several results which give the necessary and sufficient conditions for separation by a function from many important classes. For example: by convex function [2],

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by affine function [11], by monotonic function [4], by subadditive and sublinear function [10] and many other.

2 Separation by increasing or decreasing function

We start with separation theorems for increasing and decreasing functions. A corresponding theorem for monotonic functions was proved in [4].

Theorem 2.1 *Real functions f and g, defined on a real interval I satisfy the inequality*

$$f(x) \leq g(y), \tag{2.1}$$

for all $x, y \in I,\ x \leq y$ if and only if there exists an increasing function $h\colon I \to \mathbb{R}$ such that

$$f(x) \leq h(x) \leq g(x), \quad x \in I. \tag{2.2}$$

Proof Obviously, if there is an increasing function h satisfying (2.2) then the inequality (2.1) holds true. Conversely, assume that f and g satisfy (2.1). Let us define a function $h\colon I \to \mathbb{R}$ by the formula

$$h(x) := \inf\{g(y) : y \in I,\ x \leq y\}.$$

By definition it is clear that

$$f(x) \leq h(x) \leq g(x),\ x \in I.$$

To see that h is an increasing function fix $x, y \in I, x < y$ arbitrarily. Since

$$\{z \in I : y \leq z\} \subseteq \{z \in I : x \leq z\},$$

then

$$h(x) = \inf\{g(z) : z \in I, x \leq z\} \leq \inf\{g(z) : z \in I,\ y \leq z\} = h(y),$$

which finishes the proof. □

By the same arguments as in the previous proof we obtain a corresponding theorem for decreasing functions.

Theorem 2.2 *Real functions f and g, defined on a real interval I satisfy the inequality*

$$f(y) \leq g(x),$$

for all $x, y \in I,\ x \leq y$ if and only if there exists a decreasing function $h\colon I \to \mathbb{R}$ such that

$$f(x) \leq h(x) \leq g(x), \quad x \in I.$$

As an application we get the following stability result for increasing functions.

Theorem 2.3 *Let $\varepsilon > 0$. If a function $f\colon I \to \mathbb{R}$ is an ε-increasing i.e. for all $x, y \in I,\ x \leq y$ the inequality*

$$f(x) \leq f(y) + \varepsilon$$

holds, then there exists an increasing function $h\colon I \to \mathbb{R}$ such that

$$f(x) \leq h(x) \leq f(x) + \varepsilon, \quad x \in I.$$

3 Separation by function of bounded variation

Now, we will consider the separation problem for functions of bounded variation. Let $\mathcal{P}_{[a,b]}$ denote the set of partitions of the interval $[a, b]$ i.e.

$$\mathcal{P}_{[a,b]} := \bigcup_{n=1}^{\infty}\{(x_0, x_1, \ldots, x_n) : a = x_0 < x_1 < \ldots < x_n = b\}.$$

Let us recall that a total variation $W_a^b(f)$ of a function $f\colon [a, b] \to \mathbb{R}$ over $[a, b]$ is defined by the formula

$$W_a^b(f) := \sup\left\{\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| : (x_0, \ldots, x_n) \in \mathcal{P}_{[a,b]}\right\}.$$

A function f is of bounded variation on $[a, b]$ if $W_a^b(f) < \infty$.

The concept of bounded variation functions was introduced in 1881 by Camille Jordan [6] for real functions defined on a closed interval $[a, b] \subset \mathbb{R}$ in the study of Fourier series. He proved that a function is of bounded variation if and only if it can be represented as the difference of two increasing functions. This representation is known as the *Jordan decomposition*. We refer the interested reader to the comprehensive book on functions of bounded variation by Appell, Banaś and Merentes [1].

Now, we will focus our attention on the problem of separation by a function of bounded variation. Let us observe that if there exists a function $h\colon [a, b] \to \mathbb{R}$ such that

$$g(x) \le h(x) \le f(x), \quad x \in [a, b],$$

then for any partition $(x_0, \ldots, x_n) \in \mathcal{P}_{[a,b]}$ we have

$$\begin{aligned}|h(x_j) - h(x_{j-1})| &= \max\{h(x_j) - h(x_{j-1}), h(x_{j-1}) - h(x_j)\}\\ &\le \max\{f(x_j) - g(x_{j-1}), f(x_{j-1}) - g(x_j)\}.\end{aligned}$$

