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Exact Truthmaker Semantics for Modal Logics

Dongwoo Kim[1]

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Abstract
The present paper attempts to provide an exact truthmaker semantical analysis of modalized propositions. According to the present proposal, an exact truthmaker for "Necessarily P" is a state that bans every exact truthmaker for "Not P", and an exact truthmaker for "Possibly P" is a state that allows an exact truthmaker for P. Based on this proposal, a formal semantics will be developed; and the soundness and completeness results for a well-known family of the systems of normal modal propositional logic will be established. It shall be seen that the present analysis offers an exactification of the standard Kripke semantics in the sense that it analyzes the accessibility relation between possible worlds in terms of the banning and allowing relations between the constituent states, and thereby gives an account of "truth at a possible world" in terms of exact truthmaking.

Keywords Modal logic · Truthmaker semantics · Kripke semantics · Possible world · Exactification

1 Introduction

The present paper attempts to provide an exact truthmaker semantical analysis of modalized propositions. A *truthmaker* for a proposition P is a state in virtue of which P is true. A truthmaker for P is said to be *exact* if it is entirely relevant to the truth of P, and *inexact* if it contains as part an exact truthmaker for P. What then is an exact truthmaker for $\Box P$ ("Necessarily P") and for $\Diamond P$ ("Possibly P)? The present proposal is that an exact truthmaker for $\Box P$ is a state that *bans*, or *precludes*, the exact truthmakers for $\neg P$ ("Not P"), and an exact truthmaker for $\Diamond P$ is a state that *allows*, or *countenances*, an exact truthmaker for P. Based on this analysis, a formal semantics will be developed; and the soundness and completeness results for the system K of

✉ Dongwoo Kim
dw.kim@kaist.ac.kr

1 The School of Digital Humanities and Computational Social Sciences, Korea Advanced Institute of Science and Technology, Daejeon, South Korea

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modal logic will be established. The results will then be extended to a well-known family of the systems of normal modal propositional logic, such as D, T, 4, B, K4, S4, and S5. Along the way, it shall be seen that the present semantics offers an *exactification* of the Kripke semantics [5, 6] in Fine's [10] sense: it analyzes the accessibility relation between possible worlds in terms of the banning and allowing relations between the constituent states, and thereby gives an account of "truth at a world" in terms of exact truthmaking.

2 Basic Ideas of Truthmaker Semantics

It may help to begin with a brief review of the basic ideas of truthmaker semantics. The *truthmaker principle* states that every true proposition is made true by something [2, 20]. That something by which a proposition is made true is called a *truthmaker*, or *verifier*, for the proposition. For example, consider the proposition that the Empire State Building is between 33rd and 34th Streets in Manhattan. This proposition is true because of the presence of the building in the designated location. This may thus reasonably be said to be a verifier for the proposition.

Of false propositions, there are two different treatments. According to the *unilateral* truthmaker principle, a false proposition is one that has no verifiers. On the other hand, the *bilateral* truthmaker principle states that every false proposition is made false by something, i.e., a *falsemaker*, or *falsifier*. For example, the proposition that the Empire State Building is on 42nd Street in Manhattan is made false by its absence from that area. Without further argument, we shall adopt the bilateral truthmaker principle.[1]

We shall generically call a *state* whatever plays the role of a verifier or falsifier. It is standardly assumed that states have mereological structure; that is, a state may be part of another, and any two states can be put together into a single state, i.e., their *fusion* [13]. We shall also assume that every state is part of itself. When a state s is part of another state t, we shall say that t *extends* s. Aside from this, we shall take an abstract approach to states and make no special assumptions about their nature. It is important to note that we are *not* working with the factual conception of statehood according to which a state is something that in fact obtains. For we would like to say, for example, that the proposition that the Empire State Building is on 42nd Street in Manhattan—which is in fact false—would be verified by the presence of the building there—a state which does not in fact obtain. We shall also allow "impossible" states, such as the presence and absence of the Empire State Building between 33rd and 34th Streets. Strictly speaking, we should say that a verifier for a proposition is a state that *would* make the proposition true if it obtained; and similarly for the notion of a falsifier.

A verifier for a proposition P is said to be *exact* if it is entirely relevant to the truth of P; and *inexact* if it is partially relevant. Note that the notion of inexact verification can be defined in terms of exact verification and the parthood relation between states thus: a state s is an *inexact verifier* for a proposition P if and only if s extends an exact verifier for P; and similarly for the notion of inexact falsification. We do not

[1] See Fine [12] for a discussion on some of the issues relating to the unilateral and bilateral principles.

require an inexact verifier for P to be a proper extension of an exact verifier. So, every exact verifier for P is also an inexact verifier. Like remarks apply to exact and inexact falsifications.

By way of illustration, consider the presence of the Empire State Building between 33rd and 34th Streets in Manhattan. This state can reasonably be taken to be an exact verifier for the proposition that the Empire State Building is between 33rd and 34th Streets in Manhattan. And any state extending it, such as the presence of the building together with five pigeons on the top, would be considered an inexact verifier for the proposition; for only some part of it is relevant to the truth of the proposition. Here one might ask how exactly the notion of relevance is supposed to be understood. This question, though of philosophical interest, will not be important for the purpose of this paper. So, we shall content ourselves with this illustrative example and leave the notion of relevance at an intuitive level.

3 Exact Truthmaker Semantical Analysis of Modalized Propositions

We shall now turn to our main question: what are exact verifiers and falsifiers for modalized propositions, i.e., propositions of the forms $\Box P$ and $\Diamond P$? It might be in better accordance with our abstract approach to the nature of states to ask what the logical functions of exact verifiers and falsifiers for modalized propositions should be.

The informal basis of the present analysis is that some states may have implications on the modal status of others. We shall call such states *modal states*. For an intuitive example of a modal state, let s be the identity of water to H_2O and t be a state in which water is composed of XYZ molecules. Putting some philosophical complications aside, t is metaphysically impossible because of s; or, equivalently, s makes t metaphysically impossible. For another example, let u be the moral law that one ought not kill an innocent person, w be a state where Jack kills an innocent person. It is plausible to think that w is morally impermissible because of u. In cases like these, we shall say that a modal state *bans*, or *precludes*, another state.

Notice that we can draw the distinction between exact and inexact banning relations in just the same way that we did for verification and falsification. A modal state s *exactly bans* another state t when it is entirely relevant to the impossibility of t. The examples given above can all be considered as instances of exact banning. Then the notion of inexact banning can be defined as before: a state s *inexactly bans* t if and only if s extends a modal state that exactly bans t. Note that s need not itself be a modal state in the case of inexact banning. That the domain of inexact banning need not be restricted to modal states should be obvious upon reflection. It seems to be highly counterintuitive to say that the identity of water to H_2O precludes water's being composed of XYZ molecules as impossible, but its fusion with another state does not. In general, if s contains as part a modal state that bans t, then it should be sufficient for s to ban t in the inexact sense.

It might be thought that the notion of exactness here deviates from the standard understanding. Typically, exactness is understood as relevance of a state in its entirety to the truth or falsity of a proposition. In the case of banning, on the other hand, exactness is explained as relevance of a state in its entirety to the impossibility of another state. But it is important to recall that states are supposed to serve as exact verifiers or falsifiers for propositions. So, when s exactly bans t, s is entirely relevant to the impossibility of a proposition P for which t is an exact verifier and thus to the falsity of the modalized proposition that P is possible. Hence the notion of exactness with respect to the banning relation still aligns well with the standard understanding.[2]

With the notion of banning in place, we can formulate a preliminary truthmaker semantical analysis of necessitated propositions thus: $\Box P$ is true if and only if every exact falsifier for P is banned by some modal state. Observe that this analysis is not "exact" yet. For it only states the condition under which $\Box P$ is true, but it does not provide an exact verifier for $\Box P$. Here it might be thought that we could simply define an exact verifier for $\Box P$ to be a modal state that exactly bans every exact falsifier for P; in other words, an exact verifier for $\Box P$ is a modal state that exactly precludes every way in which P might be false. But this simple solution does not work for two related reasons. First, P may have more than one exact falsifier in the most general case. At least offhand, second, there does not have to be a single modal state that exactly bans all exact falsifiers for P; it seems to be conceivable that different modal states ban different exact falsifiers for P.

Let us consider how these cases might be accommodated. Let $T = \{t_1, t_2, \cdots\}$ be the set of exact falsifiers for P. For each exact falsifier t_i for P, we choose a modal state s_i that exactly bans t_i. Then we have a set $S = \{s_1, s_2, \cdots\}$ of modal states. Now, consider the fusion s of modal states in S. s contains as part, for any exact falsifier t_i for P, a modal state s_i that exactly bans t_i. So, intuitively, s is a state in which every way of falsifying P is precluded as impossible. Hence s may plausibly be taken to be a verifier for $\Box P$. Moreover, every chosen modal state s_i is entirely relevant to the necessity of P because it exactly bans an exact falsifier t_i for P; and s is obtained by putting just those modal states together. In this regard, s itself may be considered entirely relevant to the truth of $\Box P$ and hence plausibly be taken as an exact verifier for $\Box P$.

