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# One Variable Relevant Logics are S5ish

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## Abstract

Here I show that the one-variable fragment of several first-order relevant logics corresponds to certain **S5**ish extensions of the underlying propositional relevant logic. In particular, given a fairly standard translation between modal and one-variable languages and a permuting propositional relevant logic **L**, a formula $\mathcal{A}$ of the one-variable fragment is a theorem of **LQ** (**QL**) iff its translation is a theorem of **L5** (**L.5**). The proof is model-theoretic. In one direction, semantics based on the Mares-Goldblatt [15] semantics for quantified **L** are transformed into ternary (plus two binary) relational semantics for **S5**-like extensions of **L** (for a general presentation, see Seki [26, 27]). In the other direction, a valuation is given for the full first-order relevant logic based on **L** into a model for a suitable **S5** extension of **L**. I also discuss this work's relation to finding a complete axiomatization of the constant domain, non-general frame ternary relational semantics for which **RQ** is incomplete [11].


**Keywords** First-Order Relevant Logic · One-Variable Fragment · Modal Relevant Logic

## 1 Introduction

The relation between one-variable fragments of quantified logics and **S5**-ish modal logics with the same propositional base is a common subject of inquiry. For classical logic, this relation is demonstrated well in Mints [19]. For Corsi logic this relation has been shown in Caicedo et al. [4], who have also dealt with one-variable fragments of some intermediate logics [5]. So far, at least in print, there has been a overalllack of

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results and interest in the one-variable fragment of relevant logics. However, there has been a related result by Cintula et al. [6] that gives a proof-theoretic demonstration of a correspondence between the one-variable fragments of certain first-order logics and the **S5**-ish extensions of their propositional base. In particular, their results are for logics extending the full Lambek calculus with exchange, some extensions of which are extensions of logics considered in this paper.

The paper aims to accomplish two related tasks. The first is to motivate the project to the philosopher and the mathematician. The second is to prove a semantic equivalence result between the one-variable fragment of **LQ** (**QL**) and the modal relevant logic **L5** (**L.5**).

There is philosophical and mathematical interest in Fine's [11] incompleteness result for **RQ**, a first-order relevant logic based on **R**, with respect to the most straightforward way of generalizing the ternary relational semantic of Sylvan (né Routley) and Meyer [24] where quantifiers are given the usual Tarskian interpretation. (We'll call this failed semantic approach the CD Semantics.) There is still no complete axiomatization of the logic for the CD semantics. Moreover, **RQ** has received a natural, constant domain semantics from Mares and Goldblatt [15]. Fine's incompleteness result shows that a particular formula is valid in the CD semantics, yet it is not a theorem of **RQ**. This formula, which the reader will encounter below, is (with a little squinting or harmless substitution) a formula in the one-variable fragment of **RQ**. Thus, the link to modal logic may provide insight into the problems of (1) finding a proof system for the CD Semantics, and (2) determining just what goes wrong in the CD Semantics (from the point of view of **RQ**).

The paper is divided as follows: we first set out the preliminaries including axiomatizations and semantics for first-order and modal relevant logics, translations between modal and one-variable fragment languages, and select formulas and their translations. The section that follows gives the equivalence results using frame-based semantics. We conclude by expanding on the relation of this work to other topics in relevant logic, most notably Fine's incompleteness result, and by suggesting further work.

## 2 Preliminaries

We define a modal propositional language, a first-order language, and first-order language with a single variable. For modal propositional logics, we assume an at most denumerable set of a atomic propositions or propositional variables, denoted as lowercase letters from $p$ through $s$, with or without subscripts. For any propositional logic, the set of (well-formed) formulas is defined in the usual way using $\boldsymbol{t}$ (intensional truth constant), $\neg$ (negation), $\wedge$ (truth-functional conjunction), $\vee$ (truth-functional disjunction), $\circ$ (intensional conjunction) $\rightarrow$ (conditional), $\Box$ (necessity) and $\Diamond$ (possibility). We call this defined language a *modal language*. Without the modalities, we call the language simply a *propositional language*.

For first-order logics, we assume a denumerable set of variables $Var$ with a fixed ordering. Variables will be denoted by lowercase letters near the end of the Latin alphabet, often with integer decoration representing their place in the assumed fixed ordering (e.g. $x, y, z, y_4$). A *signature* is a set $\mathcal{L}$ consisting of a non-empty but at most denumerable set $Pred$ of predicate symbols and an at most denumerable set $Con$ of individual constant symbols. Each predicate symbol is of the form $P^n$, where $n$ is the arity of the predicate. The arity is often omitted. I shall denote individual constants by $c$, with or without subscripts. A *term* is denoted by $\tau$, with or without integer decoration. An $\mathcal{L}$-*term*, for signature $\mathcal{L}$, is the union of the variables and constants of $\mathcal{L}$. A term is *closed* when it contains no variables, otherwise it is open.

For a given signature $\mathcal{L}$, the atomic formulas (atomic $\mathcal{L}$-formulas) are those of the form $P^n(\tau_1, \ldots, \tau_n)$, where $P^n \in \mathcal{L}$ and $\tau_1, \ldots, \tau_n$ are $\mathcal{L}$-terms. The set of well-formed formulas of a first-order logic with signature $\mathcal{L}$ is defined in the usual way, extending the propositional connectives with the cases for $\forall x$ (universal quantification) and $\exists x$ (existential quantification), for each variable $x$. We will use calligraphic, uppercase Latin letters to range over the set of well formed formulas (of $wff$, when the language is clear) for both propositional and first-order logics.

We call this defined language a *full first order language*. When we restrict ourselves to a single variable (we will choose $x$ for this variable by *fiat*), reject zero-ary predicate letters (with the exception of $\boldsymbol{t}$), and have the empty set worth of constants, we call this a *first order language in the one-variable fragment*, although similar phrases are employed.[1] We write $\mathfrak{L}_{(x)}$ for the one-variable fragment for the variable $x$. To avoid unnecessary typing, we will often use *first order language*, when it is reasonably clear that we mean "in the one-variable fragment".

An instance of a variable $x$ is *bound* in the formula $\mathcal{A}$ if either (1) the instance is the $x$ of an expression $\forall x$ or $\exists x$ occurring in $\mathcal{A}$, or (2) the instance of $x$ occurs within the scope of a quantifier, $\forall x$ or $\exists x$. A instance is *free* when it is not bound, and a formula with no free variables is called a *sentence*. A term $\tau$ is *free for* (or freely substitutable for) $x$ in $\mathcal{A}$ if, for every variable $y$ occurring in $\tau$, there are no free occurrences of $x$ in $\mathcal{A}$ that are in the scope of a quantifier $\forall y$ or $\exists y$.

We shall write $\mathcal{A}[\tau/x]$ for the result of replacing every free occurrence of $x$ in $\mathcal{A}$ with the term $\tau$. Similarly, we will use $\mathcal{A}[\tau_0/v_0, \ldots, \tau_n/v_n]$ for the result of simultaneously replacing $v_0$ through $v_n$ with $\tau_0$ through $\tau_n$ respectively.

A variable assignment, $g \in U^\omega$, assigns an element of the domain $U$ for each variable. In detail, we order the variables and associate each position in that ordering with an element of the domain, using $gn$ to denote the $n$th element of the ordering. That is, a variable assignment is a denumerable list of elements in the domain. An $x$-variant of a variable assignment $g$ differs from $g$ in at most the assignment to the variable $x$, and the set of all $x$-variants of $g$ is denoted $xg$. We write $g[j/n]$ (or $g[j/x]$)

[1] We note that the inclusion of only unary predicate symbols, to the exclusion of (non-constant) zero-ary predicate symbols, is standard in the literature of one-variable fragments and their modal logic counterparts: e.g., see [4–6]. (Strictly speaking, this is the *monadic* fragment, which is equivalent to the one-variable fragment when there are no zero-ary predicate symbols.) The proofs below would appear to work with the addition of zero-ary predicates with translations that differ from $h$ and $g$ (defined below in Section 2.5). A zero-ary predicate must be translated into something that is equivalent to its boxed variant: i.e., $p \leftrightarrow \Box p$ is a theorem. This is possible if we translate zero-ary predicates to a formula of the form $\Box p$. However, in doing so we lose the facts that $g(h(\mathcal{A})) = \mathcal{A}$ and $h(g(\mathcal{A})) = \mathcal{A}$, which are desirable in that they help display the equivalence. The special propositional constant $\boldsymbol{t}$, however, is already treated sufficiently in the modal logic: $\Box \boldsymbol{t} \leftrightarrow \boldsymbol{t}$ is a theorem.

to represent the variable assignment just like $g$, except that the $n$-th element in $g$ (or $x$) is replaced by the element $j$.

### 2.1 First Order Relevant Logics Logics

Here we provide both an axiomatization and frame-based semantics for the propositional relevant logic extending **B**, and first order logics and their restriction to the one-variable fragments (with variable $x$). The frame-base semantics (for the first-order logic) is that introduced in [15] for **QR** and **RQ** (and extended to a wide range of first-order (modal) relevant logics in Ferenz [9]), which is based on the ternary relational semantics established in [22–24] by Sylvan and Meyer.

### 2.2 Axiomatic Presentations

We can define a range of logics from the following axioms schemes and rule schemes.

(A1) $\mathcal{A} \to \mathcal{A}$
(A2) $\mathcal{A} \to (\mathcal{A} \vee \mathcal{B})$
(A3) $\mathcal{B} \to (\mathcal{A} \vee \mathcal{B})$
(A4) $(\mathcal{A} \wedge \mathcal{B}) \to \mathcal{A}$
(A5) $(\mathcal{A} \wedge \mathcal{B}) \to \mathcal{B}$
(A6) $\mathcal{A} \wedge (\mathcal{B} \vee \mathcal{C}) \to ((\mathcal{A} \wedge \mathcal{B}) \vee (\mathcal{A} \wedge \mathcal{C}))$
(A7) $((\mathcal{A} \to \mathcal{B}) \wedge (\mathcal{A} \to \mathcal{C})) \to (\mathcal{A} \to (\mathcal{B} \wedge \mathcal{C}))$
(A8) $((\mathcal{A} \to \mathcal{C}) \wedge (\mathcal{B} \to \mathcal{C})) \to ((\mathcal{A} \vee \mathcal{B}) \to \mathcal{C})$
(A9) $\neg\neg\mathcal{A} \to \mathcal{A}$
(A10) $(\mathcal{A} \to \neg\mathcal{B}) \to (\mathcal{B} \to \neg\mathcal{A})$
(A11) $(\mathcal{A} \to \mathcal{B}) \to ((\mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{C}))$
(A12) $(\mathcal{A} \to \mathcal{B}) \to ((\mathcal{C} \to \mathcal{A}) \to (\mathcal{C} \to \mathcal{B}))$
(A13) $(\mathcal{A} \to (\mathcal{A} \to \mathcal{B})) \to (\mathcal{A} \to \mathcal{B})$
(A14) $\mathcal{A} \to ((\mathcal{A} \to \mathcal{B}) \to \mathcal{B})$
(A15) $\mathcal{A} \to (\mathcal{A} \to \mathcal{A})$
(A16) $(\mathcal{A} \to (\mathcal{B} \to \mathcal{C})) \to (\mathcal{B} \to (\mathcal{A} \to \mathcal{C}))$

