A new proof of a generalization of Gerzon's bound

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Abstract

In this paper we give a short, new proof of a natural generalization of Gerzon's bound. This bound improves the Delsarte, Goethals and Seidel's upper bound in a special case. Our proof is a simple application of the linear algebra bound method.

Keywords Gerzon's bound · Distance problem · Linear algebra bound method

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1 Introduction

In this paper we give a linear algebraic proof of the known upper bound for the size of some special spherical s -distance sets. This result generalizes Gerzon's general upper bound for the size of equiangular spherical set.

In the sequel, $\mathbb{R}[x_1, \dots, x_n] = \mathbb{R}[\mathbf{x}]$ denotes the ring of polynomials in commuting variables $x_1, \dots, x_n$ over $\mathbb{R}$.

Let $\mathcal{G} \subseteq \mathbb{R}^n$ be an arbitrary set. Denote by $d(\mathcal{G})$ the set of (non-zero) distances among points of $\mathcal{G}$:

$$d(\mathcal{G}) := \{d(\mathbf{p}_1, \mathbf{p}_2) : \mathbf{p}_1, \mathbf{p}_2 \in \mathcal{G}, \mathbf{p}_1 \neq \mathbf{p}_2\}.$$

An s -distance set is any subset $\mathcal{H} \subseteq \mathbb{R}^n$ such that $|d(\mathcal{H})| \leq s$.

Let $(\mathbf{x}, \mathbf{y})$ stand for the standard scalar product. Let $s(\mathcal{G})$ denote the set of scalar products between the distinct points of $\mathcal{G}$:

$$s(\mathcal{G}) := \{(\mathbf{p}_1, \mathbf{p}_2) : \mathbf{p}_1, \mathbf{p}_2 \in \mathcal{G}, \mathbf{p}_1 \neq \mathbf{p}_2\}.$$

A *spherical* s -distance set means any subset $\mathcal{G} \subseteq \mathbb{S}^{n-1}$ such that $|s(\mathcal{G})| \leq s$.

Let $n, s \geq 1$ be integers. Define

$$M(n, s) := \binom{n+s-1}{s} + \binom{n+s-2}{s-1}.$$

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Delsarte, Goethals and Seidel investigated the spherical s -distance sets. They proved a general upper bound for the size of a spherical s -distance set in [1].

Theorem 1.1 (Delsarte et al. [1]) *Suppose that $\mathcal{F} \subseteq \mathbb{S}^{n-1}$ is a set satisfying $|s(\mathcal{F})| \leq s$. Then*

$$|\mathcal{F}| \leq M(n, s).$$

Barg and Musin gave an improved upper bound for the size of a spherical s -distance set in a special case in [2]. Their proof builds upon Delsarte's ideas (see [1]) and they used Gegenbauer polynomials in their argument.

Theorem 1.2 (Barg and Musin [2]) *Let $n \geq 1$ be a positive integer and let $s > 0$ be an even integer. Let $\mathcal{S} \subseteq \mathbb{S}^{n-1}$ denote a spherical s -distance set with inner products $t_1, \dots, t_s$ such that*

$$t_1 + \dots + t_s \geq 0.$$

Then

$$|\mathcal{S}| \leq M(n, s-2) + \frac{n+2s-2}{s} \binom{n+s-1}{s-1}.$$

We point out the following special case of Theorem 1.1.

Corollary 1.3 *Suppose that $\mathcal{F} \subseteq \mathbb{S}^{n-1}$ is a set satisfying $|s(\mathcal{F})| \leq 2$. Then*

$$|\mathcal{S}| \leq \frac{n(n+3)}{2}.$$

An *equiangular spherical set* means a two-distance spherical set with scalar products α and $-\alpha$. Let $M(n)$ denote the maximum cardinality of an equiangular spherical set. There is a very extensive literature devoted to the determination of the precise value of $M(n)$ (see [3,4]). Gerzon gave the first general upper bound for $M(n)$

Theorem 1.4 (Gerzon [3, Theorem 8]) *Let $n \geq 1$ be a positive integer. Then*

$$M(n) \leq \frac{n(n+1)}{2}.$$

Musin proved a stronger version of Gerzon's Theorem in [5, Theorem 1]. He used the linear algebra bound method in his proof.

