On the arithmetic Kakeya conjecture of Katz and Tao

Ben Green¹ · Imre Z. Ruzsa²

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Abstract

The arithmetic Kakeya conjecture, formulated by Katz and Tao (Math Res Lett 6(5–6):625–630, 1999), is a statement about addition of finite sets. It is known to imply a form of the Kakeya conjecture, namely that the upper Minkowski dimension of a Besicovitch set in $\mathbf{R}^n$ is n . In this note we discuss this conjecture, giving a number of equivalent forms of it. We show that a natural finite field variant of it does hold. We also give some lower bounds.

Keywords Kakeya problem · Arithmetic progression

Contents

1 Introduction and statement of results	135
2 Progressions, projections and entropy	138
3 Finite fields	143
4 A problem of Erdős and Selfridge	145
5 Small unions of progressions	147
6 Entropy inequalities in positive characteristic	148
Appendix A. Covering by translates	150
References	151

1 Introduction and statement of results

The arithmetic Kakeya conjecture, sometimes known as the sums-differences conjecture, was formulated by Katz and Tao around fifteen years ago. It is a purely additive-combinatorial

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✉ Ben Green
ben.green@maths.ox.ac.uk

Imre Z. Ruzsa
ruzsa@renyi.hu

¹ Mathematical Institute, Radcliffe Observatory Quarter, Woodstock Rd, Oxford OX2 6GG, UK

² Alfred Rényi Institute, Reáltanoda utca 13-15., Budapest 1053, Hungary

statement which, if true, would have a deep geometric consequence—that the Minkowski dimension of Besicovitch sets in $\mathbf{R}^n$ is n . This is the celebrated Kakeya conjecture, discussed at length in many places: for an introduction see [21].

The arithmetic Kakeya conjecture is mentioned explicitly¹ in [20]. One of the main aims of this paper is to give a number of equivalent forms of the conjecture. Here is probably the simplest formulation. It is not the original one of Katz and Tao, which is Conjecture 1.3 below.

Conjecture 1.1 *Let k, N be positive integers. Write $F_k(N)$ for the size of the smallest set of integers containing, for each $d \in \{1, \dots, N\}$, a k -term arithmetic progression with common difference d . Then*

$$\lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{\log F_k(N)}{\log N} = 1.$$

This conjecture was raised by the second author as [17, Conjecture 4.2], but no links to the Kakeya problem were mentioned there.

We turn now to arguably the most natural of our formulations, concerning the entropy of random variables. As usual, the entropy $\mathbf{H}$ of a random variable X with finite range is defined by

$$\mathbf{H}(X) := - \sum_x \mathbf{P}(X = x) \log \mathbf{P}(X = x),$$

where x ranges over all values taken by X .

Conjecture 1.2 *For any $\varepsilon > 0$ there are² $r_1, \dots, r_k \in \mathbf{Q}$, none equal to -1 , such that for any two real-valued random variables X and Y taking only finitely many values we have*

$$\mathbf{H}(X - Y) \leq (1 + \varepsilon) \sup_j \mathbf{H}(X + r_j Y).$$

Next we give the original form of the conjecture discussed by Katz and Tao. Let $A \subset \mathbf{Z} \times \mathbf{Z}$ be a finite set. For rational r we write $\pi_r(A) := \{x + ry : (x, y) \in A\}$. We also write $\pi_\infty(A) := \{y : (x, y) \in A\}$.

Conjecture 1.3 *Let $\varepsilon > 0$ be arbitrary. Then there are $r_1, \dots, r_k \in \mathbf{Q} \cup \{\infty\}$, none equal to -1 , such that $\#\pi_{-1}(A) \leq \sup_i \#\pi_{r_i}(A)^{1+\varepsilon}$ for all finite sets $A \subset \mathbf{Z} \times \mathbf{Z}$.*

Our fourth conjecture has not, so far as we are aware, appeared explicitly in the literature before. It is in fact a whole family of conjectures, one for each natural number n ; however, we will later show that all of these are equivalent.

¹ In the earlier paper [13, p. 234] of Katz and Tao, the authors only go so far as to suggest that it is “not too outrageous tentatively to conjecture” this statement. In fact, the conjecture made in [20] is over fields of “sufficiently large characteristic” (or characteristic zero) whereas this paper provides evidence that it is natural, and simpler, to work only in characteristic zero. We believe that in any case the statements are equivalent but have not bothered to check this carefully.

² It is often convenient to “work projectively” and allow the r_i to take values in $\mathbf{Q} \cup \{\infty\}$, where we define $X + \infty Y = Y$. The two versions of Conjecture 1.2 are equivalent to one another, as may easily be seen by applying a projective transformation such as $X' = (a + 1)X$, $Y' = aX + Y$ which preserves $X - Y$ but moves other rational combinations around.

Conjecture 1.4 (*n*) Let k be a positive integer. If p is a prime, let $f_{k,n}(p)$ denote the size of the smallest set containing, for every $d \in \mathbf{F}_p^n \setminus \{0\}$, a k -term progression with common difference d . Then

$$\lim_{k \rightarrow \infty} \lim_{p \rightarrow \infty} \frac{\log f_{k,n}(p)}{\log p} = n.$$

Remarks Note that $f_{p,n}(p)$ is the size of the smallest *Besicovitch set* in $\mathbf{F}_p^n$, that is to say a set containing a full line in every direction. Since $f_{p,n}(p) \geq f_{k,n}(p)$ whenever $p \geq k$, Conjecture 1.4 (*n*) trivially implies that

$$\lim_{p \rightarrow \infty} \frac{f_{p,n}(p)}{\log p} = n,$$

i.e. any Besicovitch set in $\mathbf{F}_p^n$ has size $p^{n-o_{p \rightarrow \infty}(1)}$. This is known to be true, a celebrated result of Dvir [5]. However, the only known arguments use the “polynomial method” (see, for example, [11, 22] for modern introductions). This very strongly hints that any proof of Conjecture 1.4 (and hence, by our main theorem, of the other conjectures) would have to use some form of the polynomial method.

Our fifth and final conjecture is included mainly for historical interest, as it relates very closely to a question asked by Erdős and Selfridge in the 1970s, well before the current wave of interest in the Kakeya problem and related matters.

Conjecture 1.5 Fix a positive integer k . Then, uniformly for all positive integers N , all finite sets $p_1 < \dots < p_N$ of primes and all intervals $I \subset \mathbf{N}$ of length kp_N , we have

$$\#(I \cap \bigcup_{i=1}^N p_i \mathbf{Z}) \gg_k N^{1-\gamma_k}.$$

where $\gamma_k \rightarrow 0$ as $k \rightarrow \infty$.

Remark Erdős and Selfridge [8, §6] in fact asked whether or not one can take $\gamma_k = 0$. The second-named author [16] showed that the answer is negative, and in fact we must have $\gamma_k \geq \frac{1}{k}$. We note that Proposition 4.1 and Theorem 1.7 combine to give the much better bound $\gamma_k \gg \frac{1}{\log \log k}$.