($\forall$E) $\forall x \mathcal{A} \to \mathcal{A}[\tau/x]$, where $\tau$ is free for $x$ in $\mathcal{A}$
($\exists$I) $\mathcal{A}[\tau/x] \to \exists x \mathcal{A}$, where $\tau$ is free for $x$ in $\mathcal{A}$
(EC1) $\forall x(\mathcal{A} \vee \mathcal{B}^x) \to \forall x \mathcal{A} \vee \mathcal{B}^x$
(EC2) $\mathcal{A}^x \wedge \exists x \mathcal{B} \to \exists x(\mathcal{A}^x \wedge \mathcal{B})$

(MP) $\mathcal{A}, \mathcal{A} \to \mathcal{B} \Rrightarrow \mathcal{B}$
(ADJ) $\mathcal{A}, \mathcal{B} \Rrightarrow \mathcal{A} \wedge \mathcal{B}$
(Prefix) $\mathcal{A} \to \mathcal{B} \Rrightarrow (\mathcal{C} \to \mathcal{A}) \to (\mathcal{C} \to \mathcal{B})$
(Suffix) $\mathcal{A} \to \mathcal{B} \Rrightarrow (\mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{C})$
(RCont) $\mathcal{A} \to \neg\mathcal{B} \Rrightarrow \mathcal{B} \to \neg\mathcal{A}$
(R$\forall$I) $\mathcal{A}^x \to \mathcal{B} \Rrightarrow \mathcal{A}^x \to \forall x \mathcal{B}$
(R$\exists$E) $\mathcal{A} \to \mathcal{B}^x \Rrightarrow \exists x \mathcal{A} \to \mathcal{B}^x$
(R$\circ$) $\mathcal{A} \to (\mathcal{B} \to \mathcal{C}) \Lleftarrow\Rrightarrow (\mathcal{A} \circ \mathcal{B}) \to \mathcal{C}$
(R$\leftarrow$) $\mathcal{A} \to (\mathcal{B} \to \mathcal{C}) \Lleftarrow\Rrightarrow \mathcal{B} \to (\mathcal{C} \leftarrow \mathcal{A})$

(R$t$) $\quad t \rightarrow \mathcal{A} \Leftarrow\!\Rightarrow \mathcal{A}$

In the schemes listed above, $\mathcal{A}^x$ indicates that $x$ does not occur free in the formula $\mathcal{A}$. Further, $\Rrightarrow$ indicates a rule of proof in the sense of [14, 28], so $\mathcal{A}_1, \ldots, \mathcal{A}_n \Rrightarrow \mathcal{B}$ should be read to mean that if each $\mathcal{A}_i$ is derivable, then so is $\mathcal{B}$. We define an axiomatic derivation to be a sequence of formulas, each of which is either an instance of an axiom scheme or follows from previous formulas in the sequence by one of the rule schemes. Logics will be identified with their set of theorems (formulas provable from the axioms alone), and we will write $\vdash_{\mathbf{L}} \mathcal{A}$ for $\mathcal{A}$ is a theorem of **L**, dropping the subscript when doing so introduces no ambiguity.

In a propositional language, we define the base affixing logic **B** as (A1)–(A9), (MP), (ADJ), (Prefix), (Suffix), (RCont). The main results apply to logic with permutation, so we will stick to these logic. Most don't have standard names in the literature, though **R** and **RQ** are exceptions. Thus, for logics weaker than **R** with permutation (A16), we will typically use a super-scripted '$P$'. Thus, we have $\mathbf{B}^P = \mathbf{B} +$ (A16). We define the permuting logics:[2]

$$\mathbf{DW}^P = \mathbf{B}^P + (\mathrm{A10}) \qquad \mathbf{R} = \mathbf{RW} + (\mathrm{A13})$$
$$\mathbf{RW} = \mathbf{DW}^P + (\mathrm{A11}) + (\mathrm{A12}) \qquad \mathbf{RM} = \mathbf{R} + (\mathrm{A15})$$

Note that, as defined, these logics are in a language with $t$, $\circ$, and $\leftarrow$. We explicitly include $\leftarrow$ and its governing rule, despite that it is definable in systems with permutation, because it allows us to have a more succinct axiom system for first-order logics. We will not use any logics here that does not contain leftarrow and fusion, and so there is no decoration on the name of a logic to denote this. However, this is an important distinction: e.g., some first-order relevant logics (needing neighbourhood semantics) are not conservatively extended by fusion or leftarrow [31].

For any propositional logic **L** (defined above), in a first-order language we define two first-order extensions as follows:

$$\mathbf{QL} = \mathbf{L} + (\forall\mathrm{E}) + (\exists\mathrm{I}) + (\mathrm{R}\forall\mathrm{I}) + (\mathrm{R}\exists\mathrm{E})$$
$$\mathbf{LQ} = \mathbf{QL} + (\mathrm{EC1}) + (\mathrm{EC2})$$

The logics $\mathbf{LQ}^x$ and $\mathbf{QL}^x$, the one-variable fragments of **LQ** and **QL**, are defined to be the theorems of **LQ** and **QL**, restricted to the one-variable language.

Due to the presence of the $\leftarrow$ and $\circ$, we have easy proofs of the following theorem (schemes):

(1) $\forall x(\mathcal{A}^x \rightarrow \mathcal{B}) \rightarrow (\mathcal{A}^x \rightarrow \forall x\mathcal{B})$
(2) $\forall x(\mathcal{A} \rightarrow \mathcal{B}^x) \rightarrow (\exists x\mathcal{A} \rightarrow \mathcal{B}^x)$

Without the leftarrow, we would include (2) as an axiom scheme, and similarly for (1).

[2] Note that there are several ways to define these logic. In addition, the reader is directed to [25, Sections 4.1] for a more thorough list of axioms to extend $\mathbf{B}^P$ with.

### 2.2.1 Semantics

We rehearse the general frame-based semantics for first-order logics using the Mares-Goldblatt interpretation of the quantifiers. This interpretation was introduced in [15] for **QR** and **RQ**, and extended to a wider range of relevant logics in Ferenz [9] (and with neighbourhood semantics in Tedder and Ferenz [31]).

First, we introduce frames for propositional relevant logics.

**Definition 1** (Frames and Models for Propositional Logics) A *ternary relational frame for* $\boldsymbol{B}$ ($\mathbf{B}$-frame) is a tuple $\mathfrak{F} = \langle W, N, R, * \rangle$ where $\emptyset \neq N \subset W$, $R \subseteq W^3$, $* : W \longrightarrow W$, and we further define, for each $a, b \in W$, $a \leq b =_{df} \exists x \in N(Rxab)$. Moreover, the following conditions are satisfied:

(c1) $\leq$ is a preorder on $W$;
(c2) $N$ is an ($\leq$-)upset;
(c3) $Ra'b'c'$, $a' \leq a$, $b' \leq b$, and $c \leq c'$ imply $Rabc$;
(c4) $a \leq b$ implies $b^* \leq a^*$;
(c5) $a^{**} = a$

A *model for* $\boldsymbol{B}$ is an $\mathbf{B}$-frame with a valuation function $||-||$ that assigns an upset $||p|| \subseteq W$ to each propositional variable $p$. We define operations $\neg$, $\rightarrow$, $\leftarrow$, and $\circ$ on subsets of $W$ as follows:

$$\begin{aligned}
&\neg X =_{df} \{a \in K : \alpha^* \notin X\}\\
&X \rightarrow Y =_{df} \{a \in K : \forall b, c \in K(Rabc \;\&\; b \in X \Rightarrow c \in Y)\}\\
&X \leftarrow Y =_{df} \{a \in K : \forall b, c \in K(Rbac \;\&\; b \in X \Rightarrow c \in Y)\}\\
&X \circ Y =_{df} \{a \in K : \exists b, c \in K(Rbca \;\&\; b \in X \;\&\; c \in Y)\}
\end{aligned}$$

The assignment $||-||$ is extended to all formulas by the following:

$$\begin{aligned}
||\boldsymbol{t}|| &= N & ||\neg\mathcal{A}|| &= \neg||\mathcal{A}||\\
||\mathcal{A} \wedge \mathcal{B}|| &= ||\mathcal{A}|| \cap ||\mathcal{B}|| & ||\mathcal{A} \vee \mathcal{B}|| &= ||\mathcal{A}|| \cup ||\mathcal{B}||\\
||\mathcal{A} \rightarrow \mathcal{B}|| &= |\mathcal{A}|| \rightarrow ||\mathcal{B}|| & ||\mathcal{A} \leftarrow \mathcal{B}|| &= ||\mathcal{A}|| \leftarrow ||\mathcal{B}||\\
||\mathcal{A} \circ \mathcal{B}|| &= ||\mathcal{A}|| \circ ||\mathcal{B}||
\end{aligned}$$

We briefly list a select few semantic conditions, where $(cX)$ corresponds to the axiom/rule scheme $(X)$. For a more complete list, the reader is directed, e.g., to [25, Sections 4.1, 4.4].

(c10) $Rabc \Rightarrow Rac^*b^*$
(c11) $R^2abcd \Rightarrow Rb(ac)d$
(c12) $R^2abcd \Rightarrow Ra(bc)d$
(c13) $Rabc \rightarrow R^2abbc$
(c14) $Rabc \rightarrow Rbac$
(c15) $Rabc \Rightarrow a \leq c$ or $b \leq c$
(c16) $R^2abcd \Rightarrow R^2acbd$

In the above list, $R^2abcd$ is defined as $\exists x(Rabx \;\&\; Rxcd)$ and $Ra(bc)d$ as $\exists x(Rbcx \;\&\; Raxd)$.

