Symmetric Shannon capacity is the independence number minus 1

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Abstract

A symmetric variant of the Shannon capacity of graphs is defined and computed.

Keywords Shannon capacity · Independence number · Graph product

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1 Introduction and motivation

The Shannon capacity $\Theta(G)$ (introduced in 1956 by Shannon [5]) is a notoriously hard-to-compute graph parameter defined as the asymptotic growth rate of the independence number of strong direct powers of a graph G :

$$\Theta(G) = \lim_{k \rightarrow \infty} \left(\alpha(G^{\boxtimes k}) \right)^{1/k}$$

(the limit always exists due to the supermultiplicativity of $\alpha(G^{\boxtimes k})$). Even for the 5-cycle the exact value was only determined in 1979 by Lovász [4], and the exact value for the 7-cycle is unknown (asymptotic results exist for large odd cycles, see Bohman [2,3]). As to approximation, Alon and Lubetzky [1] show that the naïve strategy of estimating $\Theta(G)$ by $\alpha(G^{\boxtimes k})$ for any fixed k can produce unboundedly high relative error; in fact, there may be arbitrarily high polynomial jumps in the value of $\alpha(G^{\boxtimes k})$ at any finite set of k 's.

In this paper, we use a modification of Shannon capacity where instead of the strong direct power $G^{\boxtimes k}$ we consider its quotient $G[k] = G^{\boxtimes k} / S_k$ by the natural index-permuting action of the symmetric group S_k . Since the size of $G[k]$ (and consequently of $\alpha(G[k])$ as well) grows only polynomially, we define

$$F(G) = \lim_{k \rightarrow \infty} \frac{\log \alpha(G[k])}{\log k}$$

as a measure of asymptotic growth of $\alpha(G[k])$. Our main result is the following:

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Theorem For any finite graph G we have $F(G) = \alpha(G) - 1$.

Section 2 fixes our notation and Section 4 contains the proof of the Theorem. Section 3 contains a treatment of the particular case of the 5-cycle in a way that does not seem to generalize to all graphs but yields more precise intermediate estimates.

2 Setup

Let G be a simple graph on the vertex set $V = \{v_1, \dots, v_n\}$. Then the vertex set $V[k]$ of the graph $G[k] = G^{\boxtimes k}/S_k$ consists of partitions of k into n labeled nonnegative integers:

$$V[k] = \{f : V \rightarrow \mathbb{Z}_{\geq 0} : f(v_1) + \dots + f(v_n) = k\}.$$

We will consider the elements of $V[k]$ also as equivalence classes of maps from a set of cardinality k to V with two maps being equivalent if they only differ in a permutation of the k elements—configurations of k identical pebbles in the vertices of G . Two vertices $f, g \in V[k]$ are connected if one can move some pebbles in the configuration corresponding to f to neighbours in G to obtain the configuration corresponding to g (were the pebbles distinguishable, this construction would give the strong direct product $G^{\boxtimes k}$).

A priori it is not clear whether the limit defining $F(G)$ exists at all: in contrast to the case of distinguishable pebbles, there is no obvious superadditivity property for $\log \alpha(G[k])$ that would ensure convergence. It is however easy to see that

$$\alpha(G[k]) \geq \binom{k + \alpha(G) - 1}{\alpha(G) - 1}$$

—the configurations supported on a fixed independent set $W \subset V$ are clearly independent in $G[k]$ —and that

$$\alpha(G[k]) \leq \binom{k + \theta(G) - 1}{\theta(G) - 1},$$

where $\theta(G)$, the clique covering number of G , is the minimal number of cliques covering the vertices of G —for any clique covering $V = V_1 \cup \dots \cup V_\theta$ whenever the configurations $f, g \in V[k]$ have the same number of pebbles on the vertices of all V_j , they can be moved into one another by using only the edges within the V_j . This argument shows that if $\theta(G) = \alpha(G)$, then $\alpha(G[k]) = \binom{k + \alpha(G) - 1}{\alpha(G) - 1}$ and $F(G) = \alpha(G) - 1$ does indeed hold.

The simplest graph for which $\theta(G) > \alpha(G)$ (and thus the estimates above do not determine $\alpha(G[k])$) is the 5-cycle C_5 ; the following section discusses this particular graph.

3 The case of C_5

It can be checked that for $k \leq 9$ we have $\alpha(C_5[k]) = k + 1$, but for general k we can only prove the following, weaker result:

Proposition 3.1 For any natural number k we have $\alpha(C_5[k]) \leq \left\lfloor \frac{5(k+2)(k+1)}{2(k+5)} \right\rfloor$.

Proof For simplicity let us index the vertices of C_5 by natural numbers from 1 to 5 in such a way that vertices v_j and v_{j+1} are connected by an edge labelled e_j for $j = 1, \dots, 4$, and the vertices v_1 and v_5 are also connected by the edge e_5 .

Fig. 1 Labelling of C_5

Consider *edge configurations* of weight k on C_5 : functions $\psi : E(C_5) \rightarrow \mathbb{Z}_{\geq 0}$ such that $\psi(e_1) + \dots + \psi(e_5) = k$, also considered as ways to distribute k indistinguishable pebbles among the edges of C_5 . We declare an edge configuration ψ and a vertex configuration f to be adjacent if one can obtain ψ from f by moving each pebble to an adjacent edge; this defines a (bipartite) graph structure on the set of vertex and edge configurations. Note that if f and g are vertex configurations, then they are neighbours in $C_5[k]$ exactly if there exists an edge configuration ψ adjacent to both of them. As a consequence, if $A \subset V[k]$ is an independent set, then the sets of edge configurations adjacent to the elements of A are disjoint.

