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A note about maximal almost-invariant subspaces and maximal hyperinvariant subspaces

Cui Chen[1] · Ze-Hua Zhou[2] · Ya Wang[1]

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Abstract
In this paper, we show that for $T \in B(\mathcal{H})$, if $\mathcal{M}$ is almost-invariant for T, then every maximal almost-invariant subspace of $\mathcal{M}$ is of codimension 1 in $\mathcal{M}$, where $\mathcal{H}$ is a separable, infinite-dimensional Hilbert space. We also describe the maximal hyperinvariant subspaces for normal operators with all the dimensions of eigenspaces at most 1 acting on $\mathcal{H}$. Our result is that for each hyperinvariant subspace, all its maximal hyperinvariant subspaces are also of codimension 1 in it.

Keywords Maximal almost-invariant subspaces · Maximal hyperinvariant subspaces · Codimension

Mathematics Subject Classification Primary 47A15; Secondary 47L05

1 Introduction

Let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space and denote by $B(\mathcal{H})$ the set of bounded linear operators acting on $\mathcal{H}$. For $T \in B(\mathcal{H})$ a subspace $\mathcal{M}$ of $\mathcal{H}$ is called *invariant* for T, or T-*invariant*, if it is closed and $T\mathcal{M} \subseteq \mathcal{M}$. The classical Invariant Subspace Problem, one of the most important problems in Operator Theory, is about the existence of non-trivial invariant subspaces for an operator $T \in B(\mathcal{H})$.

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✉ Ya Wang
wangyasjxsy0802@163.com

Cui Chen
chencui_cc@126.com

Ze-Hua Zhou
zehuazhoumath@aliyun.com; zhzhou@tju.edu.cn

1 Department of Mathematics, Tianjin University of Finance and Economics, Tianjin 300222, People's Republic of China

2 School of Mathematics, Tianjin University, Tianjin 300354, People's Republic of China

For an operator $T \in B(\mathcal{H})$ and an invariant subspace $\mathcal{M}$ for T, a T-invariant subspace $\mathcal{N}$ is called a *maximal invariant subspace* of $\mathcal{M}$, if $\mathcal{N} \subsetneq \mathcal{M}$ and there is no T-invariant subspace $\mathcal{L}$ such that $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$. Hedenmalm [6] obtained first the result that every maximal invariant subspace of the Bergman space is of codimension 1. For further generalizions of the Bergman space, we refer the interested readers to [1,16]. Later, Guo et al. [5] extended the result to a much more general situation. Their result is the following.

Theorem 1.1 ([5, Theorem 1.1]). *Suppose T is a Fredholm operator acting on a separable Hilbert space and $1 - TT^* \in \mathcal{S}_p$ for some $p \geq 1$. If $\mathcal{M}$ is an invariant subspace for T such that* $\dim \mathcal{M} \Theta T\mathcal{M} < \infty$, *then every maximal invariant subspace of $\mathcal{M}$ is of codimension 1 in $\mathcal{M}$.*

Here $\mathcal{S}_p$ $(p > 0)$ denotes the set of Schatten-p class operators; and for two subspaces $\mathcal{U}, \mathcal{V}$ of $\mathcal{H}$, $\mathcal{U}\Theta\mathcal{V} = \mathcal{U} \cap \mathcal{V}^{\perp}$, where $\mathcal{V}^{\perp}$ denotes the orthogonal complement space of $\mathcal{V}$ in $\mathcal{H}$.

Motivated by the above work, we intend to study the maximal almost-invariant subspaces and maximal hyperinvariant subspaces.

