On a conjecture of L. Fejes Tóth and J. Molnár about circle coverings of the plane

Dietmar Dorninger¹

Published online: 13 October 2018
© The Author(s) 2018

Abstract

Tóth and Molnár (Math Nachr 18:235–243, 1958) formulated the conjecture that for a given homogeneity q the thinnest covering of the Euclidean plane by arbitrary circles is greater or equal a function $S(q)$. Florian (Rend Semin Mat Univ Padova 31:77–86, 1961) proved that if the covering consists of only two kinds of circles then the conjecture is true supposed that $S(q) \leq S(1/q)$ what can be easily verified by a computer. In this paper we consider the general case of circles with arbitrary radii from an interval of the reals. We set up two further functions $M_0(q)$ and $M_1(q)$ and prove that the conjecture is true if $S(q)$ is less than or equal to $S(1/q)$, $M_0(q)$ and $M_1(q)$. As in the case of two kinds of circles this can be readily confirmed by computer calculations. (For $q \geq 0.6$ we even do not need the function $M_1(q)$ for computer aided comparisons.) Moreover, we obtain Florian’s result in a shorter different way.

Keywords Covering of the plane · Incongruent discs · Minimum density · Conjecture of Tóth and Molnár · Minimal density of three circles covering a triangle

1 Introduction

A circle covering $\mathbf{K}(q)$ of homogeneity q of the Euclidean plane is a countable set of closed circular discs C_i with radii r_i such that every point of the plane belongs to at least one circle of $\mathbf{K}(q)$ and $q := \inf(r_i/r_j), i, j = 1, 2, \dots$. Given q with $0 \leq q \leq 1$ it is of interest to determine the density $D(q)$ of a thinnest covering of the plane. (For a formal definition of a thinnest covering see, for example, the classical book [8].)

There have been several attempts to determine $D(q)$ or accurate bounds of it. For upper bounds of $D(q)$ cf. [2,5–8], for lower bounds [1,3,4,9,10]. The most outstanding open conjecture concerning a lower bound (considered by G. Blind in 1974 as “hopeless to prove analytically” (cf. [1])) had been proposed by Tóth and Molnár (first published 1958, [10]). It states that for $0 < q \leq 1$

Dedicated to the memory of August Florian.

✉ Dietmar Dorninger
d.dorninger@tuwien.ac.at

¹ Institute of Discrete Mathematics and Geometry, TU Wien, Wiedner Hauptstraße 8/104, 1040 Vienna, Austria

Fig. 1 Triangle T

$$D(q) \geq S(q) := \min_{0 < x \leq 1} \frac{\pi - 2(1 - q^2) \arctan(\sqrt{1 - x^2}/(x + q))}{2\sqrt{1 - x^2}(x + q)} \quad (1.1)$$

with equality for $q = 1$ and, in the limit, for $q = 0$.

Geometrically $S(q)$ is based on the following setting. The centres O_1, O_2, O_3 of three closed circular discs C_1, C_2, C_3 with radii r_1, r_2, r_3 which have exactly one common point (but no common inner point) form a triangle T with angles α, β, γ (see Fig. 1; to choose a scale b is set to be 1).

Supposing that no circle cuts into the opposite side of T the quotient of the area covered by C_1, C_2, C_3 within T and the area ΔT of T is given by

$$\delta(q) := \frac{\alpha r_1^2 + \beta r_2^2 + \gamma r_3^2}{2\Delta T}. \quad (1.2)$$

That no circle cuts into the opposite side of T can be neglected if it turns out in the course of determining the minimum of the function $\delta(q)$ that this assumption is satisfied by itself.