To make this analysis a bit more precise, we first define what it is for a set S of modal states to exactly ban a set T of states. Let M be the set of all modal states. We shall say that a set S of modal states is an *exact ban* on a set T of states if and only if S is the range of a function $f : T \rightarrow M$ mapping each t to a modal state s that exactly bans t. For illustration, let s_1 be a norm prohibiting carrying a gun, s_2 be a norm prohibiting the killing of innocent people, t_1 be John's killing an innocent person with a gun, t_2 be John's killing an innocent person with a knife, and t_3 be John's carrying a gun.[3] Let us also assume, for the sake of argument, that s_1 exactly bans t_1 and t_3 and that s_2 exactly bans t_1 and t_2. Then we can easily see that $S = \{s_1, s_2\}$ is an exact

[2] In this connection, it is worth noting that Fine [12, pp.634-635] also considers an exact notion of exclusion to provide an account of negation from the unilateral truthmaker principle. For any two states s and t, roughly, a state s excludes t just in case their fusion is impossible (according to some prior notion of impossibility). Exclusion is analogous to banning in that it is subject to conditions that we would naturally want for banning, e.g., the upward closure condition to be defined below. But it should be obvious that their intended interpretations are different.

[3] I thank an anonymous referee from this journal for pressing me to clarify the notion of an exact ban with an intuitive example and also providing this particular example. The responsibility for any errors or issues arising from the use of this example in the present context rests solely with the present author.

ban on $T = \{t_1, t_2, t_3\}$ because S is the range of a function f such that $f(t_1) = s_1$, $f(t_2) = s_2$, and $f(t_3) = s_1$.

Notice also that $S = \{s_1, s_2\}$ is not an exact ban on $T^* = \{t_1\}$ since S is not the range of any function from T^* to M. In fact, this is precisely what we would naturally expect of the notion of an exact ban between sets because having both s_1 and s_2 in an exact ban on T^* would be redundant; either one is sufficient to exactly ban t_1.

According to the present definition, finally, both $S = \{s_1, s_2\}$ and $S' = \{s_1\}$ are exact bans on $T^{'} = \{t_1, t_3\}$. S is the range of a function $f_1 : T' \to M$ such that $f_1(t_1) = s_2$ and $f_1(t_3) = s_1$, and S' is the range of a function $f_2 : T' \to M$ such that $f_2(t_1) = f_2(t_3) = s_1$. Here one might object that only S' could reasonably be considered an exact ban on $T^{'}$ on the *minimalist* grounds that we should only count smallest verifiers (falsifiers, bans, etc.) as exact. The problem with this objection is that we cannot in general presuppose that there always exists smallest verifiers (falsifiers, bans, etc.). Also, we may reasonably think of f_1 and f_2 as representing two different ways of exactly banning every member of T'; we may then say that each member of S is entirely relevant to the impermissibility of a member of T'. Despite its non-minimality, therefore, S can still be regarded as an exact ban on T'.[4]

With the notion of an exact ban, we can now formulate an exact truthmaker semantical analysis of necessitated propositions. Recall that an exact verifier s for $\Box P$ ought to contain as part, for any exact falsifier t for P, a modal state s_t that exactly bans t. So, we can define an exact verifier s for $\Box P$ to be the fusion of all modal states in an exact ban on the exact falsifiers for P. For any set S of states, the *fusion* of S will be the fusion of all states in S. Then the present exact truthmaker semantical analysis of necessitated propositions can be formulated as follows:

(E$\Box$) s is an *exact verifier* for $\Box P$ $\Leftrightarrow$ s is the fusion of an exact ban on the exact falsifiers for P.[5]

Let us then turn to $\Diamond P$. One might simply think that $\Diamond P$ should be true just in case not all exact falsifiers for P are banned. This may seem reasonable in light of the duality between necessity and possibility (i.e., the equivalence between $\Diamond P$ and $\neg\Box\neg P$). But there are two problems with this. First, it leaves unclear what an exact verifier for $\Diamond P$ might be. Second, just because a certain state s does not ban another state t it does not necessarily follow that s thinks t possible; s might simply be irrelevant to the modal status of t and hence have nothing to do with whether t is possible or not.

A natural solution to these problems from the bilateral point of view is to introduce another modal relation between states which stands to the banning relation as verification stands to falsification. Let us say that a modal state s *allows*, or *countenances*, another state t when s makes t possible. Again, we will require the allowing relation to be exact. For some intuitive examples of exact allowing, we may think that Jack's having a certain genetic make-up exactly allows him to grow taller than 6 feet; and one might be allowed to cause a certain harm because it is a necessary means to bringing

[4] See Fine [13, p.564] for a critical discussion of the minimalist account of exact truthmaking. I thank an anonymous referee from this journal for pressing me to discuss the present definition of an exact ban in relation to minimalism.

[5] Here and below I use the symbol $\Leftrightarrow$ to mean *if and only if* in metalanguage, and the symbol $\Rightarrow$ to mean *if then* in the obvious way.

about a good result. Then the inexact allowing relation can be defined in the usual way. Instead of giving a merely negative account of $\Diamond P$, then, we can give a positive account of the possibility operator thus:

(E$\Diamond$) s is an *exact verifier* for $\Diamond P$ $\Leftrightarrow$ s is a modal state exactly allowing an exact verifier for P.

Given these analyses of exact verification of modalized propositions, we can easily give an account of exact falsification thereof using the duality between necessity and possibility and the basic fact about exact verification and falsification that the exact verifiers for $\neg P$ are the exact falsifiers for P. For exact falsification for $\Box P$, we have by the duality:

s exactly falsifies $\Box P$	$\Leftrightarrow$	s exactly falsifies $\neg\Diamond\neg P$;	by the basic fact,
	$\Leftrightarrow$	s exactly verifies $\Diamond\neg P$;	by the analysis of $\Diamond$,
	$\Leftrightarrow$	s exactly allows an exact verifier for $\neg P$;	by the basic fact,
	$\Leftrightarrow$	s exactly allows an exact falsifier for P.	

And similarly for exact falsification for $\Diamond P$: by the duality,

s exactly falsifies $\Diamond P$	$\Leftrightarrow$	s exactly falsifies $\neg\Box\neg P$;	by the basic fact,
	$\Leftrightarrow$	s exactly verifies $\Box\neg P$;	by the analysis of $\Box$,
	$\Leftrightarrow$	s is the fusion of an exact ban on the exact falsifiers for $\neg P$;	by the basic fact,
	$\Leftrightarrow$	s is the fusion of an exact ban on the exact verifiers for P.	

We thus have a complete exact truthmaker semantical analysis of modalized propositions.

4 Exactification

It may be helpful here to consider the present analysis in terms of *exactification* in Fine's [10, p.551] sense. It is the idea that given any inexact verifier (falsifier) for a proposition, there must be an underlying exact verifier (falsifier).

By way of illustration, let us consider the standard Boolean semantics for classical logic. A Boolean valuation in the standard semantics can itself be considered a state that verifies those propositions that are true—and falsifies those that are false—under the valuation.[6] Here the relevant notions of verification and falsification are inexact,

[6] One might wonder how a valuation itself could be seen as a state. Recall that we are taking an abstract approach to the metaphysical nature of states, according to which states are whatever play the roles of verifier and falsifier. From this perspective, the idea of taking a valuation to be a state is not intrinsically objectionable. If one finds it objectionable, however, then we could simply speak in terms of possible worlds. For a Boolean valuation can be regarded as representing a possible world in which every proposition is verified or falsified but not both.

as Fine [10, pp.551-552] notes, because a Boolean valuation assigns truth-values to *all* propositions; so, given any proposition P, the Boolean valuation—conceived as a state—may have parts that are irrelevant to the truth and falsity of P. According to exactification, then, each Boolean valuation can be represented as a state that extends an exact verifier for each formula that is true—and an exact falsifier for each formula that is false—under the Boolean valuation.

We thus have the problem of specifying the underlying notions of exact verification and falsification. The standard solution to this problem was provided by van Fraassen [19] and rediscovered in the truthmaker semantics literature by Fine [13]. With the help of the basic mereology of states, we can recursively identify the exact verifiers and falsifiers for truth-functional propositions as follows: letting s be a state and A and B be any propositions,

s exactly verifies $\neg A$	$\Leftrightarrow$	s exactly falsifies A,
s exactly falsifies $\neg A$	$\Leftrightarrow$	s exactly verifies A;
s exactly verifies $A \wedge B$	$\Leftrightarrow$	s is the fusion of an exact verifier for A and an exact verifier for B;
s exactly falsifies $A \wedge B$	$\Leftrightarrow$	s exactly falsifies A or s exactly falsifies B;
s exactly verifies $A \vee B$	$\Leftrightarrow$	s exactly verifies A or s exactly verifies B,
s exactly falsifies $A \vee B$	$\Leftrightarrow$	s is the fusion of an exact falsifier for A and an exact falsifier for B;
s exactly verifies $A \supset B$	$\Leftrightarrow$	s exactly falsifies A or s exactly verifies B,
s exactly falsifies $A \supset B$	$\Leftrightarrow$	s is the fusion of an exact verifier for A and an exact falsifier for B.

