Periodica Mathematica Hungarica (2023) 87:57–74
https://doi.org/10.1007/s10998-022-00499-x

Cesàro means with varying parameters of Walsh–Fourier series

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Accepted: 11 June 2022 / Published online: 4 November 2022
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Abstract
In this paper we discuss some convergence and divergence properties of subsequences of Cesàro means with varying parameters of Walsh–Fourier series. We give necessary and sufficient conditions for the convergence regarding the weighted variation of numbers.

Keywords Walsh–Fourier series · Cesàro means · Convergence in norm

Mathematics Subject Classification 42C10

1 Walsh functions

We shall denote the set of all nonnegative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $\mathbb{I} := [0, 1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \le p \le 2^n$.

Denote the dyadic enpansion of $n \in \mathbb{N}$ and $x \in \mathbb{I}$ by

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n)\, 2^j,\ \varepsilon_j(n) = 0, 1,$$

and

$$x = \sum_{j=0}^{\infty} \frac{x_j}{2^{j+1}},\ x_j = 0, 1.$$

György Gát is supported by the project EFOP-3.6.1-16-2016-00022 and EFOP-3.6.2-16-2017-00015 supported by the European Union, co-financed by the European Social Fund. Ushangi Goginava is very thankful to the United Arab Emirates University (UAEU) for the Start-up Grant 12S100.

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Denote by $\dot{+}$ the logical addition on $\mathbb{I}$. That is, for any $x, y \in \mathbb{I}$ and $k, n \in \mathbb{N}$,

$$x \dot{+} y := \sum_{n=0}^{\infty} |x_n - y_n| \, 2^{-(n+1)},$$

$$k \oplus n := \sum_{i=0}^{\infty} |\varepsilon_i(k) - \varepsilon_i(n)| \, 2^i, \tag{1.1}$$

and by the definition of w_n we have

$$w_{k \oplus n} = w_k w_n. \tag{1.2}$$

The sets $I_n(x) := \{y \in \mathbb{I} : y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$ for $x \in \mathbb{I}$, $I_n := I_n(0)$ for $0 < n \in \mathbb{N}$ and $I_0(x) := \mathbb{I}$ are the dyadic intervals of $\mathbb{I}$. For $0 < n \in \mathbb{N}$ let $|n| := \max\left\{j \in \mathbb{N} : n_j \neq 0\right\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Rademacher system is defined by

$$\rho_n(x) := (-1)^{x_n} \quad (x \in \mathbb{I}, n \in \mathbb{N}).$$

The Walsh–Paley system is defined as the sequence of the Walsh–Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (\rho_k(x))^{\varepsilon_k(n)} = (-1)^{\sum_{k=0}^{|n|} \varepsilon_k(n) x_k} \quad (x \in \mathbb{I}, n \in \mathbb{N}).$$

The Walsh–Dirichlet kernel is defined by

$$D_n(x) := \sum_{k=0}^{n-1} w_k(x) \ (n \in \mathbb{N}), \ D_0 = 0.$$

Recall that (see [23])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n(0), \\ 0, & \text{if } x \in \mathbb{I} \backslash I_n(0). \end{cases} \tag{1.3}$$

As usual, denote by $L_1(\mathbb{I})$ the set of measurable functions defined on $\mathbb{I}$ for which

$$\|f\|_1 := \int_{\mathbb{I}} |f(t)| \, dt < \infty.$$

Let $f \in L_1(\mathbb{I})$. The partial sums of the Walsh–Fourier series are defined as follows:

$$S_M(x, f) := \sum_{i=0}^{M-1} \widehat{f}(i) \, w_i(x),$$

where the number

$$\widehat{f}(i) = \int_{\mathbb{I}} f(t) \, w_i(t) \, dt$$

is said to be the ith Walsh–Fourier coefficient of the function f. Set

$$E_n(x, f) := S_{2^n}(x, f).$$

The maximal function is defined by

$$E^*(x, f) := \sup_{n \in \mathbb{N}} E_n(x, |f|).$$

2 Cesàro means with varying parameters

The (C, α_n) means of the Walsh–Fourier series of the function f is given by

$$\sigma_n^{\alpha_n}(f, x) := \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{n} A_{n-j}^{\alpha_n - 1} S_j(f, x) = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n-1} A_{n-j}^{\alpha_n} \widehat{f}(j)\, w_j(x),$$

where

$$A_n^{\alpha_n} := \frac{(1+\alpha_n)\cdots(n+\alpha_n)}{n!}$$

for any $n \in \mathbf{N}$, $\alpha_n \neq -1, -2, \ldots$. It is known that [29]

$$A_n^{\alpha_n} = \sum_{k=0}^{n} A_k^{\alpha_n - 1}, \; A_n^{\alpha_n - 1} = \frac{\alpha_n}{\alpha_n + n} A_n^{\alpha_n}. \tag{2.1}$$

The (C, α_n) kernel is defined by

$$K_n^{\alpha_n} = \frac{1}{A_{n-1}^{\alpha}} \sum_{j=1}^{n} A_{n-j}^{\alpha_n - 1} D_j = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n-1} A_{n-j-1}^{\alpha_n} w_j.$$

The idea of Cesàro means with variable parameters of numerical sequences is due to Kaplan [17] and the introduction of these (C, α_n) means of Fourier series is due to Akhobadze [2], [3] who investigated the behavior of the L_1-norm convergence of $\sigma_n^{\alpha_n}(f) \to f$ for the trigonometric system.

The a.e. divergence of Cesàro means with varying parameters of Walsh–Fourier series was investigated by Tetunashvili [25]. Abu Joudeh and Gát in [1] proved the almost everywhere convergence (with some restrictions) of the Cesàro (C, α_n) means of integrable functions.

The first result with respect to the a.e. convergence of the Walsh–Fejér means $\sigma_n^{\alpha_n} f$ for all integrable function f with constant sequence $\alpha_n = \alpha > 0$ is due to Fine [6] (see also Weisz [27]). On the rate of convergence of Cesàro means in this constant case see the papers of Yano [28] and Fridli [8]. Approximation properties of Cesàro means of negative order with constant sequence was investigated by the second author in [14]. That is why we investigate only the case when $\alpha_n \in (0, 1)$ for every n. It is true that in order for the definition of Cesàro mean to make sense we should only suppose that the paremeter is not a negative integer, but for negative parameters there are available divergence results (see, e.g., [14]) and for parameters at least 1 we have the very well-known almost everywhere and norm convergence results of these means. In other words, the only interesting situation is when $\alpha_n \in (0, 1)$ for every n.