A subspace $\mathcal{M}$ of $\mathcal{H}$ is called *almost-invariant* for T (or T-*almost invariant*) if it is closed and $T\mathcal{M} \subseteq \mathcal{M} + \mathcal{F}$ for some finite-dimensional subspace $\mathcal{F}$ of $\mathcal{H}$. This concept was first introduced in [2]. The minimal dimension of such a subspace $\mathcal{F}$ is referred to as the *defect* of $\mathcal{M}$ for T. It is obvious that every finite-dimensional or finite-codimensional subspace is almost-invariant under T. So we only need to consider a *half-space*, that is, a subspace of $\mathcal{H}$ which is infinite-dimensional and infinite-codimensional. For further information about almost-invariant subspaces, we refer the interested readers to [2,12–14].

In a similar way, we give the definition of maximal almost-invariant subspace.

Definition 1.2 For an operator $T \in B(\mathcal{H})$ and an almost-invariant subspace $\mathcal{M}$ for T, a T-almost invariant subspace $\mathcal{N}$ is called a *maximal almost-invariant subspace* of $\mathcal{M}$, if $\mathcal{N} \subsetneq \mathcal{M}$ and there is no T-almost invariant subspace $\mathcal{L}$ such that $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$.

We will prove that, given an operator $T \in B(\mathcal{H})$, for any T-almost invariant half-space $\mathcal{M}$ every maximal almost-invariant subspace of $\mathcal{M}$ is of codimension 1 in $\mathcal{M}$.

A subspace $\mathcal{M}$ of $\mathcal{H}$ is called *hyperinvariant* for T, or T-*hyperinvariant*, if it is closed and $W\mathcal{M} \subseteq \mathcal{M}$ for each $W \in \{T\}'$. Here $\{T\}'$ denotes the commutant of T given by

$$\{T\}' = \{W \in B(\mathcal{H}) : WT = TW\}.$$

There are many unsolved problems in the theory of invariant subspaces, hence these problems need close attention. In this paper, we first deal with hyperinvariant subspaces. For a further discussion about hyperinvariant subspaces, we recommend to the interested readers the recent papers [4,7–11,15].

We also define maximal hyperinvariant subspaces analogously.

Definition 1.3 For an operator $T \in B(\mathcal{H})$ and a hyperinvariant subspace $\mathcal{M}$ for T, a T-hyperinvariant subspace $\mathcal{N}$ is called a *maximal hyperinvariant subspace* of $\mathcal{M}$, if $\mathcal{N} \subsetneq \mathcal{M}$ and there is no T-hyperinvariant subspace $\mathcal{L}$ such that $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$.

An operator $T \in B(\mathcal{H})$, is said to be normal if $T^*T = TT^*$.

Our conclusion about maximal hyperinvariant subspaces is in the setting of a Hilbert space $\mathcal{H}$. We will show that for a normal operator $T \in B(\mathcal{H})$ and a T-hyperinvariant subspace $\mathcal{M}$, if all the dimensions of eigenspaces of T are at most 1, then every maximal hyperinvariant subspace of $\mathcal{M}$ is of codimension 1 in $\mathcal{M}$.

Throughout the paper, for a closed subspace $\mathcal{E}$ of $\mathcal{H}$, $P_{\mathcal{E}}$ denotes the orthogonal projection from $\mathcal{H}$ to $\mathcal{E}$ and $T|_{\mathcal{E}}$ is the operator T restricted to $\mathcal{E}$.

2 Maximal almost-invariant subspaces

In this section, we give a characterization of maximal almost-invariant subspaces. The main result can be formulated as follows.

Theorem 2.1 *For $T \in B(\mathcal{H})$, if $\mathcal{M}$ is a T-almost invariant half-space, then every maximal almost-invariant subspace of $\mathcal{M}$ is of codimension 1 in $\mathcal{M}$.*

The first step in the proof of Theorem 2.1 is the following lemma.