A moment's reflection should reveal the plausibility of these clauses; and, as Fine [10] notes, it is not difficult to show, under some suitable assumptions, that a statement is true (false) under a Boolean valuation if and only if there is a state containing an exact verifier (falsifier) for the statement. In this regard, these clauses can plausibly be said to exactify the standard Boolean semantics in the sense that they specify the underlying notion of exact verification and falsification.

Now, how does the present analysis of modalized propositions exactify the Kripke semantics? A possible world in the Kripke semantics may reasonably be conceived as a state verifying the propositions that are true, and falsifying those that are false, in the world. So conceived, each possible world can play the role of a verifier and falsifier. Here again, the relevant notions of verification and falsification are inexact. According to exactification, therefore, each possible world may be represented as a state extending an exact verifier for each proposition that is true—and an exact falsifier for each proposition that is false—in the possible world. So, again, we face the problem of specifying the notions of exact verification and falsification underlying the standard Kripke semantics.

The Kripke semantical clauses for modalized propositions are given in terms of the accessibility relation between possible worlds thus:

(K$\Box$) $\Box P$ is true at a possible world w $\Leftrightarrow$ P is true at all possible worlds accessible to w;

(K$\Diamond$) $\Diamond P$ is true at a possible world w $\Leftrightarrow$ P is true at some possible world accessible to w.

The current proposal attempts to exactify these clauses by giving an account of the accessibility relation between possible worlds in terms of the allowing and banning relations between the constituent states.

To see how, let us first consider some of the conditions on the allowing and banning relations that possible worlds and their constituent states should satisfy. For any state s (either modal or non-modal), define $\overline{\alpha}(s)$ to be the set of states that are inexactly allowed by s and $\overline{\beta}(s)$ to be the set of states that are inexactly banned by s. Intuitively, $\overline{\alpha}(s)$ is the set of states that s thinks possible, and $\overline{\beta}(s)$ is the set of states that s thinks impossible. There are a couple of conditions that are highly plausible for $\overline{\alpha}(s)$ and $\overline{\beta}(s)$. First, $\overline{\alpha}(s)$ should be *downward closed*, meaning that

$$t \in \overline{\alpha}(s) \text{ and } t' \text{ is part of } t \quad \Rightarrow \quad t' \in \overline{\alpha}(s).$$

Second, $\overline{\beta}(s)$ should be *upward closed*:

$$t \in \overline{\beta}(s) \text{ and } t \text{ is part of } t' \quad \Rightarrow \quad t' \in \overline{\alpha}(s).$$

Here one might argue for a stronger requirement that the exact allowing relation itself be downward closed. This requirement is objectionable, however. For, when s exactly allows t and t' is part of t, it is conceivable that only a proper part of s is relevant to the possibility of t'; in such a case, we would have to say that t' is inexactly—but not exactly—allowed by s. So, the requirement is not acceptable as long as we want to maintain the distinction between the exact and inexact allowing relations. We address this worry by requiring the downward closure condition only on the inexact allowing relation. A similar worry might arise if we require the strong upward closure condition on the exact banning relation, although it seems to have much less force.[7]

There is another set of conditions concerning the modal behavior of possible worlds. We say that a state s is *modally sound* if and only if there is no state t such that

$$t \in \overline{\alpha}(s) \cap \overline{\beta}(s);$$

[7] It might be objected that even the weaker requirements are not so obvious in deontic cases. For example, suppose that you are in a state s where there are two buttons, say A and B, such that you administer an electric shock to a pupil if you push either one of them, but nothing happens if you push them both. In the state s, we may reasonably think, you are allowed to push both buttons, but not one; or, you are banned from pushing one button but not from pushing both. Since pushing one button is part of pushing two, the objection goes, even the inexact allowing relation is not downward closed; nor is the inexact banning relation upward closed. The objection is mistaken, as one can easily see, because what is not allowed is to push *exactly* one button, i.e., to push A and not B or to push B and not A. And these are not part of pushing both A and B. But if you are allowed to push both A and B, however, you are of course allowed to push A and to push B.

otherwise, s is *modally unsound*. Also, s is said to be *modally complete* just in case for all states t,

$$t \in \overline{\alpha}(s) \cup \overline{\beta}(s),$$

and to be *modally incomplete* otherwise. Clearly, the possible worlds in the Kripke semantics are modally sound and complete. For they are either accessible or inaccessible to one another, but never both.

In the standard Kripke semantics, the accessibility relation is intended to capture relative possibility between possible worlds; so, when w' is accessible to w, it means that w' is possible relative to w. Since we intend to capture relative possibility between states in terms of the allowing relation, it is natural that for any worlds w and w',

$$w' \text{ is accessible to } w \quad \Leftrightarrow \quad w' \in \overline{\alpha}(w).$$

Hence we have:

$$\begin{aligned} w' \text{ is not accessible to } w \quad &\Leftrightarrow \quad w' \notin \overline{\alpha}(w); \quad \text{then, by the modal completeness of } w, \\ &\Leftrightarrow \quad w' \in \overline{\beta}(w), \end{aligned}$$

as one would desire.[8] Note also that the closure conditions on the inexact allowing and banning relations imply that for any states s and t,

$$\begin{aligned} t \in \overline{\alpha}(s) \quad &\Leftrightarrow \quad \text{every part of } t \text{ is in } \overline{\alpha}(s); \\ t \in \overline{\beta}(s) \quad &\Leftrightarrow \quad \text{some part of } t \text{ is in } \overline{\beta}(s). \end{aligned}$$

So, it follows: for any worlds w and w',

$$\begin{aligned} w' \text{ is accessible to } w \ &\Leftrightarrow \ \text{every part of } w' \text{ is in } \overline{\alpha}(w); \\ &\Leftrightarrow \ \text{every part of } w' \text{ is exactly allowed by some part of } w; \\ w' \text{ is not accessible to } w \ &\Leftrightarrow \ \text{some part of } w' \text{ is in } \overline{\beta}(w); \\ &\Leftrightarrow \ \text{some part of } w' \text{ is exactly banned by some part of } w. \end{aligned}$$

We thus have an analysis of the accessibility relation between possible worlds in terms of the exact allowing and banning relations between their constituent states.

Now, there is yet another condition that ought to be mentioned before we can establish the equivalence of the present analysis of modalized propositions to the standard Kripke semantical clauses. Recall that (K$\Diamond$) states that $\Diamond P$ is true at w if and only if P is true at some world accessible to w. So, according to the Kripke semantics, whatever is possible at w should be realized at a *world* accessible to w. In

[8] It might be thought that w' is accessible to w if and only if w' is exactly allowed by w, and that w' is inaccessible to w if and only if w' is exactly banned by w. This analysis is objectionable because it presupposes that possible worlds are themselves modal states. We use the inexact notions of allowing and banning to get around this problem.

this sense, we may say that possible worlds in the Kripke semantics are *robust* about the possibilities. To give a more precise formulation in terms of the present analysis, we have: for any possible world w and any state t,

(R) whenever $t \in \overline{\alpha}(w)$, there is a possible world w' such that w' extends t and $w' \in \overline{\alpha}(w)$.

One might think that the robustness condition, when conceived as applied to states, lacks intuitive or theoretical grounds. For, even if an exact verifier t for P is allowed by a possible world w, it is conceivable that every possible world w' extending t contains as part a state banned by w. In such cases, it would seem that w countenances a way in which P might be true without having any accessible world at which P is true. Of course, cases of this sort are excluded by (K$\Diamond$) in the Kripke semantics; but why should we place an analogous condition for states?

In response, note first that the robustness condition has some intuitive appeal. To see this, let w be a possible world and t be a state. Suppose that every world w' extending t is impossible relative to w. Then it is not unreasonable to think that t should not be considered as a possibility relative to w in the first place. In other words, t is a possibility relative to w only if t is realized in a world possible relative to w. But this amounts to requiring the robustness condition on w.