**Definition 2** (Frames and Models for **LQ/QL**) Suppose that **L** is a propositional relevant logic (restricted to those defined above). A *Mares-Goldblatt frame for **QL*** (an **QL**-frame) is a tuple $\mathfrak{F} = \langle W, N, R, *, U, Prop, PropFun \rangle$, where $\langle W, N, R, * \rangle$ is an **L**-frame, $U$ is a non-empty set, and, defining the 'upsets' as $\wp(K)^{\uparrow} = \{X \in \wp(K) : \forall a, b, \in K(a \in X \ \& \ a \leq b) \Rightarrow b \in X\}$, we have that $Prop \subseteq \wp(K)^{\uparrow}$, $PropFun \subseteq \{\varphi : U^{\omega} \longrightarrow Prop\}$. Moreover, the following conditions are satisfied:

(cq1) $Prop$ contains $N$, and is closed under $\cap, \cup, \neg, \rightarrow, \leftarrow, \circ$;

(cq2) $PropFun$ contains a constant function $\varphi_N$ ($\varphi_N f = N$), and is closed under $\cap, \cup, \neg, \rightarrow, \leftarrow, \circ, \forall_n$ and $\exists_n$, for every $n \in \omega$, where

(a) $(\neg\varphi)f = \neg(\varphi f)$

(b) $(\varphi \otimes \psi)f = \varphi f \otimes \psi f$, for each $\otimes \in \{\cap, \cup, \rightarrow, \circ, \leftarrow\}$

(c) $(\forall_n \varphi)f = \bigsqcap_{g \in x_n f} \varphi g = \bigcup\{X \in Prop \mid X \subseteq \bigcap_{g \in x_n f} \varphi g\}$

(d) $(\exists_n \varphi)f = \bigsqcup_{g \in x_n f} \varphi g = \bigcap\{X \in Prop \mid \bigcup_{g \in x_n f} \varphi g \subseteq X\}$

The **LQ**-frames are defined as the **QL**-frames that further satisfying the following. For every $\varphi \in PropFun$, $X, Y \in Prop$, $n \in \omega$, and $f \in U^{\omega}$:

(cEC1) $X - Y \subseteq \bigcap_{j \in U} \varphi(f[j/n])$ only if $X - Y \subseteq (\forall_n \varphi)f$

(cEC2) $\bigcup_{j \in U} \varphi(f[j/n]) \subseteq X \cup \overline{Y}$ only if $|\exists_n \varphi| f \subseteq X \cup \overline{Y}$

A *pre-model for **QL/LQ*** is a tuple $\mathfrak{M} = \langle F, |-| \rangle$ such that $F$ is a Mares-Goldblatt frame for **QL/LQ** and $|-|$ is a valuation function that assigns:

(i) an individual $|c| \in U$ to each constant symbol $c$;
(ii) a function $|P^n| : U^n \longrightarrow \wp(K)$ to each $n$-ary predicate symbol $P^n$; and
(iii) a propositional function $|\mathcal{A}| : U^{\omega} \longrightarrow \wp(K)$ to each formula $\mathcal{A}$ such that, when $\mathcal{A}$ is atomic, for every $f \in U^{\omega}$:

$$|P^n \tau_1, \ldots, \tau_n| f = |P^n|(|\tau_1| f, \ldots |\tau_n| f)$$

where "$|\tau| f$" is $fn$ when $\tau$ is the variable $x_n$, and $|c|$ when $\tau$ is constant symbol $c$. Moreover, when $\mathcal{A}$ is not atomic (or $\boldsymbol{t}$), the valuation is extended as follows, for every $f \in U^{\omega}$:

$$|\boldsymbol{t}| f = \varphi_N f \qquad |\mathcal{A} \leftarrow \mathcal{B}| f = |\mathcal{A}| f \leftarrow |\mathcal{B}| f$$
$$|\neg\mathcal{A}| f = \neg|\mathcal{A}| f \qquad |\mathcal{A} \circ \mathcal{B}| f = |\mathcal{A}| f \circ |\mathcal{B}| f$$
$$|\mathcal{A} \wedge \mathcal{B}| f = |\mathcal{A}| f \cap |\mathcal{B}| f \qquad |\forall x_n \mathcal{A}| f = \forall_n |\mathcal{A}| f$$
$$|\mathcal{A} \vee \mathcal{B}| f = |\mathcal{A}| f \cup |\mathcal{B}| f \qquad |\exists x_n \mathcal{A}| f = \exists_n |\mathcal{A}| f$$
$$|\mathcal{A} \rightarrow \mathcal{B}| f = |\mathcal{A}| f \rightarrow |\mathcal{B}| f$$

A *model for **QL/LQ*** is a pre-model for **QL/LQ** that assigns an element of $Prop$ to each atomic formula. A formula $\mathcal{A}$ is *satisfied* by a variable assignment $f$ in a model

$\mathfrak{M}$, written $\mathfrak{M}, f \vDash \mathcal{A}$, when $N \subseteq |\mathcal{A}| f$. A formula is *valid in a model* $\mathfrak{M}$ ($\mathfrak{M} \vDash \mathcal{A}$) when it is satisfied by every variable assignment in that model; *valid in a frame* $\mathfrak{F}$ ($\mathfrak{F} \vDash \mathcal{A}$) when it is valid in every model based on that frame; *valid in a class of frames* $\mathbb{C}$ ($\mathbb{C} \vDash \mathcal{A}$) when it is valid in every frame in that class.

In the full first-order language, **QL/LQ** is sound and complete with respect to the above semantics [9]. As a corollary, a formula $\mathcal{A}$ in the one-variable fragment is valid in the class of all models for **LQ** iff it is a theorem of **LQ**.

### 2.2.2 Mares-Goldblatt Quantifier Interpretation

The Mares-Goldblatt interpretation of the quantifier using admissible propositions and $\sqcap / \sqcup$, introduced in a pair of papers [15, 16], has both philosophical and formal features. To focus on the universal quantifier, $\forall x \varphi$ is (modeled by) the weakest proposition that implies all of the instances of $\varphi$.[3] The major difference from the usual approaches is that this need not been the generalized intersection of all instances of $\varphi$, and that the generalized intersection need not be an admissible proposition. Ferenz [8] describes the difference between $\sqcap$ and the generalized intersection as a failure of the generalized intersection to have the *additional information* that the listed intersectants are exhaustive of the domain. Thus, we can think of the $\sqcap$ as having more information (sometimes) than the generalized intersection.

Formally this philosophical insight reflects powerful machinery used in obtaining completeness results for quantified modal logics. Using this interpretation of the quantifiers, a completeness proof needs only assume that the prime filters in the canonical model are just that: prime. It does not need to assume that they are $\omega$-complete, where a theory is $\omega$-complete when the theory contains $\forall x \varphi$ if it contains every instance of $\varphi$. In Goldblatt [13], this is put to work to obtain a wide range of completeness results for quantified modal classical logics.

The incompleteness results of Fine [11] for the generalized intersection approach left first-order relevant logics with only more complicated (and harder to interpret) formal semantics (notably Fine [10] and Brady [2, 3]). That is, until the introduction of the Mares-Goldblatt interpretation of the quantifiers. This approach is both philosophically natural, easy to formalize, and leads to completeness results for first-order relevant logics.

## 2.3 Modal Propositional Logics

Modal relevant logics are numerous. However, although the naming conventions are fairly standard and track interesting and relevant distinctions (see Ferenz [9] for a discussion of the naming conventions and their formal correspondences), these naming conventions give long names with many decorations to each logic. This is especially so for quantified modal relevant logics. However, for a given proposotional relevant

[3] Here we say that a proposition $X$ entails a proposition $Y$ when $X \subseteq Y$, and that $Y$ is *weaker* that $X$.

logic **L**, our interest here is only in a single pair of modal relevant logic extending **L**: two kinds of **S5**ish extension of **L**. Moreover, for some logics with strong enough negation, one of these **S5**ish extensions contains the classical **S5**, written in $\{\wedge, \vee, \neg\}$ is its set of theorems.

We will hence subsequently define **L5** and **L.5** for each **L** defined above. These names are in accordance with naming conventions with the following minor comments: (1) the phrase **S5**-ish extension of **R** invoked many intuitions that are pulled apart in the context of **R** (and hence over weaker relevant logics) (see Standefer [29] for details), and (2) we suppress the urge to add additional decoration to denote the presence of $\circ$, $\leftarrow$, $\boldsymbol{t}$ and the two primitive modalities.

Note, however, that naming conventions are not completely standard. Standefer [29] uses the name '**RS5**' for the system defined by removing the axiom (BD)/(DB) from our presentation of **R5**: that is, our **R.5**. For many modal relevant logics, lacking (BD)/(DB) would impose a 'dot' in the name. Again, see Ferenz [9] for an explanation of the mostly standard use of the dot-notation, as well as motivation for using the dot to track an important distinction (the containment of the classical counterpart).

**Definition 3** Given a propositional relevant logic **L**, *the modal relevant logics* **L.5** *and* **L5** are defined by taking the axiom and rule schemes of **L**, in a modal language (with $\Box$ primitive and $\Diamond$ defined by $\Diamond\mathcal{A} = \neg\Box\neg\mathcal{A}$), and adding the following axiom and rule schemes:

| | | | |
|---|---|---|---|
| (K) | $\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$ | ($\mathrm{K}_\Diamond$) | $\Box(\mathcal{A} \to \mathcal{B}) \to (\Diamond\mathcal{A} \to \Diamond\mathcal{B})$ |
| ($\Box$-$\wedge$) | $\Box(\mathcal{A} \wedge \mathcal{B}) \leftrightarrow (\Box\mathcal{A} \wedge \Box\mathcal{B})$ | ($\Diamond$-$\vee$) | $\Diamond(\mathcal{A} \vee \mathcal{B}) \leftrightarrow (\Diamond\mathcal{A} \vee \Diamond\mathcal{B})$ |
| (T) | $\Box\mathcal{A} \to \mathcal{A}$ | ($\mathrm{T}_\Diamond$) | $\mathcal{A} \to \Diamond\mathcal{A}$ |
| (4) | $\Box\mathcal{A} \to \Box\Box\mathcal{A}$ | ($4_\Diamond$) | $\Diamond\Diamond\mathcal{A} \to \Diamond\mathcal{A}$ |
| (BD) | $\Box(\mathcal{A} \vee \mathcal{B}) \to (\Box\mathcal{A} \vee \Diamond\mathcal{B})$ | (DB) | $(\Diamond\mathcal{A} \wedge \Box\mathcal{B}) \to \Diamond(\mathcal{A} \wedge \mathcal{B})$ |
| (B) | $\mathcal{A} \to \Box\Diamond\mathcal{A}$ | (Nec) | $\mathcal{A} \Rrightarrow \Box\mathcal{A}$ |

Note that the axiomatization given above can be fairly redundant over some relevant logics, especially because we define the $\Diamond$. E.g., over **R**, with a defined dual modality, each $\Diamond$-heavy axiom is equivalent its usual $\Box$-heavy dual. The dual of (DB) we will call (BD), and often refer to these as *modal confinement* axioms.[4]

The following lemma will prove useful, and highlights that the main results of this paper apply to quite an interesting class of relevant logics. Note that the right-hand side is equivalent to the left-hand side prefixed by a modality.

**Lemma 1** *Let* **L** *be a relevant logic extending* $\boldsymbol{B}^P$. *The following formula schemes and rule scheme are derivable in* **L5**:[5]

[4] (BD) and (DB) are named after Dunn and Belnap. The axioms were suggested by Belnap to ensure that **R4** contained all the theorems of **S4** under suitable translation, and they play an important role in Dunn's positive modal logics [7].