This leads us to the following definitions. For arbitrary two functions $f, g\colon [a, b] \to \mathbb{R}$, we define

$$M_{f,g}(x) := \max\{f(x), g(x)\}, \quad m_{f,g}(x) := \min\{f(x), g(x)\},$$

and

$$\beta_{f,g}(x, y) := \max\left\{M_{f,g}(x) - m_{f,g}(y), M_{f,g}(y) - m_{f,g}(x)\right\}. \tag{3.1}$$

It is easy to see that, if $g(x) \le f(x),\ x \in [a, b]$, then

$$\beta_{f,g}(x, y) = \max\{f(x) - g(y), f(y) - g(x)\}.$$

Proposition 3.1 *Let $f, g\colon [a, b] \to \mathbb{R}$ and let $\beta_{f,g}$ be a function given by the formula (3.1). Then for all $x, y, z \in [a, b]$ the following conditions hold:*

(a) $\beta_{f,g}(x, y) = \beta_{f,g}(y, x)$,
(b) $\beta_{f,g}(x, x) = |f(x) - g(x)|$,
(c) $\beta_{f,f}(x, y) = |f(x) - f(y)|$,
(d) $\beta_{f,g}(x, z) \le \beta_{f,g}(x, y) + \beta_{f,g}(y, z)$,
(e) $\max\{|f(x) - f(y)|, |g(x) - g(y)|\} \le \beta_{f,g}(x, y)$.

Proof The assertions (a)–(c) and (e) follow immediately from the definition of $\beta_{f,g}$. We prove assertion (d). Assume contrary, that is, for some $x, y, z \in I$ we have

$$\beta_{f,g}(x,z) + \beta_{f,g}(z,y) < \beta_{f,g}(x,y).$$

From one side,

$$\begin{aligned}\beta_{f,g}(x,y) > &\max\{M_{f,g}(z) - m_{f,g}(y), M_{f,g}(y) - m_{f,g}(z)\}\\ &+ \max\{M_{f,g}(z) - m_{f,g}(x), M_{f,g}(x) - m_{f,g}(z)\}\\ \geq &M_{f,g}(z) - m_{f,g}(y) + M_{f,g}(x) - m_{f,g}(z)\\ \geq &M_{f,g}(x) - m_{f,g}(y),\end{aligned}$$

and on the other side

$$\begin{aligned}\beta_{f,g}(x,y) > &\max\{M_{f,g}(z) - m_{f,g}(y), M_{f,g}(y) - m_{f,g}(z)\}\\ &+ \max\{M_{f,g}(z) - m_{f,g}(x), M_{f,g}(x) - m_{f,g}(z)\}\\ \geq &M_{f,g}(y) - m_{f,g}(z) + M_{f,g}(z) - m_{f,g}(x)\\ \geq &M_{f,g}(y) - m_{f,g}(x).\end{aligned}$$

Consequently,

$$\beta_{f,g}(x,y) > \max\{M_{f,g}(x) - m_{f,g}(y), M_{f,g}(y) - m_{f,g}(x)\} = \beta_{f,g}(x,y)$$

and this contradiction finishes the proof. □

The above proposition leads to the following concept of variation involving two functions. Let $f, g\colon [a,b] \to \mathbb{R}$ be two given functions. We define

$$V_a^b(f,g) := \sup\left\{\sum_{j=1}^{n} \beta_{f,g}(x_{j-1}, x_j) : (x_0, \ldots, x_n) \in \mathcal{P}_{[a,b]}\right\}$$

and $V_a^a(f,g) = |f(a) - g(a)|$. Clearly, $V_a^b(f,g) = V_a^b(g,f) \geq 0$, moreover, $V_a^b(f,f) = W_a^b(f)$.

As we will see the above defining variation $V_a^b(f,g)$ depending on two functions has similar properties to usual variation.

Theorem 3.2 *Let $f, g\colon [a,b] \to \mathbb{R}$ and $c \in (a,b)$ be an arbitrary point. Then $V_a^b(f,g) < \infty$ if and only if $V_a^c(f,g) < \infty$ and $V_c^b(f,g) < \infty$. Furthermore, if $V_a^b(f,g) < \infty$ then*

$$V_a^b(f,g) = V_a^c(f,g) + V_c^b(f,g).$$

Proof Assume that $V_a^b(f,g) < \infty$. First, we will prove that

$$V_a^b(f,g) \geq V_a^c(f,g) + V_c^b(f,g).$$

Let $(x_0, \ldots, x_p) \in \mathcal{P}_{[a,c]}$ and $(y_0, \ldots, y_q) \in \mathcal{P}_{[c,b]}$ be arbitrary partitions. Clearly, $(z_0, \ldots, z_{p+q}) \in \mathcal{P}_{[a,b]}$, where

$$z_j = \begin{cases} x_j, & j = 0, \ldots, p \\ y_{j-p}, & j = p, \ldots, p+q \end{cases}$$