It is easy to see that

$$\sigma_n^{\alpha_n}(f, x) = \left(f * K_n^{\alpha_n}\right)(x) = \int_{\mathbb{I}} f(t)\, K_n^{\alpha_n}(x+t)\, dt.$$

The Fejér kernel is defined by

$$K_n(t) := \frac{1}{n} \sum_{k=1}^{n} D_k(t).$$

The following estimate was proved by Akhobadze [2, 3]. If $k, n \in \mathbb{N}$, then

$$\begin{aligned} & c_1 (1+\alpha_n)(2+\alpha_n) k^{\alpha_n} < A_k^{\alpha_n} < c_2 (1+\alpha_n)(2+\alpha_n) k^{\alpha_n}, \\ & \quad \text{when } -2 < \alpha_n < -1; \end{aligned} \tag{2.2}$$

$$c_1\,(1+\alpha_n)\,k^{\alpha_n} < A_k^{\alpha_n} < c_2\,(1+\alpha_n)\,k^{\alpha_n},\ \text{when } -1<\alpha_n<0; \tag{2.3}$$
$$c_1\,(d)\,k^{\alpha_n} < A_k^{\alpha_n} < c_2\,(d)\,k^{\alpha_n},\ \text{when } 0<\alpha_n\le d. \tag{2.4}$$

3 L_1-estimation for the kernel $K_n^{\alpha_n}$

For $n=\sum_{j=0}^{\infty}\varepsilon_j\,(n)\,2^j$, $\varepsilon_j\,(n)=0,1$, we define

$$n^{(s)}:=\sum_{j=s}^{\infty}\varepsilon_j\,(n)\,2^j,\ n_{(s)}:=n-n^{(s+1)}=\sum_{j=0}^{s}\varepsilon_j\,(n)\,2^j,\ n_{(-1)}:=0.$$

Theorem 3.1 *If $\alpha_n\in(0,1)$ and $n\in\mathbb{N}$, then there exist positive constants c_1 and c_2 (independent of n) such that*

$$\frac{c_1}{n^{\alpha_n}}\sum_{k=0}^{|n|}|\varepsilon_k\,(n)-\varepsilon_{k+1}\,(n)|\,2^{k\alpha_n}\le\left\|K_n^{\alpha_n}\right\|_1\le\frac{c_2}{n^{\alpha_n}}\sum_{k=0}^{|n|}|\varepsilon_k\,(n)-\varepsilon_{k+1}\,(n)|\,2^{k\alpha_n}.$$

Proof of Theorem 3.1 We can write

$$\begin{aligned}A_{n-1}^{\alpha_n}K_n^{\alpha_n}&=\sum_{j=0}^{n-1}A_{n-j-1}^{\alpha_n}w_j=\sum_{s=0}^{|n|}\varepsilon_s\,(n)\sum_{j=n^{(s+1)}}^{n^{(s)}-1}A_{n-j-1}^{\alpha_n}w_j\\&=\sum_{s=0}^{|n|}\varepsilon_s\,(n)\sum_{j=0}^{2^s-1}A_{n_{(s)}-j-1}^{\alpha_n}w_{j+n^{(s+1)}}\\&=\sum_{s=0}^{|n|}\varepsilon_s\,(n)\,w_{n^{(s+1)}}\sum_{j=0}^{2^s-1}A_{n_{(s)}-j-1}^{\alpha_n}w_j.\end{aligned}\tag{3.1}$$

Since in the case of $\varepsilon_s(n)=1$

$$n_{(s)}-j-1=n_{(s-1)}+2^s-1-j,$$

and

$$2^s-1-j=\left(2^s-1\right)\oplus j,$$

from (1.2) and (3.1) we obtain

$$\begin{aligned}A_{n-1}^{\alpha_n}K_n^{\alpha_n}&=\sum_{s=0}^{|n|}\varepsilon_s\,(n)\,w_{n^{(s+1)}}\sum_{j=0}^{2^s-1}A_{n_{(s-1)}+j}^{\alpha_n}w_{(2^s-1)\oplus j}\\&=\sum_{s=0}^{|n|}\varepsilon_s\,(n)\,w_{n^{(s)}-1}\sum_{j=0}^{2^s-1}A_{n_{(s-1)}+j}^{\alpha_n}w_j.\end{aligned}\tag{3.2}$$

Applying Abel's transformation twice we get

$$\sum_{j=0}^{2^s-1}A_{n_{(s-1)}+j}^{\alpha_n}w_j=\sum_{j=1}^{2^s-1}A_{n_{(s-1)}+j}^{\alpha_n-2}jK_j-A_{n_{(s)}-1}^{\alpha_n-1}2^sK_{2^s}+A_{n_{(s)}-1}^{\alpha_n}D_{2^s}.\tag{3.3}$$

Combining (3.2)–(3.3) we obtain

$$
\begin{aligned}
K_n^{\alpha_n} =& \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n)\, w_{n^{(s)}-1} \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-2}\, j K_j \\
& - \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n)\, w_{n^{(s)}-1} A_{n_{(s)}-1}^{\alpha_n-1} 2^s K_{2^s} \\
& + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n)\, w_{n^{(s)}-1} A_{n_{(s)}-1}^{\alpha_n} D_{2^s} \\
=:& Q_1(n) + Q_2(n) + Q_3(n).
\end{aligned} \tag{3.4}
$$

From the estimates (2.2)–(2.4) we get

$$
\begin{aligned}
& \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{j=1}^{2^s-1} j \left| A_{n_{(s-1)}+j}^{\alpha_n-2} \right| \\
& \le \frac{c\alpha_n}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{j=1}^{2^s-1} j \left(n_{(s-1)} + j\right)^{\alpha_n-2} \\
& \le \frac{c\alpha_n}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{j=1}^{2^s-1} \left(n_{(s-1)} + j\right)^{\alpha_n-1} \\
& \le \frac{c}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) \sum_{j=1}^{2^s-1} A_{n_{(s-1)}+j}^{\alpha_n-1} \\
& = \frac{c}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) \left(A_{n_{(s)}}^{\alpha_n} - A_{n_{(s-1)}}^{\alpha_n} \right) \\
& \le c < \infty.
\end{aligned}
$$

Since $\sup_n \|K_n\|_1 < 2$ (see [24]; it even holds $\sup_n \|K_n\|_1 = 17/15$, see [26]) from (3.4) we infer

$$
\|Q_l(n)\|_1 \le c < \infty, l = 1, 2. \tag{3.5}
$$

Next, we find an upper estimate for $\|Q_3(n)\|_1$. We have

$$
w_n Q_3(n) = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n)\, w_n w_{n^{(s)}-1} A_{n_{(s)}-1}^{\alpha_n} D_{2^s}.
$$

From (1.2) (we can suppose $\varepsilon_s(n) = 1$ otherwise nothing is to be added), we get

$$
\begin{aligned}
w_n w_{n^{(s)}-1} &= w_n w_{n^{(s+1)}+2^s-1} = w_n w_{n^{(s+1)}} w_{2^s-1} \\
&= w_{n \oplus n^{(s+1)}} w_{2^s-1} = w_{n_{(s)}} w_{2^s-1} \\
&= w_{2^s} w_{n_{(s-1)}} w_{2^s-1} = w_{2^s} w_{n_{(s-1)\oplus(2^s-1)}}.
\end{aligned}
$$