In addition to this intuitive ground, there is also a compelling reason to think that the robustness condition is indispensable for the purpose of exactification. Since we think of possible worlds themselves as verifiers, according to exactification, the truth of $\Diamond P$ at w requires that there is an exact verifier for $\Diamond P$ *within* w. In the Kripke semantics, on the other hand, the truth of $\Diamond P$ at w requires the existence of a (potentially different) possible world w' at which P is true. Hence any attempt to exactify (K$\Diamond$) faces the problem of finding a state *within* w that witnesses the existence of a possible world w' extending an exact verifier for P. And, of course, the solution in its most general and abstract form would be to posit states that do the job. This is exactly what we do when postulating modal states and imposing the robustness condition: whenever a state t is possible relative to a world w—that is, $t \in \overline{\alpha}(w)$—there is a modal state s within w such that s countenances t as part of some world w' which is itself possible relative to w. This solution, to be sure, leaves the substantive question of what modal states are. But no particular conception of modal states is required for the purpose of exactification, except that they should be subject to the robustness condition or to another condition to the same effect. In this regard, the robustness condition seems to be indispensable for the exactification of the Kripke semantics.

We are now in a position to establish the equivalence of the present analysis of modalized propositions to the standard Kripke semantical clauses. As in the case of non-modal classical propositional logic, it suffices to show that $\Box P$ is true at a possible world if and only if the world extends an exact verifier for $\Box P$. To first verify the left-to-right direction, suppose that $\Box P$ is true at a possible world w; that is, P is true at all worlds w' accessible to w. Assume for *reductio* that w does not extend the fusion of an exact ban on the exact falsifiers for P. This would imply that some exact falsifier t for

P is not in $\overline{\beta}(w)$. Since w is modally complete, then, t would be in $\overline{\alpha}(w)$. By (R), then, there would be a world w' such that w' extends t and w' is in $\overline{\alpha}(w)$. By our analysis of the accessibility relation, then, it would follow that there is a world accessible to w, namely w', at which P is false; contradiction. By rejecting the *reductio* assumption, therefore, we conclude that w extends the fusion of an exact ban on the exact falsifiers for P.

Now conversely, suppose that w extends the fusion of an exact ban on the exact falsifiers for P. Then every exact falsifier for P is in $\overline{\beta}(w)$. Then it follows by the upward closure condition that any possible world w' extending an exact falsifier for P is also in $\overline{\beta}(w)$. This implies by the preceding analysis of the accessibility relation that w' is not accessible from w. It thus follows that every world w' at which P is false is not accessible to w. By a similar argument, we can show that the equivalence holds for $\Diamond P$ also.

So, the proposed exact truthmaker semantics for modalized propositions is equivalent to the Kripke semantics under certain natural assumptions about the modal behavior of possible worlds and their constituent states. In the light of this, the present analysis may be viewed as exactifying the fundamental notion of the Kripke semantics, namely that of truth at a possible world.

Here one could raise an objection that it is misleading to say that the present analysis offers an exact account of the basic semantical notions of the Kripke semantics; for it uses the inexact allowing and banning relations to describe the modal behaviors of possible worlds. The use of inexact notions, however, is not in and of itself an objection to exactness of the present account. To see this, notice that when viewed from the perspective of exact truthmaker semantics, the accessibility relation itself is most naturally understood as an inexact notion because it depends on the modal behaviors of their parts. Parts of possible worlds may stand in exact allowing or banning relations to one another. However, we cannot just assume that possible worlds themselves must stand in the exact modal relations to one another. In the general case, therefore, the modal behaviors of possible worlds ought to be described in terms of inexact notions. And this does not undermine exactness of the present analysis because it ultimately explains the modal relations between possible worlds on the basis of the exact modal relations between their parts as we have seen above. Hence the present account can still be said to give an exact account of the modal behaviors of possible worlds.

The emerging picture of the relation between the two semantics is quite straightforward. In the Kripke semantics, the possible worlds are conceived merely as indices that bear an accessibility relation to each another. Under the current proposal, the possible worlds are given internal structures of states that bear the allowing and banning relations to each other. It thus gives an account of the possible worlds and the accessibility relation in terms of their internal structures and the way the constituent states are related. This seems to be well-aligned with the way we think about possible worlds in many applications. Imagine, for example, that we are tossing coins, say A and B, simultaneously. Each coin will land either on its head or tail. So there are four 'possible worlds' in total. These possible worlds are constituted by the states of each coin. For example, the possible world $(A : head, B : tail)$ would most naturally be considered as consisting of the states $(A : head)$ and $(B : tail)$. If we work under the notion of possibility such that it is impossible for a coin to show up both faces at the same time, then we may think of the state $(A : head)$ as a modal state that bans the state $(A : tail)$ and vice versa. In this way, the present truthmaker semantics captures the intuitive way that we think about possible worlds.

5 Consequence

Let us consider how we may give an exact truthmaker semantical account of basic logical notions of modal logic, such as the consequence relation and validity. For the sake of simplicity, we shall mainly be concerned with the case of non-modal propositional logic here; the situation is similar for the case of modal propositional logic.

The consequence relation is standardly understood in terms of *truth preservation*: a proposition A is a consequence of a set Γ of propositions if and only if A is true in every case in which all propositions in Γ are true. Within the context of classical logic, the relevant cases are canvassed by Boolean valuations, which are both sound in that they never assign both true and false to a proposition and complete in that they assign either true or false to every proposition. Hence the consequence relation can also be characterized, equivalently, as *the absence of counterexamples*; that is, A is a consequence of Γ if and only if there are no cases where all propositions in Γ are true but A false.

But this equivalence no longer holds once we start thinking in terms of states from the bilateral point of view. For, on this view, states need neither be sound nor complete. To see this, recall first that it is part of basic mereology of states that any two states can be put together into their fusion. So, when a proposition A has both an exact verifier and an exact falsifier, then their fusion both verifies and falsifies A. Hence states may be unsound in the sense that they both verify and falsify some propositions. Moreover, according to the bilateral truthmaker principle, unverifiedness does not imply falsifiedness and vice versa. So, a state may be incomplete in the sense that it neither verifies or falsifies a proposition; it might well just be irrelevant to the truth or falsity of the proposition. Thus the logical behavior of states is different from that of Boolean valuations in ways that are crucial to our understanding of the consequence relation. And this consideration raises the question of how exactly the consequence relation should be analyzed within the framework of truthmaker semantics.

A simple solution to this question would be to define the consequence relation in terms of a restricted class of states whose logical behavior resembles that of Boolean valuations. Let us say that a state s is *atomically sound* if and only if s does not extend both an exact verifier and an exact falsifier for an atomic proposition; and *atomically unsound* otherwise. We also say that s is *atomically complete* if and only if, for any atomic proposition P, s extends either an exact verifier or an exact falsifier for P; and *atomically incomplete* otherwise. Obviously, atomically sound and complete states behave just like Boolean valuations in that they either verify or falsify every truth-functional proposition, but never both. So, there is a natural correspondence between the atomically sound and complete states and the Boolean valuations.[9]

We may thus give an easy truthmaker semantical analysis of the consequence relation as follows: for any proposition A and a set Γ of propositions,

(C1) A is a consequence of Γ if and only if, for every atomically sound and complete state s, if s extends an exact verifier for every proposition B in Γ then s also extends an exact verifier for A.

It can easily be checked that (C1) is equivalent to the usual definition of the consequence relation in the standard Boolean semantics.

One problem with this simple analysis is that it is at odds with the main philosophical motivation behind the truthmaker semantics, as is forcefully expressed by Fine [12, p.645]:

> One remarkable aspect of the present theory of truthmaker content is that possible worlds completely drop out of the picture. ... One might jokingly remark that the possible worlds approach is fine but for two features: the first is that possible worlds are worlds, i.e., complete rather than partial; and the second is that they are possible [i.e., sound]. Drop both requirements, impose a mereological structure on the resulting states, and we obtain a framework that is of much more help in developing an adequate theory of content.

[9] Recall that one notable feature of the present semantics is that there can be atomically inconsistent, and atomically incomplete, states. Due to this feature, there is a natural correspondence between the current semantics and the four-valued semantics as defined in Belnap [8], where each formula can be assigned one of the following four values: True, False, Both, and Neither. For each state s, we may define the corresponding four-valued assignment φ_s for propositional atoms P by setting

$$\varphi_s(P) = \begin{cases} \{T\} & \text{if } s \text{ extends only an exact verifier, but not an exact falsifier, for } P; \\ \{F\} & \text{if } s \text{ extends only an exact falsifier, but not an exact verifier, for } P; \\ \{T, F\} & \text{if } s \text{ extends both an exact verifier and an exact falsifier for } P; \\ \emptyset & \text{if } s \text{ extends neither an exact verifier, nor an exact falsifier, for } P. \end{cases}$$

A *Belnapian* valuation $\overline{\varphi_s}$ is a valuation extending φ_s to all formulas according to the scheme as given in [8]. Then it is not difficult to see that for all propositions A, s extends an exact verifier for A if and only if $T \in \overline{\varphi_s}(A)$, and s extends an exact falsifier for A if and only if $F \in \overline{\varphi_s}(A)$. In this sense, each state s in a model corresponds to a valuation $\overline{\varphi_s}$ in the four-valued semantics. From this perspective, the present truthmaker semantics provides a unifying semantical framework in which the consequences of some of the best-known multi-valued logics, such as FDE, the strong three-valued logic K3 of Kleene [3, 4], and the logic of paradox LP of Priest [7], can be defined thus:

(FDE) For any state s, if s extends an exact verifier for every formula B in Γ, then s also extends an exact verifier for A.