As previously stated, our main result is the equivalence of the five conjectures stated above.

Theorem 1.6 Conjectures 1.1, 1.2, 1.3, 1.4(*n*) (for each $n = 1, 2, 3, \dots$) and 1.5 are all equivalent.

Let us make some further remarks.

- (1) Once Theorem 1.6 is proven, it seems natural to use the term “arithmetic Kakeya conjecture” to refer to any one of the five conjectures.
- (2) It is known that Conjecture 1.3 (and hence all the other conjectures) implies that the upper Minkowski dimension of any Besicovitch set³ in $\mathbf{R}^n$ is n , a statement often referred to as the *Kakeya conjecture*. This follows by a straightforward generalisation of the “slicing”

³ That is, a compact subset of $\mathbf{R}^n$ containing a unit line segment in every direction.

argument of Bourgain [4]: a sketch of this may be found in [21]. However, Bourgain [2,3] observed that, in the notation of Conjecture 1.1, if the statement

$$\lim_{N \rightarrow \infty} \frac{\log F_{N^\eta}(N)}{\log N} \geq 1 \quad (1.1)$$

is true for all $\eta > 0$ then the Kakeya conjecture follows. Since $F_k(N)$ is a nondecreasing function of k , (1.1) is immediately implied by Conjecture 1.1, whilst an implication in the reverse direction seems very unlikely without resolving both conjectures. In this sense, the arithmetic Kakeya conjecture should be considered a *strictly* harder problem than the Kakeya conjecture.

- (3) The equivalence of Conjectures 1.2 and 1.3 was proven by the second author in [18] (see also [14]). We are not aware of any references for the other implications.

Now we discuss the other results in the paper. First, we establish a lower bound showing that the convergence in Theorem 1, if it occurs, is very slow.

Theorem 1.7 *In the notation of Conjecture 1.1, we have*

$$\lim_{N \rightarrow \infty} \frac{\log F_k(N)}{\log N} \leq 1 - \frac{c}{\log \log k},$$

where the constant $c > 0$ is absolute.

Second, we show that a finite field variant of Conjecture 1.2 is true. Write $\mathbf{F}_p^\infty := \bigcup_{n=1}^\infty \mathbf{F}_p^n$.

Theorem 1.8 *Suppose that X and Y are two $\mathbf{F}_p^\infty$ -valued random variables, both taking only finitely many values. Then*

$$\mathbf{H}(X - Y) \leq \left(1 + O\left(\frac{1}{\log p}\right)\right) \sup_{r \in \mathbf{F}_p \cup \{\infty\} \setminus \{-1\}} \mathbf{H}(X + rY).$$

Here, the constant in the $O()$ notation is absolute.

The $O(\frac{1}{\log p})$ term is best possible, as we remark in Sect. 6.

We neither discuss nor make progress on partial results towards any of Conjectures 1.1, 1.2, 1.3, 1.4 or 1.5. We believe that the best value of ε for which Conjecture 1.2 is known is $\varepsilon \approx 0.67513 \dots$, which is equivalent to a result obtained in [13]. (The precise value here is $\alpha - 1$, where α solves $\alpha^3 - 4\alpha + 2 = 0$.) This bound is now 15 years old.

Notation. Most of our notation is quite standard. We use $\#X$ for the cardinality of a set X . Occasionally, if A is a set in some abelian group and k is an integer we will write $k \cdot A$ to mean $\{ka : a \in A\}$.

2 Progressions, projections and entropy

In this section we establish around half of Theorem 1.6 by proving that the first three conjectures mentioned in the introduction are equivalent. Whilst at a local level the arguments are a mix of fairly unexciting linear algebra and standard tools such as Freiman isomorphisms, random projections and taking tensor powers, the large number of them makes the proof of Theorem 1.6 somewhat lengthy.

It is convenient to proceed by first showing that Conjectures 1.1, 1.3 and 1.2 are equivalent. In the course of doing so, and for later use, it is convenient to introduce a further conjecture, apparently stronger than Conjecture 1.1 but, as it turns out, equivalent to it.

Conjecture 1'. Let k be a positive integer. Write $F'_k(N)$ for the cardinality of the smallest set $A \subset \mathbf{Z}$ which contains an arithmetic progression of length k and common difference d , for N different values of d . Then

$$\lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{\log F'_k(N)}{\log N} = 1.$$

It is obvious that Conjecture 1' implies Conjecture 1.1, because $F'_k(N) \leq F_k(N)$. It turns out that the reverse implication holds as well. In fact, we claim that the following is true.

Proposition 2.1 *We have $F_k(N) \ll k^3 \log N \cdot F'_k(N)$.*

Proof Suppose we have a set

$$A_0 = \bigcup_{i=1}^N \bigcup_{j=0}^{k-1} \{a_i + jd_i\},$$

where the d_i are distinct. We claim that there is a set A_1 , $\#A_1 \ll k^3 \log N \cdot \#A_0$, containing an arithmetic progression of length k and common difference d for all $d \in \{1, \dots, N\}$. This obviously implies the result.

Pick $\theta \in (0, 1)$ uniformly at random, and define the function

$$\phi_\theta : \mathbf{Z} \rightarrow \{0, 1, \dots, N-1\}$$

by

$$\phi_\theta(x) := \lfloor N\{\theta x\} \rfloor.$$

Here, $\{t\} = t - \lfloor t \rfloor$, so $0 \leq \{t\} < 1$.

Note that if $i \neq j$ then

$$\mathbf{P}_\theta(\phi_\theta(d_i) = \phi_\theta(d_j)) \leq \mathbf{P}_\theta(\theta(d_i - d_j) \in \left(-\frac{1}{N}, \frac{1}{N}\right) \pmod{1}) = \frac{2}{N}.$$

It follows that the expected number of pairs (i, j) with $i < j$ for which $\phi_\theta(d_i) = \phi_\theta(d_j)$ is at most $\frac{2}{N} \binom{N}{2} = N - 1$. By linearity of expectation, there is some choice of θ for which, setting $d'_i := \phi_\theta(d_i)$, there are at most $N - 1$ pairs (i, j) with $i < j$ and $d'_i = d'_j$. If $n \in \{0, 1, \dots, N-1\}$, write $f(n)$ for the number of i with $d'_i = n$. Then it follows that $\sum_n \binom{f(n)}{2} \leq N - 1$, from which we obtain, since $\sum_n f(n) = N$, that $\sum_n f(n)^2 \leq 3N$. By Cauchy–Schwarz,

$$N^2 = \left(\sum_n f(n) \right)^2 \leq \#\{n : f(n) \neq 0\} \sum_n f(n)^2,$$

and therefore there are at least $N/3$ values of n for which $f(n) \neq 0$, or in other words there are at least $N/3$ distinct values amongst the d'_i .