Lemma 2.2 *Let $T \in B(\mathcal{H})$. Suppose $\mathcal{M}$ and $\mathcal{N}$ are two T-almost invariant half-spaces with $\mathcal{N} \subsetneq \mathcal{M}$ and $\dim \mathcal{M}\Theta\mathcal{N} \geq 2$. Put $S = P_{M\Theta N}T|_{M\Theta N}$. Then T has an almost invariant half-space $\mathcal{L}$ such that $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$ if and only if there exists an S-almost invariant subspace $\mathcal{L}_0$ such that $0 \subsetneq \mathcal{L}_0 \subsetneq \mathcal{M}\Theta\mathcal{N}$.*

Proof Since $\mathcal{M}, \mathcal{N}$ are both T-almost invariant half-spaces, we write $T\mathcal{M} \subseteq \mathcal{M} + \mathcal{F}_1$ and $T\mathcal{N} \subseteq \mathcal{N} + \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are finite-dimensional subspaces of $\mathcal{H}$. Now, if $\mathcal{L}$ is almost-invariant for T with $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$, we assume that $T\mathcal{L} \subseteq \mathcal{L} + \mathcal{F}_3$ for some finite-dimensional subspace $\mathcal{F}_3$. Put $\mathcal{L}_0 = \mathcal{L}\Theta\mathcal{N}$; then it is clear that $0 \subsetneq \mathcal{L}_0 \subsetneq \mathcal{M}\Theta\mathcal{N}$, and

$$S\mathcal{L}_0 = P_{\mathcal{M}\Theta\mathcal{N}}T|_{\mathcal{M}\Theta\mathcal{N}}(\mathcal{L}\Theta\mathcal{N}) \subseteq (\mathcal{L} + \mathcal{F}_3)\Theta\mathcal{N} \subseteq \mathcal{L}\Theta\mathcal{N} + \mathcal{F}_3 \subseteq \mathcal{L}_0 + \mathcal{F}_3,$$

that is $\mathcal{L}_0$ is S-almost invariant.

Conversely, we assume that there exists a subspace $\mathcal{L}_0$ with $0 \subsetneq \mathcal{L}_0 \subsetneq \mathcal{M}\Theta\mathcal{N}$ that is S-almost invariant by $S\mathcal{L}_0 \subseteq \mathcal{L}_0 + \mathcal{F}_4$ for some finite-dimensional subspace $\mathcal{F}_4$. Put $\mathcal{L} = \mathcal{L}_0 + \mathcal{N}$, then $\mathcal{L}$ is half-space and $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$. Next we prove that $\mathcal{L}$ is T-almost invariant. Noting that $S\mathcal{L}_0 \subseteq \mathcal{L}_0 + \mathcal{F}_4$, i.e., $P_{\mathcal{M}\Theta\mathcal{N}}T|_{\mathcal{M}\Theta\mathcal{N}}\mathcal{L}_0 \subseteq \mathcal{L}_0 + \mathcal{F}_4$, we have $P_{\mathcal{N}^\perp}T\mathcal{L}_0 \subseteq \mathcal{L}_0 + \mathcal{F}_4 + \mathcal{F}_1$. Then $T\mathcal{L}_0 = P_{\mathcal{N}}T\mathcal{L}_0 + P_{\mathcal{N}^\perp}T\mathcal{L}_0 \subseteq \mathcal{N} + \mathcal{L}_0 + \mathcal{F}_4 + \mathcal{F}_1$, thus

$$T\mathcal{L} = T\mathcal{L}_0 + T\mathcal{N} \subseteq \mathcal{L} + \mathcal{F}_4 + \mathcal{F}_1 + \mathcal{N} + \mathcal{F}_2 \subseteq \mathcal{L} + \widehat{\mathcal{F}},$$

for some finite-dimensional subspace $\widehat{\mathcal{F}}$, that is, $\mathcal{L}$ is T-almost invariant. So the assertion of this lemma is proved. □

The following lemma, proved by Popov and Tcaciuc in [14], is quite important to get the main result of this section.

Lemma 2.3 *Let T be a bounded operator on an infinite-dimensional, reflexive Banach space $\mathcal{X}$. Then $\mathcal{X}$ admits an almost-invariant half-space with defect 1.*

Using this lemma, we can prove the following result, which is the key idea of the proof of Theorem 2.1.