L. Fejes Tóth and J. Molnár had shown in [10] that $\min \delta(q) \leq D(q)$ and assumed that $\min \delta(q) = S(q)$.

In this paper we consider a representation of $\delta(q)$ together with its adhering boundary conditions derived by the author in [2]. In this representation $\delta(q)$ is expressed as a function of q, β, r_1 and γ (see Fig. 1). We prove that for given q and β $\delta(q, \beta, r_1, \gamma)$ has a unique stationary point at which $\delta(q, \beta, r_1, \gamma)$ assumes its minimum $\delta_0(q, \beta)$. However, this minimum is not always assumed within the limits of the domain B defined by all boundary conditions. If the minimum is attained within B computer calculations show that $M_0(q) := \min_{\beta} \delta_0(q, \beta) > S(q)$. As for the boundary values of B we prove that in order to find $\min \delta(q)$ only two further functions remain which have to be compared to $S(q)$ by computer aid. First, if $q < 0.6$ and $|O_1, O_3| = r_1 + r_3$, the minimum of the covering density will yield a function $M_1(q)$ which has to be taken into account. Second, if $0 < q \leq 1$ and two radii of the circles C_1, C_2, C_3 coincide we will prove that $\min \delta(q) = S(q)$ or $\min \delta(q) = S(1/q)$, this way recovering a result by Florian [3], but in a totally different (and shorter) way. As already pointed out in [3] computer calculations establish that $S(q) \leq S(1/q)$. Showing by computer aid that also $M_1(q) > S(q)$ for $0 < q < 0.6$ we therefore obtain that $S(q)$ is the smallest of the four functions $M_0(q), M_1(q), S(1/q)$ and $S(q)$ for any homogeneity q , this way confirming the conjecture of Tóth and Molnár.

2 $\delta(q)$ for constant q and β

Referring to the parameters specified in Fig. 1 we assume that $r_1 \geq r_2 \geq r_3$ and $r_3/r_1 = q$, that is $r_3 = r_1 q$. Because the circles C_1, C_2, C_3 have no common inner point we can suppose that their common intersection point lies within the triangle T . In view of $r_1 \geq r_2 \geq r_3$ we can further conclude that

$$\alpha \leq \pi/2 \quad \text{and} \quad 1/(1+q) \leq r_1 \leq 1/(1-q). \quad (2.1)$$

As derived in [2] $\delta(q)$ can be expressed by the equation:

$$\begin{aligned} \delta(q) = & r_1^2[(\pi - \gamma(1 - q^2)) \cot \gamma - (\pi - (\beta + \gamma)(1 - q^2)) \cot(\beta + \gamma)] \\ & + \beta \cot \beta - \beta \sqrt{4r_1^2 - (r_1^2(1 - q^2) + 1)^2}, \end{aligned} \quad (2.2)$$

The following notations will turn out to be very useful:

$$\begin{aligned} R &:= r_1^2, \quad p := 1 - q^2, \quad G := (\pi - \gamma p) \cot \gamma - (\pi - (\beta + \gamma)p) \cot(\beta + \gamma), \\ W &:= \sqrt{4R - (Rp + 1)^2}; \end{aligned} \quad (2.3)$$

due to $1/(1+q) \leq r_1 \leq 1/(1-q)$ the function W is well defined.

With these notations (2.2) reads as

$$\delta(q) = RG + \beta \cot(\beta) - \beta W. \quad (2.4)$$

As for boundary conditions apart from (2.1) it was shown in [2] that the following inequalities have to be satisfied:

$$Rp \cot(\beta + \gamma) \leq W - \cot \beta \leq Rp \cot \gamma \quad (2.5)$$

with the first inequality representing $r_2 \leq r_1$ and the second to $r_3 \leq r_2$.

We now agree to keep the parameters q and β fixed with $0 < q < 1$ and $0 < \beta < \pi$.

As for variables, we pick γ and R , so that we consider δ as a function $\delta(\gamma, R)$ subject to the boundary conditions (2.1) and (2.5). Condition (2.1) then assumes the form

$$\pi/2 - \beta \leq \gamma < \pi - \beta, \quad \gamma > 0 \quad \text{and} \quad 1/(1+q)^2 \leq R \leq 1/(1-q)^2. \quad (2.6)$$

In a more suggestive form the boundary condition (2.5) can be written as

$$\gamma \leq \operatorname{arccot} \left(\frac{W - \cot \beta}{Rp} \right) \leq \beta + \gamma. \quad (2.7)$$

The domain of $\delta(\gamma, R)$ defined by (2.6) and (2.7) will be referred to by B . If we only consider the rectangle defined by (2.6) we will refer to it as A . — A typical shape of B is illustrated by Fig. 2 (for which $q = 3/5$ and $\beta = 0.628$).