(K3) For any modally sound state s, if s extends an exact verifier for every formula B in Γ, then s also extends an exact verifier for A.

(LPa) For any modally complete state s, if s extends an exact verifier for every formula B in Γ, then s also extends an exact verifier for A.

(LPb) For any state s, if s does not extend an exact falsifier for every formula B in Γ, then s does not extend an exact falsifier for A either.

This analysis is similar to the one in Angelberger, Faroldi, and Korbmacher [1]. (LPb) and (C2) below are due to the present author.

In other words, the advantage of truthmaker semantics lies in the fact that it does not presuppose the existence of possible worlds, i.e., states that are sound and complete. I myself do not think that it is so important to get rid of possible worlds. More important, I think, is to give an account of various notions of the standard semantics in terms of exact verification and falsification by parts of possible worlds—and that is exactly what we have done so far under the rubric of exactification. But I do agree with Fine that it is desirable to have a semantical analysis of logical notions without necessarily assuming the existence of possible worlds. So, we face the problem of finding a truthmaker semantical account of the consequence relation with appeal, not to possible worlds, but only to their parts.

On the bilateral conception of truthmaking, a natural solution to this problem is to make use of the notion of falsification. The idea is roughly to hold premises and conclusions to different standards, so that we say that A is a consequence of Γ if and only if there is no case where all statements in Γ are *verified* but where A is *falsified*. More precisely:

(C2) A is a *consequence* of Γ if and only if for all atomically sound states s, if s extends an exact verifier for every proposition B in Γ, then no exact falsifier for A is extended by s.[10]

Or, equivalently, A is a consequence of Γ if and only if there is no atomically sound state s such that s extends both an exact verifier for every proposition B in Γ and an exact falsifier for A. (C2) can easily be shown to be the standard definition of the consequence relation in the Boolean semantics. For this, it suffices to check that (C2) is equivalent to (C1). Assume that (C2) holds. Let s be an atomically sound and complete state extending an exact verifier for every proposition B in Γ. Then it is immediate from the assumption that s does not extend any exact falsifier for A. Since s is atomically complete, on the other hand, s must extend either an exact verifier, or an exact falsifier, for A. So, s extends an exact verifier for A. The other direction is trivial, so we omit it. We thus arrive at an alternative exact truthmaker semantical analysis of the consequence relation.

Note that (C2) still assumes the existence of atomically sound states. However, this is not objectionable because the problem was to explain the consequence relation using only parts of possible worlds, which can reasonably be assumed to be atomically sound. Furthermore, (C2) does not deviate much from the standard understanding of the consequence relation. It captures the intuitive conception of the consequence relation as the absence of counterexamples in a way that is naturally in line with the bilateral conception of truthmaking.

I do not mean to assert that (C2) is the only plausible analysis of the consequence relation of the classical propositional logic. As Fine [10, p.569] notes, we can conceive of many different approaches depending on one's intuitive understanding of consequence (truth-preservation vs. absence of counterexample), on the form of verification and falsification (exact vs. inexact), and on how verification and falsification relates to

[10] This "mixed" scheme has received much attention in the recent literature on the sorites and liar paradoxes. It should also be noted that the analysis below originates from the so-called strict-tolerant logic. See Cobreros *et al.* [9] and Fitting [15] for the basic ideas behind the mixed scheme and for the ways it relates to the strict-tolerant and other many-valued logics.

the notion of truth and falsity. It would go beyond the scope of this paper to discuss all possible approaches. However, one approach is worthy of note, so as not to be confused with the present one. One might be interested in defining what we may call the *exact* logical notions. For a rough example, one might define the notion of exact equivalence thus: two propositions are *exactly equivalent* if and only if they have the same exact verifiers and falsifiers. Similarly, various logical notions, including the consequence relation, might be defined directly in terms of exact verification and falsification.[11] It may further be asked that how these exact logical notions relate to one another and also to the standard ones. All these problems, I think, are of great importance for the development of truthmaker semantics. But such attempts to define "exact logics", as it were, should not be confused with the present project of exactification which aims at giving an account of the standard logical notions in terms of the underlying notions of exact verification and falsification.

Finally, let me briefly indicate how the considerations so far apply to an exact truthmaker semantical account of consequence in modal logic. In the Kripke semantics, the notion of consequence may be defined as the absence of counterexamples thus: a formula A is a consequence of a set Γ of formulas (in modal logic L) if and only if there is no possible world where every formula in Γ is true and A false (in all models of L). Now, recall that according to the current analysis, possible worlds are conceived as modally sound and complete states that are robust about the possibilities; let us call such states *worldly*. For the purpose of developing a formal semantics for modal logic, we may just define the consequence relation in modal logic in terms of worldly states thus:

(MC1) A is a consequence of Γ (in modal logic L) if and only if, for every worldly state s, if s extends an exact verifier for every proposition B in Γ then s also extends an exact verifier for A (in all relevant models for L).

But again, this analysis is not quite satisfactory if we want to work only with parts of possible worlds. The problem here is obviously analogous to the one we faced in giving an account of the consequence relation for non-modal propositional logic; and so, it can be resolved in essentially the same way as follows:

(MC2) A is a consequence of Γ (in modal logic L) if and only if, for every modally sound state s, if s extends an exact verifier for every proposition B in Γ then no exact falsifier for A is extended by s (in all relevant models for L).[12]

The similarity between (C2) and (MC2) should be obvious. In both cases, we drop the completeness requirement and give a mixed account of the consequence relation using verification and falsification by parts of possible worlds. We shall show that this account can be used to establish the soundness and completeness results for various standard systems of modal classical propositional logic without necessarily presupposing the existence of possible worlds.

[11] See, for example, van Fraassen [19, p.485] and Fine [10, pp.556-557] and [12, p.669] for various such notions of consequence.

[12] It should be noted that this informal analysis is *not* intended to be formally precise; in fact, it is slightly more general than the formal one as provided in Definition 4 below. I suspect that the two may turn out to be equivalent, in which case the formal semantics could be simplified quite a bit. In this connection, please see the discussion about the notions of modal boundary and absolute possibility in p.21 below.

6 Formal Exposition

A *partial order* is an ordered pair $\langle \mathcal{S}, \sqsubseteq \rangle$, where $\mathcal{S}$ is any set and $\sqsubseteq$ is a reflexive, transitive, and anti-symmetric binary relation on $\mathcal{S}$. A *state space* is defined to be a partial order $\langle \mathcal{S}, \sqsubseteq \rangle$ which is complete in the sense that every $S \subseteq \mathcal{S}$ has a least upper bound (in symbol, $\bigsqcup S$). Intuitively, $\mathcal{S}$ is the set of states and $\sqsubseteq$ is parthood relation on states. When $s \sqsubseteq t$, we say that s is *part* of t, or equivalently, that t *extends* s. The *fusion* $\bigsqcup S$ of S is the smallest state extending all members of S. The completeness requirement guarantees the existence of the least and greatest states in $\mathcal{S}$, which we shall call, respectively, the *bottom state* $\bot = \bigsqcup \emptyset$ and the *top state* $\top = \bigsqcup \mathcal{S}$. When $S = \{s_1, ..., s_n\}$, we shall sometimes write $s_1 \sqcup ... \sqcup s_n$ to mean $\bigsqcup S$; so, for example, $s \sqcup t = \bigsqcup \{s, t\}$. For any sets S and T of states, we let

$$S \sqcup T = \{s \sqcup t : (\exists s)(\exists t)(s \in S \text{ and } t \in T)\}.$$

Notice that $S \sqcup T = \emptyset$ if and only if either S or T is empty. In case $S = \{s\}$,

$$S \sqcup T = \{s \sqcup t : t \in T\}.$$

A *modal space* (in short, *m-space*) is defined to be an ordered triple $\Sigma = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$, where $\langle \mathcal{S}, \sqsubseteq \rangle$ is a state space and μ is a function from $M \subseteq \mathcal{S}$ into $\mathcal{P}(\mathcal{S}) \times \mathcal{P}(\mathcal{S})$. Each $s \in M$ is called a *modal state* and μ assigns to each $s \in M$ a pair $\langle \alpha(s), \beta(s) \rangle$ of subsets of $\mathcal{S}$. When $t \in \alpha(s)$, we say that t is *exactly allowed* by s; and when $t \in \beta(s)$, t is said to be *exactly banned* by s. In writing $\alpha(s)$ and $\beta(s)$, s is assumed to be a modal state. For each state $s \in \mathcal{S}$, the *modal profile* of s is $\langle \overline{\alpha}(s), \overline{\beta}(s) \rangle$, where

$$\overline{\alpha}(s) = \{t \in \mathcal{S} : (\exists s' \in M)(s' \sqsubseteq s \text{ and } t \in \alpha(s'))\};$$
$$\overline{\beta}(s) = \{t \in \mathcal{S} : (\exists s' \in M)(s' \sqsubseteq s \text{ and } t \in \beta(s'))\}.$$

When $t \in \overline{\alpha}(s)$, t is said to be *inexactly allowed* by s; and when $t \in \overline{\beta}(s)$, we say that t is *inexactly banned* by s. A state s is *modally sound* if and only if $\overline{\alpha}(s) \cap \overline{\beta}(s) = \emptyset$, and *modally complete* if and only if $\overline{\alpha}(s) \cup \overline{\beta}(s) = \mathcal{S}$.