Since $n_{(s-1)\oplus(2^s-1)} < 2^s$, from (1.3) we obtain

$$
D_{2^s} w_{n_{(s-1)\oplus(2^s-1)}} = D_{2^s}.
$$

Consequently,

$$
\begin{aligned}
w_n Q_3(n) =& \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) A_{n_{(s)}-1}^{\alpha_n} w_{2^s} D_{2^s} \\
=& \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=0}^{|n|} \varepsilon_s(n) A_{n_{(s)}-1}^{\alpha_n} \left(D_{2^{s+1}} - D_{2^s}\right) \\
=& \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=1}^{|n|} \left(\varepsilon_{s-1}(n) - \varepsilon_s(n)\right) A_{n_{(s)}-1}^{\alpha_n} D_{2^s} + \\
& + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=1}^{|n|} \varepsilon_s(n) \left(A_{n_{(s-1)}-1}^{\alpha_n} - A_{n_{(s)}-1}^{\alpha_n}\right) D_{2^s} + \\
& + \frac{1}{A_{n-1}^{\alpha_n}} \varepsilon_{|n|}(n) A_{n_{(|n|-1)}-1}^{\alpha_n} D_{2^{|n|+1}} - \\
& - \frac{1}{A_{n-1}^{\alpha_n}} \varepsilon_0(n) A_{n_{(0)}-1}^{\alpha_n} D_1 \\
=:& Q_{31}(n) + Q_{32}(n) + Q_{33}(n) + Q_{34}(n).
\end{aligned} \tag{3.6}
$$

From (2.2)–(2.4) and (1.3) we get

$$
\|Q_{32}(n)\|_1 \le \frac{1}{A_{n-1}^{\alpha_n}} \sum_{s=1}^{|n|} \left(A_{n_{(s)}-1}^{\alpha_n} - A_{n_{(s-1)}-1}^{\alpha_n}\right) \le c < \infty,
$$

$$
\|Q_{33}(n)\|_1, \|Q_{34}(n)\|_1 \le c < \infty, \tag{3.7}
$$

$$
\|Q_{31}(n)\|_1 \le \frac{c}{n^{\alpha_n}} \sum_{s=1}^{|n|} \left|\varepsilon_{s-1}(n) - \varepsilon_s(n)\right| 2^{s\alpha_n}. \tag{3.8}
$$

Combining (3.4)–(3.8) we conclude that

$$
\left\|K_n^{\alpha_n}\right\|_1 \le \frac{c_1}{n^{\alpha_n}} \sum_{k=1}^{|n|} \left|\varepsilon_k(n) - \varepsilon_{k+1}(n)\right| 2^{k\alpha_n}. \tag{3.9}
$$

Hence the upper estimate is proved.

Now, we find a lower estimate for $\|Q_{31}(n)\|_1$. Let a_i and $b_i, i = 1, \ldots, s$, be strictly increasing sequences, i.e.,

$$
0 \le a_1 \le b_1 < a_2 \le b_2 < \cdots < a_s \le b_s < a_{s+1} = \infty
$$

for which

$$
\varepsilon_j(n) = \begin{cases} 1, & a_i \le j \le b_i, \\ 0, & b_i < j < a_{i+1}. \end{cases} \tag{3.10}
$$

Then it is evident that $b_j + 1 \le a_{j+1}$. But we may suppose even more, namely

$$
b_j + 2 \le a_{j+1} \tag{3.11}
$$

and then the intervals defined below are disjoint sets. That is, set

$$
A_k := \left(\frac{1}{2^{a_k+1}}, \frac{1}{2^{a_k}}\right), B_k := \left(\frac{1}{2^{b_k+2}}, \frac{1}{2^{b_k+1}}\right), k = 1, \ldots, s.
$$

Let $x \in A_k$. Then we can write

$$\begin{aligned} |Q_{31}(n)(x)| &= \left| \frac{-1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{s} A_{n_{(a_j)}-1}^{\alpha_n} D_{2^{a_j}}(x) + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{s} A_{n_{(b_j+1)}-1}^{\alpha_n} D_{2^{b_j+1}}(x) \right| \\ &= \left| \frac{-1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{k} A_{n_{(a_j)}-1}^{\alpha_n} 2^{a_j} + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{k-1} A_{n_{(b_j+1)}-1}^{\alpha_n} 2^{b_j+1} \right| \\ &= \left| \frac{1}{A_{n-1}^{\alpha_n}} A_{n_{(a_k)}-1}^{\alpha_n} 2^{a_k} - \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{k-1} \left(A_{n_{(b_j+1)}-1}^{\alpha_n} 2^{b_j+1} - A_{n_{(a_j)}-1}^{\alpha_n} 2^{a_j} \right) \right|. \end{aligned} \tag{3.12}$$

Since

$$b_{k-1} + 1 \le a_k - 1 < a_k,$$

we can write

$$\begin{aligned} &\frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{k-1} \left(A_{n_{(b_j+1)}-1}^{\alpha_n} 2^{b_j+1} - A_{n_{(a_j)}-1}^{\alpha_n} 2^{a_j} \right) \\ &\le \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{k-1} \left(A_{n_{(b_j+1)}-1}^{\alpha_n} 2^{b_j+1} - A_{n_{(b_{j-1}+1)}-1}^{\alpha_n} 2^{b_{j-1}+1} \right) \\ &\le \frac{1}{A_{n-1}^{\alpha_n}} A_{n_{(b_{k-1}+1)}-1}^{\alpha_n} 2^{b_{k-1}+1}. \end{aligned}$$

Consequently, from (3.12) we obtain

$$\begin{aligned} |Q_{31}(n)(x)| &\ge \frac{1}{A_{n-1}^{\alpha_n}} \left(A_{n_{(a_k)}-1}^{\alpha_n} 2^{a_k} - A_{n_{(b_{k-1}+1)}-1}^{\alpha_n} 2^{b_{k-1}+1} \right) \\ &\ge \frac{1}{A_{n-1}^{\alpha_n}} \left(A_{n_{(a_k)}-1}^{\alpha_n} 2^{a_k} - A_{n_{(a_k)}-1}^{\alpha_n} 2^{a_k-1} \right) \\ &= \frac{A_{n_{(a_k)}-1}^{\alpha_n}}{A_{n-1}^{\alpha_n}} \left(2^{a_k} - 2^{a_k-1} \right) \\ &= \frac{1}{2} \frac{A_{n_{(a_k)}-1}^{\alpha_n}}{A_{n-1}^{\alpha_n}} 2^{a_k}. \end{aligned}$$

Hence,

$$\int_{A_k} |Q_{31}(n)| \ge \frac{A_{n_{(a_k)}-1}^{\alpha_n}}{4} \frac{1}{A_{n-1}^{\alpha_n}}. \tag{3.13}$$

Let $x \in B_k$. Since $n_{(b_j+1)} = n_{(b_j)}$ and $n_{(a_j)} \le n_{(b_j)}$, we have

$$\begin{aligned} w_n Q_{31}(n)(x) &= \frac{-1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{k} A_{n_{(a_j)}-1}^{\alpha_n} 2^{a_j} + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{k} A_{n_{(b_j+1)}-1}^{\alpha_n} 2^{b_j+1} \\ &= \frac{-1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{k} A_{n_{(a_j)}-1}^{\alpha_n} 2^{a_j} + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{k} A_{n_{(b_j)}-1}^{\alpha_n} 2^{b_j+1} \end{aligned}$$

$$\geq \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{k} A_{n_{(b_j)}-1}^{\alpha_n} 2^{b_j+1} - \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{k} A_{n_{(b_j)}-1}^{\alpha_n} 2^{b_j}$$

$$\geq \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{k} A_{n_{(b_j)}-1}^{\alpha_n} 2^{b_j}$$

$$\geq \frac{1}{A_{n-1}^{\alpha_n}} A_{n_{(b_k)}-1}^{\alpha_n} 2^{b_k}.$$

Hence

$$\int_{B_k} |Q_{31}(n)| \geq \frac{A_{n_{(b_k)}-1}^{\alpha_n}}{4} \frac{1}{A_{n-1}^{\alpha_n}}. \tag{3.14}$$