Now consider the set $A_2 := \phi_\theta(A_0)$. Obviously $\#A_2 \leq \#A_0$. Whilst A_2 itself does not obviously contain any long progressions, we observe that

$$\phi_\theta(a_i + (j+1)d) - \phi_\theta(a_i + jd) - d'_i \in \{0, 1\} - \{0, N\}$$

(In fact, $\phi_\theta(x+y) - \phi_\theta(x) - \phi_\theta(y) \in \{0, 1\} - \{0, N\}$ for every x, y .) By a simple induction,

$$\phi_\theta(a_i) + jd'_i - \phi_\theta(a_i + jd) \in \{0, 1, \dots, k-1\} - \{0, N, \dots, (k-1)N\}$$

for $j = 0, 1, \dots, k-1$, and so the set $A_3 := A_2 + \{0, 1, \dots, k-1\} - \{0, N, \dots, (k-1)N\}$ contains a progression of length k and common difference d'_i , for all i . Note that $\#A_3 \leq k^2 \#A_0$.

By taking random translates (see Lemma 6.2 for details) and the fact that there are $\geq N/3$ distinct d'_i , there is some set T of integers, $\#T \ll \log N$, such that every element of $\{1, \dots, N\}$ can be written as $d'_i + t$ with $t \in T$. Set

$$A_1 := A_3 + \{0, 1, \dots, k-1\} \cdot T.$$

We have $\#A_1 \leq k \cdot \#T \cdot \#A_3 \ll k^3 \log N \cdot \#A_0$. It is easy to see that A_1 contains an arithmetic progression of length k and common difference $d'_i + t$, for all i and for all $t \in T$, and hence contains an arithmetic progression of length k and common difference d for all $d \in \{1, \dots, N\}$. This concludes the proof of Proposition 2.1. $\square$

Now we turn to the proof that Conjectures 1', 1.2 and 1.3 are equivalent.

Conjecture 1' implies Conjecture 1.3. Suppose that Conjecture 1.3 is false. Then there is some $\varepsilon > 0$ such that, for every k , there is a set $A_k \subset \mathbf{Z} \times \mathbf{Z}$ such that

$$\#\pi_{-1}(A_k) > \max_{r \in H_k \setminus \{-1\}} \#\pi_r(A_k)^{1+\varepsilon}, \quad (2.1)$$

where H_k denotes the set of rationals with height at most k , that is to say

$$H_k := \left\{ \frac{a}{b} : |a|, |b| \leq k \right\} \cup \{\infty\}.$$

Our first step is to use a “tensor power” argument to show that there are arbitrarily large sets with the same property; in fact, we shall argue that for every j there is a set $A_{k,j} \subset \mathbf{Z} \times \mathbf{Z}$ such that

$$\#\pi_{-1}(A_{k,j}) \geq j \max_{r \in H_k \setminus \{-1\}} \#\pi_r(A_{k,j})^{1+\varepsilon}. \quad (2.2)$$

This is simple if the $A_{k,j}$ are allowed to be subsets of $\mathbf{Z}^n$. Indeed we may define $A_k^{(n)}$ to be the set

$$\left\{ ((a_1, a_2, \dots, a_n), (a'_1, a'_2, \dots, a'_n)) \in \mathbf{Z}^n \times \mathbf{Z}^n : (a_i, a'_i) \in A_k \text{ for all } i \right\}.$$

Then, writing $\pi_r^{(n)} : \mathbf{Z}^n \times \mathbf{Z}^n \rightarrow \mathbf{Z}^n$, for the map sending (x, y) to $x + ry$ (or, when $r = \infty$, to y) we have

$$\#\pi_r^{(n)}(A_k^{(n)}) = (\#\pi_r(A_k))^n$$

for all r, n . In particular, by choosing n large enough (depending on j) we have, since the inequality (2.1) is strict,

$$\#\pi_{-1}^{(n)}(A_k^{(n)}) \geq j \max_{r \in H_k \setminus \{-1\}} \#\pi_r^{(n)}(A_k^{(n)})^{1+\varepsilon}. \quad (2.3)$$

To create a subset of $\mathbf{Z} \times \mathbf{Z}$ from $A_k^{(n)}$, we take an integer t and apply a map $\psi_t : \mathbf{Z}^n \times \mathbf{Z}^n \rightarrow \mathbf{Z} \times \mathbf{Z}$ of the form

$$\psi_t(x, y) = ((t, t^2, \dots, t^n) \cdot x, (t, t^2, \dots, t^n) \cdot y),$$

where the dot denotes the usual inner product. Set $A := \psi_t(A_k^{(n)})$. Choose t such that for $r \in H_k$ and $(x, y), (x', y') \in A_k^{(n)}$ we have

$$(\pi_r^{(n)}(x, y) - \pi_r^{(n)}(x', y')) \cdot (t, t^2, \dots, t^n) \neq 0 \quad (2.4)$$

unless $\pi_r^{(n)}(x, y) = \pi_r^{(n)}(x', y')$. There is such a t , because for each of the finite number of choices of x, y, x', y', r the left-hand side of (2.4) is a nontrivial polynomial equation in t . For such a choice of t it follows that $\pi_r(\psi_t(x, y)) = \pi_r(\psi_t(x', y'))$ if and only if $\pi_r^{(n)}(x, y) = \pi_r^{(n)}(x', y')$, and so

$$\#\pi_r(A) = \#\pi_r^{(n)}(A_k^{(n)})$$

for all r . This establishes the existence of the sets $A_{k,j}$ satisfying (2.2).

For each j, k , consider the set $S_{k,j} \subset \mathbf{Q}$ defined by

$$S_{k,j} := \bigcup_{1 \leq i \leq k} \bigcup_{r \in H_k \setminus \{-1\}} \frac{i}{k} \cdot \pi_r(A_{k,j}).$$

Then

$$\#S_{k,j} \leq k \cdot \#H_k \cdot \max_{r \in H_k^+} \#\pi_r(A_{k,j}) \ll_k (j^{-1} \#\pi_{-1}(A_{k,j}))^{1/(1+\varepsilon)}.$$

On the other hand, suppose that $d \in -\pi_{-1}(A_{k,j})$. This means that $d = y - x$ for some $(x, y) \in A_{k,j}$. If $0 \leq i \leq k - 1$, we have

$$x + \frac{id}{k} = \frac{k-i}{k} \left(x + \frac{i}{k-i} y \right).$$

Since $x + \frac{id}{k} \in \pi_{i/(k-i)}(A_{k,j}) \subset \bigcup_{r \in H_k^+} \pi_r(A_{k,j})$, it follows that $x + \frac{id}{k} \in S_{k,j}$ for $i = 0, 1, \dots, k-1$, that is to say $S_{k,j}$ contains a progression of length k and common difference $\frac{d}{k}$. Thus, writing $N_j := \#\pi_{-1}(A_{k,j})$, we see that $S_{k,j}$ is a set of size $\ll (j^{-1} N_j)^{1/(1+\varepsilon)}$ containing progressions of length k with at least N_j distinct common differences. Since, evidently, $\#S_{k,j} \geq k$, the presence of the factor j^{-1} forces $N_j \rightarrow \infty$ as $j \rightarrow \infty$. By multiplying through by an appropriate integer, we may find sets $\tilde{S}_{k,j} \subset \mathbf{Z}$ with the same property, contrary to Conjecture 1'.