[5] Note that the exact modality on the right-hand-side or each biconditional is irrelevant. From $\Diamond\mathcal{C} \to \mathcal{C}$ we obtain $\neg\Box\neg\mathcal{C} \to \mathcal{C}$ using duality, the (K) axiom and 1. Then using contraposition, (T), and double negation equivalence we obtain $\mathcal{C} \to \Box\mathcal{C}$

1. $\neg\Box\mathcal{A} \leftrightarrow \Box\neg\Box\mathcal{A}$
2. $\neg\Diamond\mathcal{A} \leftrightarrow \Box\neg\Diamond\mathcal{A}$
3. $\Box\mathcal{A} \wedge \Box\mathcal{B} \leftrightarrow \Box(\Box\mathcal{A} \wedge \Box\mathcal{B})$
4. $\Box\mathcal{A} \wedge \Diamond\mathcal{B} \leftrightarrow \Box(\Box\mathcal{A} \wedge \Diamond\mathcal{B})$
5. $\Diamond\mathcal{A} \wedge \Diamond\mathcal{B} \leftrightarrow \Box(\Diamond\mathcal{A} \wedge \Diamond\mathcal{B})$
6. $\Diamond\mathcal{A} \vee \Diamond\mathcal{B} \leftrightarrow \Diamond(\Diamond\mathcal{A} \vee \Diamond\mathcal{B})$
7. $\Diamond\mathcal{A} \vee \Box\mathcal{B} \leftrightarrow \Diamond(\Diamond\mathcal{A} \vee \Box\mathcal{B})$
8. $\Box\mathcal{A} \vee \Box\mathcal{B} \leftrightarrow \Diamond(\Box\mathcal{A} \vee \Box\mathcal{B})$
9. $\Box\mathcal{A} \circ \Box\mathcal{B} \leftrightarrow \Box(\Box\mathcal{A} \circ \Box\mathcal{B})$
10. $\Box\mathcal{A} \circ \Diamond\mathcal{B} \leftrightarrow \Box(\Box\mathcal{A} \circ \Diamond\mathcal{B})$
11. $\Diamond\mathcal{A} \circ \Diamond\mathcal{B} \leftrightarrow \Box(\Diamond\mathcal{A} \circ \Diamond\mathcal{B})$
12. $\Box\mathcal{A} \rightarrow \Box\mathcal{B} \leftrightarrow \Diamond(\Box\mathcal{A} \rightarrow \Box\mathcal{B})$
13. $\Box\mathcal{A} \rightarrow \Diamond\mathcal{B} \leftrightarrow \Diamond(\Box\mathcal{A} \rightarrow \Diamond\mathcal{B})$
14. $\Diamond\mathcal{A} \rightarrow \Diamond\mathcal{B} \leftrightarrow \Diamond(\Diamond\mathcal{A} \rightarrow \Diamond\mathcal{B})$
15. $\Box\mathcal{A} \leftarrow \Box\mathcal{B} \leftrightarrow \Diamond(\Box\mathcal{A} \leftarrow \Box\mathcal{B})$
16. $\Box\mathcal{A} \leftarrow \Diamond\mathcal{B} \leftrightarrow \Diamond(\Box\mathcal{A} \leftarrow \Diamond\mathcal{B})$
17. $\Diamond\mathcal{A} \leftarrow \Diamond\mathcal{B} \leftrightarrow \Diamond(\Diamond\mathcal{A} \leftarrow \Diamond\mathcal{B})$
18. $\mathcal{A}^{\Box} \rightarrow \mathcal{B} \Rrightarrow \mathcal{A}^{\Box} \rightarrow \Box\mathcal{B}$

*Note that $\mathcal{A}^{\Box}$ indicates that every propositional variable in $\mathcal{A}$ is in the scope of some modal operator: $\mathcal{A}$ is* modally closed.

***Proof*** We prove several items, with a focus on the more interesting cases. In the following, $\otimes, \oplus \in \{\Box, \Diamond, \neg\Box, \neg\Diamond\}$.

(3)–(5): It is straightforward to show $\Box \otimes \mathcal{A} \wedge \Box \oplus \mathcal{B} = \otimes\mathcal{A} \wedge \oplus\mathcal{B}$.

(9)–(11): consider the follow derivation scheme:

| | | |
|---|---|---|
| 1. | $\otimes\mathcal{A} \rightarrow (\oplus\mathcal{B} \rightarrow \otimes\mathcal{A} \circ \oplus\mathcal{B})$ | Theorem |
| 2. | $\Box \otimes \mathcal{A} \rightarrow (\Box \oplus \mathcal{B} \rightarrow \Box(\otimes\mathcal{A} \circ \oplus\mathcal{B}))$ | 1. (K) (MP) |
| 3. | $\otimes\mathcal{A} \rightarrow (\oplus\mathcal{B} \rightarrow \Box(\otimes\mathcal{A} \circ \oplus\mathcal{B}))$ | 2. Equivalences |
| 4. | $(\otimes\mathcal{A} \circ \oplus\mathcal{B}) \rightarrow \Box(\otimes\mathcal{A} \circ \oplus\mathcal{B})$ | 3. (r$\circ$) |

The other direction of the biconditional is just the (T) axiom.

(12)–(14): consider the following derivation scheme:

| | | |
|---|---|---|
| 1. | $(\otimes\mathcal{A} \rightarrow \oplus\mathcal{B}) \rightarrow (\otimes\mathcal{A} \rightarrow \oplus\mathcal{B})$ | Axiom |
| 2. | $\otimes\mathcal{A} \rightarrow ((\otimes\mathcal{A} \rightarrow \oplus\mathcal{B}) \rightarrow \oplus\mathcal{B})$ | 1. (A16), MP |
| 3. | $\Box \otimes \mathcal{A} \rightarrow \Box((\otimes\mathcal{A} \rightarrow \oplus\mathcal{B}) \rightarrow \oplus\mathcal{B})$ | 2. Nec, $K_{\Box}$, MP |
| 4. | $\Box \otimes \mathcal{A} \rightarrow (\Diamond(\otimes\mathcal{A} \rightarrow \oplus\mathcal{B}) \rightarrow \Diamond \oplus \mathcal{B})$ | 3. $K_{\Diamond}$, Transitivity |
| 5. | $\otimes\mathcal{A} \rightarrow (\Diamond(\otimes\mathcal{A} \rightarrow \oplus\mathcal{B}) \rightarrow \oplus\mathcal{B})$ | 4. Replacement of Provable Equivalents |
| 6. | $\Diamond(\otimes\mathcal{A} \rightarrow \oplus\mathcal{B}) \rightarrow (\otimes\mathcal{A} \rightarrow \oplus\mathcal{B})$ | 5. (A16), MP |

The other direction is just an instance of the $T_{\Diamond}$ axiom.[6]

(15)–(17): There is a similar derivation to the previous cases, but with two applications of the (R$\leftarrow$) rule.

(18): Suppose that there is a proof of $\mathcal{A} \rightarrow \mathcal{B}$ where every propositional variable of $\mathcal{A}$ is in the scope of a modality. By the previous cases in this lemma, $\mathcal{A} = \Box\mathcal{A}$ or $\mathcal{A} = \Diamond\mathcal{A}$ (rather both, as shown in the previous footnote). To generalize, notate this by $\mathcal{A} = \oplus\mathcal{A}$. Then by assumption we obtain a proof of $\oplus\mathcal{A} \rightarrow \mathcal{B}$, and then $\Box \oplus \mathcal{A} \rightarrow \Box\mathcal{B}$ by (K), (Nec), and (MP). But we have that $\Box \oplus \mathcal{A} \leftrightarrow \mathcal{A}$, and so we thus have a derivation of the theorem $\mathcal{A} \rightarrow \Box\mathcal{B}$. □

[6] In logics in which $\circ$ and $\rightarrow$ are interdefinable, such as in **R**, we obtain another route to showing cases (12)–(14):

$$\begin{aligned} \otimes\mathcal{A} \rightarrow \otimes\mathcal{B} &= \neg(\otimes\mathcal{A} \circ \neg \otimes \mathcal{B}) && \text{Definition} \\ &= \neg\Box(\otimes\mathcal{A} \circ \neg \otimes \mathcal{B}) && \text{Applying (9)–(11)} \\ &= \Diamond\neg(\otimes\mathcal{A} \circ \neg \otimes \mathcal{B}) && \text{Duality} \\ &= \Diamond(\otimes\mathcal{A} \rightarrow \otimes\mathcal{B}) && \text{Definition} \end{aligned}$$

Unfortunately, the presence or permutation appears to be necessary over **B**. In fact, not even the related assertion axiom (A14) is sufficient.[7] The reader is left to generate such a countermodel for logics with assertion (and lacking permuation), an exercise which is made simple using MaGIC. We record that absent (both assertion and) permutation we obtain counter-examples.

**Lemma 2** *In the logic* ***E5****, neither both* $\Diamond(\otimes\mathcal{A} \to \oplus\mathcal{B}) \leftrightarrow (\otimes\mathcal{A} \to \oplus\mathcal{B})$ *nor* $\Box(\otimes\mathcal{A} \to \oplus\mathcal{B}) \leftrightarrow (\otimes\mathcal{A} \to \oplus\mathcal{B})$ *are theorems.*

***Proof*** We provide a counterexample found using MaGIC.[8] Take the matrix to consist of the integers 0 through 4, the usual ordering, where $\boldsymbol{t} = 1$. Define the conditional, negation, and modalities as in the following tables, and $a \circ b = \bigwedge\{c : a \leq b \to c\}$:

| $\to$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 2 | 2 | 2 | 2 | 2 |
| 1 | 0 | 1 | 2 | 2 | 2 |
| 2 | 0 | 0 | 2 | 2 | 2 |
| 3 | 0 | 0 | 0 | 1 | 2 |
| 4 | 0 | 0 | 0 | 0 | 2 |

| | $\neg$ | $\Box$ | $\Diamond$ |
|---|---|---|---|
| 0 | 4 | 0 | 0 |
| 1 | 3 | 1 | 1 |
| 2 | 2 | 1 | 3 |
| 3 | 1 | 3 | 3 |
| 4 | 0 | 4 | 4 |

It is easy to check that $\Diamond(\Box\mathcal{A} \to \Box\mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B}) = 0$ when $\mathcal{A} = \mathcal{B} = 0$. Moreover, this matrix validates **E5**, and hence the formula in question is invalid in logics contained in **E5**. The case is similar for $(\Box\mathcal{A} \to \Box\mathcal{B}) \to \Box(\Box\mathcal{A} \to \Box\mathcal{B})$ □

A countermodel can also be given in **B**+(A14).

Near the end of the paper we further discuss this need for permutation, and the possibility of extending **L5** with another axiom scheme (when it lacks permutation).

#### 2.3.1 Semantics for Modal Relevant Logics

Here we follow Seki [27] in defining frames and models for **L5** and **L.5**; however, we do not use bounded frames, as we do not consider extensions which require them. We use both $S_\Diamond$ and $S_\Box$, although strictly speaking only one is required. That is, we define $S_\Diamond$ in terms of $S_\Box$.