Since $A_i, B_i, i = 1, \ldots, s$, are pairwise disjoint from (3.13) and (3.14), we have

$$\int_{\mathbb{I}} |Q_{31}(n)| \geq \sum_{k=1}^{s} \left(\int_{A_k} |Q_{31}(n)| + \int_{B_k} |Q_{31}(n)| \right)$$

$$\geq \frac{1}{4A_{n-1}^{\alpha_n}} \sum_{k=1}^{s} \left(A_{n_{(a_k)}-1}^{\alpha_n} + A_{n_{(b_k)}-1}^{\alpha_n} \right).$$

Combining (3.5)–(3.14) we complete the proof of Theorem 3.1. □

4 Uniform and *L*-convergence of (C, α_n) means

The Hardy space $H_1(\mathbb{I})$ consists all functions f that satisfy

$$\|f\|_{H_1} := \left\|E^*(f)\right\|_1 < \infty.$$

For a nonnegative integer n let

$$V(n) := \sum_{i=0}^{\infty} |\varepsilon_i(n) - \varepsilon_{i+1}(n)| + \varepsilon_0(n)$$

and

$$V(n, \alpha) := \frac{1}{n^{\alpha_n}} \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| \, 2^{k\alpha_n}.$$

Let $C_w(\mathbb{I})$ denote the space of uniformly continuous functions on $\mathbb{I}$ with the supremum norm

$$\|f\|_{C_w} := \sup_{x \in \mathbb{I}} |f(x)| \quad (f \in C_w(\mathbb{I})).$$

Let $X = X(\mathbb{I})$ be either the space $L_1(\mathbb{I})$, or the space of uniformly continuous functions, that is, $C_w(\mathbb{I})$. The corresponding norm is denoted by $\|\cdot\|_X$.

We remind the reader that $C_w(\mathbb{I})$ is the collection of all functions $f : \mathbb{I} \to \mathbb{R}$ that are uniformly continuous from the dyadic topology of $\mathbb{I}$ to the usual topology $\mathbb{R}$, or for short: uniformly w-continuous.

The modulus of continuity, when $X = C_w(\mathbb{I})$, and the integrated modulus of continuity, where $X = L_1(\mathbb{I})$ are defined by

$$\omega\left(\frac{1}{2^n}, f\right)_X = \sup_{h \in I_n} \| f(\cdot \dotplus h) - f(\cdot)\|_X .$$

For Walsh–Fourier series Fine [5] has obtained a sufficient condition for the uniform convergence which is in complete analogy with the Dini–Lipshitz condition (see also [23]). Similar results are true for the space of integrable functions $L_1(\mathbb{I})$ [22]. Gulicev [16] has estimated the rate of uniform convergence of a Walsh–Fourier series using the Lebesgue constant and the modulus of continuity. Uniform convergence of Walsh–Fourier series of the functions of classes of generalized bounded variation was investigated by the second author [13]. This problem has been considered for the Vilenkin group by Fridli [7] and Gát [9]. Lukomskii [19] considered uniform and L_1 -convergence of subsequences of partial sums of Walsh–Fourier series. In particular, he proved that the condition $\sup_A V(m_A) < \infty$ is necessary and sufficient for the uniform and L_1-convergence of subsequences of partial sums $S_{m_A}(f)$ of Walsh–Fourier series. In [4, 10–12, 15, 18–21] the X -norm convergence of subsequences of Walsh–Fourier series is investigated.

In this section we discuss some convergence and divergence properties of subsequences of Cesàro means with varying parameters of Walsh–Fourier series. The following are true.

Theorem 4.1 *If $f \in X(\mathbb{I})$ and $\alpha_n \in (0, 1)$, then*

$$\begin{aligned}\left\|\sigma_n^{\alpha_n}(f) - f\right\|_X \leq & c_1 \omega\left(\frac{1}{2^{|n|}}, f\right)_X V(n,\alpha) \\ & + c_2 \alpha_n \sum_{r=0}^{|n|-2} 2^{r-|n|} \omega\left(\frac{1}{2^r}, f\right)_X \\ & + c_3 \omega\left(\frac{1}{2^{|n|-1}}, f\right)_X .\end{aligned}$$

Theorem 4.1 implies

Theorem 4.2 *If for a function $f \in X(\mathbb{I})$ and a subsequence $\{m_n : n \in \mathbb{N}\}$ we have*

$$\omega\left(\frac{1}{m_n}, f\right)_X = o\left(\frac{1}{V(m_n,\alpha)}\right),$$

then the subsequence $\sigma_{m_n}^{\alpha_n}(f)$ converges in X-norm.

Theorem 4.3 *Let $\{m_n : n \in \mathbb{N}\}$ be such that*

$$\sup_n V(m_n,\alpha) = \infty.$$

Then there exists $\{p_k : k \in \mathbb{N}\}$ and a function $g \in X(\mathbb{I})$ such that

$$\omega\left(\frac{1}{m_{p_k}}, g\right)_X = O\left(\frac{1}{V(m_{p_k},\alpha)}\right)$$

and

$$\left\|\sigma_{m_{p_k}}^{\alpha_{p_k}}(g) - g\right\|_1 \nrightarrow 0 \text{ as } k \to \infty.$$

Theorem 4.4 *a) Let* $\{m_n : n \in \mathbb{N}\}$ *be such that*

$$\sup_n V(m_n, \alpha) < \infty.$$

Then the operator $\sigma_{m_n}^{\alpha_n}(f)$ *is bounded from the space* $L_1(\mathbb{I})$ *to the space* $L_1(\mathbb{I})$*;*
b) Let $\{m_n : n \in \mathbb{N}\}$ *be such that*

$$\sup_n V(m_n, \alpha) = \infty.$$

Then there exists $f \in H_1(\mathbb{I}) \subset L_1(\mathbb{I})$ *such that*

$$\sup_n \left\| \sigma_{m_n}^{\alpha_n}(f) \right\|_1 = \infty.$$

Proof of Theorem 4.1 We can write

$$\begin{aligned}
&\sigma_n^{\alpha_n}(x, f) - f(x) \\
&= \int_{\mathbb{I}} [f(x \dotplus t) - f(x)] \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{n} A_{n-j}^{\alpha_n - 1} D_j(t)\, dt \\
&= \int_{\mathbb{I}} [f(x \dotplus t) - f(x)] \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{2^{|n|-1}-1} A_{n-j}^{\alpha_n - 1} D_j(t)\, dt \\
&\quad + \int_{\mathbb{I}} [f(x \dotplus t) - f(x)] \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=2^{|n|-1}}^{2^{|n|}-1} A_{n-j}^{\alpha_n - 1} D_j(t)\, dt \\
&\quad + \int_{\mathbb{I}} [f(x \dotplus t) - f(x)] \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=2^{|n|}}^{n} A_{n-j}^{\alpha_n - 1} D_j(t)\, dt \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{4.1}$$