Conjecture 1.3 implies Conjecture 1.2. This implication is essentially given in [18]. The notation there takes a little unpicking and the proof is short, so we repeat the argument.

Let $\varepsilon > 0$, and suppose that $r_1, \dots, r_k \in \mathbf{Q}_{\geq 0} \cup \{\infty\} \setminus \{-1\}$ are such that

$$\#\pi_{-1}(A) \leq \sup_i \#\pi_{r_i}(A)^{1+\varepsilon} \quad (2.5)$$

for all finite sets $A \subset \mathbf{Z} \times \mathbf{Z}$. We claim that

$$\mathbf{H}(X - Y) \leq (1 + \varepsilon) \sup_j \mathbf{H}(X + r_j Y). \quad (2.6)$$

for all $\mathbf{Z}$ -valued random variables X, Y , both taking only finitely many values. (Let us remind the reader that, by convention, $\mathbf{H}(X + \infty Y) = \mathbf{H}(Y)$.)

We begin with a couple of observations. The first is that (2.5) is automatically true for sets $A \subset \mathbf{Z}^n \times \mathbf{Z}^n$, for any n . This follows from the case $n = 1$ by applying a suitable map $\psi_t : \mathbf{Z}^n \times \mathbf{Z}^n \rightarrow \mathbf{Z} \times \mathbf{Z}$, exactly as in the argument following (2.3) above.

The second observation is that, by a simple limiting argument, we may assume that there is some q such that $q\mathbf{P}((X, Y) = (x, y)) \in \mathbf{Z}$ for all (x, y) : if we can prove the result for such (X, Y) the same inequality for arbitrary (X, Y) with finite range follows by letting $q \rightarrow \infty$.

Now let m be very large, and construct a set $A \subset \mathbf{Z}^{mq} \times \mathbf{Z}^{mq}$ as follows. Let it consist of all pairs $((x_1, \dots, x_{mq}), (y_1, \dots, y_{mq})) \in \mathbf{Z}^{mq} \times \mathbf{Z}^{mq}$ for which

$$\#\{i : (x_i, y_i) = (x, y)\} = mq\mathbf{P}((X, Y) = (x, y)).$$

Let us calculate $\#\pi_r(A)$. After a moment's thought we see that

$$\pi_r(A) = \{(z_1, \dots, z_{mq}) : \#\{i : z_i = z\} = mq \mathbf{P}(X + rY = z)\}.$$

(Here, we interpret $\mathbf{P}(X + \infty Y = z)$ as $\mathbf{P}(Y = z)$.) Writing $n = mq$ and $p_z = \mathbf{P}(X + rY = z)$ for short, it follows that

$$\#\pi_r(A) = \frac{n!}{\prod_z (np_z)!}.$$

Note that the product over z is finite, and that each np_z is an integer. Taking logs and using the fact that $\log N! = N \log N - N + o(N)$, we have

$$\log \pi_r(A) = -n \sum_z p_z \log p_z + o(n) = n\mathbf{H}(X + rY) + o(n).$$

We may assume that the $o(n)$ term is uniform in $r \in \{r_1, \dots, r_k\}$ (since this is a finite set); of course, it also depends on X, Y , but we are thinking of these as fixed for the duration of the argument.

Taking logs of (2.5) (which is valid for $A \subset \mathbf{Z}^n \times \mathbf{Z}^n$, as remarked), we conclude that

$$n\mathbf{H}(X - Y) \leq (1 + \varepsilon)n \sup_i \mathbf{H}(X + r_i Y) + o(n).$$

Now we may simply divide through by n and let $n \rightarrow \infty$ to conclude the claim (2.6).

Conjecture 1.2 implies Conjecture 1.1. This is relatively easy. Assume Conjecture 1.2. Let $\varepsilon > 0$ be arbitrary, and select $r_1, \dots, r_m \in \mathbf{Q} \cup \{\infty\} \setminus \{-1\}$ so that we have

$$\mathbf{H}(X - Y) \leq (1 + \varepsilon) \sup_i \mathbf{H}(X + r_i Y). \quad (2.7)$$

Let Q, M be positive integers to be specified later (depending on $r_1, \dots, r_m$) and suppose that $A \subset \mathbf{Z}$ contains an arithmetic progression of length $k = 2MQ$ and common difference d , for every $d \in \{1, \dots, N\}$. Define $\mathbf{Z}$ -valued random variables X, Y as follows: pick d uniformly at random, and let $\{a(d), \dots, a(d) + (k - 1)d\}$ be the progression in A for which $a(d)$ is minimal (choosing $a(d)$ minimal is not important, but is one way of making a definite choice). Set $X = a(d) + MQd$ and $Y = a(d) + (M + 1)Qd$.

Then $X - Y$ is uniformly distributed on the set $\{-Q, -2Q, \dots, -NQ\}$, and so

$$\mathbf{H}(X - Y) = \log N. \quad (2.8)$$

On the other hand,

$$\mathbf{H}(X + r_j Y) = \mathbf{H}\left(\frac{X + r_j Y}{1 + r_j}\right) = \mathbf{H}\left(a(d) + \left(QM + \frac{Qr_j}{1 + r_j}\right)d\right).$$

By choosing Q and then M suitably, we may ensure that all the $Qr_j/(1 + r_j)$ are integers of magnitude $< QM$, which means that

$$a(d) + \left(QM + \frac{Qr_j}{1 + r_j}\right)d \in \{a(d), \dots, a(d) + (k - 1)d\} \subset A.$$

That is, $X + r_j Y$ takes values in $(1 + r_j) \cdot A$. Since $\mathbf{H}(W) \leq \log m$ for any random variable W taking values in a set of size m , this implies that

$$\mathbf{H}(X + r_j Y) \leq \log \#A.$$

Combining this with (2.7) and (2.8) we obtain

$$\log N \leq (1 + \varepsilon) \log \#A,$$

or in other words

$$\#A \geq N^{1/(1+\varepsilon)}.$$

Since ε was arbitrary, the implication follows.

This completes the proof that Conjectures 1.1, 1', 1.2 and 1.3 are equivalent.

3 Finite fields

Next we turn to Conjecture 1.4 (n). To demonstrate its equivalence to the first three conjectures, it suffices to show that for each n we have Conjecture 1' $\Rightarrow$ Conjecture 1.4 (n) $\Rightarrow$ Conjecture 1.1.