Then

$$V_a^b(f,g) \geq \sum_{j=1}^{p+q} \beta_{f,g}(z_{j-1},z_j) = \sum_{j=1}^{p} \beta_{f,g}(x_{j-1},x_j) + \sum_{j=1}^{q} \beta_{f,g}(y_{j-1},y_j).$$

Taking the supremum over all partitions $(x_0,\ldots,x_p) \in \mathcal{P}_{[a,c]}$ and $(y_0,\ldots,y_q) \in \mathcal{P}_{[c,b]}$ we get

$$V_a^b(f,g) \geq V_a^c(f,g) + V_c^b(f,g).$$

Now, let $(x_0,\ldots,x_n) \in \mathcal{P}_{[a,b]}$ be an arbitrary partition. Since $c \in (a,b)$ then there exists a $k \in \{1,\ldots,n\}$ such that

$$x_{k-1} \leq c \leq x_k.$$

Let define the new partitions

$$y_0 = x_0 = a,\ \ldots,\ y_{k-1} = x_{k-1},\ y_k = c$$

and

$$z_0 = c,\ z_1 = x_k,\ \ldots,\ z_{n-k+1} = x_n = b.$$

Obviously $(y_0,\ldots,y_k) \in \mathcal{P}_{[a,c]}$ and $(z_0,\ldots,z_{n-k+1}) \in \mathcal{P}_{[c,b]}$, so

$$\sum_{j=1}^{n} \beta_{f,g}(x_{j-1},x_j) \leq \sum_{j=1}^{k} \beta_{f,g}(y_{j-1},y_j) + \sum_{j=1}^{n-k+1} \beta_{f,g}(z_{j-1},z_j) \leq V_a^c(f,g) + V_c^b(f,g).$$

Taking the supremum over all partitions $(x_0,\ldots,x_n) \in \mathcal{P}_{[a,b]}$, we obtain the reverse inequality

$$V_a^b(f,g) \leq V_a^c(f,g) + V_c^b(f,g).$$

The proof is finished. □

Corollary 3.3 *Let $f,g\colon [a,b] \to \mathbb{R}$ be such functions that $V_a^b(f,g) < \infty$. Then the function*

$$[a,b] \ni x \longrightarrow V_a^x(f,g)$$

is nondecreasing.

Proof Fix $x,y \in [a,b],\ x < y$ arbitrarily. By the previous theorem we get

$$V_a^y(f,g) - V_a^x(f,g) = V_x^y(f,g) \geq 0.$$

□

Now, using the concept of bounded variation involving two functions we obtain the following separation theorem for functions of bounded variation.

Theorem 3.4 *Let $f,g\colon [a,b] \to \mathbb{R}$. Then for every function $h\colon [a,b] \to \mathbb{R}$ satisfying the inequalities*

$$m_{f,g}(x) \leq h(x) \leq M_{f,g}(x), \quad x \in [a,b] \tag{3.2}$$

we get

$$W_a^b(h) \leq V_a^b(f,g),$$

in particular,

$$\max\{W_a^b(f), W_a^b(g)\} \leq V_a^b(f, g).$$

Proof It follows immediately from inequalities (3.2) that for any partition $(x_0, \ldots, x_n) \in \mathcal{P}_{[a,b]}$ we have

$$\sum_{j=1}^{n} |h(x_j) - h(x_{j-1})| \leq \sum_{j=1}^{n} \beta_{f,g}(x_{j-1}, x_j) \leq V_a^b(f, g),$$

Consequently, by taking the supremum over all partitions $(x_0, \ldots, x_n) \in \mathcal{P}_{[a,b]}$ we obtain

$$W_a^b(h) \leq V_a^b(f, g).$$

Therefore, since f and g satisfies (3.2) we get

$$\max\{W_a^b(f), W_a^b(g)\} \leq V_a^b(f, g).$$

□

Remark 3.5 It follows from the above theorem that if $V_a^b(f, g) < \infty$ and

$$f(x) \leq g(x), \quad x \in [a, b]$$

then there always exists a function of bounded variation $h\colon [a, b] \to \mathbb{R}$ which lies between f and g. Indeed, it is enough to put $h := f$ or $h := g$. Then the separation property is trivially fulfilled.