Lemma 2.1 *G is a positive, strongly convex function of γ , $\gamma \in (0, \pi - \beta)$. It assumes its minimum for $\gamma = \gamma_0$ which is the unique solution of the equation*

$$p \cot \gamma_0 + \frac{\pi - \gamma_0 p}{\sin^2 \gamma_0} = p \cot(\beta + \gamma_0) + \frac{\pi - (\beta + \gamma_0)p}{\sin^2(\beta + \gamma_0)}. \quad (2.8)$$

Moreover, $\gamma_0 \geq \pi/2 - \beta$.

Fig. 2 Domain A and $B \subset A$

Proof From $\delta(q) = RG + \beta \cot \beta - \beta W \geq 1$ and $\beta \cot \beta < 1$ we infer $G > \beta W/R \geq 0$.

Next we consider the function $g(x) := (\pi - xp) \cot(x)$ for $0 < x < \pi$ which constitutes $G = g(\gamma) - g(\beta + \gamma)$, further the first, second and third derivatives of $g(x)$:

$$g'(x) = -p \cot(x) - \frac{\pi - xp}{\sin^2(x)}, \quad g''(x) = \frac{2}{\sin^2(x)} (p + (\pi - xp) \cot(x)) \quad (2.9)$$

$$g'''(x) = \frac{-2}{\sin^4(x)} \left((2 + \cos(2x))(\pi - px) + \frac{3p}{2} \sin(2x) \right). \quad (2.10)$$

$g'''(x) \leq 0$: If $x \leq \pi/2$ this is obvious. For $\pi/2 < x < \pi$ it suffices to show that the function $f(x) := (2 + \cos(2x))(\pi - x) + 3/2 \sin(2x) \geq 0$.

From $f'(x) = -2((\pi - x) \sin(2x) + 1 - \cos(2x)) = -2 \sin(2x)((\pi - x) - \tan(\pi - x)) \leq 0$ we obtain that f is decreasing and therefore, because of $f(\pi/2) = \pi/2$ and $f(\pi) = 0$ we can conclude that $f \geq 0$.

Due to $g''' \leq 0$ the second derivative g'' is decreasing, hence $G''(\gamma) = g''(\gamma) - g''(\beta + \gamma) > 0$, i.e., G is strongly convex.

Equation (2.8) follows from $G'(\gamma) = g'(\gamma) - g'(\beta + \gamma) = 0$. Because $g''(x) \geq 0$ for $x \leq \pi/2$ the first derivative of g is increasing so that $\gamma_0 \leq \pi/2$ and $\gamma_0 + \beta \leq \pi/2$ cannot hold together, from which we get that in any case $\gamma_0 \geq \pi/2 - \beta$. $\square$

Theorem 2.2 *The function $\delta(\gamma, R)$ has exactly one stationary point $(\gamma_0, R_0) \in A$ at which it assumes its minimum within the domain A .*

Proof $\frac{\partial \delta}{\partial \gamma} = 0$ implies Eq. (2.8) from which we implicitly obtain the unique point γ_0 and $G_0 := G(\gamma_0)$.

From $\frac{\partial \delta}{\partial R} = 0$ we infer $G = \beta \frac{2-(Rp+1)p}{W}$. Solving this equation with respect to R we get

$$R_{1/2} = \frac{1}{p^2} \left(1 + q^2 \pm 2q \frac{G}{\sqrt{p^2 \beta^2 + G^2}} \right).$$

$R = R_1$ implies

$$0 < G = \frac{\beta}{W} \left(2 - p - (1 + q^2 + 2q \frac{G}{\sqrt{p^2 \beta^2 + G^2}}) \right) = \frac{\beta}{W} \left(\frac{-2qG}{\sqrt{p^2 \beta^2 + G^2}} \right) < 0,$$

a contradiction. So we choose the second root R_2 for R . By taking into account γ_0 and setting $R_0 := R_2$ we thus obtain

$$R_0 = \frac{1}{p^2} \left(1 + q^2 - 2q \frac{G_0}{\sqrt{p^2 \beta^2 + G_0^2}} \right). \quad (2.11)$$

$p^2 = (1 - q^2)^2 = (1 - q)^2(1 + q)^2$, hence

$$\begin{aligned} \frac{1}{(1 + q)^2} &= \frac{1 - 2q + q^2}{p^2} \leq \frac{1}{p^2} \left(1 + q^2 - 2q \frac{G_0}{\sqrt{p^2 \beta^2 + G_0^2}} \right) = R_0 \\ &\leq \frac{1}{p^2} \left(1 + q^2 + 2q \frac{G}{\sqrt{p^2 \beta^2 + G_0^2}} \right) \leq \frac{1}{(1 - q)^2}. \end{aligned}$$

According to the properties of γ_0 stated in Lemma 2.1 this shows that $(\gamma_0, R_0) \in A$.