Let an m-space $\Sigma = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$ be given. For any set $T \subseteq \mathcal{S}$, we say that a set S of modal states is an *exact ban* on T if and only if $S = \mathrm{ran}(f)$ for some $f : T \to M$ such that $t \in \beta(f(t))$. Given any set $T \subseteq \mathcal{S}$, we let

$$\mathcal{A}(T) = \{x \in \mathcal{S} : (\exists t \in T)(t \in \alpha(x))\};$$
$$\mathcal{B}(T) = \{x \in \mathcal{S} : (\exists S \subseteq \mathcal{S})(S \text{ is an exact ban on } T \text{ and } x = \bigsqcup S\}.$$

Informally, $\mathcal{A}(T)$ is the set of states that think something in T is possible and $\mathcal{B}(T)$ is the set of states that think nothing in T is possible.

Let $\Sigma = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$ be an m-space. A state s is said to be a *modal boundary* of Σ if and only if s is modally sound and every proper extension of s is modally unsound.

Then we define the set $\mathcal{S}^{\Diamond}$ of *absolute possibilities* of Σ as follows:

$$\mathcal{S}^{\Diamond} = \{t \in \mathcal{S} : (\exists s \in \mathcal{S})(s \text{ is a modal boundary and } t \sqsubseteq s)\}.$$[13]

Intuitively, the modal boundaries are the largest parts of possible worlds that Σ can see. And a state s is considered an absolute possibility if it is part of some modal boundary.

To get an intuitive grip on the notion of modal boundary, let us introduce a few auxiliary notions. Given an m-space $\Sigma = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$, we say that a set $C \subseteq \mathcal{S}$ of states is a *chain* if and only if every pair of states in C is comparable (i.e., for all $s, t \in C$ either $s \sqsubseteq t$ or $t \sqsubseteq s$). A chain C is *modally sound* if and only if every $s \in C$ is modally sound. A modally sound chain C is *maximal* if and only if $C \nsubseteq C'$ for another modally sound chain C'. Now suppose that every modally sound maximal chain $C = \{s_1, s_2, ...\}$ of Σ is such that $\bigsqcup C = s_1 \sqcup s_2 \sqcup ...$ is modally unsound. This implies that no modally sound state in Σ can reasonably be considered part of a possible world. To see this, note first that for any modally sound maximal chain $C = \{s_1, s_2, ...\}$ of Σ, if $\bigsqcup C = s_1 \sqcup s_2 \sqcup ...$ is modally unsound then there is no possible world w extending all s_i's. For, if there was a possible world w extending every $s_i \in C$, then $\bigsqcup C$ would also be part of w and hence modally sound. Since any modally sound state s is a member of some modally sound chain of Σ, it follows that no modally sound state s is part of a possible world. We can avoid this consequence by requiring that there are modal boundaries of Σ. For, then, there will exist at least some modally sound maximal chain C such that $\bigsqcup C$ is modally sound; and this guarantees that there are parts of possible worlds in Σ. It is worth noting here that modal boundaries, as we shall soon see with examples, need not themselves be possible worlds.

For any state $s \in \mathcal{S}$ and for any sets $T \subseteq \mathcal{S}$ and $U \subseteq \mathcal{S}$ of states, we say that T and U are *s-incompatible* if and only if $T \sqcup U \subseteq \overline{\beta}(s)$. Informally, T and U are s-incompatible when s thinks that no pairwise fusions of T and U are possible. Then we say that a state $t \in \mathcal{S}$ is *modally s-incompatible* if and only if there is a set U of states such that $\{t\}$ and $\mathcal{A}(U) \cup \mathcal{B}(U)$ are s-incompatible. That is, t is modally s-incompatible if and only if

$$\{t\} \sqcup (\mathcal{A}(U) \cup \mathcal{B}(U)) \subseteq \overline{\beta}(s),$$

for some $U \subseteq \mathcal{S}$. To see the intuitive motivation behind the notion of modal incompatibility, let us assume that t is modally s-incompatible. Consider what this implies in relation to possible worlds. Recall first that according to the present analysis, every possible world w is modally complete. So, w must think that either something in U is possible or else everything in U is impossible. In other words, w must extend some

[13] The definition is inspired by Fine's notion of a modalized state space in [12, 13]. A *modalized state space* is $\langle \mathcal{S}, \sqsubseteq, S \rangle$ where $\langle \mathcal{S}, \sqsubseteq \rangle$ is a state space and S is a subset of $\mathcal{S}$ that is closed under parthood relation: for all state s and t, if $s \in S$ and $t \sqsubseteq s$ then $t \in S$. Intuitively, a modalized state space is a state space with a designated set S of states that are considered possible in the "absolute" sense. The above definition of absolute possibility essentially provides an analysis of Fine's notion of modalized state space in terms of the allowing and banning relations.

$u \in \mathcal{A}(U)$ and $\mathcal{B}(U)$. Since t is modally s-incompatible, however, it means that s thinks it impossible to fuse t with any such u. Hence s ought to think that t is not part of a world accessible to it.

With the notion of modal incompatibility in place, we are now in a position to provide one of the main definitions of the present semantics:

Definition 1 An m-space $\Sigma = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$ is said to be *normal* if and only if it satisfies the following conditions:

(**N1**) $\mathcal{S}^{\Diamond} \neq \emptyset$.

(**N2**) The inexact allowing relation is downward closed: for all states s, t, and t',

$$t \in \overline{\alpha}(s) \text{ and } t' \sqsubseteq t \quad \Rightarrow \quad t' \in \overline{\alpha}(s).$$

(**N3**) The inexact banning relation is upward closed: for all states s, t, and t',

$$t \in \overline{\beta}(s) \text{ and } t \sqsubseteq t' \quad \Rightarrow \quad t' \in \overline{\beta}(s).$$

(**N4**) For any modal boundary s of Σ and any $t \in \mathcal{S}^{\Diamond}$, if $t \in \overline{\alpha}(s)$, then there is a modal boundary t' such that $t \sqsubseteq t'$ and $t' \in \overline{\alpha}(s)$.

(**N5i**) For any nonempty $U \subseteq \mathcal{S}$, $\mathcal{A}(U) \cup \mathcal{B}(U) \neq \emptyset$.

(**N5ii**) For any modal boundary s of Σ and any $t \in \mathcal{S}^{\Diamond}$, if t is modally s-incompatible, then $t \in \overline{\beta}(s)$.

Some comments are in order. First, (**N1**) ensures that the existence of modal boundaries and hence of modally sound states in Σ. Second, (**N4**) is a version of robustness condition on the modal boundaries of Σ; but it does not require the modal boundaries to be modally complete as we shall soon see with examples below. Third, (**N5ii**) should seem to be plausible in light of the intuitive motivation behind the notion of modal incompatibility.

Fourth, we require (**N5i**) to ensure that modalized propositions are not devoid of content. In the standard bilateralist truthmaker semantics, the content of a proposition is characterized by the ordered pair of its exact verifiers and falsifiers.[14] So, a proposition is said to be *contentless* when it has neither. Now, we would naturally want to avoid contentless propositions when devising a formal semantics for classical logical systems, because these systems are not designed to model inferences involving such propositions.[15] Without (**N5i**), however, modalized propositions risk being

[14] See, for example, Fine [10, p.564]

[15] It may help to consider the case of the Boolean semantics for classical propositional logic. For any set Γ of propositions; we say that a Boolean valuation v is *adequate* for Γ if and only if v assigns a truth value to every propositional atom appearing in some formula of Γ. Then we define the notion of consequence thus: A is a consequence of Γ if and only if, for every valuation v that is adequate for Γ; A, v assigns true to A whenever v assigns true to every member of Γ. We do not typically make the adequacy condition on valuations explicit because we generally consider valuations that assign truth-values to every propositional atoms. But it is not difficult to see that the requirement is strictly necessary to give a correct definition of classical consequence. To see this, let $\Gamma = \emptyset$ and $A = P \vee \neg P$, and consider a valuation v that does not assign a truth-value to P. v is obviously inadequate for Γ; A. Also, v trivially assigns true to every member of Γ. But it does not assign true to A because it does not assign a truth-value to P at all. So, without the adequacy requirement, $P \vee \neg P$ would not be valid. In this way, our standard semantics for classical propositional logic makes an assumption that is analogous to (**N5i**).

contentless. For a simple example, $\Box P$ has no exact verifier or falsifier when no exact falsifier for P is banned or allowed. So, for the present purpose of providing a formal truthmaker semantics for standard systems of classical modal propositional logic, we require condition (**N5i**) for normal m-models to ensure that every proposition has content (see Proposition 1 below).[16]

Finally, note that (**N5i**) is equivalent to

(**N5i***) For all states s, there exists a state t such that $s \in \alpha(t) \cup \beta(t)$.