Conjecture 1' implies Conjecture 1.4 (n). Suppose that $A_1 \subset \mathbf{F}_p^n$ is a set containing a k -term arithmetic progression with common difference d , for every $d \in \mathbf{F}_p^n$. Define the “unwrapping” map $\psi : \mathbf{F}_p \rightarrow \mathbf{Z}$ to be the inverse of the natural projection map from $\{0, \dots, p-1\}$ to $\mathbf{F}_p$. Define a map $\psi^{(n)} : \mathbf{F}_p^n \rightarrow \mathbf{Z}^n$ by setting $\psi^{(n)}(x_1, \dots, x_n) := (\psi(x_1), \dots, \psi(x_n))$.

For each $d \in \mathbf{F}_p^n$, select a progression $\{x(d) + \lambda d, \lambda = 0, 1, \dots, k-1\}$, lying in A_1 . Let $A_2 \subset \mathbf{Z}^n$ be the union of all progressions $\{\psi^{(n)}(x(d)) + \lambda \psi^{(n)}(d) : \lambda = 0, 1, \dots, k-1\}$. By construction, $A_2 \subset \{0, 1, \dots, k(p-1)\}^n$, and $\pi^{(n)}(A_2) \subset A_1$, where $\pi^{(n)} : \mathbf{Z}^n \rightarrow \mathbf{F}_p^n$ is the natural map. Since $\{0, 1, \dots, k(p-1)\}$ is covered by k discrete intervals of length p , on each of which the projection map $\pi : \mathbf{Z} \rightarrow \mathbf{F}_p$ is injective, we see that $\#A_2 \leq k^n \#A_1$.

By construction, A_2 contains a progression of length k and common difference d for p^n distinct values of d . Whilst A_2 is a subset of $\mathbf{Z}^n$, we can create a subset of $\mathbf{Z}$ with the same properties by looking at the image of A_2 under the map $f : \mathbf{Z}^n \rightarrow \mathbf{Z}$ defined by $f(x_1, \dots, x_n) = \sum_{i=1}^n (10kp)^i x_i$. It follows that $\#A_2 \geq F'_k(p^n)$, and hence $\#A_1 \geq k^{-n} F'_k(p^n)$. In the notation of Conjecture 1.4, this means that $f_{n,k}(p) \geq k^{-n} F'_k(p^n)$. It follows that

$$\lim_{p \rightarrow \infty} \frac{\log f_{n,k}(p)}{\log p} \geq n \lim_{p \rightarrow \infty} \frac{\log F'_k(p^n)}{\log p^n},$$

and so

$$\lim_{k \rightarrow \infty} \lim_{p \rightarrow \infty} \frac{\log f_{n,k}(p)}{\log p} \geq n \lim_{k \rightarrow \infty} \lim_{p \rightarrow \infty} \frac{\log F'_k(p^n)}{\log p^n}.$$

Assuming Conjecture 1' (taking $N = p^n$), the right hand side here is precisely n . This implies Conjecture 1.4.

Conjecture 1.4 implies Conjecture 1.1. Suppose we have a set $A_1 \subset \mathbf{Z}$ containing a progression of length k and common difference d for each $d \in \{1, \dots, N\}$. Partition $\mathbf{Z}$ into intervals $I_j := 10kjN + \{1, \dots, 10kN\}$, $j \in \mathbf{Z}$. Any progression of length k and common difference $d \in \{1, \dots, N\}$ is either wholly contained in some I_j , or else is split into two progressions, one in I_j and the other in I_{j+1} , with one of these having length at least $k/2$. It follows that the set $A_2 \subset \mathbf{Z}$ defined by⁴

$$A_2 = \bigcup_j \{(A_1 \cap I_j) - 10kjN\}$$

⁴ This “cut-and-move” trick is quite standard in the study of the Kakeya problem.

contains a progression of length at least $k/2$ and common difference d , for all $d \in \{1, \dots, N\}$. Manifestly $\#A_2 \leq \#A_1$, and by construction A_2 has the additional property that

$$A_2 \subset \{1, \dots, 10kN\}. \quad (3.1)$$

Using A_2 , we construct a set $A_3 \subset \mathbf{Z}^n$. We will later use this to construct a further set $A_4 \subset \mathbf{F}_p^n$, for a suitable prime p , by projection. To define A_3 , let $M := \lfloor N^{1/n} \rfloor$. Select $t \in \{-10kN, \dots, 20kN - 1\}$ uniformly at random, and define

$$A_3(t) := \left\{ (x_1, \dots, x_n) \in \{0, \dots, M-1\}^n : \sum_{i=1}^n M^{i-1} x_i \in A_2 + t \right\}.$$

Suppose that $d = \sum_{i=1}^n M^{i-1} d_i$ with $0 \leq d_i \leq M/2k$ for all i . There are at least $(M/4k)^n$ such values of d , and all lie in $\{0, \dots, N\}$. For each such d there is, by assumption, a progression $\{x(d) + \lambda d : \lambda = 0, 1, \dots, \lfloor k/2 \rfloor - 1\}$ lying in A_2 . The progression $\{x(d) + t + \lambda d : \lambda = 0, 1, \dots, \lfloor k/2 \rfloor - 1\}$ then lies in $A_2 + t$. Write

$$S := \left\{ \sum_{i=1}^n M^{i-1} s_i : 0 \leq s_i < M/2 \text{ for all } i \right\}.$$

If it so happens that $t \in -x(d) + S$ then $A_3(t)$ contains a progression of length k and common difference $(d_1, \dots, d_n)$, namely $\{(s_1, \dots, s_n) + \lambda(d_1, \dots, d_n) : \lambda \in \{0, 1, \dots, k-1\}\}$, where $x(d) + t = \sum_{i=1}^n M^{i-1} s_i$.

Since $0 \leq x(d) \leq 10kN$ and $S \subset \{0, 1, \dots, M^n\}$, $-x(d) + S \subset \{-10kN, \dots, 20kN - 1\}$. It follows that

$$\mathbf{P}(t \in -x(d) + S) = \frac{1}{30kN} \#S \geq \frac{1}{30kN} \left(\frac{M}{2}\right)^n \gg_{k,n} 1.$$

(Recall here that t is chosen uniformly at random on $\{-10kN, \dots, 20kN - 1\}$.) Summing over the $(M/2k)^n \gg_{k,n} N$ choices of d , we see that the expected number of d for which $t \in -x(d) + S$ is $\gg_{k,n} N$. Fix some choice of t such that $t \in -x(d) + S$ for $\gg_{k,n} N \gg_{k,n} M^n$ values of d , and write $A_3 := A_3(t)$. Then by construction we have

$$\#A_3 \leq \#A_2 \leq \#A_1, \quad (3.2)$$

whilst A_3 contains a progression of length $\geq k/2$ and common difference d for all d in some set $\mathcal{D} \subset \{0, \dots, M-1\}^n$, $\#\mathcal{D} \gg_{k,n} M^n$.