For J_3 we have

$$\begin{aligned}
J_3 &= \int_{\mathbb{I}} [f(x \dotplus t) - S_{2^{|n|}}(x \dotplus t, f)] \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=2^{|n|}}^{n} A_{n-j}^{\alpha_n - 1} D_j(t)\, dt \\
&\quad + \int_{\mathbb{I}} [S_{2^{|n|}}(x \dotplus t, f) - S_{2^{|n|}}(x, f)] \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=2^{|n|}}^{n} A_{n-j}^{\alpha_n - 1} D_j(t)\, dt \\
&\quad + \int_{\mathbb{I}} \left[S_{2^{|n|}}(x, f) - f(x)\right] \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=2^{|n|}}^{n} A_{n-j}^{\alpha_n - 1} D_j(t)\, dt \\
&=: J_{31} + J_{32} + J_{33}.
\end{aligned} \tag{4.2}$$

Now we can write

$$\begin{aligned}
J_{32} &= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=2^{|n|}}^{n} A_{n-j}^{\alpha_n - 1} \left(S_j * S_{2^{|n|}}\right)(x; f) - \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=2^{|n|}}^{n} A_{n-j}^{\alpha_n - 1} S_{2^{|n|}}(x; f) \\
&= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=2^{|n|}}^{n} A_{n-j}^{\alpha_n - 1} S_{2^{|n|}}(x; f) - \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=2^{|n|}}^{n} A_{n-j}^{\alpha_n - 1} S_{2^{|n|}}(x; f) = 0.
\end{aligned} \tag{4.3}$$

From Minkowski's inequality and Theorem 3.1 we get

$$\begin{aligned}\|J_{31}\|_X &\le \omega\left(\frac{1}{2^{|n|}}, f\right)_X \int_{\mathbb{I}} \frac{1}{A_{n-1}^{\alpha_n}} \left|\sum_{j=2^{|n|}}^{n} A_{n-j}^{\alpha_n-1} D_j(t)\right| dt \\ &\le \omega\left(\frac{1}{2^{|n|}}, f\right)_X (1+V(n,\alpha)). \end{aligned} \tag{4.4}$$

Analogously, we can prove

$$\|J_{33}\|_X \le \omega\left(\frac{1}{2^{|n|}}, f\right)_X (1+V(n,\alpha)). \tag{4.5}$$

Combining (4.2)–(4.5) we obtain

$$\|J_3\|_X \le 2\omega\left(\frac{1}{2^{|n|}}, f\right)_X (1+V(n,\alpha)). \tag{4.6}$$

We can write

$$\begin{aligned} J_1 =& \frac{1}{A_{n-1}^{\alpha_n}} \sum_{r=0}^{|n|-2} \int_{\mathbb{I}} [f(x \dot{+} t) - f(x)] \sum_{j=2^r}^{2^{r+1}-1} A_{n-j}^{\alpha_n-1} D_j(t)\, dt \\ =& \frac{1}{A_{n-1}^{\alpha_n}} \sum_{r=0}^{|n|-2} \int_{\mathbb{I}} [f(x \dot{+} t) - S_{2^r}(x \dot{+} t, f)] \sum_{j=2^r}^{2^{r+1}-1} A_{n-j}^{\alpha_n-1} D_j(t)\, dt \\ &+ \frac{1}{A_{n-1}^{\alpha_n}} \sum_{r=0}^{|n|-2} \int_{\mathbb{I}} [S_{2^r}(x, f) - f(x)] \sum_{j=2^r}^{2^{r+1}-1} A_{n-j}^{\alpha_n-1} D_j(t)\, dt \\ =&: J_{11} + J_{12}. \end{aligned} \tag{4.7}$$

Using Minkowski's inequality we get

$$\|J_{11}\|_X \le \sum_{r=0}^{|n|-2} \omega\left(\frac{1}{2^r}, f\right)_X \frac{1}{A_{n-1}^{\alpha_n}} \int_{\mathbb{I}} \left|\sum_{j=2^r}^{2^{r+1}-1} A_{n-j}^{\alpha_n-1} D_j(t)\right| dt.$$

Since

$$\begin{aligned} &\sum_{j=2^r}^{2^{r+1}-1} A_{n-j}^{\alpha_n-1} D_j(t) \\ &= \sum_{j=2^r}^{2^{r+1}-2} A_{n-j}^{\alpha_n-2} j K_j(t) + A_{n-2^{r+1}+1}^{\alpha_n-1} \left(2^{r+1}-1\right) K_{2^{r+1}-1}(t) \\ &\quad - A_{n-2^r}^{\alpha_n-1} \left(2^r - 1\right) K_{2^r-1}(t) \end{aligned}$$

and

$$\sup_n \|K_n\|_1 < 2,$$

from (2.2)–(2.4) we get ($r < |n| - 1$)

$$\frac{1}{A_{n-1}^{\alpha_n}} \int_{\mathbb{I}} \left| \sum_{j=2^r}^{2^{r+1}-1} A_{n-j}^{\alpha_n - 1} D_j(t) \right| dt$$

$$\leq \frac{2}{A_{n-1}^{\alpha_n}} \left\{ \sum_{j=2^r}^{2^{r+1}-2} \left| A_{n-j}^{\alpha_n - 2} \right| j + A_{n-2^{r+1}+1}^{\alpha_n - 1} \left(2^{r+1} - 1\right) + A_{n-2^r}^{\alpha_n - 1} \left(2^r - 1\right) \right\}$$

$$\leq \frac{c\left(1-\alpha_n\right)}{n^{\alpha_n}} \left\{ \sum_{j=2^r}^{2^{r+1}-2} (n-j)^{\alpha_n - 2} j + \left(n - 2^{r+1} + 1\right)^{\alpha_n - 1} \left(2^{r+1} - 1\right) \right.$$

$$\left. + \left(n - 2^r\right)^{\alpha_n - 1} \left(2^r - 1\right) \right\}$$

$$\leq \frac{c\left(1-\alpha_n\right)\alpha_n}{n^{\alpha_n}} \frac{2^{2r}}{\left(n - 2^{|n|-1}\right)^{2-\alpha_n}} + \frac{c 2^r \left(1-\alpha_n\right)\alpha_n}{n^{\alpha_n}\left(n - 2^{|n|-1}\right)^{1-\alpha_n}}$$

$$\leq c\alpha_n 2^{r-|n|}.$$

Consequently,

$$\|J_{11}\|_X \leq c\alpha_n \sum_{r=0}^{|n|-2} 2^{r-|n|} \omega\left(\frac{1}{2^r}, f\right). \tag{4.8}$$

Analogously, we can prove

$$\|J_{12}\|_X \leq c\alpha_n \sum_{r=0}^{|n|-2} 2^{r-|n|} \omega\left(\frac{1}{2^r}, f\right). \tag{4.9}$$

