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CORRECTION

Correction to: *X*-coordinates of Pell equations as sums of two Tribonacci numbers

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Abstract
In this work, we correct an oversight from [1].

1 Introduction

For a positive squarefree positive integer d and the Pell equation $X^2 - dY^2 = \pm 1$, where $X, Y \in \mathbb{Z}^+$, it is well known that all its solutions (X, Y) have the form $X + Y\sqrt{d} = X_k + Y_k\sqrt{d} = (X_1 + Y_1\sqrt{d})^k$ for some $k \in \mathbb{Z}^+$, where (X_1, Y_1) is the smallest positive integer solution. Let $\{T_n\}_{n\geq 0}$ be the Tribonacci sequence given by $T_0 = 0,\ T_1 = T_2 = 1,\ T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for all $n \geq 0$. Let $U = \{T_n + T_m : n \geq m \geq 0\}$ be the set of non-negative integers which are sums of two Tribonacci numbers. In [1], we looked at Pell equations $X^2 - dY^2 = \pm 1$ such that the containment $X_\ell \in U$ has at least two positive integer solutions ℓ. The following result was proved.

Theorem 1.1 *For each squarefree integer d, there is at most one positive integer ℓ such that $X_\ell \in U$ except for $d \in \{2, 3, 5, 15, 26\}$.*

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Furthermore, for each $d \in \{2, 3, 5, 15, 26\}$, all solutions ℓ to $X_\ell \in U$ were given together with the representations of these X_ℓ's as sums of two Tribonacci numbers. Unfortunately, there was an oversight in [1], which we now correct.

The following intermediate result is Lemma 4.1 in [1].

Lemma 1.2 *Let (m_i, n_i, ℓ_i) be two solutions of $T_{m_i} + T_{n_i} = X_{\ell_i}$, with $0 \le m_i < n_i$ for $i = 1, 2$ and $1 \le \ell_1 < \ell_2$. Then*

$$m_1 < n_1 \le 1535, \quad \ell_1 \le 1070 \quad \text{and} \quad n_2 < 2.5 \cdot 10^{42}.$$

The rest of the argument in [1] were just reductions of the above parameters. The first step of the reduction consisted in finding all the solutions to

$$X_{\ell_1} = F_{n_1} + F_{m_1}, \quad \ell_1 \in [1, 1070] \quad 2 \le m_1 < n_1 \le 1535.$$

Unfortunately, the case $\ell_1 = 1$ was omitted in [1]. Here, we discuss the missed case $\ell_1 = 1$.

In order to reduce the above bound on n_2 from Lemma 1.2, we do not consider the equation $P^{\pm}_{\ell_1}(X_1) = X_1$ since there is no polynomial equation to solve; instead, we consider each minimal solution $\delta := \delta(X_1, \epsilon)$ of Pell equation $X^2 - dY^2 = \epsilon = \pm 1$, for each $X_1 = T_{m_1} + T_{n_1}$, according to the bounds in Lemma 1.2. Thus, after some reductions using the Baker–Davenport method on the linear form in logarithms Γ_1 and Γ_2 from [1, inequalities 3.9 and 3.12], for $(m, n, \ell) = (m_2, n_2, \ell_2)$, one shows that the only range for the variables to be considered is

$$\ell_1 = 1, \quad 1 \le m_1 < n_1 \le 1811, \quad 1 \le m_2 < n_2 \le 3210, \quad \text{and} \quad 2 \le \ell_2 \le 2220. \tag{1}$$

Now, with this new bound on n_2, by the same procedure (LLL algorithm and continued fractions) used on the linear form in logarithms Γ_3, Γ_4 and Γ_5 in [1, inequalities 3.15–3.26], we reduce again the bound on n_1 given in Lemma 1.2. Then, further cycles of reductions (for n_2 with the new bound of n_1) on Γ_1 and Γ_2 yield the following result.

Lemma 1.3 *Let (m_i, n_i, ℓ_i) be two solutions of $T_{m_i} + T_{n_i} = X_{\ell_i}$, with $0 \le m_i < n_i$ for $i = 1, 2$. If $\ell_1 = 1$, then $1 \le m_1 < n_1 \le 160$, $1 \le m_2 < n_2 < 250$ and $2 \le \ell_2 \le 175$.*

An exhaustive search in this last range finds no new solutions. Hence, albeit the work in [1] missed one branch of computations which are described in this note, this does not affect the final result Theorem 1.1.

Reference

1. E.F. Bravo, C.A. Gómez, F. Luca, X-coordinates of Pell equations as sums of two Tribonacci numbers. Period. Math. Hung. **77**(2), 175–190 (2018)