The following example below shows that if the condition $V_a^b(f, g) < \infty$ is not satisfied then it can happen that there is no function of bounded variation between f and g so this condition is sufficient for separation by a function of bounded variation. From this example it also follows that the condition $V_a^b(f, g) < \infty$ is not necessary for separation by a function of bounded variation.

Example 3.6 Let $\varepsilon > 0$ and let $f, g : [0, 2] \to \mathbb{R}$ be given by the formulas

$$g(x) = \begin{cases} \sin(\frac{\pi}{x}) - \varepsilon, & x \in (0, 2], \\ 0, & x = 0, \end{cases} \qquad f(x) = \begin{cases} \sin\left(\frac{\pi}{x}\right), & x \in (0, 2], \\ 0, & x = 0. \end{cases}$$

Consider a sequence of partitions

$$\Pi_n = \left(0, \frac{2}{2n-1}, \frac{2}{2n-3}, \ldots, \frac{2}{5}, \frac{2}{3}, 2\right) \in \mathcal{P}_{[0,2]}.$$

Clearly, $g \leq f$, moreover, for any $h : \ g \leq h \leq f$ and $\varepsilon \in (0, 2)$ we have

$$V_0^2(f, g) \geq W_0^2(h) \geq \sum_{j=1}^{n} (2 - \varepsilon) = n(2 - \varepsilon) \xrightarrow{n \to \infty} \infty.$$

On the other hand if $\varepsilon \in [2, \infty)$ then the function $h \equiv -1$ is of bounded variation, and

$$g(x) \leq h(x) \leq f(x), \ x \in [0, 2].$$

As an immediate consequence of Theorem 3.4 we get

Corollary 3.7 *Let* $f, g\colon [a, b] \to \mathbb{R}$. *Then*

(a) *if* $V_a^b(f, g) < \infty$ *then* $W_a^b(f) < \infty$ *and* $W_a^b(g) < \infty$,
(b) *if* $W_a^b(f) = \infty$ *or* $W_a^b(g) = \infty$ *then* $V_a^b(f, g) = \infty$.

The following example shows that the converse to the assertion (a) from the previous corollary does not hold in general.

Example 3.8 Consider two constant functions $f, g : [0, 1] \to \mathbb{R}$, $f(x) = 0$, $g(x) = 1$, $x \in [0, 1]$. Clearly, $W_a^b(f) = W_a^b(g) = 0$. On the other hand for an arbitrary sequence of partitions

$$\pi_n = (x_0^n, x_1^n, \dots, , x_{k_n}^n),\ n \in \mathbb{N}$$

we have

$$\beta_{f,g}(x_{j-1}^n, x_j^n) = 1, \quad n \in \mathbb{N}.$$

Consequently,

$$k_n = \sum_{j=1}^{k_n} \beta_{f,g}(x_{j-1}^n, x_j^n) \le V_a^b(f, g), \quad n \in \mathbb{N},$$

so $V_a^b(f, g) = \infty$.

The above example shows that if $V_a^b(f, g) < \infty$ then the functions f and g can not be different on a big set. The next theorem gives a characterization of functions f and g for which $V_a^b(f, g) < \infty$. In order to present this theorem we need the following notation: for a function $f : [a, b] \to \mathbb{R}$ let D_f denotes the set of all discontinuity points of f, i.e.

$$D_f = \{x \in [a, b] : f \text{ is discontinuous at } x\}.$$

It is well known that if f is a function of bounded variation then D_f is at most countable set. Moreover, each function of bounded variation has one sides limits at any point of $[a, b]$.

The following theorem gives necessary and sufficient conditions under which two functions $f, g : [a, b] \to \mathbb{R}$ enjoy the property

$$V_a^b(f, g) < \infty.$$

Theorem 3.9 *Functions* $f, g : [a, b] \to \mathbb{R}$ *satisfy the condition*

$$V_a^b(f, g) < \infty,$$

if and only if they are both of bounded variation and

$$f(x) = g(x), \quad x \in [a, b] \setminus (D_f \cup D_g).$$

Furthermore,

$$\sum_{k=1}^{\infty} |f(x_k) - g(x_k)| < \infty,$$

in the case where $D_f \cup D_g = \{x_k : k \in \mathbb{N}\}$ *is an infinite sequence of discontinuity points of* f *and* g.