Now choose a prime p with $M \leq p < 2M$, and let $A_4 \subset \mathbf{F}_p^n$ be the image of A_3 under the natural projection $\pi^{(n)} : \mathbf{Z}^n \rightarrow \mathbf{F}_p^n$. We have

$$\#A_4 = \#A_3, \quad (3.3)$$

and moreover A_4 contains a progression of length k and common difference d for all $d \in \pi^{(n)}(\mathcal{D})$, that is to say for $\gg_{n,k} N \gg_{n,k} p^n$ values of d . By a standard argument (taking random translations of $\pi^{(n)}(\mathcal{D})$, see Corollary 6.4 for details) there is a further set $A_5 \subset \mathbf{F}_p^n$,

$$\#A_5 \ll_{n,k} (\log p) \#A_4, \quad (3.4)$$

containing a progression of length k and common difference d , for all $d \in \mathbf{F}_p^n \setminus \{0\}$. Tracing back through (3.4), (3.3), (3.2) we see that

$$F_k(N) \gg_{k,n} \frac{1}{\log p} f_{n,k}(p),$$

where $p = p(N) \sim N^{1/n}$ is some prime. It follows that

$$\lim_{N \rightarrow \infty} \frac{\log F_k(N)}{\log N} \geq \lim_{N \rightarrow \infty} \frac{\log f_{n,k}(p(N))}{n \log p(N)}.$$

Assuming Conjecture 1.4 (n), the limit on the right is 1. This concludes the proof that Conjecture 1.4 (n) implies Conjecture 1.1.

Before leaving this topic, we remark that it is quite possible that very strong bounds such as

$$f_{p^n,1}(p) \geq p/2 \quad (3.5)$$

are true, provided $p \geq p_0(\eta)$ is large enough. This issue is strongly hinted at, if not explicitly conjectured, in [1]. It is pointed out there that such bounds imply vastly more than is currently known about the purely arithmetic problem of bounding the least quadratic nonresidue modulo p .

Whilst a bound of the form (3.5) is not known to imply the arithmetic Kakeya conjecture (the progressions are of length p^η , rather than of bounded size), the arguments of Bourgain may be adapted to show that it does imply the Euclidean Kakeya conjecture. Further details may be found in lecture notes of the first author [9, Section 10].

It is quite interesting that the innocent-looking statement (3.5) implies two famous unsolved problems in completely different mathematical areas.

4 A problem of Erdős and Selfridge

Finally, we turn to Conjecture 1.5. In fact, we prove the following rather tight connection between Conjecture 1' and Conjecture 1.5.

Proposition 4.1 *Write $G_k(N)$ for the minimum, over all choices $p_1 < \dots < p_N$ of primes and all intervals I of length kp_N , of $\#(I \cap \bigcup_{i=1}^N p_i \mathbf{Z})$. Then $F'_k(N) \leq G_k(N) \leq kF'_k(N)$. In particular, Conjectures 1' and 1.5 are equivalent.*

Proof Suppose first we have a set of primes $p_1 < \dots < p_N$ and an interval I of length kp_N so that $\#A = G_k(N)$, where $A = \bigcup_{i=1}^N \{x \in I : p_i | x\}$. Note that A obviously contains a progression of length k and common difference p_i , for each i , and therefore $F'_k(N) \leq G_k(N)$.

In the other direction, suppose we have a set A attaining the bound $F'_k(N)$, that is to say $\#A = F'_k(N)$ and A contains, for $i = 1, \dots, N$, a progression $\{a_i + jd_i : j = 0, 1, \dots, k-1\}$. By translating if necessary, we may assume that A consists of positive integers. Let $\delta \in (0, \frac{1}{2})$ be a quantity to be specified shortly. By the theorem of the first author and T. Tao [10, Theorem 1.2], we may find positive u and v such that all of the numbers $v, u+v, \dots, d_N u+v$ are prime and lie in some interval $[(1-\delta)X, X]$, $X \geq 100$. Set $p_i := d_i u + v$. Note that $\frac{v}{u+v} \geq 1-\delta$, which rearranges as $\frac{v}{u} \geq \frac{1}{\delta} - 1$, hence

$$\frac{v}{u} > 4 \max A \quad (4.1)$$

provided that δ is chosen sufficiently small. Note also that

$$\frac{p_i}{p_N} \geq \frac{v}{v + u d_N} = \frac{1}{1 + \frac{u}{v} d_N} \geq 1 - \frac{1}{4k} \quad (4.2)$$

if δ is small enough. In particular if δ is small enough then we have

$$p_i > \frac{3}{4}p_N \geq \frac{1}{2}p_N + \frac{1}{4}v > \frac{1}{2}p_N + u \max A \quad (4.3)$$

by (4.1).

Define $A' := u \cdot A + \{0, v, 2v, \dots, (k-1)v\}$. The cardinality of A' satisfies $\#A' \leq kF'_k(N)$, and

$$A' \supset \bigcup_{i=1}^N \{ua_i + jp_i : j = 0, 1, \dots, k-1\} \quad (4.4)$$

for $i = 1, \dots, N$. By the Chinese remainder theorem we may find w so that $p_i | w + ua_i$ for $i = 1, \dots, N$.

Set $I := w - \lfloor \frac{1}{2}p_N \rfloor + \{1, 2, \dots, kp_N\}$. Obviously I is an interval of length kp_N . Let $i \in \{1, \dots, N\}$. We claim that $w + ua_i + jp_i \in I$ for an integer j if and only if $j \in \{0, 1, \dots, k-1\}$. Since $w + ua_i + p_i \mathbf{Z} = p_i \mathbf{Z}$, this implies that

$$I \cap p_i \mathbf{Z} = \{w + ua_i + jp_i : j = 0, 1, \dots, k-1\},$$

and hence by (4.4)

$$I \cap \bigcup_{i=1}^N p_i \mathbf{Z} \subset w + A',$$

whence

$$G_k(N) \leq \# \left(I \cap \bigcup_{i=1}^N p_i \mathbf{Z} \right) \leq \#A' \leq kF'_k(N).$$

It remains to prove the claim. To prove the *if* implication, it suffices in view of (4.4) to show that $w + A' \subset I$. However it is obvious that $\min(w + A') \geq \min I$ (since all elements of A' are positive) and moreover

$$\begin{aligned} \max(w + A') &\leq w + u \max A + (k-1)v \\ &< w + \left(k - \frac{1}{2}\right)v \quad \text{by (4.1)} \\ &\leq w + \left(k - \frac{1}{2}\right)p_N \\ &\leq \max I. \end{aligned}$$

This establishes the *if* direction of the claim. To establish the *only if* direction, it suffices to show that

$$w + ua_i - p_i < \min I \quad (4.5)$$

and that

$$w + ua_i + kp_i > \max I. \quad (4.6)$$

However by (4.3) we have

$$w + ua_i - p_i < w + u(a_i - \max A) - \frac{1}{2}p_N \leq w - \frac{1}{2}p_N \leq \min I,$$

so (4.5) does hold. Also,

$$\begin{aligned} w + ua_i + kp_i &> w + kp_i \text{ since } A \subset \mathbb{N} \\ &\geq w + \left(k - \frac{1}{4}\right) p_N \text{ by (4.2)} \\ &\geq w + kp_N - \left\lfloor \frac{1}{2} p_N \right\rfloor = \max I, \end{aligned}$$

the last step being a consequence of the fact that $p_N \geq (1 - \delta)X \geq 50$. Thus (4.6) also holds, and this completes the proof of the claim. $\square$

Remark The use of the theorem of the first author and Tao is a little excessive. One could do without it using simpler arguments if one was prepared to settle for logarithmic losses.