Lemma 2.4 *Let $T \in B(\mathcal{H})$. Suppose $\mathcal{M}$ and $\mathcal{N}$ are two T-almost invariant half-spaces with $\mathcal{N} \subsetneq \mathcal{M}$ and $\dim \mathcal{M}\Theta\mathcal{N} \geq 2$. Then there is a T-almost invariant half-space $\mathcal{L}$ such that $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$.*

Proof Set $T\mathcal{M} \subseteq \mathcal{M} + \mathcal{F}_1$ and $T\mathcal{N} \subseteq \mathcal{N} + \mathcal{F}_2$ as in the proof of Lemma 2.2. Firstly, assuming that $\dim \mathcal{M}\Theta\mathcal{N} < \infty$, we can choose half-space $\mathcal{L}$ such that $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$ since $\dim \mathcal{M}\Theta\mathcal{N} \geq 2$. Moreover, for each half-space $\mathcal{L}$ with $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$, we have

$$T\mathcal{L} \subseteq T\mathcal{M} \subseteq T(\mathcal{M}\Theta\mathcal{N}) + T\mathcal{N} \subseteq T(\mathcal{M}\Theta\mathcal{N}) + \mathcal{N} + \mathcal{F}_2 \subseteq \mathcal{L} + \widetilde{\mathcal{F}},$$

for some finite-dimensional subspace $\widetilde{\mathcal{F}}$ of $\mathcal{H}$ since $\dim \mathcal{M}\Theta\mathcal{N} < \infty$.

Now we assume $\dim \mathcal{M}\Theta\mathcal{N} = \infty$. Consider the operator $S = P_{\mathcal{M}\Theta\mathcal{N}}T|_{\mathcal{M}\Theta\mathcal{N}}$. Since $S = P_{\mathcal{M}\Theta\mathcal{N}}T|_{\mathcal{M}\Theta\mathcal{N}} \in B(\mathcal{M}\Theta\mathcal{N})$ and $\mathcal{M}\Theta\mathcal{N}$ is an infinite-dimensional, reflexive Banach space, by Lemma 2.3, $\mathcal{M}\Theta\mathcal{N}$ admits an S-almost invariant half-space with defect 1. Therefore, using Lemma 2.2, there exists a T-almost invariant subspace $\mathcal{L}$ such that $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$. This completes the proof. □

Now the characterization of maximal almost-invariant subspaces is a direct consequence of this lemma.

At the end of this section, we want to pose a question to the interested readers. In the proof, the finite-dimensional subspaces making sure that $\mathcal{N}$, $\mathcal{L}$, $\mathcal{M}$ are T-almost invariant may not have the same dimension or even be the same subspace. Of course, here we mean such a finite-dimensional subspace with minimal dimension to make sure $\mathcal{N}$, $\mathcal{L}$, $\mathcal{M}$ are T-almost invariant. Hence, it is natural to ask the following question:

Question 2.5 *Is there a separable, infinite-dimensional Hilbert space $\mathcal{H}$, and an operator $T \in B(\mathcal{H})$ such that there exist three half-spaces $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$ that are T-almost invariant subspaces with the same defect or even the same finite-dimensional subspace are T-almost invariant subspaces?*

3 Maximal hyperinvariant subspaces

In the case when we consider the maximal hyperinvariant subspaces, we focus on the normal operators acting on a separable, infinite-dimensional Hilbert space $\mathcal{H}$. The main result relies on the following lemma, which is proved in a similar way to [5, Lemma 2.3].

Lemma 3.1 *Let $T \in B(\mathcal{H})$, $\mathcal{M}$ and $\mathcal{N}$ be two T-hyperinvariant subspaces with $\mathcal{N} \subsetneq \mathcal{M}$ and* $\dim \mathcal{M}\Theta\mathcal{N} \geq 2$. *Put $S = P_{\mathcal{M}\Theta\mathcal{N}}T|_{\mathcal{M}\Theta\mathcal{N}}$. Then if S has a non-trivial hyperinvariant subspace then T has a hyperinvariant subspace $\mathcal{L}$ such that $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$.*

Moreover, if in additon T is normal, then the existence of a T-hyperinvariant subspace $\mathcal{L}$ with $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$ implies the existence of a non-trivial hyperinvariant subspace for S.