Proof If

$$V_a^b(f, g) < \infty,$$

then by Corollary 3.7 we have

$$\max\{W_a^b(f), W_a^b(g)\} \leq V_a^b(f, g),$$

hence f and g are both of bounded variation. Fix an arbitrary point $x_0 \in [a, b] \setminus (D_f \cup D_g)$. We show that

$$f(x_0) = g(x_0).$$

Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of points of interval $[a, b]$ such that

$$x_n < x_{n+1}, \quad n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} x_n = x_0.$$

Clearly,

$$(a, x_1, x_2, \ldots, x_n, b) \in \mathcal{P}_{[a,b]}, \quad n \in \mathbb{N}.$$

By the assumption,

$$\sum_{n=1}^{\infty} |f(x_n) - g(x_{n-1})| < \infty,$$

then

$$\lim_{n\to\infty} |f(x_n) - g(x_{n-1})| = 0,$$

and since both functions f and g are continuous at x_0 we obtain

$$f(x_0) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x_0).$$

In the case where

$$D_f \cup D_g = \{x_k : k \in \mathbb{N}\},$$

is an infinite set, using the inequalities

$$|f(x) - g(x)| \leq |f(x) - f(y)| + |f(y) - g(x)| \leq |f(x) - f(y)| + \beta_{f,g}(x, y),$$

we get

$$\begin{aligned}\sum_{k=1}^{\infty} |f(x_k) - g(x_k)| &\leq \sum_{k=1}^{\infty} |f(x_k) - f(x_{k-1})| + \sum_{k=1}^{\infty} \beta_{f,g}(x_k, x_{k-1}) \\ &\leq W_a^b(f) + V_a^b(f, g) < \infty.\end{aligned}$$

Conversely, assume that

$$\max\{W_a^b(f), W_a^b(g)\} < \infty, \quad f(x) = g(x),\ x \in [a, b] \setminus (D_f \cup D_g).$$

Moreover, in the case where the set of discontinuity points of f and g:

$$D_f \cup D_g = \{x_k : k \in \mathbb{N}\}$$

is infinite, we suppose that the condition

$$\sum_{k=1}^{\infty} |f(x_k) - g(x_k)| < \infty$$

holds. We will show that

$$V_a^b(f, g) < \infty.$$

It is easy to prove that the inequality

$$\beta_{f,g}(x, y) \leq \max\{|g(x) - g(y)|, |f(x) - f(y)|\} + |g(y) - f(y)|.$$

holds. Using this inequality we get

$$\begin{aligned}
&\sum_{k=1}^{n} \beta_{f,g}(x_k, x_{k-1}) \\
&\leq \sum_{k=1}^{n} \max\{|g(x_k) - g(x_{k-1})|, |f(x_k) - f(x_{k-1})|\} + \sum_{k=1}^{n} |f(x_k) - g(x_k)| \\
&\leq W_a^b(f) + W_a^b(g) + \sum_{k=1}^{\infty} |f(x_k) - g(x_k)| < \infty.
\end{aligned}$$

Taking the supremum over all partitions $(x_0, x_1, \ldots, x_n) \in \mathcal{P}_{[a,b]}$ we infer that

$$V_a^b(f, g) < \infty,$$

which finishes the proof. □

The above theorem shows that the first approach proposed in the present paper, although natural, leads us to the unusual phenomenon, namely, the bounded joint variation of f and g forces these functions to be essentially the same. This behavior seems unnatural. Therefore we are going to propose an another approach to the separation problem by a function of bounded variation. The starting point will be the following theorem which gives a necessary and sufficient condition that a given map has a bounded variation which is true even in the case where the map takes values in metric space.