Combining (4.7)–(4.9) we get

$$\|J_1\|_X \leq c\alpha_n \sum_{r=0}^{|n|-2} 2^{r-|n|} \omega\left(\frac{1}{2^r}, f\right). \tag{4.10}$$

Finally, we estimate J_2. Since $D_{2^m - j} = D_{2^m} - w_{2^m - 1} D_j$, $j = 0, 1, \ldots, 2^m - 1$, we have

$$\int_{\mathbb{I}} \frac{1}{A_{n-1}^{\alpha_n}} \left| \sum_{j=2^{|n|-1}}^{2^{|n|}-1} A_{n-j}^{\alpha_n - 1} D_j(t) \right| dt$$

$$= \int_{\mathbb{I}} \frac{1}{A_{n-1}^{\alpha_n}} \left| \sum_{j=1}^{2^{|n|-1}} A_{n(|n|-1)+j}^{\alpha_n - 1} D_{2^{|n|}-j}(t) \right| dt$$

$$\leq \int_{\mathbb{I}} \frac{1}{A_{n-1}^{\alpha_n}} \left| \sum_{j=1}^{2^{|n|-1}} A_{n(|n|-1)+j}^{\alpha_n - 1} D_{2^{|n|}}(t) \right| dt$$

$$+ \int_{\mathbb{I}} \frac{1}{A_{n-1}^{\alpha_n}} \left| A_{n(|n|-1)+j}^{\alpha_n - 1} D_j(t) \right| dt$$

$$\leq c_1 + \int_{\mathbb{I}} \frac{1}{A_{n-1}^{\alpha_n}} \left| A_{n(|n|-1)+j}^{\alpha_n - 1} D_j(t) \right| dt$$

$$\leq c_1 + \int_{\mathbb{I}} \frac{1}{A_{n-1}^{\alpha_n}} \left| \sum_{j=1}^{2^{|n|-1}-1} A_{n(|n|-1)+j}^{\alpha_n - 2} j K_j (t) \right| dt$$

$$+ \int_{\mathbb{I}} \frac{A_{n(|n|-1)+2^{|n|-1}}^{\alpha_n - 1}}{A_{n-1}^{\alpha_n}} 2^{|n|-1} \left| K_{2^{|n|-1}} (t) \right| dt$$

$$\leq c_1 + \frac{c_2 \sum_{j=0}^{2^{|n|-1}} A_{n(|n|-1)+j}^{\alpha_n - 1}}{A_{n-1}^{\alpha_n}} \leq c < \infty.$$

Hence,

$$\| J_2 \|_X \leq c\omega \left(\frac{1}{2^{|n|-1}}, f \right)_X . \tag{4.11}$$

Combining (4.1), (4.6), (4.10) and (4.11) we complete the proof of Theorem 4.1. □

Proof of Theorem 4.3 Since $\sup_n V(m_n, \alpha) = \infty$, there exists $\{p_k : k \in \mathbb{N}\}$ such that

$$\left| m_{p_k} \right| > \left| m_{p_{k-1}} \right| + 2 \log(k+1); \tag{4.12}$$

$$V(m_{p_k}, \alpha) \geq 2k V(m_{p_{k-1}}, \alpha). \tag{4.13}$$

At first we consider the case $X(\mathbb{I}) = L_1(\mathbb{I})$. We set

$$g(x) := \sum_{j=1}^{\infty} g_j(x), \quad g_j(x) := \frac{D_{2^{|m_{p_j}|+1}}(x)}{V(m_{p_j}, \alpha)}.$$

If $y \in I_{|m_{p_k}|}$, then for $l = 1, 2, \ldots, k-1$ we obtain

$$D_{2^{|m_{p_l}|}} (x \dotplus y) = D_{2^{|m_{p_l}|}} (x). \tag{4.14}$$

Then, from (1.3) and (4.13),

$$\int_{\mathbb{I}} |g(x \dotplus y) - g(x)| \, dx$$

$$\leq \sum_{j=k}^{\infty} \frac{1}{V\left(m_{p_j}, \alpha\right)} \int_{\mathbb{I}} \left| D_{2^{|m_{p_j}|+1}} (x \dotplus y) - D_{2^{|m_{p_j}|+1}} (x) \right| dx$$

$$\leq \frac{c}{V\left(m_{p_k}, \alpha\right)}.$$

Consequently,

$$\omega\left(\frac{1}{m_{p_k}}, g\right)_1 = O\left(\frac{1}{V\left(m_{p_p}, \alpha\right)}\right).$$

Further,

$$\left\| \sigma_{m_{p_k}}^{\alpha_{p_k}} (g) - g \right\|_1$$

$$\geq \left\| \sigma_{m_{p_k}}^{\alpha_{p_k}} \left(\sum_{j=k}^{\infty} g_j \right) \right\|_1 - \sum_{j=k}^{\infty} \| g_j \|_1 - \left\| \sigma_{m_{p_k}}^{\alpha_{p_k}} \left(\sum_{j=1}^{k-1} g_j \right) - \sum_{j=1}^{k-1} g_j \right\|_1 . \tag{4.15}$$

Since for $j \geq k$

$$\sigma_{m_{p_k}}^{\alpha_{p_k}}\left(g_j\right) = g_j * K_{m_{p_k}}^{\alpha_{p_k}} = K_{m_{p_k}}^{\alpha_{p_k}} * g_j = \frac{1}{V\left(m_{p_j},\alpha\right)} K_{m_{p_k}}^{\alpha_{p_k}} * D_{2^{|m_{p_j}|+1}}$$
$$= \frac{1}{V\left(m_{p_j},\alpha\right)} S_{2^{|m_{p_j}|+1}}\left(K_{m_{p_k}}^{\alpha_{p_k}}\right) = \frac{1}{V\left(m_{p_j},\alpha\right)} K_{m_{p_k}}^{\alpha_{p_k}},$$

from Theorem 3.1 we obtain

$$\left\|\sigma_{m_{p_k}}^{\alpha_{p_k}}\left(\sum_{j=k}^{\infty} g_j\right)\right\|_1 = \sum_{j=k}^{\infty} \frac{1}{V\left(m_{p_j},\alpha\right)} \left\|K_{m_{p_k}}^{\alpha_{p_k}}\right\|_1 \geq \frac{\left\|K_{m_{p_k}}^{\alpha_{p_k}}\right\|_1}{V\left(m_{p_k},\alpha\right)} \geq c_0 > 0. \tag{4.16}$$

From (1.3) and (4.13) we get

$$\sum_{j=k}^{\infty} \left\|g_j\right\|_1 \leq 2\sum_{j=k}^{\infty} \frac{1}{V\left(m_{p_j},\alpha\right)} \leq \frac{3}{V\left(m_{p_k},\alpha\right)}. \tag{4.17}$$

Using Theorem 4.1, from (4.12) and (4.14) we obtain for $j < k$

$$\left\|\sigma_{m_{p_k}}^{\alpha_{p_k}}\left(g_j\right) - g_j\right\| \leq c\alpha_{p_k} \sum_{r=0}^{|m_{p_k}|-2} 2^{r-|m_{p_k}|}\omega\left(g_j,\frac{1}{2^r}\right)_1$$
$$\leq c\alpha_{p_k} \sum_{r=0}^{|m_{p_j}|-2} 2^{r-|m_{p_k}|}\omega\left(g_j,\frac{1}{2^r}\right)_1$$
$$\leq \frac{c}{2^{|m_{p_k}|}} \sum_{r=0}^{|m_{p_j}|} 2^r$$
$$\leq \frac{c}{k^2}.$$