5 Small unions of progressions

In this section we prove Theorem 1.7. Write $3 = p_1 < p_2 < \dots$ for the odd primes, and set $Q := \prod_{i=1}^m p_i$, where $m = \lceil 10 \log k \rceil$. Note that $Q = k^{O(\log \log k)}$.

Define a set S to be the union of all progressions $\{x_d + jd : j = 0, 1, \dots, k-1\}$ where, for $d \in \{1, \dots, Q-1\}$, x_d is the unique element of $\{1, \dots, Q\}$ congruent to $d^2 \pmod{Q}$. Evidently, S contains a progression of length k and common difference d , for all $d \in \{0, 1, \dots, Q-1\}$.

Fix $j \in \{0, 1, \dots, k-1\}$. For each i we have

$$x_d + jd \equiv d^2 + jd \equiv \left(d + \frac{j}{2}\right)^2 - \frac{j^2}{4} \pmod{p_i},$$

and so $x_d + jd \pmod{p_i}$ takes values in a set of size $\frac{1}{2}(p_i + 1)$ as d varies. Therefore $x_d + jd \pmod{Q}$ takes values in a set of size $\prod_{i=1}^m \frac{1}{2}(p_i + 1)$. Since, additionally, $0 < x_d + jd \leq kQ$, $x_d + jd$ takes values in a set of size $k \prod_{i=1}^m \frac{1}{2}(p_i + 1)$. Therefore

$$\#S \leq k^2 \prod_{i=1}^m \frac{1}{2}(p_i + 1) = k^2 2^{-m} Q \prod_{i=1}^m \left(1 + \frac{1}{p_i}\right).$$

Recalling that $m \sim 10 \log k$, and using the bound $\prod_{i=1}^m (1 + \frac{1}{p_i}) \ll \log m \ll k$, we see that

$$\#S \ll k^{-7} Q$$

and so

$$\#S \leq Q^{1 - \frac{c}{\log \log k}}$$

if k is sufficiently large, for some absolute $c > 0$.

Now let n be an arbitrary positive integer, set $N_n := Q^n$, and consider the set

$$A_n := \{s_0 + s_1 Q + \dots + s_{n-1} Q^{n-1} : s_0, \dots, s_{n-1} \in S\}.$$

Then $\#A_n \leq (\#S)^n \leq N_n^{1 - \frac{c}{\log \log k}}$. The set A_n contains a progression of length k and common difference $d_0 + d_1 Q + \dots + d_{n-1} Q^{n-1}$ for any choice of $d_i \in \{0, 1, \dots, Q-1\}$, or in other words for all $d \in \{0, \dots, N_n - 1\}$.

Finally, suppose N is an arbitrary positive integer. Choose n minimal so that $N_n > N$, and set $A := A_n$. Then A contains a progression of length k and common difference d , for all $d \in \{1, \dots, N\}$. Moreover,

$$\#A \leq N_n^{1 - \frac{c}{\log \log k}} \leq (QN)^{1 - \frac{c}{\log \log k}} \ll_k N^{1 - \frac{c}{\log \log k}}.$$

The result follows.

6 Entropy inequalities in positive characteristic

In this section we give the proof of Theorem 1.8. Suppose that X and Y are two $\mathbf{F}_p^\infty$ -valued random variables, both taking finitely many values. Suppose that

$$\mathbf{H}(X - Y) \geq (1 + \varepsilon) \sup_{r \neq -1} \mathbf{H}(X + rY). \quad (6.1)$$

Our aim is to prove that $\varepsilon = O(\frac{1}{\log p})$, which immediately implies Theorem 1.8.

The initial phases of the argument mirror the deduction of Conjecture 1.2 from Conjecture 1.3. We may assume that there is some q such that $q\mathbf{P}((X, Y) = (x, y)) \in \mathbf{Z}$ for all (x, y) ; if (6.1) can be established in this case, uniformly in q , then the general result follows by an easy approximation argument on letting $q \rightarrow \infty$.

Now let m be very large, write $n = mq$, and construct a set $B^{(n)} \subset (\mathbf{F}_p^\infty)^{qm} \times (\mathbf{F}_p^\infty)^{qm}$ as follows. Let it consist of all pairs $((x_1, \dots, x_{mq}), (y_1, \dots, y_{mq}))$ for which

$$\#\{i : (x_i, y_i) = (x, y)\} = mq\mathbf{P}((X, Y) = (x, y)).$$

By arguments essentially the same as we saw before,

$$\mathbf{H}(X + rY) = \frac{1}{n} \log \pi_r(B^{(n)}) + o_{n \rightarrow \infty}(1).$$

Hence, taking m sufficiently large (and observing that $(\mathbf{F}_p^\infty)^{qm}$ is isomorphic to $\mathbf{F}_p^\infty$ as a vector space), we obtain arbitrarily large sets $B \subset \mathbf{F}_p^\infty \times \mathbf{F}_p^\infty$ such that

$$\#\pi_{-1}(B) \geq \sup_{r \neq -1} (\#\pi_r(B))^{1 + \varepsilon/2}. \quad (6.2)$$

Note in particular that $\pi_{-1}(B)$ becomes arbitrarily large.

For such a B , we construct a finite set $A \subset \mathbf{F}_p^\infty$ as follows. If $(x, y) \in B$ and $x \neq y$, include the entire progression (line) through x and y in A . The points on this line are $\frac{x+ry}{1+r}$, for $r \neq -1$, and y . Therefore

$$A \subset \pi_\infty(B) \cup \bigcup_{r \neq -1} \frac{1}{1+r} \cdot \pi_r(B),$$

and therefore

$$\#A \leq p \sup_{r \neq -1} \pi_r(B).$$

On the other hand, A contains a progression of length p (line) and common difference d , for every $d \in \pi_{-1}(B) \setminus \{0\}$. Thus, writing $N := \pi_{-1}(B) - 1$, we have

$$\#A \ll_p N^{\frac{1}{1+\varepsilon/2}}. \quad (6.3)$$

On the other hand we have the following result, whose proof we will supply shortly.

