Knots not concordant to L-space knots

Ramazan Yozgyur¹

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Abstract

In this short note we use methods of Friedl, Livingston and Zentner to show that there are knots that are not algebraically concordant to a connected sum of positive and negative L-space knots.

Keywords Knots · Concordant knots · L-space knot

1 Introduction

A closed oriented 3-manifold Y is called an L-space if it is a rational homology sphere and its Heegaard–Floer homology has minimal possible rank i.e. $\widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|$. A knot $K \subset S^3$ is called (positive) L-space knot if a surgery with a sufficiently large positive coefficient is an L-space.

L-space knots were introduced by Ozsváth and Szabó in [10] in their attempt to classify knots such that a surgery on them gives a lens space. In particular, they proved the following result.

Theorem 1.1 [10, Corollary 1.3] *If Δ is the Alexander polynomial of an L-space knot, then $\exists a_1, \dots, a_n$ such that $a_0 < a_1 < \dots < a_n$ and*

$$\Delta = t^{a_0} - t^{a_1} + t^{a_2} \dots + t^{a_n} \quad (1.1)$$

The class of L-space knots includes all torus knots. More generally, all algebraic knots are L-space knots; see [6, 7]. Moreover, in [8] Hedden proves the following result:

Theorem 1.2 *An L-space knot is strongly quasipositive and fibred.*

We refer to [12] for the definition and the properties of strongly quasipositive knots.

We have the following result of Borodzik and Feller.

Theorem 1.3 (see [1]) *Every link L is topologically concordant to a strongly quasipositive link L' .*

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✉ Ramazan Yozgyur
ryozgyur@mimuw.edu.pl

¹ Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland

In this light, the main theorem of the paper seems a bit surprising.

Theorem 1.4 *There are knots that are not topologically concordant to any combinations of L-space knots and their mirrors.*

2 Proof of Theorem 1.4

Let us recall the following fact, which can be found e.g. in [11, Section 8.1].

Theorem 2.1 (Cauchy's bound) *Suppose $P = \alpha_0 + \alpha_1 t + \cdots + \alpha_m t^m$ is a complex polynomial with $\alpha_m \neq 0$.*

For any root z of P we have $|z| < 1 + \max_{j < m} \frac{|\alpha_j|}{|\alpha_m|}$.

From Cauchy's bound we obtain the following proposition.

Proposition 2.2 *The Alexander polynomial of an L-space knot has roots with modulus at most 2.*

Remark 2.3 From the multiplicativity of the Alexander polynomial under connected sums we infer also that if K is a connected sum of L-space knots and their mirrors, then $\Delta_K(t)$ has all roots inside of the disk of radius 2. Note that by [9], a non-trivial connected sum of L-space knots is not an L-space knot anymore.

Following [4], for $n = 1, 2, \dots$ we define

$$P_n = 1 + nt - (2n + 1)t^2 + nt^3 + t^4. \quad (2.1)$$

We need some properties of roots of this polynomial.

Theorem 2.4 (see [4, Lemma 4]) *The polynomial P_n is irreducible over $\mathbb{Q}[t]$ and it has two roots on the unit circle.*

Denote these two roots by θ_n^+ and θ_n^- . We will use a more specific control for one of the other roots of P_n .

Lemma 2.5 *For $n \geq 1$, the polynomial P_n has a root with modulus greater than 2.*

Proof We have $P_n(-n-1) = 1 - 2n - 3n^2 - n^3 < 0$. On the other hand, $P(-(n+2)) = 13 + 10n + 2n^2 > 0$. Therefore P_n has a real root in the interval $(-n-2, -n-1)$. $\square$

Theorem 2.6 *Let K be connected sum of L-space knots and some mirrors of L-space knots, then Δ_K cannot vanish at θ_n^+ .*

Proof Since the Alexander polynomial of a connected sum of knots is the product of the Alexander polynomial of each knot, it is enough to prove the theorem for K being an L-space knot.

Suppose $\Delta_K(\theta_n^+) = 0$. Then $\gcd(\Delta_K, P_n)$ has positive degree and it divides P_n . As P_n is irreducible over $\mathbb{Q}[t]$, it follows that $P_n | \Delta_K$. But P_n has a root outside a disk of radius 2 and all the roots of Δ_K are inside this disk. $\square$

Theorem 2.7 Suppose K is a knot that is concordant to a knot K' which is a connected sum of L-space knots and mirrors of L-space knots. Then the order of the root of Δ_K at θ_n^+ is an even number.

Proof As K and K' are concordant, an easy corollary of Fox-Milnor theorem [5] implies that there exist polynomials $f, g \in \mathbb{Z}[t, t^{-1}]$ such that

$$\Delta_K(t) f(t) f(t^{-1}) = \Delta_{K'}(t) g(t) g(t^{-1}). \quad (2.2)$$

Claim. If $\xi(t) \in \mathbb{Z}[t, t^{-1}]$ vanishes at θ_n^+ , then $\xi^{-1}(t)$ vanishes at θ_n^+ with the same order.

To prove the claim, note that if ξ vanishes at θ_n^+ up to order s , then it is divisible by $(t - \theta_n^+)^s$ and also by $(t - \theta_n^-)^s$ (because ξ has real coefficients and $\overline{\theta_n^+} = \theta_n^-$). As $\theta_n^+ \theta_n^- = 1$ we have

$$(t^{-1} - \theta_n^+)(t^{-1} - \theta_n^-) = t^{-2}(t - \theta_n^+)(t - \theta_n^-).$$

From the above identity the claim follows readily.

From the claim we conclude that the order of the root of $\Delta_{K'} g(t) g(t^{-1})$ at θ_n^+ is an even number. Using the claim once again, this time for $f(t)$, we see that (2.2) implies that $\Delta_K(t)$ vanishes at θ_n^+ up to an even power (maybe zero). $\square$

To conclude the proof of Theorem 1.4 we will show that there exist knots such that their Alexander polynomial vanishes at θ_n^+ with an odd order. As P_n is a symmetric polynomial and $P_n(1) = 1$, for any n there exist a knot K_n such that $\Delta_{K_n} = P_n$, see [13]. Furthermore, the knot K_n from [4, Figure 1] has Alexander polynomial P_n .

Example 2.8 A notable example of a knot that is not concordant to a combination of L-space knots is the knot 12n642. According to KnotInfo web page [2], its Alexander polynomial is P_7 . On the other hand, 12n642 is strongly quasipositive and fibered. It is also almost positive in the sense of [3].

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