Consequently,

$$\sum_{j=1}^{k-1} \left\|\sigma_{m_{p_k}}^{\alpha_{p_k}}\left(g_j\right) - g_j\right\|_1 \leq \frac{c}{k}. \tag{4.18}$$

Combining (4.15)–(4.18) we obtain

$$\overline{\lim_{k\to\infty}} \left\|\sigma_{m_{p_k}}^{\alpha_{p_k}}(g) - g\right\|_1 > 0.$$

Now, we discuss the case $X(\mathbb{I}) = C_w(\mathbb{I})$. Let the conditions (4.12) and (4.13) be satisfied. We set

$$h(x) := \sum_{j=1}^{\infty} \frac{h_j(x)}{V(m_{p_j},\alpha)},$$

where

$$h_j(x) := \operatorname{sgn}\left(K_{m_{p_j}}^{\alpha_{p_j}}\right).$$

It is evident that $h_0 \in C_w(\mathbb{I})$. If $y \in I_{|m_{p_k}|}$, then for $j = 1, 2, \ldots, k-1$ we obtain

$$h_j\left(x \dotplus y\right) - h_j\left(x\right) = 0$$

and from (4.13) we get

$$|h(x\dotplus y)-h(x)|\leq 2\sum_{j=k}^{\infty}\frac{1}{V\left(m_{p_j},\alpha\right)}=O\left(\frac{1}{V\left(m_{p_k},\alpha\right)}\right).$$

Hence

$$\omega\left(\frac{1}{m_{p_k}},h\right)_C=O\left(\frac{1}{V\left(m_{p_k},\alpha\right)}\right).$$

Clearly

$$\begin{aligned}&\left|\sigma_{m_{p_k}}^{\alpha_{p_k}}(0,h)-h(0)\right|\\&\geq\frac{\left|\sigma_{m_{p_k}}^{\alpha_{p_k}}(0,h_k)\right|}{V\left(m_{p_k},\alpha\right)}-\sum_{j=k}^{\infty}\frac{\left|h_j(0)\right|}{V\left(m_{p_j},\alpha\right)}-\sum_{j=k+1}^{\infty}\frac{\left|\sigma_{m_{p_k}}^{\alpha_{p_k}}\left(0,h_j\right)\right|}{V\left(m_{p_j},\alpha\right)}\\&\quad-\sum_{j=1}^{k-1}\frac{\left|\sigma_{m_{p_k}}^{\alpha_{p_k}}\left(0,h_j\right)-h_j(0)\right|}{V\left(m_{p_j},\alpha\right)}\\&=:R_1-R_2-R_3-R_4.\end{aligned}\tag{4.19}$$

Using Theorem 3.1, from (4.12) and (4.13) we obtain

$$R_1=\frac{\left\|K_{m_{p_k}}^{\alpha_{p_k}}\right\|_1}{V\left(m_{p_k},\alpha\right)}\geq c>0,\tag{4.20}$$

$$R_2\leq\frac{c}{V\left(m_{p_k},\alpha\right)},\tag{4.21}$$

$$R_3\leq\sum_{j=k+1}^{\infty}\frac{\left\|K_{m_{p_k}}^{\alpha_{p_k}}\right\|_1}{V\left(m_{p_j},\alpha\right)}\leq\frac{cV\left(m_{p_k},\alpha\right)}{V\left(m_{p_{k+1}},\alpha\right)}\leq\frac{c}{k},\tag{4.22}$$

$$\begin{aligned}R_4&\leq c\sum_{j=1}^{k-1}\sum_{r=0}^{|m_{p_k}|-2}2^{r-|m_{p_k}|}\omega\left(\frac{1}{2^r},h_j\right)_C\\&\leq c\sum_{j=1}^{k-1}\sum_{r=0}^{|m_{p_j}|}2^{r-|m_{p_k}|}\omega\left(\frac{1}{2^r},h_j\right)_C\\&\leq\frac{ck2^{|m_{p_{k-1}}|}}{2^{|m_{p_k}|}}\leq\frac{c}{k}.\end{aligned}\tag{4.23}$$

Combining (4.19)–(4.23) we complete the proof of Theorem 4.3. □

Proof of Theorem 4.4 The validity of part a) immediately follows from Theorem 3.1. Now, we prove part b). Since $\sup_a V(m_a,\alpha)=\infty$, without loss of generality we can suppose

$$V(m_a,\alpha)\geq a^4.\tag{4.24}$$

Set

$$f:=\sum_{a=1}^{\infty}\lambda_a f_a,,$$

where

$$\lambda_a := \frac{1}{\sqrt{V(m_a,\alpha)}}$$

and

$$f_a := D_{2^{|m_a|+1}} - D_{2^{|m_a|}}.$$

We can write

$$S_{2^n}(f_a) = S_{2^n}\left(D_{2^{|m_a|+1}} - D_{2^{|m_a|}}\right) = \begin{cases} 0, & \text{if } n \le |m_a|, \\ f_a, & \text{if } n > |m_a|. \end{cases}$$

Hence,

$$S_{2^n}(f) = \sum_{a=1}^{\infty} \lambda_a S_{2^n}(f_a) = \sum_{\{a:|m_a|<n\}} \lambda_a f_a$$

and

$$\sup_n |S_{2^n}(f)| \le \sum_a \lambda_a |f_a| = \sum_a \lambda_a D_{2^{|m_a|}}.$$

Applying (1.3) and (4.24) we conclude that

$$\left\| \sup_n |S_{2^n}(f)| \right\|_1 \le \sum_a \lambda_a \le \sum_a \frac{1}{a^2} < \infty.$$

Hence $f \in H_1(\mathbb{I})$.

We can write

$$\sigma_{m_a}^{\alpha_a}(f) = \sigma_{m_a}^{\alpha_a}(\lambda_a f_a) + \sum_{j=0}^{a-1} \lambda_j \sigma_{m_a}^{\alpha_a}(f_j) + \sum_{j=a+1}^{\infty} \lambda_j \sigma_{m_a}^{\alpha_a}(f_j). \tag{4.25}$$

Let $j > a$. Then it is easy to see that

$$\begin{aligned} \sigma_{m_a}^{\alpha_a}(f_j) &= f_j * K_{m_a}^{\alpha_a} = \left(D_{2^{|m_j|+1}} - D_{2^{|m_j|}}\right) * K_{m_a}^{\alpha_a} \\ &= S_{2^{|m_j|+1}}\left(K_{m_a}^{\alpha_a}\right) - S_{2^{|m_j|}}\left(K_{m_a}^{\alpha_a}\right) = 0. \end{aligned} \tag{4.26}$$