Theorem 1.5 (Musin [5, Theorem 1]) *Let $\mathcal{S}$ be a spherical two-distance set with inner products a and b . Suppose that $a+b \geq 0$. Then*

$$|\mathcal{S}| \leq \frac{n(n+1)}{2}.$$

De Caen gave a lower bound for the size of an equiangular spherical set.

Theorem 1.6 (de Caen [6]) *Let $t > 0$ be a positive integer. For each $n = 3 \cdot 2^{2t-1} - 1$ there exists an equiangular spherical set of $\frac{2}{9}(n+1)^2$ vectors.*

Our main result is an alternative proof of a natural generalization of Gerzon's bound, which improves the Delsarte, Goethals and Seidel's upper bound in a special case. Our proof uses the linear algebra bound method. The following statement was proved in [7] Theorem 6.1. The original proof builds upon matrix techniques and the addition formula for Jacobi polynomials.

Theorem 1.7 (Delsarte et al. [7, Theorem 6.1]) *Let $s = 2t > 0$ be an even number and $n > 0$ be a positive integer. Let $\mathcal{S}$ denote a spherical s -distance set with inner products $a_1, \dots, a_t, -a_1, \dots, -a_t$. Suppose that $0 < a_i < 1$ for each i . Then*

$$|\mathcal{S}| \leq \binom{n+s-1}{s}.$$

2 Preliminaries

We prove our main result using the linear algebra bound method and the Determinant Criterion (see [8, Proposition 2.7]). We recall here this principle for the reader's convenience.

Proposition 2.1 (Determinant Criterion) *Let $\mathbb{F}$ be an arbitrary field. Let $f_i : \Omega \rightarrow \mathbb{F}$ be functions and $\mathbf{v}_j \in \Omega$ elements for each $1 \leq i, j \leq m$ such that the $m \times m$ matrix $B = (f_i(\mathbf{v}_j))_{i,j=1}^m$ is non-singular. Then $f_1, \dots, f_m$ are linearly independent.*

Consider the set of vectors

$$\mathcal{M}(n, s) := \left\{ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n : \alpha_1 \leq 1, \sum_{i=1}^n \alpha_i \text{ is even}, \sum_{i=1}^n \alpha_i \leq s \right\}.$$

Define the set

$$\mathcal{N}(n, s) = \left\{ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n : \sum_{i=1}^n \alpha_i \leq s \right\}.$$

Lemma 2.2 *Let $n, s \geq 1$ be integers. Then*

$$|\mathcal{N}(n, s)| = \binom{n+s}{s}.$$

Proof For a simple proof of this fact see [9, Section 9.2 Lemma 4]. $\square$

We need the following combinatorial statement.

Lemma 2.3 *Let $n > 0$ be a positive integer and $s > 0$ be an even integer. Then*

$$|\mathcal{M}(n, s)| = \binom{n+s-1}{s}.$$

Proof It is easy to check that there exists a bijection $f : \mathcal{M}(n, s) \rightarrow \mathcal{N}(n-1, s)$, since s is even. Namely, let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{M}(n, s)$ be an arbitrary element. Define $f(\alpha) := (\alpha_2, \dots, \alpha_n)$. It is easy to verify that $f(\mathcal{M}(n, s)) \subseteq \mathcal{N}(n-1, s)$ and f is a bijection.

Hence $|\mathcal{M}(n, s)| = |\mathcal{N}(n-1, s)|$ and Lemma 2.2 gives the result. $\square$

3 Proof of Theorem 1.7

Consider the real polynomial

$$g(x_1, \dots, x_n) = \left(\sum_{m=1}^n x_m^2 \right) - 1 \in \mathbb{R}[x_1, \dots, x_n].$$

Let $\mathcal{S} = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ denote a spherical s -distance set with inner products $a_1, \dots, a_t, -a_1, \dots, -a_t$. Here $r = |\mathcal{S}|$. Define the polynomials

$$P_i(x_1, \dots, x_n) := \prod_{m=1}^t \left((\langle \mathbf{x}, \mathbf{v}_i \rangle)^2 - (a_m)^2 \right) \in \mathbb{R}[\mathbf{x}]$$

for each $1 \leq i \leq r$. Clearly $\deg(P_i) \leq s = 2t$ for each $1 \leq i \leq r$.