Proof Suppose that $\mathcal{L}_0$ is a nontrivial hyperinvariant subspace for S. It is clear that $0 \subsetneq \mathcal{L}_0 \subsetneq \mathcal{M}\Theta\mathcal{N}$. Setting $\mathcal{L} = \mathcal{L}_0 + \mathcal{N}$, we have $\mathcal{N} \subsetneq \mathcal{L} \subsetneq \mathcal{M}$. Next we prove that $\mathcal{L}$ is hyperinvariant for T. For any $W \in \{T\}'$, we first prove $P_{\mathcal{M}\Theta\mathcal{N}}W|_{\mathcal{M}\Theta\mathcal{N}} \in \{S\}'$. Indeed,

$$\begin{aligned}
P_{\mathcal{M}\Theta\mathcal{N}}W|_{\mathcal{M}\Theta\mathcal{N}}P_{\mathcal{M}\Theta\mathcal{N}}T|_{\mathcal{M}\Theta\mathcal{N}} &= P_{\mathcal{M}\Theta\mathcal{N}}WP_{\mathcal{M}\Theta\mathcal{N}}TP_{\mathcal{M}\Theta\mathcal{N}} \\
&= P_{\mathcal{M}\Theta\mathcal{N}}W(P_{\mathcal{M}} - P_{\mathcal{N}})TP_{\mathcal{M}\Theta\mathcal{N}} \\
&= P_{\mathcal{M}\Theta\mathcal{N}}WP_{\mathcal{M}}TP_{\mathcal{M}\Theta\mathcal{N}} - P_{\mathcal{M}\Theta\mathcal{N}}WP_{\mathcal{N}}TP_{\mathcal{M}\Theta\mathcal{N}} \\
&= P_{\mathcal{M}\Theta\mathcal{N}}WTP_{\mathcal{M}\Theta\mathcal{N}},
\end{aligned}$$

here we used $P_{\mathcal{M}\Theta\mathcal{N}}WP_{\mathcal{N}}TP_{\mathcal{M}\Theta\mathcal{N}} = 0$ since $\mathcal{N}$ is hyperinvariant for T and $W \in \{T\}'$.

In a similar way, we can obtain

$$\begin{aligned}
P_{\mathcal{M}\Theta\mathcal{N}}T|_{\mathcal{M}\Theta\mathcal{N}}P_{\mathcal{M}\Theta\mathcal{N}}W|_{\mathcal{M}\Theta\mathcal{N}} &= P_{\mathcal{M}\Theta\mathcal{N}}TP_{\mathcal{M}\Theta\mathcal{N}}WP_{\mathcal{M}\Theta\mathcal{N}} \\
&= P_{\mathcal{M}\Theta\mathcal{N}}T(P_{\mathcal{M}} - P_{\mathcal{N}})WP_{\mathcal{M}\Theta\mathcal{N}} \\
&= P_{\mathcal{M}\Theta\mathcal{N}}TP_{\mathcal{M}}WP_{\mathcal{M}\Theta\mathcal{N}} - P_{\mathcal{M}\Theta\mathcal{N}}TP_{\mathcal{N}}WP_{\mathcal{M}\Theta\mathcal{N}} \\
&= P_{\mathcal{M}\Theta\mathcal{N}}TWP_{\mathcal{M}\Theta\mathcal{N}}.
\end{aligned}$$

Since $WT = TW$, thus

$$P_{\mathcal{M}\Theta\mathcal{N}}W|_{\mathcal{M}\Theta\mathcal{N}}P_{\mathcal{M}\Theta\mathcal{N}}T|_{\mathcal{M}\Theta\mathcal{N}} = P_{\mathcal{M}\Theta\mathcal{N}}T|_{\mathcal{M}\Theta\mathcal{N}}P_{\mathcal{M}\Theta\mathcal{N}}W|_{\mathcal{M}\Theta\mathcal{N}},$$

that is $P_{\mathcal{M}\Theta\mathcal{N}}W|_{\mathcal{M}\Theta\mathcal{N}} \in \{S\}'$.