**Definition 4** (Frames and Models for **L5** and **L.5**) A **L.5**-frame is a tuple $\mathfrak{F} = \langle W, N, R, *, S_\Box, Prop\rangle$ where $\langle W, N, R, *\rangle$ is an **L**-frame, $Prop$ is a subset of the upsets of the frame, $S_\Box$ is a binary relation, we set $S_\Diamond$ by $S_\Box ab$ iff $S_\Diamond a^*b^*$, and the following conditions hold:

- $(s1)$ $S_\Box bc$ and $a \leq b$ imply $S_\Box ac$
- $(s2)$ $S_\Box aa$
- $(s3)$ $S_\Box ab$ and $a \in N$ imply $b \in N$
- $(s4)$ $S_\Box ab$ and $S_\Box bc$ imply $S_\Box ac$
- $(s5)$ $S_\Box ab$ implies $S_\Box b^*a^*$ (i.e., $S_\Diamond ba$)
- $(s1)'$ $S_\Diamond bc$ and $a \leq b$ imply $S_\Diamond ac$
- $(s2)'$ $S_\Diamond aa$
- $(s4)'$ $S_\Diamond ab$ and $S_\Diamond bc$ imply $S_\Diamond ac$
- $(s5)'$ $S_\Diamond ab$ implies $S_\Diamond b^*a^*$ (i.e., $S_\Box ba$)

[7] (A14) is equivalent to (A16) in the presence of (A11) and (A12).

[8] The author thanks Andrew Tedder for running MaGIC on this and similar cases.

(s6) $\exists x(Rabx \,\&\, S_{\Box}xc)$ implies $\exists x, y(S_{\Box}ax \,\&\, S_{\Box}by \,\&\, Rxyc)$
(s6)′ $\exists x(Rbxd \,\&\, S_{\Diamond}xc)$ implies $\exists x, y(S_{\Box}bx \,\&\, S_{\Diamond}dy \,\&\, Rxcy)$
(p1) $Prop$ contains $N$ and is closed under $\cap, \cup, \neg, \to, \circ, \Diamond, \Box$, where we define:[9]

(i) $\Box X =_{df} \{a \in W : S_{\Box}ab \Rightarrow b \in X\}$
(ii) $\Diamond X =_{df} \{a \in W : S_{\Diamond}ab \,\&\, b \in X\}$

An **L5**-frame is an **L.5**-frame that further satisfies:

(s7) $S_{\Box}ab$ implies $\exists x \leq b(S_{\Box}ax \,\&\, S_{\Diamond}ax)$
(s7)′ $S_{\Diamond}ab$ implies $\exists x \leq b(S_{\Box}ax \,\&\, S_{\Diamond}ax)$

Note that in many logics, particularly logics with an axiom form of contraposition, most of the 'primed' conditions are redundant. In fact, in any logic we consider, some are already redundant.

A ***L5**- or **L.5**-model* is a tuple $\langle \mathfrak{F}, ||-|| \rangle$ where $\mathfrak{F}$ is an **L5**- or**L.5**-frame and $||-||$ is a valuation function that assigns to each propositional variable an element of $Prop$ (and $\boldsymbol{t}$ to $N$). The valuation is extended as above, with the following new cases:

$$||\Box\mathcal{A}|| = \Box||\mathcal{A}|| \qquad\qquad ||\Diamond\mathcal{A}|| = \Diamond||\mathcal{A}||$$

A formula $\mathcal{A}$ is valid in a model when $N \subseteq ||\mathcal{A}||$. Validity in a frame and class of frames is defined as usual. We state the following fact:

**Lemma 3** *The logic **L5** (**L.5**) is sound and complete w.r.t. the class of **L5**-frames (**L.5**-frames).*

For the proof, see Seki [27] (and for logics without $\boldsymbol{t}$ see Fuhrmann [12]).

### 2.4 Theories, Pair Extension, Squeeze

**Definition 5** (Theories) Let $\mathbb{L}$ be a modal or first-order relevant logic (defined above). Where $\Gamma$,$\Delta$, and $\Sigma$ are sets of $\mathbb{L}$-formulas:

(i) $\Gamma \gg_{\mathbb{L}} \Delta$ is defined to mean that there are some $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \Gamma$ and $\mathcal{B}_1, \ldots, \mathcal{B}_m \in \Delta$ such that $(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_n) \vdash (\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m)$ is a theorem of $\mathbb{L}$.
(ii) $\Gamma \gg_{\mathbb{L}} A$ is shorthand for $\Gamma \gg_{\mathbb{L}} \{A\}$
(iii) When $\Gamma \not\gg_{\mathbb{L}} \Delta$, we say the pair $(\Gamma, \Delta)$ is an $\mathbb{L}$-independent pair.
(iv) A set of formulas $\Gamma$ is *an $\mathbb{L}$-theory* when, if $\Gamma \gg_{\mathbb{L}} \mathcal{A}$, then $\mathcal{A} \in \Gamma$.
(v) A theory $\Gamma$ is *prime* if and only if, if $\mathcal{A} \vee \mathcal{B} \in \Gamma$, then either $\mathcal{A} \in \Gamma$ or $\mathcal{B} \in \Gamma$.
(vi) A theory $\Gamma$ is $\mathbb{L}$-*regular* if and only if it contains every theorem of $\mathbb{L}$.
(vii) We define ternary $R'$ and binary $S'_{\Box}$ and $S'_{\Diamond}$ by the following:

(a) $R'\Gamma\Delta\Sigma$ iff $\{\mathcal{A} \circ \mathcal{B} : \mathcal{A} \in \Gamma \,\&\, \mathcal{B} \in \Delta\} \subseteq \Sigma$.
(b) $S'_{\Box}\Gamma\Delta$ iff $\{\mathcal{A} : \Box\mathcal{A} \in \Gamma\} \subseteq \Delta$
(c) $S'_{\Diamond}\Gamma\Delta$ iff $\{\Diamond\mathcal{A} : \mathcal{A} \in \Delta\} \subseteq \Gamma$

[9] It is straightforward to derive that $\neg\Diamond\neg X = \Box X$.

**Lemma 4** *(Extensions, Pair-Extensions, Squeeze)Let $\mathbb{L}$ be either a modal or first-order relevant logic, and let $\Gamma$, $\Delta$, $\Sigma$ be appropriate sets of formulas.*

*(i)* *If $(\Gamma, \Delta)$ is an $\mathbb{L}$-independent pair, then there is a prime $\mathbb{L}$-theory $\Gamma' \supseteq \Gamma$ such that $(\Gamma', \Delta)$ is an $\mathbb{L}$-independent pair.*
*(ii)* *If $(\Gamma, \Delta)$ is an $\mathbb{L}$-independent pair and $\Gamma \cup \Delta = wff$, then $\Gamma$ is prime*
*(iii)* *If $\Sigma$ is prime and $R'\Gamma\Delta\Sigma$, then there exist prime $\mathbb{L}$-theories $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$ such that $R'\Gamma'\Delta'\Sigma$.*
*(iv)* *If $\Sigma$ is a prime $\mathbb{L}$-theory and $\Gamma$ and $\Delta$ $\mathbb{L}$-theories, $R'\Sigma\Gamma\Delta$, and $\mathcal{A} \notin \Delta$, then there is a prime $\mathbb{L}$-theories $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$ such that $R'\Sigma\Gamma'\Delta'$.*
*(v)* *If $\Sigma$ is prime and $\mathcal{A} \to \mathcal{B} \notin \Sigma$, then there exist prime theories $\Gamma$ and $\Delta$ such that $R'\Sigma\Gamma\Delta$ where $\mathcal{A} \in \Gamma$ and $\mathcal{B} \notin \Delta$.*
*(vi)* *Suppose that $S'_{\Box}\Gamma\Delta$ and $\mathcal{A} \notin \Delta$, for prime $\mathbb{L}$-theory $\Gamma$ and $\mathbb{L}$-theory $\Delta$. Then there is a prime $\Delta'$ such that $S'_{\Box}\Gamma\Delta'$.*
*(vii)* *Suppose that $\Gamma$ is a prime $\mathbb{L}$-theory and $\Box\mathcal{A} \notin \Gamma$. Then there is a prime $\mathbb{L}$-theory $\Delta$ such that $\mathcal{A} \notin \Delta$ and $S_{\Box}\Gamma\Delta$ .*

***Proof*** The logics in question are pair extension acceptable (see, e.g., [21, 5.1–5.2] or [1, pp. 123–126]). The remainder of the proof is quite standard in the literature. □

### 2.5 Translations

We define the following two translation functions between first-order and modal languages, to facilitate the proof of equivalence between the first-order and modal relevant logics (with permutation):[10]

$$
\begin{aligned}
h(p) &= P(x) & g(P(x)) &= p\\
h(\boldsymbol{t}) &= \boldsymbol{t} & g(\boldsymbol{t}) &= \boldsymbol{t}\\
h(\mathcal{A} \wedge \mathcal{B}) &= h(\mathcal{A}) \wedge h(\mathcal{B}) & g(\mathcal{A} \wedge \mathcal{B}) &= g(\mathcal{A}) \wedge g(\mathcal{B})\\
h(\mathcal{A} \vee \mathcal{B}) &= h(\mathcal{A}) \vee h(\mathcal{B}) & g(\mathcal{A} \vee \mathcal{B}) &= g(\mathcal{A}) \vee g(\mathcal{B})\\
h(\mathcal{A} \to \mathcal{B}) &= h(\mathcal{A}) \to h(\mathcal{B}) & g(\mathcal{A} \to \mathcal{B}) &= g(\mathcal{A}) \to g(\mathcal{B})\\
h(\mathcal{A} \circ \mathcal{B}) &= h(\mathcal{A}) \circ h(\mathcal{B}) & g(\mathcal{A} \circ \mathcal{B}) &= g(\mathcal{A}) \circ g(\mathcal{B})\\
h(\mathcal{A} \leftarrow \mathcal{B}) &= h(\mathcal{A}) \leftarrow h(\mathcal{B}) & g(\mathcal{A} \leftarrow \mathcal{B}) &= g(\mathcal{A}) \leftarrow g(\mathcal{B})\\
h(\neg\mathcal{A}) &= \neg h(\mathcal{A}) & g(\neg\mathcal{A}) &= \neg g(\mathcal{A})\\
h(\Box\mathcal{A}) &= \forall x h(\mathcal{A}) & g(\forall x\mathcal{A}) &= \Box g(\mathcal{A})\\
h(\Diamond\mathcal{A}) &= \exists x h(\mathcal{A}) & g(\exists x\mathcal{A}) &= \Diamond g(\mathcal{A})
\end{aligned}
$$

It is easy to show that $h(g(\mathcal{A})) = \mathcal{A}$ and $g(h(\mathcal{A})) = \mathcal{A}$. Using these translation functions, we can examine several interesting formulas and their translations. First, let's consider the modal axioms (BD) and (DB), which are used by Dunn [7] to

[10] We match the predicate letters and propositional variables one-to-one, a fact which we will implicitly use.

axiomatize positive modal logics. Take an instance of the axiom (BD):

$$\begin{aligned} h((BD)) &= h(\Box(p \vee q) \to (\Box p \vee \Diamond q)) \\ &= \forall x(p(x) \vee q(x)) \to (\forall x p(x) \vee \exists x q(x)) \end{aligned}$$

That is, $h((BD))$ is very close to the axiom (EC) which added to **QL** results in **LQ**. In the other direction, start with an instance of (EC) in the one-variable fragment:[11]

$$\begin{aligned} g(EC) &= g(\forall x(Px \vee \forall x Q(x)) \to (\forall x Px \vee \exists x \forall x Q(x))) \\ &= \Box(p \vee \Box q) \to (\Box p \vee \Diamond \Box q) \end{aligned}$$

A similar relation between (EC) and (DB) (and their duals) is shown for some intermediate intuitionistic modal and predicate logics, as reported by Ono [20] (see also Suzuki [30]).