Theorem 3.10 *Let (X, d) be a metric space. Then a function $f : [a, b] \to X$ is of bounded variation i.e.*

$$W_a^b(f) := \sup \left\{ \sum_{j=1}^{n} d(f(x_j), f(x_{j-1})) : (x_0, \ldots, x_n) \in \mathcal{P}_{[a,b]} \right\} < \infty$$

if and only if there exists an increasing function $k : [a, b] \to \mathbb{R}$ such that for all $x, y \in [a, b],\ x \leq y$

$$d(f(y), f(x)) \leq k(y) - k(x).$$

Proof Assume that $W_a^b(f) < \infty$. Let us define a function $k : [a, b] \to \mathbb{R}$ by the formula

$$k(x) := W_a^x(f).$$

Then for all $x, y \in [a, b], x \leq y$ we have

$$d(f(y), f(x)) \leq W_x^y(f) = W_a^y(f) - W_a^x(f) = k(y) - k(x).$$

Conversely, assume that there exists an increasing function $k\colon [a,b] \to \mathbb{R}$ such that for all $x, y \in [a,b]$, $x \leq y$ the inequality

$$d(f(y), f(x)) \leq k(y) - k(x)$$

holds. For arbitrary $(x_0, \ldots, x_n) \in \mathcal{P}_{[a,b]}$ we have

$$\sum_{j=1}^{n} d(f(x_j), f(x_{j-1})) \leq \sum_{j=1}^{n} (k(x_j) - k(x_{j-1})) = k(b) - k(a),$$

and taking the supremum over all partitions $(x_0, \ldots, x_n) \in \mathcal{P}_{[a,b]}$ we get

$$W_a^b(f) \leq k(b) - k(a) < \infty.$$

□

The above theorem allow us to consider the partial order $\preceq$ in the space $\bar{X} := X \times \mathbb{R}$ where (X, d) is a metric space defined in the following manner

$$\bar{Y_1} := (Y_1, y_1), \bar{Y_2} := (Y_2, y_2) \in \bar{X}, \quad \bar{Y_1} \preceq \bar{Y_2} \Leftrightarrow d(Y_1, Y_2) \leq y_2 - y_1.$$

A slightly different order was used by Ekeland in [3] to prove his famous variational principle which has a number applications in convex analysis. In the case where $(X, \|\cdot\|)$ is a normed space the above mentioned partially order is generated by so called *Lorentz cone*:

$$C := \{(Y, y) \in \bar{X} : \|Y\| \leq y\},$$

that is, for $\bar{Y_1}, \bar{Y_2} \in \bar{X}$

$$\bar{Y_1} \preceq_C \bar{Y_2} \Leftrightarrow \bar{Y_2} - \bar{Y_1} \in C \Leftrightarrow \|Y_1 - Y_2\| \leq y_2 - y_1.$$

This partial order is compatible with the linear structure of $\bar{X}$, i.e.

- $x \preceq_C y \Rightarrow x + z \preceq_C y + z \quad$ for $x, y, z \in \bar{X}$,
- $x \preceq_C y \Rightarrow \alpha x \preceq_C \alpha y \quad$ for $x, y \in \bar{X},\ \alpha \geq 0$,

where the addition and scalar multiplication are defined coordinatewise.

The partial order generated by the Lorentz cone plays a fundamental role in the optimization theory and Jordan algebras. In this paper we will consider a very particular case where a normed space is $(\mathbb{R}, |\cdot|)$.

By virtue of Theorem 3.10 (applied for $d(x, y) = |x - y|$) a function $F\colon [a,b] \to \mathbb{R}$ is of bounded variation if and only if there exists an increasing function $f\colon [a,b] \to \mathbb{R}$ such that the inequality

$$|F(y) - F(x)| \leq f(y) - f(x)$$

holds for all $x, y \in [a,b],\ x \leq y$. By analogy to the notion of delta-convexity (see [12]) we say that F is a delta-increasing function with a control function f. Thus, the set of delta-increasing functions on $[a,b]$ forms the smallest linear space containing the increasing functions on $[a,b]$.

Note that, defining for given maps $F\colon [a,b] \to \mathbb{R}$ and $f\colon [a,b] \to \mathbb{R}$ the map $\overline{F}\colon [a,b] \to \mathbb{R} \times \mathbb{R}$ via the formula

$$\overline{F}(x) := (F(x), f(x)), \quad x \in [a,b],$$

we can rewrite the above inequality by the formula

$$x, y \in [a,b],\ x \leq y \Rightarrow \overline{F}(x) \preceq \overline{F}(y).$$

The above remark shows that the notion of bounded variation generalizes the notion of usual monotonicity by replacing the classic inequality by the relation of partial order induced by the Lorentz cone. The results for usual monotonicity are obtained by putting $F = 0$.

The separation theorem for functions of bounded variation with respect to a partial order generated by the Lorentz cone reads as follows.