$\frac{\partial^2 \delta}{\partial \gamma^2}(\gamma_0, R_0) = R_0 \frac{\partial^2 G}{\partial \gamma^2} > 0$ by Lemma 2.1. Moreover, $\frac{\partial^2 \delta}{\partial \gamma \partial R}(\gamma_0, R_0) = \frac{\partial^2 \delta}{\partial R \partial \gamma}(\gamma_0, R_0) = \frac{\partial G}{\partial \gamma}(\gamma_0) = 0$.

$$\frac{\partial^2 \delta}{\partial R^2} = \dots = \frac{\beta}{W} \left(p^2 + \left(\frac{2 - (Rp + 1)p}{W} \right)^2 \right) > 0.$$

Therefore the Hesse determinant $H(\gamma_0, R_0) = R_0 \frac{\partial^2 G}{\partial \gamma^2}(\gamma_0, R_0) \frac{\partial^2 G}{\partial R^2}(\gamma_0, R_0) > 0$ which confirms that $\delta(\gamma, R)$ has a relative minimum in its unique stationary point (γ_0, R_0) . Because $\delta(\gamma, R)$ is continuous we have found its absolute minimum $\delta_0(q, \beta)$ on A . $\square$

In Fig. 3 the function $\delta(\gamma, R)$ is plotted for $q = 0.4$ and $\beta = \pi/3$.

Substituting R_0 in W one obtains

$$W_0 = \frac{2q\beta}{\sqrt{p^2 \beta^2 + G_0^2}}, \quad (2.12)$$

and for $\delta_0(q, \beta) = \delta(\gamma_0, R_0) = R_0 G_0 + \beta(\cot \beta - W_0)$ a short calculation shows that

$$\delta_0(q, \beta) = \frac{1 + q^2}{p^2} G_0 - \frac{2q}{p^2} \sqrt{\beta^2 p^2 + G_0^2} + \beta \cot \beta. \quad (2.13)$$

Now we add the boundary conditions (2.7) to the conditions (2.6) calling forth two possibilities, namely $(\gamma_0, R_0) \in B$ and $(\gamma_0, R_0) \notin B$.

If (γ_0, R_0) satisfies condition (2.6) then the minimum $\delta(\gamma_0, R_0)$ will be assumed within B in which case we introduce the following function:

Fig. 3 $\delta(\gamma, R)$

Definition 2.3 Let $M_0(q) := \min_{\beta} \delta_0(q, \beta)$ for $\gamma_0 \leq \operatorname{arccot}(\frac{W_0 - \cot(\beta)}{R_0 p}) \leq \beta + \gamma_0$, besides the assumption that γ_0 and W_0 can both be expressed by G_0 according to (2.11) and (2.12), respectively.

If $(\gamma_0, R_0) \notin B$, we have to check the following boundaries of B , what we will do aside the assumption that $(\gamma_0, R_0) \notin B$:

- (a) $\gamma = \pi/2 - \beta$, if $\beta \leq \pi/2$ and $\gamma = 0$ for $\beta > \pi/2$,
- (b) $\gamma = \pi - \beta$,
- (c) $R = 1/(1+q)^2$,
- (d) $R = 1/(1-q)^2$,
- (e) $Rp \cot(\beta + \gamma) = W - \cot \beta$, which stands for $r_1 = r_2$ and
- (f) $Rp \cot \gamma = W - \cot \beta$, which represents the case $r_3 = r_1 q = r_2$.

According to Theorem 4.1 and its proof in [2] $\min \delta(q)$ can only occur within triangles T with $\alpha \leq \gamma$ or $\alpha = \beta$ or $r_2 = r_3$ (the latter leading to case (f)). For discussing the cases (a) and (b) we will assume these conditions.