Let $j < a$. Then from Theorem 4.1 we have

$$\begin{aligned} \left\|\sigma_{m_a}^{\alpha_a}(f_j)\right\|_1 &\le \left\|\sigma_{m_a}^{\alpha_a}(f_j) - f_j\right\|_1 + 2 \\ &\le \sum_{r=0}^{|m_a|} 2^{r-|m_a|} \omega\left(f_j, \frac{1}{2^r}\right)_1 + 2 \\ &\le \sum_{r=0}^{|m_j|} 2^{r-|m_a|} \omega\left(f_j, \frac{1}{2^r}\right)_1 + 2 \\ &\le \frac{2^{|m_{a-1}|}}{2^{|m_a|}} + 2 < 3. \end{aligned} \tag{4.27}$$

Finally, we estimate $\sigma_{m_a}^{\alpha_a}(f_a)$. Let $m_a = 2^{|m_a|} + m_a'$, $0 \le m_a' < 2^{|m_a|}$. Then we can write

$$\left|\sigma_{m_a}^{\alpha_a}(f_a)\right| = \frac{1}{A_{m_a-1}^{\alpha_a}} \left| \sum_{j=0}^{m_a'-1} A_{m_a'-1-j}^{\alpha_a} w_j \right|$$

and consequently, from Theorem 3.1 and by (2.2)–(2.4), we get

$$\begin{aligned}\left\|\sigma_{m_a}^{\alpha_a}(f_a)\right\|_1 &= \frac{1}{A_{m_a-1}^{\alpha_a}}\left\|\sum_{j=0}^{m_a'-1} A_{m_a'-1-j}^{\alpha_a} w_j\right\|_1\\ &\geq \frac{c}{A_{m_a-1}^{\alpha_a}}\sum_{j=0}^{|m_a'|}\left|\varepsilon_j\left(m_a'\right)-\varepsilon_{j+1}\left(m_a'\right)\right| 2^{j\alpha_a}\\ &\geq \frac{c}{2^{|m_a|\alpha_a}}\sum_{j=0}^{|m_a|}\left|\varepsilon_j\left(m_a\right)-\varepsilon_{j+1}\left(m_a\right)\right| 2^{j\alpha_a}\\ &\geq cV\left(m_a,\alpha\right). \qquad (4.28)\end{aligned}$$

Combining (4.25)–(4.28) we conclude that

$$\sup_a \left\|\sigma_{m_a}^{\alpha_a}(f)\right\|_1 = \infty.$$

□

Acknowledgements The authors are deeply indebted to the anonymous referee for his/her valuable help.

Funding Open access funding provided by University of Debrecen.

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References

1. A.A. Abu Joudeh, G. Gát, Convergence of Cesàro means with varying parameters of Walsh-Fourier series. Miskolc Math. Notes **19**(1), 303–317 (2018)
2. T. Akhobadze, On the convergence of generalized Cesàro means of trigonometric Fourier series. II. Acta Math. Hungar. **115**(1–2), 79–100 (2007)
3. T. Akhobadze, On the convergence of generalized Cesàro means of trigonometric Fourier series. I. Acta Math. Hungar. **115**(1–2), 59–78 (2007)
4. I. Blahota, G. Gát, Norm summability of Nörlund logarithmic means on unbounded Vilenkin groups. Anal. Theory Appl. **24**(1), 1–17 (2008)
5. N.J. Fine, On the Walsh functions. Trans. Am. Math. Soc. **65**, 372–414 (1949)
6. J. Fine, Cesàro summability of Walsh–Fourier series. Proc. Nat. Acad. Sci. USA **41**, 558–591 (1955)
7. S. Fridli, Approximation by Vilenkin–Fourier sums. Acta Math. Hungar. **47**(1–2), 33–44 (1986)
8. S. Fridli, On the rate of convergence of Cesàro means of Walsh–Fourier series. J. Approx. Theory **76**(1), 31–53 (1994)
9. G. Gát, Best approximation by Vilenkin-like systems. Acta Math. Acad. Paedagog. Nyházi. (N.S.) **17**(3), 161–169 (2001)
10. G. Gát, U. Goginava, Uniform and L-convergence of logarithmic means of Walsh-Fourier series. Acta Math. Sin. (Engl. Ser.) **22**(2), 497–506 (2006)
11. G. Gát, U. Goginava, On the divergence of Nörlund logarithmic means of Walsh-Fourier series. Acta Math. Sin. (Engl. Ser.) **25**(6), 903–916 (2009)
12. G. Gát, U. Goginava, Norm convergence of logarithmic means on unbounded Vilenkin groups. Banach J. Math. Anal. **12**(2), 422–438 (2018)

13. U. Goginava, On the uniform convergence of Walsh–Fourier series. Acta Math. Hungar. **93**(1–2), 59–70 (2001)
14. U. Goginava, On the approximation properties of Cesàro means of negative order of Walsh–Fourier series. J. Approx. Theory **115**(1), 9–20 (2002)
15. U. Goginava, G. Tkebuchava, Convergence of subsequences of partial sums and logarithmic means of Walsh–Fourier series. Acta Sci. Math. (Szeged) **72**(1–2), 159–177 (2006)
16. N.V. Gulićev, Approximation to continuous functions by Walsh–Fourier sums. Anal. Math. **6**(4), 269–280 (1980)
17. I.B. Kaplan, Cesàro means of variable order. Izv. Vyssh. Uchebn. Zaved. Mat. **18**(5), 62–73 (1960)
18. S.V. Konyagin, On a subsequence of Fourier-Walsh partial sums (Russian); translated from Mat. Zametki **54** (1993), no. 4, 69–75, 158 Math. Notes **54** (1993), no. 3-4, 1026–1030 (1994)
19. S.F. Lukomskii, Convergence of multiple Walsh series in measure and in L. East J. Approx. **3**(3), 317–332 (1997)
20. N. Memić, Almost everywhere convergence of some subsequences of the Nörlund logarithmic means of Walsh-Fourier series. Anal. Math. **41**(1–2), 45–54 (2015)
21. K. Nagy, Almost everywhere convergence of a subsequence of the Nörlund logarithmic means of Walsh–Kaczmarz–Fourier series. J. Math. Inequal. **3**(4), 499–510 (2009)
22. C.W. Onneweer, On L_1-convergence of Walsh–Fourier series. Int. J. Math. Math. Sci. **1**(1), 47–56 (1978)
23. F. Schipp, W.R. Wade, P. Simon, J. Pál, *Walsh series: An introduction to dyadic harmonic analysis* (Adam Hilger, Bristol and New York, 1990), (English)
24. F. Schipp, W.R. Wade, P. Simon, J. Pál, *Walsh Series. An Introduction to Dyadic Harmonic Analysis* (Adam Hilger, Bristol-New York, 1990)
25. Sh. Tetunashvili, On divergence of Fourier series by some methods of summability. J. Funct. Spaces Appl. Art. ID 542607 (2012)
26. R. Toledo, On the boundedness of the L_1-norm of Walsh–Fejér kernels. J. Math. Anal. Appl. **457**(1), 153–178 (2018)
27. F. Weisz, (C, α) summability of Walsh–Fourier series. Anal. Math. **27**, 141–156 (2001)
28. S. Yano, On approximation by Walsh functions. Proc. Amer. Math. Soc. **2**, 962–967 (1951)
29. A. Zygmund, *Trigonometric Series*, vol. 1 (Cambridge Univ. Press, Cambridge, 1959)

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