A large motivation for examining the one-variable fragment of **RQ** comes from Fine [11], in which **RQ** is shown to be incomplete with respect to the more straightforward way of constructing constant domain ternary relational semantics.[12] Starting with the usual ternary relational frames for propositional relevant logics given above, one obtains constant domain, (non-general) Tarskian frames by adding an set $U$ of individuals, and interpreting the universal quantifier (in the Tarskian way) as the generalized intersection of all 'instances'. Fine showed that these frames determine a logic stronger than **RQ**. Moreover, to this day the following question is unanswered: *what is the logic (axiomatically or proof-theoretically) of these frames*? This question is at least partially difficult because the first clue is given by the formula that Fine shows valid in all of these models, which is invalid in **RQ** and does not wear its meaning on its sleeve. The formula in question is $A_0 \to A_1$, such that

$$\begin{aligned} A_0 &:= (p \to \exists x Ex) \wedge \forall y((p \to Fy) \vee (Gy \to Hy)) \\ A_1 &:= (\forall z((Ez \wedge Fz) \to q) \wedge \forall u((Eu \to q) \vee Gu)) \to (\exists v Hv \vee (p \to q)) \end{aligned}$$

This formula is of interest to us because there are corresponding formulas in the one-variable fragment of **RQ**. By noting the absence of scope overlap, we may replace each propositional variable (zero-ary predicate) with a new sentence (assuming no zero-ary predicates in this fragment), and further write out the formula using a single variable. We thus obtain the following examples:

$$\begin{aligned} A_0' &:= (\forall x Px \to \exists x Ex) \wedge \forall x((\forall x Px \to Fx) \vee (Gx \to Hx)) \\ A_1' &:= (\forall x((Ex \wedge Fx) \to \forall x Qx) \wedge \forall x((Ex \to \forall x Qx) \vee Gx)) \to \\ &\qquad\qquad \to (\exists x Hx \vee (\forall x Px \to \forall x Qx)) \end{aligned}$$

[11] We remind the reader we omit zero-ary predicates from the one-variable fragment, so we make suitable substitutions with appropriate "$\forall x Q(x)$" or "$\exists x Q(x)$".

[12] A thanks to George Metcalfe, whose comments on a related conference presentation prompted this inquiry into the one-variable fragment of relevant logics.

$$A_0'' := (\exists x Px \to \exists x Ex) \wedge \forall x((\exists x Px \to Fx) \vee (Gx \to Hx))$$
$$A_1'' := (\forall x((Ex \wedge Fx) \to \exists x Qx) \wedge \forall x((Ex \to \exists x Qx) \vee Gx)) \to$$
$$\to (\exists x Hx \vee (\exists x Px \to \exists x Qx))$$

These formulas, $A_0' \to A_1'$ and $A_0'' \to A_1''$, henceforth $F'$ and $F''$ are in the one-variable fragment of **RQ** (with ***t*** but no zero-ary predicate symbols).

The reader will note that Fine's formula is strictly not a formula in the one-variable fragment because it contains propositional variables (i.e. zero-ary predicates), and similarly for the the (scheme of) (EC). However, taken as a formula scheme, Fine's formula has many instantiations in the one-variable fragment, of which we have displayed but two.

The formula $F'$, a one-variable variant of Fine's formula above, can be translated into a modal language using the translation function $g$.[13] As a result, we obtain $g(F') = g(A_0') \to g(A_1')$, where

$$g(A_0') := (\Box P \to \Diamond E) \wedge \Box((\Box P \to F) \vee (G \to H))$$
$$g(A_1') := (\Box((E \wedge F) \to \Box Q) \wedge \Box((E \to \Box Q) \vee G)) \to (\Diamond H \vee (\Box P \to \Box Q))$$

Similarly, for $F''$ we obtain the following:

$$g(A_0'') := (\Diamond P \to \Diamond E) \wedge \Box((\Diamond P \to F) \vee (G \to H))$$
$$g(A_1'') := (\Box((E \wedge F) \to \Diamond Q) \wedge \Box((E \to \Diamond Q) \vee G)) \to (\Diamond H \vee (\Diamond P \to \Diamond Q))$$

One significant difference between first order relevant and first-order classical logic is the lack of theoremhood for two formula (schemes) required for prenex normal forms. This was shown by Meyer [18, p. 278], who also shows why the formulas in question should not be valid in a relevant logic. These formulas are the following:

$$K_1' := (p \to \exists x Fx) \to \exists x(p \to Fx)$$
$$K_2' := (\forall x Fx \to p) \to \exists x(Fx \to p)$$

To put these into the first-order fragment with no zero-ary predicates, we replace $p$ with $\forall x Px$, obtaining $K_1$ and $K_2$. By applying the translation above to the latter we obtain the following modal formulas:[14]

$$g(K_1) := (\Box p \to \Diamond f) \to \Diamond(\Box p \to f)$$
$$g(K_2) := (\Box f \to \Box p) \to \Diamond(f \to \Box p)$$

13 So far, the author has been unable to find a (finite) counter-example to these translated formulas in **R5**, with the aid of MaGIC. That there is a counter-example follows as a corallary from the main result of the paper. Fine's proof that **RQ** does not imply $F$, however, relies on an infinte counter-example.

14 We can, e.g., use the matrix for **E5** above to show that these formulas are not valid. For $g(K_1)$ set $p = 3$, $f = 2$; for $g(K_2)$ set $p = f = 2$.

We could also replace $p$ with $\exists x Px$, giving $K_1^\Diamond$ and $K_2^\Diamond$. The modal translation outputs the following:

$$g(K_1^\Diamond) := (\Diamond p \to \Diamond f) \to \Diamond(\Diamond p \to f)$$
$$g(K_2^\Diamond) := (\Box f \to \Diamond p) \to \Diamond(f \to \Diamond p)$$

### 2.6 (EC) and h((BD))

**Lemma 5** *(EC) and h((BD)) are equivalent over **QB** (=**BQ**−(EC)), in the one-variable fragment.*

***Proof*** (EC) implies h((BD)):

| | | |
|---|---|---|
| (1) | $\forall x(Ax \vee Bx) \to (Ax \vee Bx)$ | (∀E) |
| (2) | $(Ax \vee Bx) \to (Ax \vee \exists x Bx)$ | Theorem |
| (3) | $\forall x(Ax \vee Bx) \to \forall x(Ax \vee \exists x Bx)$ | (1), (2), Trans, (r∀I) |
| (4) | $\forall x(Ax \vee Bx) \to (\forall x Ax \vee \exists x Bx)$ | (3), $(EC)$, Trans |

h((BD)) implies (EC):

| | | |
|---|---|---|
| (1) | $\forall x(Ax \vee \exists x Bx) \to (\forall x Ax \vee \exists x \exists x Bx)$ | Instance of h((BD)) |
| (2) | $(\forall x Ax \vee \exists x \exists x Bx) \to (\forall x Ax \vee \exists x Bx)$ | Theorem |
| (3) | $\forall x(Ax \vee \exists x Bx) \to (\forall x Ax \vee \exists x Bx)$ | (1),(2), Trans |

□

## 3 Equivalence

The first main theorem of the paper is that, for any logic **L** extending $\mathbf{B}^P$ (henceforth any permuting logic), for any first order formula $\mathcal{A}$, $\mathcal{A}$ is valid in the class of frames for **QL** (**LQ**) if and only if its translation $g(\mathcal{A})$ is valid in the class of frames for **L.5** (**L5**).

### 3.1 Modal validity implies first-order validity

For this subsection we always suppose that the logic $L$ is a permuting logic extending **B**. For any formula $\mathcal{A}$ in the one-variable fragment that is not a theorem of **LQ** (**QL**), there is a model on which that formula is not satisfied. This implies that there is a canonical model construction for a language that contains only the predicate letters occurring in the formula (and denumerably many constants) which also fails to satisfy the formula in question. We turn this canonical model into a model for **L5** (**L.5**), and this model will fail to satisfy the modal translation of the formula $\mathcal{A}$. The canonical model construction is employed to find a workaround to a problem of merely transforming the original first-order model that fails to satisfy the formula. This is due to condition (s7), which is existentially hungry. In the constructed model, (s7) is shown to hold via arguments similar to those employed first in Mares and Meyer [17] (and modified for the multi-relation frames in Ferenz [9] based on the work of Seki [26, 27]).

The following lemma is recorded, and is a mere corollary of the usual completeness proof.

**Lemma 6** *Given any first-order formula $\mathcal{A}$ in the one-variable fragment, if $\mathcal{A}$ is not valid in the class of **LQ**-frames (**QL**-frames) — i.e. there is an **LQ**-model (**QL**-model) in which it is not valid — then there is a canonical model for the language restricted to the predicates occurring in $\mathcal{A}$. In particular, we note the following:*

1. *$W$ is the set of prime theories; $N$ of regular prime theories;*
2. *$\langle a, b, c\rangle \in R$ iff $\{\mathcal{A} \circ \mathcal{B} : \mathcal{A} \in a \;\&\; \mathcal{B} \in b\} \subseteq c$;*
3. *$a^* = \{\mathcal{A} : \neg\mathcal{A} \notin a\}$;*
4. *$U$ is an infinite set of constants;*
5. *For closed $\mathcal{A}$, $||\mathcal{A}||^{\forall} = \{a \in W : \mathcal{A} \in a\}$;*[15]
6. *$Prop = \{||\mathcal{A}||^{\forall} : \mathcal{A}$ is closed$\}$;*
7. *$PropFun$ is the set of functions $\varphi_{\mathcal{A}} : U^{\omega} \longrightarrow Prop$, defined by $\varphi_{\mathcal{A}} f = ||\mathcal{A}^f||^{\forall}$, where $\mathcal{A}$ is a formula;*[16]
8. *For every formula $\mathcal{A}$, $\mathcal{A}^f \in a$ iff $a \in |\mathcal{A}|f$ (i.e., $||\mathcal{A}^f||^{\forall} = |\mathcal{A}|f$).*

**Lemma 7** *Given any modal formula $\mathcal{A}$, if $\mathcal{A}$ is valid in the class of **L5**-frames (**L.5**-frames), then $h(\mathcal{A})$ is valid on the class of **LQ**-frames (**QL**-frames). That is, if $\vdash_{\boldsymbol{L5}} \mathcal{A}$ ($\vdash_{\boldsymbol{L.5}} \mathcal{A}$), then $\vdash_{\boldsymbol{LQ}} h(\mathcal{A})$ ($\vdash_{\boldsymbol{QL}} h(\mathcal{A})$).*

***Proof*** Suppose that $h(\mathcal{A})$ is not valid in the class of **LQ**-frames (**QL**-frames), and that we fix $x$ as the variable in the one-variable fragment. There there is a model $\mathfrak{M}''$ and variable assignment $f$ such that $h(\mathcal{A})$ is not satisfied by $f$ on $\mathfrak{M}''$. By Lemma 6, there is a canonical model construction $\mathfrak{M}'$ on which $h(\mathcal{A})$ is not valid. Fix $f$ to be a variable assignment where $h(\mathcal{A})$ is not satisfied by $f$ on $\mathfrak{M}'$.