That is, every state is either exactly allowed or banned by a modal state. To see this, first assume that (**N5i**) holds. Let s be any state. By (**N5i**), either $\mathcal{A}(\{s\})$ is nonempty, in which case there is some t such that $s \in \alpha(t)$, or $\mathcal{B}(\{s\})$ is nonempty, in which case there is some t such that $s \in \beta(t)$. Either way, therefore, $s \in \alpha(t) \cup \beta(t)$ for some t. Hence (**N5i***) holds. Conversely, suppose that (**N5i***) holds. Let U be any set of states. Now either each state in U is exactly banned by some state or not. In the former case, $\mathcal{B}(U)$ is not empty. In the latter case, some state u in U is not exactly banned by any state; by (**N5i***), then, u is exactly allowed by some state and so $\mathcal{A}(U)$ is not empty. Either way, therefore, $\mathcal{A}(U) \cup \mathcal{B}(U) \neq \emptyset$. So (**N5i**) holds.

Let P_1, P_2, ... be a countably infinite list of *propositional atoms*. The *well-formed formulas* (or simply, *formulas*) are constructed in the usual way, using truth-functional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and the modal operator $\Box$ (necessity). We shall use P, Q, R, ... as metavariables for propositional atoms, and A, B, C, ... for formulas in general. For any formulas A and B, $A \supset B$ is defined to be $(\neg A \vee B)$, and $\Diamond A$ to be $\neg\Box\neg A$. For any set Γ of formulas, we let:

$$At(\Gamma) = \{P : \ P \text{ is a propositional atom occurring in some formula in } \Gamma\};$$
$$Fml(\Gamma) = \{A : \text{ every atomic subformula } P \text{ of } A \text{ is in } At(\Gamma)\}.$$

Given a set Γ of formulas, we define an *m-model* of Γ to be an ordered quadruple $\langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$, where $\langle \mathcal{S}, \sqsubseteq, \mu \rangle$ is an m-space and v is a valuation that assigns to each state $s \in \mathcal{S}$ a pair $\langle [s]^+, [s]^- \rangle$ of subsets of $At(\Gamma)$.[17] Here $[s]^+$ is the set of propositional atoms that are exactly verified by s, and $[s]^-$ is the set of propositional atoms that are exactly falsified by s. We require that for each $P \in At(\Gamma)$, there is at least one $s \in \mathcal{S}$ such that $P \in [s]^+ \cup [s]^-$.

Definition 2 Given an m-model $\mathcal{M}$ of Γ, the notions of exact verification and falsification (written $\mathcal{M}, s \Vdash^+ A$ and $\mathcal{M}, s \Vdash^- A$, respectively) can be defined recursively as follows:

[16] It would be an interesting line of inquiry to devise a formal semantics that can handle contentless propositions and see what a sound and complete logical system looks like for the semantics. Some of the ideas from the present semantics might be developed in this direction and be used to provide an interesting formal semantical and logical account of contentless propositions. Here I ought to leave it as a task for another day.

[17] The current definition of a valuation may appear to deviate from the standard one, according to which a valuation is a (possibly partial) function from a set of propositional atoms to the pairs of sets of states. Obviously, however, the two definitions are equivalent; in this connection, see the two equivalent ways of stating of the semantic clauses for propositional connectives below.

$$
\begin{array}{rcl}
\mathcal{M}, s \Vdash^{+} P & \Leftrightarrow & P \in [s]^{+}, \text{ for atomic } P; \\
\mathcal{M}, s \Vdash^{-} P & \Leftrightarrow & P \in [s]^{-}, \text{ for atomic } P; \\
\mathcal{M}, s \Vdash^{+} \neg A & \Leftrightarrow & \mathcal{M}, s \Vdash^{-} A; \\
\mathcal{M}, s \Vdash^{-} \neg A & \Leftrightarrow & \mathcal{M}, s \Vdash^{+} A; \\
\mathcal{M}, s \Vdash^{+} A \wedge B & \Leftrightarrow & s = s_1 \sqcup s_2, \text{ for some } s_1, s_2 \text{ with } \mathcal{M}, s_1 \Vdash^{+} A \\
& & \text{and } \mathcal{M}, s_2 \Vdash^{+} B; \\
\mathcal{M}, s \Vdash^{-} A \wedge B & \Leftrightarrow & \mathcal{M}, s \Vdash^{-} A \text{ or } \mathcal{M}, s \Vdash^{-} B; \\
\mathcal{M}, s \Vdash^{+} A \vee B & \Leftrightarrow & \mathcal{M}, s \Vdash^{+} A \text{ or } \mathcal{M}, s \Vdash^{+} B; \\
\mathcal{M}, s \Vdash^{-} A \vee B & \Leftrightarrow & s = s_1 \sqcup s_2 \text{ for some } s_1, s_2 \text{ with } \mathcal{M}, s_1 \Vdash^{-} A \\
& & \text{and } \mathcal{M}, s_2 \Vdash^{-} B. \\
\mathcal{M}, s \Vdash^{+} \Box B & \Leftrightarrow & s = \bigsqcup S, \text{ where } S \text{ is an exact ban on the} \\
& & \text{exact falsifiers for } B; \\
\mathcal{M}, s \Vdash^{-} \Box B & \Leftrightarrow & t \in \alpha(s) \text{ for some exact falsifier } t \text{ for } B.
\end{array}
$$

We shall often omit the mention of a model $\mathcal{M}$ when it does not sacrifice clarity.

These clauses can be simplified with some set-theoretic notation. If we let

$$
\begin{aligned}
|A|^{+} &= \{s \in \mathcal{S} : s \Vdash^{+} A\} \\
|A|^{-} &= \{s \in \mathcal{S} : s \Vdash^{-} A\},
\end{aligned}
$$

then the clauses above can be restated thus:

$$
\begin{array}{rcl}
s \Vdash^{+} P & \Leftrightarrow & s \in |P|^{+}, \text{ for atomic } P; \\
s \Vdash^{-} P & \Leftrightarrow & s \in |P|^{-}, \text{ for atomic } P; \\
s \Vdash^{+} \neg A & \Leftrightarrow & s \in |A|^{-}; \\
s \Vdash^{-} \neg A & \Leftrightarrow & s \in |A|^{+}; \\
s \Vdash^{+} A \wedge B & \Leftrightarrow & s \in |A|^{+} \sqcup |B|^{+} \\
s \Vdash^{-} A \wedge B & \Leftrightarrow & s \in |A|^{-} \text{ or } s \in |B|^{-}; \\
s \Vdash^{+} A \vee B & \Leftrightarrow & s \in |A|^{+} \text{ or } s \in |B|^{+}; \\
s \Vdash^{-} A \vee B & \Leftrightarrow & s \in |A|^{-} \sqcup |B|^{-}; \\
s \Vdash^{+} \Box A & \Leftrightarrow & s \in \mathcal{B}(|A|^{-}); \\
s \Vdash^{-} \Box A & \Leftrightarrow & s \in \mathcal{A}(|A|^{-}).
\end{array}
$$

The notions of inexact verification and falsification (in symbol, respectively, $s \triangleright^{+} A$ and $s \triangleright^{-} A$) in an m-model $\mathcal{M}$ are defined as follows:

$$
\begin{array}{rcl}
\mathcal{M}, s \triangleright^{+} A & \Leftrightarrow & (\exists t)(t \sqsubseteq s \text{ and } \mathcal{M}, t \Vdash^{+} A); \\
\mathcal{M}, s \triangleright^{-} A & \Leftrightarrow & (\exists t)(t \sqsubseteq s \text{ and } \mathcal{M}, t \Vdash^{-} A).
\end{array}
$$

In an m-model $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ of Γ, a state $s \in \mathcal{S}$ is said to be *atomically sound* if and only if there is no $P \in At(\Gamma)$ such that $s \triangleright^{+} P$ and $s \triangleright^{-} P$. s is *atomically complete* if and only if for all $P \in At(\Gamma)$, $s \triangleright^{+} P$ or $s \triangleright^{-} P$. We shall also say that

t is *atomically s-incompatible* if and only if there is some propositional atom P such that $\{t\}$ and $|P|^+ \cup |P|^-$ are s-incompatible—that is,

$$\{t\} \sqcup (|P|^+ \cup |P|^-) \subseteq \overline{\beta}(s).$$

Definition 3 Given a set Γ of formulas, an m-model $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ of Γ is *normal* if and only if the underlying m-space $\Sigma = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$ is normal and v satisfies the following two conditions:

(**V1**) Every $s \in \mathcal{S}^\Diamond$ is atomically sound.
(**V2**) For any modal boundary s of Σ and $t \in \mathcal{S}^\Diamond$, if t is atomically s-incompatible, then $t \in \overline{\beta}(s)$.