Using $\rightarrow$ to indicate that a variable approaches a point and also for a function to converge to a limit, we thus obtain

(a) If $\gamma = \pi/2 - \beta$ for $\beta \leq \pi/2$ then $\alpha \leq \gamma$ or $\alpha = \beta$ implies $\alpha = \gamma = \pi/2$ or $\alpha = \beta = \pi/2$, respectively. If both, α and γ , approach $\pi/2$, then $R \rightarrow \infty$ which because of $R \leq 1/(1-q)^2$ means that $q \rightarrow 1$. Therefore we can conclude that $\delta(q) \geq \min \delta(1)$. If α and β approach $\pi/2$ then $\gamma \rightarrow 0$ and by Eq. (2.2) we obtain $\delta(q) \rightarrow \infty$.

If $\pi/2 < \beta < \pi$ then $\gamma = 0$ implies $\alpha = 0$ for $\alpha \leq \gamma$ from which we can infer $\delta(q) \rightarrow \infty$, and with respect to $\alpha = \beta$ the outcome will be the same.

Fig. 4 $r_1 = 1/(1+q)$

(b) $\gamma = \pi - \beta$, i.e. $\alpha \rightarrow 0$:

$$\delta(q) = \lim_{\alpha \rightarrow 0} (R[(\pi - (\pi - \beta)p) \cot(\beta) + (\pi - \pi p) \cot \alpha] + \beta \cot \beta - \beta W \rightarrow +\infty.$$

As for the cases (c) and (d), since our goal is at least to find minimal values on the boundaries of B in respect to all angles β , we will make use of the assumption within conjecture (1.1) that no circle must cut into the opposite side of the triangle T . However, in case (d) this will alter the upper bound of R .

The next section will be devoted to the cases (c) and (d).

3 Two radii in a line

If $r_1 = 1/(1+q)$ and no circle cuts into an opposite side of T , r_1 and $r_3 (= r_1 q^2)$ form the segment of a straight line, and the third radius is orthogonal to this line (see Fig. 4). We will establish a representation for $\delta(q)$ anew on the basis of Fig. 4. From this drawing we can read off readily:

The area ΔT of the triangle T is given by $2\Delta T = r_2 \tan(\alpha) = r_2(1+q) = q \tan(\gamma)$, $\arctan(q) \leq \alpha \leq \pi/4$ and, looking for the smallest possible and the largest β , we get $\pi/4 + \arctan(q) \leq \beta \leq \pi/4 + \arctan(1/q)$ which is equivalent to $\arctan q \leq \alpha \leq \pi/4$.

We therefore obtain

$$\begin{aligned} \delta(q, \beta) &= \frac{1}{r_2} \left(\frac{\alpha}{(1+q)^2} + \beta r_2^2 + \frac{\gamma q^2}{(1+q)^2} \right) \\ &= \frac{1}{r_2} (\alpha r_2^2 \cot^2 \alpha + \beta r_2^2 + \gamma q^2 r_2^2 \cot^2 \alpha) \\ &= \frac{\tan \alpha}{1+q} ((\alpha + \gamma q^2) \cot^2 \alpha + \pi - \alpha - \gamma) \\ &= \frac{1}{1+q} (\alpha(\cot \alpha - \tan \alpha) + \pi \tan \alpha + \gamma(q^2 \cot \alpha - \tan \alpha)) \end{aligned} \quad (3.1)$$

$$\begin{aligned}
&= \frac{1}{1+q} \left(2\alpha \cot(2\alpha) + \pi \tan \alpha + \arctan \left(\frac{1}{q} \tan \alpha \right) (q^2 \cot \alpha - \tan \alpha) \right) \\
&=: \bar{\delta}_1(q, \alpha).
\end{aligned} \tag{3.2}$$

Theorem 3.1 If $q \geq 0.6$ then $\delta(q, \beta) \geq 1.219... > S(1)$ for every β .