We construct a modal model essentially by transforming $\mathfrak{M}'$ (modulo the variable assignment $f$). We construct the following model:

1. $W = W'$; $N = N'$; $R = R'$; $* = *'$;
2. $S_{\Box} = \{\langle a, b\rangle | \forall \mathfrak{B} \in \mathfrak{L}_{(x)}((\forall x\mathcal{B})^f \in a \Rightarrow \mathcal{B}^f \in b)\}$;
3. $S_{\Diamond} = \{\langle a, b\rangle | \forall \mathfrak{B} \in \mathfrak{L}_{(x)}(\mathcal{B}^f \in b \Rightarrow (\exists x\mathcal{B})^f \in a)\}$;
4. $Prop = PropFun'(f)$;[17]
5. $||p|| = |p(x)|f$, for each atomic $p$; the valuation is extended as usual.

We aim to show two facts: that the defined structure is a well-defined **L5**-model (**L.5**-model), and that $a \in ||h(\mathcal{A})||$ iff $h(\mathcal{A}) \in a$. Hence the following sublemmas. □

**Lemma 8** *The frame of the defined $\mathfrak{M}$ is an **L5**-frame (**L.5**-frame).*

[15] We use the notation $\mathcal{A}$, $||\mathcal{A}||^{\forall}$ to differentiate the case of truth sets for propositional and modal logics.

[16] For any formula $\mathcal{A}$ and $f \in U^{\omega}$, $\mathcal{A}^f$ is the result of simultaneously replacing every variable $x_n$ with the constant $fx_n$: i.e., it is a sentencification of $\mathcal{A}$ through the assignment $f$.

[17] $PropFun'(f)$ is defined to $\{X \in Prop' | X = \varphi f$, for some $\varphi \in PropFun'\}$.

***Proof*** It is straightforward and quite standard to show that the underlying frame is an **L**-frame and that the subset relation on theories is the $\leq$ relation on the model. $Prop$ is indeed a subset of upsets, and the $S_{\Box}$ and $S_{\Diamond}$ relations are binary on $W$. It is also easy to show that $S_{\Box}ab$ iff $S_{\Diamond}a^*b^*$. It remains to show (s1)–(s6) and ($p1$) hold, and, when we have **LQ**, that (s7) holds.

Because $\leq$ is the subset relation among theories, (s1) holds. Condition (s2) follows from the fact that $|\forall x\mathcal{B}|f \subseteq |\mathcal{B}|f$, by ($\forall$E). Assuming the antecedents of (s3), by the derivable (UG), $\forall x\mathcal{B} \in a$ for each theorem $\mathcal{B}$, and so each theorem is in $b$, as required. If $\forall x\mathcal{B} \in a$, then $\forall x\forall x\mathcal{B} \in a$ by (UG). By the antecedents of (s4), we may obtain that $\forall x\mathcal{B} \in b$ and then $\mathcal{B} \in c$, ensuring that (s4) holds. For (s5) suppose that $S_{\Box}ab$ and $b^* \in |\forall x\mathcal{B}|f$. Then $b \notin |\neg\forall x\mathcal{B}|f$. Thus $a \notin |\forall x\neg\forall x\mathcal{B}|f = |\neg\forall x\mathcal{B}|f$. And so $a^* \in |\forall x\mathcal{B}|f$, and thus $a^* \in |\mathcal{B}|f$ (by ($\forall$E)), as required.

For (s6) suppose that $Rabx$ and $S_{\Box}xc$. Let $a'' = \{\mathcal{A} : \forall x\mathcal{A} \in a\}$ and $b'' = \{\mathcal{A} : \forall x\mathcal{A} \in b\}$. By using the theorems $\forall x(\mathcal{A} \wedge \mathcal{B}) \leftrightarrow \forall x\mathcal{A} \wedge \forall x\mathcal{B}$ and $\forall x(\mathcal{A} \to \mathcal{B}) \to (\forall x\mathcal{A} \to \forall x\mathcal{B})$, we can show that $a''$ and $b''$ are **LQ**-theories (**QL**).[18] Suppose that $\mathcal{A} \in a''$and $\mathcal{B} \in b''$. Then $\forall x\mathcal{A} \in a$ and $\forall x\mathcal{B} \in b$. Thus $\forall x\mathcal{A} \circ \forall x\mathcal{B} \in x$. By the theoremhood of $\forall x\mathcal{A}\circ\forall x\mathcal{B} \to \forall x(\mathcal{A}\circ\mathcal{B})$, we have $\forall x(\mathcal{A}\circ\mathcal{B}) \in x$, which entails that $\mathcal{A} \circ \mathcal{B} \in c$. This is that $Ra''b''c$, when restricted to $R$ is liberally applied to theories. By applying Lemma 4, we extend $a''$ and $b''$ to prime theories $a'$ and $b'$, respectively, such that the consequent of (s6) holds.

For (s7), suppose that $S_{\Box}ab$. We show that there is an prime **LQ**-theory $c \leq b$ such that, for every $\mathcal{B}$, (i) if $a \in |\forall x\mathcal{B}|f$ then $c \in |\mathcal{B}|f$ and (ii) if $c \in |\mathcal{B}|f$ then $a \in |\exists x\mathcal{B}|f$.[19] Consider the pair $(\forall^{-1}a, (\text{wff} - b) \cup (\text{wff} - \exists^{-1}a))$, where $\forall^{-1}a = \{\mathcal{B} \in \text{wff} | \forall x\mathcal{B} \in a\}$ and $\exists^{-1}a = \{\mathcal{B} \in \text{wff} | \exists x\mathcal{B} \in a\}$. We first show that this is an independent pair. Suppose that it is not. Then there are $\mathcal{A}_1, \ldots, \mathcal{A}_m \in \forall^{-1}a$, and $\mathcal{B}_1, \ldots, \mathcal{B}_n \in (\text{wff} - b)$, and $\mathcal{C}_1, \ldots, \mathcal{C}_p \in (\text{wff} - \exists^{-1}a)$ where $m, n + p \geq 1$ such that:

$$\vdash_{\mathbf{LQ}} \mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_m \to \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_n \vee \mathcal{C}_1 \vee \cdots \vee \mathcal{C}_p$$

The following steps subsequently follow:

$$\begin{aligned} &\vdash_{\mathbf{LQ}} \mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_m \to \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_n \vee \exists x(\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_p) \\ &\vdash_{\mathbf{LQ}} \forall x(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_m \to \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_n \vee \exists x(\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_p)) \\ &\vdash_{\mathbf{LQ}} \forall x(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_m) \to \forall x(\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_n \vee \exists x(\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_p)) \end{aligned}$$

By extensional confinement, we also have the theorem:

$$\vdash_{\mathbf{LQ}} \forall x(\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_n \vee \exists x(\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_p)) \to (\forall x(\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_n) \vee \exists x(\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_p))$$

and so

$$\vdash_{\mathbf{LQ}} (\forall x(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_m) \to (\forall x(\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_n) \vee \exists x(\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_p))$$

[18] E.g., suppose that $\mathcal{A} \in a''$ and that $\mathcal{A} \to \mathcal{B}$ is a theorem. Then it is also a theorem that $\forall x\mathcal{A} \to \forall x\mathcal{B}$. By the primness of $a$, and the fact that $\forall x\mathcal{A} \in a$, $\mathcal{B} \in a''$, as required.

[19] The proof is similar to that found in [17] for (s7) for the canonical model constructed using modal theories. Here our models are defined using **LQ**-theories, so we present the arguments in full.

The formula $(\forall x(\mathcal{A}_1 \wedge \cdots \wedge \mathcal{A}_m) \in a$, because each $\mathcal{A}_i \in \forall^{-1}a$. By the primeness of $a$, either $\forall x(\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_n) \in a$ or $\exists x(\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_p) \in a$. The former implies that $\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_n \in b$, which contradicts our assumptions because $b$ is prime. The latter implies $\exists x\mathcal{C}_1 \vee \exists x\mathcal{C}_p \in a$. Primeness implies one of the disjuncts is in $a$, which contradicts out assumptions. Thus, the pair $(\forall^{-1}a, (\text{wff} - b) \cup (\text{wff} - \exists^{-1}a))$ is an **LQ**-independent pair. By Lemma 4, we can extend $\forall^{-1}a$ to a prime theory $c$ where $(c, (\text{wff} - b) \cup (\text{wff} - \exists^{-1}a))$ is an independent pair. It is straightforward that $S_{\Box}ac$. Suppose that $c \in |\mathcal{B}|f$ and $a \notin |\exists x\mathcal{B}|f$. But such formulas were excluded from $c$ in the application of the pair extension lemma, thus $S_{\Diamond}ac$.

For (p1), suppose that $X, Y \in Prop$. Then $X = (\varphi')f$ and $Y = (\psi')f$ for some $\varphi, \psi \in PropFun'$. For the non-modal cases, it is sufficient to show that $(\varphi')f \otimes (\psi')f = (\varphi' \otimes \psi')f$, given the definition of $Prop$, for each binary connective $\otimes$, and similarly for each unary connective. indeed, the non-modal cases are covered by the canonical model construction. We consider the case for modality.

We show that $\Box X = |\forall_x \varphi'|f$. For the left-to-right direction, suppose that $a \in \Box X$. Then $S_{\Box}ab$ implies $b \in X$, for each $b \in W$. For reductio, suppose that $a \notin |\forall_x \varphi'|f = ||\forall x\varphi'||^{\forall}$. Consider the set $c = \{\mathcal{A} : \forall x\mathcal{A} \in a\}$. We have that $S_{\Box}ac$, when the relation is relaxed to theories. Note that $(c, \{(\varphi')^f\})$ is an independent pair: if it were not, then it is straightforward to show that $\Box(\varphi')^f \in a$. By Lemma 4.(i), we can extend $c$ to a prime theory $d$ such that $S_{\Box}ad$ and $(\varphi')^f \notin d$. The former, with our first assumption, contradicts the latter, completing the reductio.

The converse is direction is nearly trivial: suppose that $a \in |\forall_x\varphi'|f = ||\forall x\varphi'||^{\forall}$. Thus $(\forall x\varphi')^f \in a$. Using $S_{\Box}ab$ we then obtain $b \in (\varphi')f$, as required. □

Thus we have an **L5**-frame, or **L.5**-frame, as desired. To show that this frame is a model, note that each atomic proposition is mapped to an element of $Prop$ (and $\boldsymbol{t}$ to $N$). The valuation is extended as usual, and this is well-defined given the frame satisfies (p1), as shown in the previous sub-lemma. We now show the crucial sub-lemma.