Since $\mathcal{L}_0$ is a hyperinvariant subspace for S, then $P_{\mathcal{M}\Theta\mathcal{N}}W|_{\mathcal{M}\Theta\mathcal{N}}\mathcal{L}_0 \subseteq \mathcal{L}_0$, hence we have $P_{\mathcal{N}^\perp}W\mathcal{L}_0 \subseteq \mathcal{L}_0$. Then it is easy to see that $W\mathcal{L}_0 = P_{\mathcal{N}}W\mathcal{L}_0 + P_{\mathcal{N}^\perp}W\mathcal{L}_0 \subseteq \mathcal{N} + \mathcal{L}_0 = \mathcal{L}$. Therefore, we conclude that

$$W\mathcal{L} = W\mathcal{L}_0 + W\mathcal{N} \subseteq \mathcal{L}.$$

That is, $\mathcal{L}$ is hyperinvariant for T by the arbitrariness of $W \in \{T\}'$, which proves the first assertion of this lemma.

Conversely, note that T is normal, i.e., $T^* \in \{T\}'$. Therefore, $\mathcal{M}, \mathcal{N}$ are both reducing subspaces of T, so is $\mathcal{M}\Theta\mathcal{N}$. Thus we conclude that $S = T|_{\mathcal{M}\Theta\mathcal{N}}$. Then the operator T has the corresponding decomposition

$$T = S \oplus T_1,$$

where T_1 is the restriction of T on $(\mathcal{M}\Theta\mathcal{N})^\perp$. Given an operator $W_0 \in \mathcal{L}(\mathcal{M}\Theta\mathcal{N})$, if $W_0S = SW_0$, set $W = W_0 \oplus I_{(\mathcal{M}\Theta\mathcal{N})^\perp}$, it follows that $W \in \{T\}'$. Thus $W\mathcal{L} \subseteq \mathcal{L}$. It is easy to prove that $W_0\mathcal{L}_0 \subseteq \mathcal{L}_0$, if $\mathcal{L}_0 = \mathcal{L}\Theta\mathcal{N}$. So the second assertion of the lemma is obtained. □

Lemma 3.2 *Let $T \in B(\mathcal{H})$ be a normal operator, and $\mathcal{M}$ and $\mathcal{N}$ two T-hyperinvariant subspaces with $\mathcal{N} \subsetneq \mathcal{M}$ and* $\dim \mathcal{M}\Theta\mathcal{N} \geq 2$. *Then $S = P_{\mathcal{M}\Theta\mathcal{N}}T|_{\mathcal{M}\Theta\mathcal{N}}$ is also normal.*

Proof Since $\mathcal{M}, \mathcal{N}$ are both T-hyperinvariant, and $T^* \in \{T\}'$, then $\mathcal{M}, \mathcal{N}$ are both reducing subspaces of T, so is $\mathcal{M}\Theta\mathcal{N}$. Hence $S = T|_{\mathcal{M}\Theta\mathcal{N}}$. Next, we will show that $T|^*_{\mathcal{M}\Theta\mathcal{N}} = T^*|_{\mathcal{M}\Theta\mathcal{N}}$. In fact,

$$\langle T|^*_{\mathcal{M}\Theta\mathcal{N}}x, y\rangle = \langle x, T|_{\mathcal{M}\Theta\mathcal{N}}y\rangle = \langle x, Ty\rangle = \langle T^*x, y\rangle = \langle T^*|_{\mathcal{M}\Theta\mathcal{N}}x, y\rangle$$

for $x, y \in \mathcal{M}\Theta\mathcal{N}$. Then the result follows from the normality of T. □

The next result can be found in [3].