Proposition 6.1 *Suppose that $A \subset \mathbf{F}_p^\infty$ is a finite set containing a progression of length p (that is, a line) and common difference d , for all d in some set of size N . Then $\#A \gg_p N^{1 - \frac{\log 2}{\log p} - o(1)}$.*

Combining Proposition 6.1 with the construction of A satisfying (6.3) immediately gives the desired upper bound $\varepsilon = O(\frac{1}{\log p})$, thereby concluding the proof of Theorem 1.8.

It remains to prove Proposition 6.1.

Proof of Proposition 6.1 Set $A_1 := A$. In its initial stages, the proof of this result goes along rather similar lines to that of Proposition 2.1, only it is rather easier. The use of random projections in a similar context may be found in [7, §3]. Let n be the smallest positive integer for which $p^n \geq N$.

Since A_1 is finite, it is contained in some copy of $\mathbf{F}_p^M$. Let $\pi : \mathbf{F}_p^M \rightarrow \mathbf{F}_p^n$ be a random linear map, selected by choosing the images of the basis vectors $e_1, \dots, e_M$ uniformly at random from $\mathbf{F}_p^n$. Set $A_2 := \pi(A_1)$; evidently $\#A_2 \leq \#A_1$. Let $\mathcal{D}$ be the set of common differences of progressions (of length p) lying in A_1 . Then A_2 contains a progression of length p and common difference $\pi(d)$, for every $d \in \mathcal{D}$.

Put some arbitrary order $<$ on $\mathcal{D}$, and suppose that $d < d'$. Then $\pi(d) = \pi(d')$ if and only if $\pi(d - d') = 0$. However, $\pi(d - d')$ is uniformly distributed in $\mathbf{F}_p^n$, and so the probability of this happening is p^{-n} . It follows that the expected number of pairs (d, d') with $d < d'$ and $\pi(d) = \pi(d')$ is $p^{-n} \binom{N}{2} \leq \frac{1}{N} \binom{N}{2} \leq N/2$. Pick some map π for which the number of such pairs is at most N . For each $v \in \mathbf{F}_p^n$, write $f(v) := \#\pi^{-1}(v)$. Then we have $\sum_v \binom{f(v)}{2} \leq N/2$, from which we obtain, since $\sum_v f(v) = N$, that $\sum_v f(v)^2 \leq 2N$. By Cauchy–Schwarz,

$$N^2 = \left(\sum_v f(v) \right)^2 \leq \#\{v : f(v) \neq 0\} \sum_v f(v)^2,$$

and therefore there are at least $N/2$ values of v for which $f(v) \neq 0$. From the choice of n it is clear that $p^n \leq pN$, and so at least $(\#\mathbf{F}_p^n)/2p$ elements of $\mathbf{F}_p^n$ lie in the image of π , or in other words are common differences of progressions in A_2 .

By a random translation argument (see Corollary 6.4), there is a set $A_3 \subset \mathbf{F}_p^n$, $\#A_3 \ll (np \log p) \#A_2$, containing a line in every direction. That is, A_3 is a finite field Besicovitch set.

Now we bring in bounds on the size of such sets of a strength which, famously, are available in the finite field setting but not in characteristic zero. By the main result of [6] we have $\#A_3 \geq (p/2)^n = (p^n)^{1 - \frac{\log 2}{\log p}} \geq N^{1 - \frac{\log 2}{\log p}}$. The proposition follows. $\square$

Remarks Note that here it was crucial to have an effective lower bound on the size of Kakeya sets for fixed p but with $n \rightarrow \infty$. For this, the celebrated work of Dvir [5] on the Kakeya problem would not suffice. However (at the cost of weakening the exponents slightly) we could have used the main result of [19], which has a slightly simpler proof than that of [6].

The $O(\frac{1}{\log p})$ term in Theorem 1.7 is sharp. To see this, pick a, b, b' independently and uniformly from $\mathbf{F}_p$, and define random variables X, Y taking values in $\mathbf{F}_p^2$ by

$$X = (a + b, ab), \quad Y = (a + b', ab').$$

Then

$$X - Y = (b - b', a(b - b')),$$

which is almost uniformly distributed on $\mathbf{F}_p^2$: a short calculation gives

$$\mathbf{H}(X - Y) = 2 \log p + O\left(\frac{\log p}{p}\right).$$

By contrast, if $r \neq -1$ then

$$\frac{X + rY}{1 + r} = \left(a + \frac{b + rb'}{1 + r}, a \cdot \frac{b + rb'}{1 + r}\right),$$

and so $X + rY$ is supported on a dilate of the set $V := \{(u + v, uv) : u, v \in \mathbf{F}_p\}$, which has cardinality $\frac{1}{2}p^2 + O(p)$. Therefore

$$\mathbf{H}(X + rY) \leq 2 \log p - \log 2 + O\left(\frac{\log p}{p}\right).$$

Cognoscenti will recognise V as being equivalent to the well-known construction of optimal Kakeya sets in $\mathbf{F}_p^2$, due to Mockenhaupt and Tao [15].

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Appendix A. Covering by translates

In this section we review some standard lemmas on random translates.

Lemma 6.2 *Suppose that $S \subset \{1, \dots, X\}$ is a set. Then there is a set T of size $\ll \frac{X}{\#S} \log X$ such that $S + T \supset \{1, \dots, X\}$.*

Proof We inductively define $t_1, t_2, \dots \in \{-X+1, \dots, X\}$ and $A_i := \{1, \dots, X\} \setminus \bigcup_{j=1}^i (S + t_j)$ such that, given the choice of $t_1, \dots, t_i$, $\#A_{i+1}$ is as small as possible. We have

$$\sum_t \#(A_i \cap (S + t)) = \#A_i \#S,$$

and so

$$\max_t \#(A_i \cap (S + t)) \geq \#A_i \frac{\#S}{2X}.$$

Therefore

$$\#A_{i+1} \leq \#A_i \left(1 - \frac{\#S}{2X}\right).$$

This process terminates with $\#A_i < 1$ (and hence $\#A_i = 0$) in $\ll \frac{X}{\#S} \log X$ steps. $\square$

Lemma 6.3 *Suppose that $S \subset \mathbf{F}_p^n$ is a set. Then there is a set $T \subset \mathbf{F}_p^n$ of size $\ll \frac{p^n}{\#S} n \log p$ such that $S + T = \mathbf{F}_p^n$.*

Proof Very similar to the previous lemma, and left as an exercise. $\square$

Corollary 6.4 Suppose that $A \subset \mathbb{F}_p^n$ is a set containing a k -term arithmetic progression with common difference d , for all d lying in some set $\mathcal{D}$ of size δp^n . Then there is a set A' , $\#A' \ll_{k,n} \log p \cdot \#A$, containing a k -term arithmetic progression with every common difference.

Proof Apply Lemma 6.3 with $S = \mathcal{D}$, and let T be the resulting set. Then take $A' = \bigcup_{x \in \{0\} \cup T \cup \dots \cup (k-1) \cdot T} (A + x)$. $\square$

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