**Lemma 9** *(Truth Lemma) Where $f \in U^{\omega}$ is the variable assignment on which the frame is based, for any modal formula $\mathcal{A}$: $a \in ||\mathcal{A}||$ iff $(h(\mathcal{A}))^f \in a$. (That is, $a \vDash_{\boldsymbol{L5/L.5}} \mathcal{A}$ iff $(h(\mathcal{A}))^f \in a$ or equivalently $||\mathcal{A}|| = ||(h(\mathcal{A}))^f||^{\forall}$.)*

***Proof*** The proof is by induction on the complexity of $\mathcal{A}$. First note that, given the canonical model construction for **LQ** (**QL**), $a \in ||(h(\mathcal{A}))^f||^{\forall}$ iff $(h(\mathcal{A}))^f \in a$ iff $a \in |h(\mathcal{A})|f$. For the base case, suppose that $\mathcal{A} = p$ (and $h(\mathcal{A}) = p(x)$.) Then $||p|| = |h(p)|f$ by the definition of the valuation, and by the previously stated fact we obtain $a \in ||p||$ iff $(p(x))^f \in a$, as required.

Using the arguments of the **LQ**- (**QL**)-construction, the inductive cases excepting the quantifier-modality cases are all covered straightforwardly. We record only the case for $\forall/\Box$.

Suppose $\mathcal{A} = \Box\mathcal{B}$. Then $||\Box\mathcal{B}|| = \Box||\mathcal{B}||^{\forall}$. By the inductive hypothesis, we have $||\mathcal{B}|| = ||(h(\mathcal{B}))^f|| = |h(\mathcal{B})|f$. By the previous lemma, we have that $\Box||\mathcal{B}||^{\forall} = |\forall_x h(\mathcal{B})|f$ and given that $x$ is the only variable that may occur in $\mathcal{B}$, we have $\forall_x h(\mathcal{B}) = h(\forall_x\mathcal{B})$, which gives us $||\Box\mathcal{B}|| = ||(h(\forall_x\mathcal{B}))^f||^{\forall}$ completes the case.

By assumption, there is a prime, regular **LQ**-theory (**QL**-theory) $a$ such that $a \notin |h(\mathcal{A})|f$. By the Truth Lemma, we have that $a \nvDash \mathcal{A}$ (rather, $a \notin ||\mathcal{A}||$). Because the constructed structure is indeed a model, this shows that $\mathcal{A}$ is neither valid in the class of **L5**-frames (**L.5**-frames), nor a theorem of **L5** (**L.5**). Thus completes this main lemma of this section.

### 3.2 First-order validity implies modal validity

**Lemma 10** *Given any first-order formula $\mathcal{A}$ in the one-variable fragment, if $\mathcal{A}$ is a theorem of **LQ** (**QL**), then $g(\mathcal{A})$ is valid in the class of **L5**-frames (**L.5**-frames), and by Fact 3 also a theorem of **L5** (**L.5**)). That is, if $\vdash_{\mathbf{LQ}} \mathcal{A}$ ($\vdash_{\mathbf{QL}} \mathcal{A}$), then $\vdash_{\mathbf{L5}} g(\mathcal{A})$ ($\vdash_{\mathbf{L.5}} g(\mathcal{A})$).*

***Proof*** The main idea of the proof is as follows: given a countermodel for the modal sentence $g(\mathcal{A})$, we evaluate the entirety of **LQ** (**QL**) into this model. The idea is to map $\mathcal{A}$ onto $g(\mathcal{A})$, when $\mathcal{A}$ is in the $x$-fragment (thus guaranteeing the valuation will invalidate $\mathcal{A}$), while simultaneously mapping *at least* all of the remaining theorems of **LQ** (**QL**) onto modal theorems. We will overshoot, making many of the non-$x$-fragment formulas in **LQ** (**QL**) valid, but such method is sufficient because all of **LQ** (**QL**) is valid and $\mathcal{A}$ is not.

First we define an ‘extension’ of $g$ that will map each formula of **LQ** (**QL**) onto a modal formula. We define such a translation, $g^x(\mathcal{A})$, as follows:

1. If the formula contains any non-$x$ variable:
   (a) Remove all quantifiers from the formula,
   (b) Replace each atomic formula $P(y)$, even when $x = y$, with $\forall x P'x$, where $P'$ is a new predicate letter not appearing in $\mathcal{A}$;
2. Apply $g$ to the resulting formula, which is clearly in the $x$-fragment.

In other worlds, from the viewpoint of $x$, the modal translation of atomic formulas containing other variables *might as well be new pseudo-atomic propositions.* The new predicate letter and the closure under a ‘$\forall x$’ with minimal scope ensure a kind of ‘atomic’ behavior with respect to the lack of interaction with the $x$-fragment’s translation.

We then define a modal as follows. Given a **L5** (**L.5**) countermodel $M'$ of $g(\mathcal{A})$, define a new valuation (of the formulas of **LQ** (**QL**)) into that very model. This model $M$ is defined by giving a valuation function $|{-}|$ from first-order formulas into $Prop'$ (where $Prop'$ is of $M'$) by the following:

$$|\mathcal{A}| = ||g^x(\mathcal{A})||$$

Given this model $M$, we show that it is appropriate for **LQ** (**QL**) (i.e., makes all of **LQ**’s (**QL’s**) theorems true, despite not being an **LQ**-model (**QL**-model) as defined above), and that it does not validate the formula $\mathcal{A}$ in the statement of the lemma (i.e., that $N \not\subseteq |\mathcal{A}|$).

**Lemma 11** *Every theorem of **LQ** (**QL**) is valid on $M$ (defined on a **L5**-model (**L.5**-model)).*

***Proof*** The proof uses a fairly standard method for soundness: induction on the length of a proof. We show that every axiom is valid in $M$ and that the rules preserve validity. The cases of the axioms of **L** are straightforward, because $g^x$ will affect identical subformulas in identical ways. We show the cases for (MP) and some of the explicitly first-order axioms and rules.

Case (MP): Suppose that $\vdash_{\mathbf{LQ}} \mathcal{A}$ and $\vdash_{\mathbf{LQ}} \mathcal{A} \to \mathcal{B}$. By the induction hypothesis, $N \subseteq ||g^x(\mathcal{A})||, ||g^x(\mathcal{A} \to \mathcal{B})||$. But then $N \subseteq ||g^x(\mathcal{B})||$ by the properties of $M$.

Case (∀E): Consider $\mathcal{B} = \forall y(\ldots Py \ldots) \to (\ldots Pz \ldots)$ with any non-$x$ variable. Either $y$ and $z$ are the same variable or they are not. In either case $g^x\mathcal{B}$ is an instance of $\mathcal{A} \to \mathcal{A}$. On the other hand, suppose that the formula is in the one-variable fragment (and thus $x = y = z$. Then it is easy to check the translation to be an instance of (T). In every case, the translation of an instance of (∀E) is a theorem of the modal logic **L.5**, and thus valid in $M$.

Case (EC1): We again deal with every combination of variables. Each axiom instance has a required quantifier in its statement. If this quantifier is not $x$, then we trivially get an instance of $\mathcal{A} \to \mathcal{A}$, again because $g^x$ affects identical subformulas identically. Now, if the variable in question is $x$, $g^x\mathcal{B} = g\mathcal{B}$, which is $\Box(\mathcal{A}^\Box \vee \mathcal{B}) \to (\mathcal{A}^\Box \vee \Box\mathcal{B})$, itself a theorem of **L5** (and not of **L.5**).

Case (R∀I): Suppose that $\vdash \mathcal{A}^y \to \mathcal{B}$. We know that $g^x(\mathcal{A}^y)$ is modally closed if $x$ does not occur free in $\mathcal{A}^y$. Assume that $x$ does not occur free in $A^y$ Since we have $N \subseteq ||g^x(\mathcal{A}^y \to \mathcal{B})|| = ||g^x(\mathcal{A}^y) \to g^x(\mathcal{B}))||$, the modal closure along with Lemma 1 entails that $N \subseteq ||g^x(\mathcal{A}^y) \to \Box g^x(\mathcal{B}))||$.

Now, $g^x(\mathcal{A}^y \to \forall y\mathcal{B})$ depends on whether $y = x$. If $y = x$, then the already demonstrated $N \subseteq ||g^x(\mathcal{A}^y) \to \Box g^x(\mathcal{B}))||$ suffices. If $y \neq x$, then $N \subseteq ||g^x(\mathcal{A}^y \to \mathcal{B})||$ is sufficient, because $g^x(\mathcal{A}^y \to \mathcal{B}) = g^x(\mathcal{A}^y \to \forall y\mathcal{B})$.

Now, if $x$ does occur free in $\mathcal{A}^y$. Then $x \neq y$. The result follows from $g^x(\mathcal{A}^y \to \mathcal{B}) = g^x(\mathcal{A}^y \to \forall y\mathcal{B})$. □

**Lemma 12** *If $g(\mathcal{A})$ is invalid on $M'$ (and hence, $\mathcal{A} \in \mathfrak{L}_{(x)}$), then $\mathcal{A}$ is invalid on $M$.*

***Proof*** Suppose that $g(\mathcal{A})$ is invalid on $M'$. That is, $N \not\subseteq ||g(\mathcal{A})||$. However, $||g(\mathcal{A})|| = ||g^x(\mathcal{A})||$, and so $N \not\subseteq |\mathcal{A}|$, as required. □

Putting the previous sublemmas in place, suppose for reductio that $\mathcal{A} \in \mathfrak{L}_{(x)}$ is a theorem of **LQ** (**QL**), but that $g(\mathcal{A})$ is not a theorem of **L5** (**L.5**). The model construction above entails that $\mathcal{A}$ is valid in $M$ (Lemma 11), but this contradicts the fact that $\mathcal{A}$ is not valid in $M$ (Lemma 12). We this contradiction we complete the lemma. □

### 3.3 Conclusion

By combining Lemmas 10 and 7, we obtain our goal, which we record in the following Theorem.

**Theorem 13** (Semantic Equivalence) *Where **L** is a* permuting *propositional relevant logic extending **B**:*

1. $\vDash_{\mathbf{LQ}^x} \mathcal{A}$ *iff* $\vDash_{\mathbf{L5}} h(\mathcal{A})$

2. $\vDash_{QL^x} \mathcal{A}$ *iff* $\vDash_{L.5} h(\mathcal{A})$

Note that, in one direction, we have to detour through the canonical model construction (at least for the logics with (EC1)/(EC2) and (DB)/(BD)). It appears to the author that this is a requirement, given the frame condition for (DB)/(BD), its existential hunger, and its striking difference from (cEC1)/(cEC2) in the first-order models. It would an improvement to show this result without the canonical detour.

Note that permutation is needed in these proofs, so that the modal logic 'matches' the vacuous quantification in the corresponding first-order logic. Permutation is one route to ensuring this matching, but another would be to require that $\mathcal{A} \leftrightarrow \Box\mathcal{A}$ and $\mathcal{A} \leftrightarrow \Diamond\mathcal{A}$, when $\mathcal{A}$ is modally closed. This is not the route we opted for, but might be a more natural route to take in the algebraic setting.

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