Proof As shown above $\delta(q, \beta) = \bar{\delta}_1(q, \alpha)$. Observing that $1/r_2 = (1+q) \cot \alpha \geq 1+q$ and $\tan \alpha \geq q$, substituting γ within Eq. (3.1) by $\pi - \beta - \alpha$ we obtain

$$\begin{aligned}
\bar{\delta}_1(q, \alpha) &= \frac{1}{r_2} \frac{\alpha(1-q^2)}{(1+q)^2} + \beta r_2 + \frac{(\pi - \beta)q^2}{(1+q)^2 r_2} \geq \frac{\alpha(1-q)}{r_2(1+q)} + \beta \frac{\tan \alpha}{1+q} + \frac{(\pi - \beta)q^2}{(1+q)} \\
&\geq \alpha \cot(\alpha)(1-q) + \beta \frac{\tan \alpha}{1+q} + \frac{\pi q^2}{1+q} - \frac{\beta q^2}{1+q} \\
&\geq \alpha \cot(\alpha)(1-q) + \frac{\beta(q - q^2)}{1+q} + \frac{\pi q^2}{1+q}.
\end{aligned}$$

Since $\alpha \cot \alpha$ is a decreasing function and $\alpha \leq \pi/4$ we further obtain

$$\bar{\delta}_1(q, \alpha) \geq \frac{\pi}{4}(1-q) + \frac{\beta q}{1+q}(1-q) + \frac{\pi q^2}{1+q} =: \bar{\epsilon}_1(q, \beta).$$

$\bar{\epsilon}_1(q, \beta)$ is an increasing function of β from which we can infer that

$$\bar{\delta}_1(q, \alpha) \geq \frac{\pi}{4}(1-q) + \frac{q(1-q)}{1+q} \left(\frac{\pi}{4} + \arctan q \right) + \frac{\pi q^2}{1+q} =: \tilde{\delta}_1(q).$$

$\tilde{\delta}_1(q)$ is an increasing function of q (easily checked for $q \geq 0.5$), and $\tilde{\delta}_1(0.6) > 1.219 > S(1)$. $\square$

Definition 3.2 For $0 < q < 0.6$ and $\bar{\delta}_1(q, \alpha)$ according to (3.2) we set $M_1(q) := \min_{\alpha} \bar{\delta}_1(q, \alpha)$ for $\arctan(q) \leq \alpha \leq \pi/4$.

Remark 3.3 As one can see by numerical calculations the minimum of $\bar{\delta}_1(q, \alpha)$ is assumed for $\alpha = \arctan q$, hence one would obtain

$$M_1(q) = \frac{1}{1+q} \left(\pi q + \frac{1-q^2}{q} \arctan q \right).$$

Next we will discuss case (d) altered by the assumption that no circle cuts into an opposite side of the triangle T . Then due to r_1 being as long as possible the situation illustrated in Fig. 5 occurs.

From Fig. 5 one immediately realizes substituting r_1 with r and $a = rq + r_2$ that $r_2 = r \cot \beta$, $a = r(q + \cot \beta)$, $2\Delta T = ar$, $\sin \gamma = r = 1/\sqrt{1+q^2}$, $\gamma = \operatorname{arccot}(q)$ and the range of β is $\pi/4 \leq \beta \leq \operatorname{arccot}(q)$.

Representing $\delta(q)$ as a function of q and β , referred to as $\delta_2(q, \beta)$, we therefore obtain:

$$\begin{aligned}
\delta_2(q, \beta) &= \frac{1}{ar} (\alpha r^2 + \beta r^2 \cot^2 \beta + \gamma r^2 q^2) = \frac{r}{a} (\pi - \beta - \gamma + \beta \cot^2 \beta + \gamma q^2) \\
&= \frac{1}{q + \cot \beta} (\pi - \beta + \beta \cot^2 \beta - (1 - q^2) \operatorname{arccot}(q)).
\end{aligned}$$

As one can easily check by means of the first partial derivative of $\delta_2(q, \beta)$, $\delta_2(q, \beta)$ is a decreasing function of β for $\pi/4 \leq \beta \leq \pi/2$. Setting $\beta = \operatorname{arccot}(q)$ and considering the

Fig. 5 $r_1 = 1/\sqrt{1+q^2}$

function $s(q, x) := \frac{\pi - 2(1-q^2) \arctan(\sqrt{1-x^2}/(x+q))}{2\sqrt{1-x^2}(x+q)}$ which constitutes $S(q)$ (see Eq. (1.1)), one gets

$$\min_{\beta} \delta_2(q, \beta) = \frac{1}{2q} (\pi - 2(1-q^2) \operatorname{arccot}(q)) = s(q, 0) \geq S(q).$$

This way we have proven.