Lemma 3.3 *A normal operator that is not a multiple of the identity has a non-trivial hyperinvariant subspace.*

Using the previous lemmas, we are now ready to give the required generalization about maximal hyperinvariant subspaces.

Theorem 3.4 *Let $T \in B(\mathcal{H})$ be a normal operator, and $\mathcal{M}$ be a T-hyperinvariant subspace. If all the dimensions of eigenspaces of T are at most 1, then every maximal hyperinvariant subspace of $\mathcal{M}$ is of codimension 1 in $\mathcal{M}$.*

Proof Note that the condition that all the dimensions of eigenspaces of T are at most 1 guarantees that $P_{\mathcal{M}\Theta\mathcal{N}}T|_{\mathcal{M}\Theta\mathcal{N}}$ is not a multiple of the identity for each T-hyperinvariant subspace $\mathcal{N}$ with $\mathcal{N} \subsetneq \mathcal{M}$ and $\dim \mathcal{M}\Theta\mathcal{N} \geq 2$. Then the assertion comes easily from the lemmas above. □

References

1. A. Atzmon, Maximal, minimal, and primary invariant subspaces. J. Funct. Anal. **185**, 155–213 (2001)
2. G. Androulakis, A.I. Popov, A. Tcaciuc, V.G. Troitsky, Almost invariant half-spaces of operators on Banach spaces. Integer. Equat. Oper. Th. **65**(4), 473–484 (2009)
3. I. Chalfnbar, J.R. Partington, *Modern Approaches to the Invariant-Subspace Problem, Cambridge Tracts in Mathematics, 188* (Cambridge University Press, Cambridge, 2011)
4. C. Folas, I.B. Jung, E. Ko, C. Pearcy, Hyperinvariant subspaces for some subnormal operators. T. Am. Math. Soc. **359**(6), 2899–2913 (2007)
5. K.Y. Guo, W. He, S.Z. Hou, Maximal invariant subspaces for a class of operators. Ark. Math. **48**, 323–333 (2010)
6. H. Hedenmalm, Maximal invariant subspaces in the Bergman space. Ark. Math. **36**, 97–101 (1998)
7. S. Hamid, C. Onica, C. Pearcy, On the hyperinvariant subspace problem. II. Indiana U. Math. J. **54**(3), 743–754 (2005)
8. H.J. Kim, Hyperinvariant subspace problem for quasinipotent operators. Integer. Equat. Oper. Th. **61**, 103–120 (2008)
9. H.J. Kim, Hyperinvariant subspaces for quasinipotent operators on Hilbert spaces. J. Math. Anal. Appl. **350**, 262–270 (2009)
10. H.J. Kim, Hyperinvariant subspaces for operators having a compact part. J. Math. Anal. Appl. **386**, 110–114 (2012)
11. M.X. Liu, On hyperinvariant subspaces of contraction operators on a Banach space whose spectrum contains the unit circle. Acta Math. Sin. Eng. Ser. **24**(9), 1471–1474 (2008)
12. L. Marcoux, A.I. Popov, H. Radjavi, On almost-invariant subspaces and approximate commutation. J. Funct. Anal. **264**, 1088–1111 (2013)
13. A.I. Popov, Almost invariant half-spaces of algebras of operators. Integer. Equat. Oper. Th. **67**, 247–256 (2010)
14. A.I. Popov, A. Tcaciuc, Every operator has almost-invariant subspaces. J. Funct. Anal. **265**, 257–265 (2013)
15. L.Y. Shi, Y.J. Wu, Y.Q. Ji, Invariant and hyperinvariant subspaces for amenable operators. J. Operat. Theor. **69**(1), 87–100 (2013)
16. T. Trent, Maximal invariant subspaces for $A_\alpha^2(D)$. Proc. Am. Math. Soc. **132**, 2429–2432 (2004)