Consider the set of vectors

$$\mathcal{E}(n, s) := \left\{ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n : \sum_i \alpha_i \text{ is even, } \sum \alpha_i \leq s \right\}$$

It is easy to verify that if we can expand P_i as a linear combination of monomials, then we get

$$P_i(x_1, \dots, x_n) = \sum_{\alpha \in \mathcal{E}(n, s)} c_\alpha x^\alpha, \quad (3.1)$$

where $c_\alpha \in \mathbb{R}$ are real coefficients for each $\alpha \in \mathcal{E}(n, s)$ and x^α denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Since $\mathbf{v}_i \in \mathbb{S}^{n-1}$, this means that the equation

$$x_1^2 = 1 - \sum_{j=2}^n x_j^2 \quad (3.2)$$

is true for each $\mathbf{v}_i$, where $1 \leq i \leq r$. Let Q_i denote the polynomial obtained by writing P_i as a linear combination of monomials and replacing, repeatedly, each occurrence of x_1^t , where $t \geq 2$, by a linear combination of other monomials, using the relations (3.2).

Since $g(\mathbf{v}_i) = 0$ for each i , hence $Q_i(\mathbf{v}_j) = P_i(\mathbf{v}_j)$ for each $1 \leq i \neq j \leq r$.

We prove that the set of polynomials $\{Q_i : 1 \leq i \leq r\}$ is linearly independent. This fact follows from the Determinant Criterion, when we define $\mathbb{F} := \mathbb{R}$, $\Omega = \mathbb{S}^{n-1}$ and $f_i := Q_i$ for each i . It is enough to prove that $Q_i(\mathbf{v}_i) = P_i(\mathbf{v}_i) \neq 0$ for each $1 \leq i \leq r$ and $Q_i(\mathbf{v}_j) = P_i(\mathbf{v}_j) = 0$ for each $1 \leq i \neq j \leq r$, since then we can apply the Determinant Criterion.

But $P_i(\mathbf{v}_i) = \prod_{m=1}^t (1 - a_m^2)$ and $P_i(\mathbf{v}_j) = 0$, because $\mathcal{S} = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is a spherical s -distance set with inner products $a_1, \dots, a_t, -a_1, \dots, -a_t$.

It is easy to check that we can write Q_i as a linear combination of monomials in the form

$$Q_i = \sum_{\alpha \in \mathcal{M}(n, s)} d_\alpha x^\alpha,$$

where $d_\alpha \in \mathbb{R}$ are the real coefficients for each $\alpha \in \mathcal{M}(n, s)$. This follows immediately from the expansion (3.1) and from the relation (3.2).

Since the polynomials $\{Q_i : 1 \leq i \leq r\}$ are linearly independent and if we expand Q_i as a linear combination of monomials, then all monomials appearing in this linear combination contained in the set of monomials

$$\{x^\alpha : \alpha \in \mathcal{M}(n, s)\}$$

for each i , hence

$$r = |\mathcal{S}| \leq |\mathcal{M}(n, s)| = \binom{n+s-1}{s},$$

by Lemma 2.3. □

4 Concluding remarks

The following Conjecture is a natural generalization of Theorem 1.5 and a strengthening of Theorem 1.2.

Conjecture 4.1 *Let $n \geq 1$ be a positive integer and let $s > 0$ be an even integer. Let $S \subseteq \mathbb{S}^{n-1}$ denote a spherical s -distance set with inner products $t_1, \dots, t_s$ such that*

$$t_1 + \dots + t_s \geq 0.$$

Then

$$|S| \leq \binom{n+s-1}{s}.$$

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