Theorem 3.4 $\min_{\beta} \delta_2(q, \beta) \geq S(q)$.

Next we will discuss the cases (e) and (f) of the boundary conditions stated above, yet without the assumption that no circle must cut into an opposite side of the triangle T , since looking for minima this fact will turn out by itself.

4 Coverings with only two kind of circles

Theorem 4.1 If $r_1 = r_2$ then $\min \delta(q) = S(q)$, and if $r_2 = r_3$ then $\min \delta(q) = S(1/q)$.

Proof According to (2.5) $r_1 = r_2$ means $Rp \cot(\beta + \gamma) = W - \cot(\beta)$, a quadratic equation in R which entails the solution ([2], (10))

$$R_{12} := \frac{\sin \alpha}{p^2 \sin \beta} \left(2 \sin \alpha \sin \beta - p \cos \gamma - \sqrt{(2 \sin \alpha \sin \beta - p \cos \gamma)^2 - p^2} \right).$$

Setting $R = R_{12}$ in $\delta(q) = RG + \beta \cot(\beta) - \beta W$ yields

$$\delta_{12}(q, \beta, \gamma) := \frac{\pi - \gamma p}{p^2 \sin \gamma} \left(q^2 \cos \gamma - \cos(2\beta + \gamma) - \sqrt{(q^2 \cos \gamma - \cos(2\beta + \gamma))^2 - p^2} \right).$$

Keeping γ fixed, within a few calculation steps one obtains that

$$\frac{\partial \delta_{12}(q, \beta, \gamma)}{\partial \beta} = -2 \frac{\sin(2\beta + \gamma)}{\sqrt{(q^2 \cos \gamma - \cos(2\beta + \gamma))^2 - p^2}} \delta_{12}(q, \beta, \gamma).$$

This shows: If $2\beta + \gamma \leq \pi$ then $\delta_{12}(q, \beta, \gamma)$ is decreasing in β and if $2\beta + \gamma \geq \pi$ then $\delta_{12}(q, \beta, \gamma)$ is increasing, hence $\delta_{12}(q, \beta, \gamma)$ is minimal for $2\beta + \gamma = \pi$ which means

$\alpha = \beta$. If this is the case then one can find by some elaborate but elementary calculations (cf. [2], (14)) that

$$\begin{aligned}\delta_{12}(q, \beta, \pi - 2\gamma) &= \dots = \frac{\pi q^2 + 2\beta(1 - q^2)}{(1 - q^2)^2 \sin 2\beta} \left(q \sin \beta - \sqrt{1 - q^2 \cos^2 \beta} \right)^2 \\ &=: \delta_{12}(q, \beta)\end{aligned}\quad (4.1)$$

If $r_2 = r_3$ then (as already derived in [2]) $W - \cot(\beta) = Rp \cot \gamma$ yields

$$R_{23} := \frac{\sin \gamma}{p^2 \sin \beta} \left(2 \sin \beta \sin \gamma - p \cos(\beta - \gamma) - \sqrt{(2 \sin \beta \sin \gamma - p \cos(\beta - \gamma))^2 - p^2} \right),$$

which, substituted for R within $\delta(q) = RG + \beta \cot(\beta) - \beta W$ and writing $\pi - \alpha - \gamma$ instead of β entails

$$\delta_{23}(q, \alpha, \gamma) := \frac{\pi q^2 + \alpha p}{p^2 \sin \alpha} \left(\cos \alpha - q^2 \cos(\alpha + 2\gamma) - \sqrt{(\cos \alpha - q^2 \cos(\alpha + 2\gamma))^2 - p^2} \right).$$

One can easily deduce that

$$\frac{\partial \delta_{23}(q, \alpha, \gamma)}{\partial \gamma} = -2 \frac{q^2 \sin(\alpha + 2\gamma)}{\sqrt{(\cos \alpha - q^2 \cos(\alpha + 2\gamma))^2 - p^2}} \delta_{23}(q, \alpha, \gamma),$$

and ascertain that $\delta_{23}(q, \alpha, \gamma)$ is minimal for $\alpha + 2\gamma = \pi$, i.e. $\beta = \gamma$. In this case (as already carried out in [2]) elementary calculations lead to

$$\delta_{2,3}(q, \alpha, (\pi - \alpha)/2) = \dots = \frac{\pi - 2\beta(1 - q^2)}{(1 - q^2)^2 \sin 2\beta} \left(\sin \beta - \sqrt{q^2 - \cos^2 \beta} \right)^2 =: \delta_{23}(q, \beta). \quad (4.2)$$