Theorem 3.11 *Let $a, b \in \mathbb{R}$, $a < b$. Then the functions $f, g, F, G\colon [a,b] \to \mathbb{R}$ satisfy the inequality*

$$\forall_{x,y\in[a,b]}\, (x \le y \Rightarrow |G(x) - F(y)| \le f(y) - g(x))\,, \tag{3.3}$$

if and only if there exist functions $h, H\colon [a,b] \to \mathbb{R}$ such that for all $x, y \in [a,b]$

(a) $|G(x) - H(x)| \le h(x) - g(x)$,
(b) $|F(x) - H(x)| \le f(x) - h(x)$,
(c) $x \le y \Rightarrow |H(x) - H(y)| \le h(y) - h(x)$.

Proof Suppose that conditions (a)–(c) hold. Fix $x, y \in [a,b]$, $x \le y$ arbitrarily. By (a), (b), (c) and the triangle inequality we get

$$\begin{aligned}|G(x) - F(y)| &\le |G(x) - H(x)| + |H(x) - H(y)| + |H(y) - F(y)|\\ &\le h(x) - g(x) + h(y) - h(x) + f(y) - h(y) = f(y) - g(x).\end{aligned}$$

Conversely, suppose that the inequality (3.3) holds. Then

$$x \le y \Rightarrow \begin{cases} G(x) + g(x) \le F(y) + f(y),\\ g(x) - G(x) \le f(y) - F(y).\end{cases}$$

By Theorem 2.1 there exist increasing functions $h_1, h_2\colon [a,b] \to \mathbb{R}$ such that

$$G(x) + g(x) \le h_1(x) \le F(x) + f(x), \quad x \in [a,b]$$

and

$$g(x) - G(x) \le h_2(x) \le f(x) - F(x), \quad x \in [a,b].$$

Let define the functions $H, h\colon [a,b] \to \mathbb{R}$ by the formulas

$$H(x) := \frac{h_1(x) - h_2(x)}{2}, \quad h(x) := \frac{h_1(x) + h_2(x)}{2}, \quad x \in [a,b].$$

We will show that conditions (a)–(c) hold true.

To prove a), by adding the expression $\frac{h_1(x)-h_2(x)}{2}$ to both sides of the inequality

$$g(x) - G(x) \le h_2(x),$$

we get

$$\frac{h_1(x) - h_2(x)}{2} - G(x) \le \frac{h_1(x) + h_2(x)}{2} - g(x).$$

Analogously, by subtracting the term $\frac{h_1(x)+h_2(x)}{2}$ from both sides of inequality

$$g(x) + G(x) \le h_1(x),$$

we get

$$g(x) - \frac{h_1(x) + h_2(x)}{2} \le \frac{h_1(x) - h_2(x)}{2} - G(x),$$

and, consequently,

$$|H(x) - G(x)| \le h(x) - g(x), \quad x \in [a, b].$$

For (b), subtracting the expression $\frac{h_1(x)+h_2(x)}{2}$ from both sides of the inequality

$$h_2(x) \le f(x) - F(x),$$

we get

$$F(x) - \frac{h_1(x) - h_2(x)}{2} \le f(x) - \frac{h_1(x) + h_2(x)}{2}.$$

Similarly, by subtracting the expression $\frac{h_1(x)-h_2(x)}{2}$ from both sides of the inequality

$$h_1(x) \le F(x) + f(x)$$

we get

$$\frac{h_1(x) + h_2(x)}{2} - f(x) \le F(x) - \frac{h_1(x) - h_2(x)}{2}.$$

Therefore,

$$|F(x) - H(x)| \le f(x) - h(x), \quad x \in [a, b].$$

Part (c) follows immediately from the definition of H and h. Indeed, for $x, y \in [a, b]$, $x \le y$ we have

$$\begin{aligned} |H(y) - H(x)| &= \left| \frac{h_1(y) - h_2(y)}{2} - \frac{h_1(x) - h_2(x)}{2} \right| = \left| \frac{h_1(y) - h_1(x)}{2} + \frac{h_2(x) - h_2(y)}{2} \right| \\ &\le \left| \frac{h_1(y) - h_1(x)}{2} \right| + \left| \frac{h_2(x) - h_2(y)}{2} \right| = \frac{h_1(y) - h_1(x)}{2} + \frac{h_2(y) - h_2(x)}{2} \\ &= \frac{h_1(y) + h_2(y)}{2} - \frac{h_1(x) + h_2(x)}{2} = h(y) - h(x). \end{aligned}$$

The proof of the theorem is finished. □

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