From this fact we get a bound on $|A|$: if every vertex configuration is adjacent to at least m edge configurations, and their total size cannot exceed $\binom{k+4}{4}$, the total number of edge configurations of weight k , then $|A| \leq \frac{\binom{k+4}{4}}{m}$. Unfortunately, there are vertex configurations that are adjacent to few edge configurations: for example, only $k+1$ edge configurations are adjacent to the vertex configuration where all the pebbles are assigned to the same vertex, and this makes the obtained upper bound on $|A|$ too high. To circumvent this problem, we count edge configurations with weight. Firstly, we count only edge configurations that assign zero weight to at least one pair of non-adjacent edges. Let $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ denote the set of such edge configurations, where S_j is the set of those edge configurations that assign zero weight to edges e_{j-1} and e_{j+1} (with indices considered modulo 5). Finally, as the sets S_j are not disjoint, we shall consider each edge configuration weighted by the number of sets S_j that contain it.

Now let $f \in V[k]$ be an arbitrary vertex configuration. Edge configurations in S_1 that are adjacent to f have no pebbles on edges e_2 and e_5 and hence must be formed by moving all pebbles in v_1 and v_2 to the edge e_1 , all pebbles in v_3 to the edge e_3 and all pebbles in v_5 to the edge e_4 . The only freedom left is the distribution of the pebbles in v_4 to e_3 and e_4 ; the number of such distributions is $f(v_4) + 1$. Repeating this argument for S_2 to S_5 , we count $f(v_1) + 1 + \dots + f(v_5) + 1 = k + 5$ edge configurations adjacent to f , each as many times as there are sets S_j that contain it. The total weight of the edge configurations in S is $|S_1| + \dots + |S_5| = 5\binom{k+2}{2}$, hence there cannot be more than $\left\lfloor \frac{5\binom{k+2}{2}}{k+5} \right\rfloor = \left\lfloor \frac{5(k+2)(k+1)}{2(k+5)} \right\rfloor$ pairwise non-neighbouring vertex configurations, as claimed. $\square$

As an immediate corollary, we get

Corollary 3.2 $F(C_5) = 1$.

Open question 3.1 Does $\alpha(C_5[k]) = k + 1$ hold for all k ?

4 General case

The main Theorem is the immediate consequence of the trivial lower bound and the following estimate:

Proposition 4.1 *For any finite graph G we have $\alpha(G[k]) = O(k^{\alpha(G)-1})$.*

Proof We prove our statement by induction on the cardinality of V . For $|V| = 1$ the claim is trivial. Assume now that the statement holds for all graphs with less vertices than $|V|$, and let $H \subset V[k]$ be an arbitrary independent vertex set in $G[k]$. Partition H into pieces $H_1, \dots, H_n$ such that in H_j the vertex v_j carries the greatest weight; in particular, this weight is at least $\frac{k}{n}$.

Denote by $N(v_j) \subset V$ the union of the vertex v_j and its neighbours in G . Each H_j we further divide into chunks $H_{j;m,b_1,\dots,b_n}$, where the chunk $H_{j;m,b_1,\dots,b_n}$ contains those elements of H_j in which the total weight of v_j and its neighbours is m and for all $i \leq n$ the weight of the vertex v_i falls into the interval $\left[b_i \frac{k}{n^2}, (b_i + 1) \frac{k}{n^2}\right)$. The index j can take n different values, m can take $k + 1$ different values, and each b_i can be assumed to be between 0 and n^2 (the rest are always empty), so altogether we get at most $n(k + 1)(n^2 + 1)^n = O(k)$ chunks (with an implied dependence on n). Pick now an arbitrary chunk $H_{j;m,b_1,\dots,b_n}$ and consider its elements as configurations on $G \setminus N(v_j)$ (with the weights on the elements of $N(v_j)$ omitted). These configurations all have weight $k - m$ and we claim that they form an independent set in $(G \setminus N(v_j))[k - m]$. Indeed, if two configurations f and g in $H_{j;m,b_1,\dots,b_n}$ either coincide or are neighbours in $(G \setminus N(v_j))[k - m]$, then there exists a pebble transport from f to g on $G \setminus N(v_j)$. On the other hand, there exists a pebble transport from f to g on $N(v_j)$ as well: for each vertex v_i that is a neighbour of v_j we send $f(v_i) - g(v_i)$ pebbles from v_i to v_j if $f(v_i) > g(v_i)$, and we send $g(v_i) - f(v_i)$ pebbles from v_j to v_i if $g(v_i) > f(v_i)$. Combining these two pebble transports yields a pebble transport from f to g .

This means that the chunk $H_{j;m,b_1,\dots,b_n}$ has size at most $O(k^{\alpha(G \setminus N(v_j)) - 1})$ by the induction hypothesis. But any independent set in $G \setminus N(v_j)$ can be extended by v_j to form an independent set in G , hence $\alpha(G \setminus N(v_j)) \leq \alpha(G) - 1$. Summing this estimate for all the $O(k)$ chunks we get that $|H| = O(k^{\alpha(G)-1})$, proving the induction step. $\square$

Open question 4.1 *Does $\alpha(G[k]) = \binom{k + \alpha(G) - 1}{\alpha(G) - 1}$ or at least*

$$\lim_{k \rightarrow \infty} \frac{\alpha(G[k])}{k^{\alpha(G)-1}} = \frac{1}{(\alpha(G) - 1)!}$$

hold?

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