As also conveyed in [2], interpreting geometrically x and q within the function $s(q, x) := \frac{\pi - 2(1 - q^2) \arctan(\sqrt{1 - x^2}/(x + q))}{2\sqrt{1 - x^2}(x + q)}$ constituting $S(q)$, straightforward calculations link β to x (see Fig. 6 for $\alpha = \beta$): One obtains $\min_{0 < \beta < \pi/2} \delta_{12}(q, \beta) = S(q)$ and $\min_{0 < \beta < \pi/2} \delta_{2,3}(q, \beta) = S(1/q)$. $\square$

Fig. 6 Geometric interpretation of $s(2, x)$

Fig. 7 $S(q)$, $M_0(q)$, $S_1(q)$, $M_1(q)$

5 Conclusions

We have come so far to know that for all $q \geq 0.6$ $\min \delta(q)$ is to be found among the minima of the densities $\delta_0(q, \beta)$ (Eq. 2.13), $\delta_{12}(q, \beta)$ (Eq. 4.1) and $\delta_{23}(q, \beta)$ (Eq. 4.2) described by the functions $M_0(q)$ (Definition 2.3), $S(q)$ and $S(1/q)$ (see Eq.(1.1)).

For $0 < q < 0.6$ we have not only to take into account the minima of $\delta_0(q, \beta)$, $\delta_{12}(q, \beta)$ and $\delta_{23}(q, \beta)$ but also the minimum of $\bar{\delta}_1(q, \alpha)$ which gives rise to $M_1(q)$ (Definition 3.2). With this in mind we can formulate the following

Theorem 5.1 *If $M_0(q)$ and $S(1/q)$ are less than or equal to $S(q)$ for $0.6 \leq q \leq 1$ — this can be verified by computer calculations — the conjecture of L. Fejes Tóth and J. Molnár can be considered as confirmed.*

If $M_0(q)$, $M_1(q)$ and $S(1/q)$ are less than or equal to $S(q)$ for $0 < q < 0.6$ — this can also be shown by computer calculations — the correctness of the conjecture ensues.

The functions $M_0(q)$, $M_1(q)$, $S(q)$ and $S(1/q)$ are visualized in Fig. 7.

Acknowledgements Open access funding provided by TU Wien (TUW).

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. G. Blind, Überdeckungen der Ebene durch inkongruente Kreise. Math. Z. **140**, 179–194 (1974)
2. D. Dorninger, Thinnest covering of the Euclidean plane with incongruent circles. Anal. Geom. Metric. Sp. **5**, 40–46 (2017). <https://doi.org/10.1515/agms-2017-0002>

3. A. Florian, Überdeckung der Ebene durch Kreise. Rend. Sem. Mat. Univ. Padova **31**, 77–86 (1961)
4. A. Florian, Zum Problem der dünnsten Kreisüberdeckung der Ebene. Acta Math. Acad. Sci. Hung. **13**, 397–400 (1962)
5. A. Heppes, J. Molnár, Recent results in discrete geometry. I. Mat. Lapok **9**(4), 330–355 (1960). (in Hungarian)
6. E. Jucovic, J. Leso, Eine Bemerkung zur Überdeckung der Ebene durch inkongruente Kreise. Mat. Fyz. Casopis Slovensk. Acad. Vied **19**, 324–328 (1966)
7. J. Molnár, Unterdeckung und Überdeckung der Ebene durch Kreise. Ann. Univ. Sci. Bp. Rolando Eotvös Sect. Math. **2**, 33–40 (1959)
8. L.F. Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum. Die Grundlehren der Mathematischen Wissenschaften*, vol. 65, 2nd edn. (Springer, Berlin, 1972)
9. G.F. Tóth, Covering the plane with two kinds of circles. Discrete Comput. Geom. **13**, 445–457 (1995)
10. L.F. Tóth, J. Molnár, Unterdeckung und Überdeckung der Ebene durch Kreise. Math. Nachr. **18**, 235–243 (1958)