(**V2**) is obviously analogous to (**N5ii**). Informally, when t is atomically s-incompatible, it means that s thinks it impossible to fuse t either with an exact verifier, or with an exact falsifier, for P. So, t is never part of a world accessible to s because every world is atomically complete in that it extends either an exact verifier or an exact falsifier for every propositional atom. So, s already has sufficient reason to preclude t. (In this connection, see Theorem 5 below.)

Before proceeding, it may help to consider a few examples of normal m-models.

Example 1 For an intuitive example, let us consider a state space $\langle \mathcal{S}, \sqsubseteq \rangle$ with four states, $\bot$, s, t, and $\top$. Let $\bot$ represent an essentialist principle that every chemical substance has a unique chemical composition as its necessary property. We also let s represent a state in which water is a chemical substance composed of H_2O molecules, and t represent a state in which water is composed of XYZ molecules, and finally $\top = s \sqcup t$. Let us assume, for the sake of simplicity, that every state exactly allows itself. Then it intuitively seems reasonable to assume that $t \in \beta(s)$ and $s \in \beta(t)$. Also, $\top \in \overline{\beta}(\bot)$ because $\top$ represents a state in which water has two different chemical composition, which $\bot$ regards as impossible.

To formalize all this, we define a state space $\langle \mathcal{S}, \sqsubseteq \rangle$ as follows:

- $\mathcal{S} = \{\bot, s, t, \top\}$;
- $\sqsubseteq = \{\langle \bot, x \rangle : x \in \mathcal{S}\} \cup \{\langle x, \top \rangle : x \in \mathcal{S}\} \cup \{\langle x, x \rangle : x \in \mathcal{S}\}$.

Then define an m-space $\Sigma_1 = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$ by letting μ be a function with domain $\mathcal{S}$ such that:

$$\begin{aligned}
\alpha(\bot) &= \{\bot\}, \quad \beta(\bot) = \{\top\};\\
\alpha(s) &= \{s\}, \quad \beta(s) = \{t\};\\
\alpha(t) &= \{t\}, \quad \beta(t) = \{s\};\\
\alpha(\top) &= \{\top\}, \quad \beta(\top) = \emptyset.
\end{aligned}$$

Σ_1 can be represented by the diagram in Fig. 1 thus:

The parthood relation is represented in the obvious way with the assumptions that each state is part of itself and that it is transitive (e.g., $\bot \sqsubseteq \top$). The exact allowing and banning relations are respectively indicated by normal arrows and by bar-arrows.

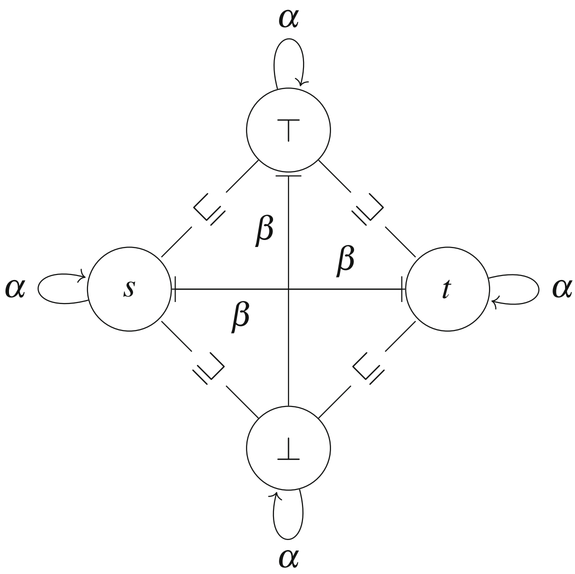

Fig. 1 Diagrammatic representation of Σ_1

The loop normal arrow around each state represents that it is exactly allowed by itself. Also, we can easily see from the bar arrows that $\top \in \beta(\bot)$, $s \in \beta(t)$, and $t \in \beta(s)$. From this diagram, we can easily calculate:

$\overline{\alpha}(\bot) = \alpha(\bot) = \{\bot\}$, and $\overline{\beta}(\bot) = \beta(\bot) = \{\top\}$
$\overline{\alpha}(s) = \alpha(s) \cup \alpha(\bot) = \{\bot, s\}$, and $\overline{\beta}(s) = \beta(s) \cup \beta(\bot) = \{\top, t\}$
$\overline{\alpha}(t) = \alpha(t) \cup \alpha(\bot) = \{\bot, t\}$, and $\overline{\beta}(t) = \beta(t) \cup \beta(\bot) = \{\top, s\}$
$\overline{\alpha}(\top) = \overline{\alpha}(s) \cup \overline{\alpha}(t) \cup \alpha(\top) = \mathcal{S}$, and $\overline{\beta}(\top) = \overline{\beta}(s) \cup \overline{\beta}(t) = \{\top, s, t\}$

We can thus see that $\top$ is modally unsound. Since both s and t are modally sound, it follows that s and t are modal boundaries of Σ_1; hence $\mathcal{S}^\Diamond = \{\bot, s, t\}$. So, Σ_1 satisfies (**N1**). We can easily check that Σ_1 satisfies both (**N2**) and (**N3**). For example, $\overline{\alpha}(s) = \{\bot, s\}$ is clearly downward closed. To see that (**N4**) is met, notice that $\overline{\alpha}(s) = \{\bot, s\}$. So, whatever is in $\overline{\alpha}(s)$ can be extended to a modal boundary, namely s itself; and the same goes for t also. It is obvious from the diagram that (**N5i**) is satisfied because every state is hit by some arrow. We finally check that (**N5ii**) is met. We consider s; the case for t is symmetric. Since $\mathcal{S}^\Diamond = \{\bot, t, s\}$ and since $t \in \overline{\beta}(s)$, it suffices to show that neither $\bot$ nor s is modally s-incompatible. Here we verify that $\bot$ is not modally s-incompatible. Assume, for contradiction, that $\bot$ is modally s-incompatible; let $U \subseteq \mathcal{S}^\Diamond$ be such that $\{\bot\} \sqcup (\mathcal{A}(U) \cup \mathcal{B}(U)) \subseteq \overline{\beta}(s)$. Notice two things. First, since $\overline{\beta}(s) = \{\top, t\}$, for any $x \in \mathcal{S}$, if $\bot \sqcup x \in \overline{\beta}(s)$ then x is either t or $\top$. Second, $\mathcal{A}(U) = U$. Therefore, $U \subseteq \{t, \top\}$. Now we argue by cases:

Case 1: $U = \emptyset$. Then $\mathcal{A}(U) \cup \mathcal{B}(U) = \{\bot\}$, and so $\{\bot\} \sqcup (\mathcal{A}(U) \cup \mathcal{B}(U)) = \{\bot\} \not\subseteq \overline{\beta}(s)$.
Case 2: $t \in U$. Then $s \in \mathcal{B}(U)$, and so $\{\bot\} \sqcup (\mathcal{A}(U) \cup \mathcal{B}(U)) \not\subseteq \overline{\beta}(s)$.
Case 3: $U = \{\top\}$. Then $\bot \in \mathcal{B}(U)$, and so $\{\bot\} \sqcup (\mathcal{A}(U) \cup \mathcal{B}(U)) \not\subseteq \overline{\beta}(s)$.

It thus follows that $\{\bot\} \sqcup (\mathcal{A}(U) \cup \mathcal{B}(U)) \not\subseteq \overline{\beta}(s)$, contradicting the assumption. In a similar fashion, we can also check that s is not modally s-incompatible.

Now, let P be the proposition that water is composed of H_2O molecules. We define a normal m-model $\mathcal{M}_1 = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ for Γ by setting $[s]^+ = \{P\}$, $[t]^- = \{P\}$, and $[s]^- = [t]^+ = \emptyset$; and for the other $x \in \mathcal{S}$, $[x]^+ = [x]^- = \emptyset$. So, $|P|^+ \cup |P|^- = \{s, t\}$. (**V1**) is clearly met. To see that (**V2**) is satisfied, it suffices to note that if $x \in \mathcal{S}^\Diamond$ is atomically s-incompatible, then $x = t \in \overline{\beta}(s)$.

Notice that $\mathcal{B}(|P|^-) = \{s\}$ because t is the sole exact falsifier for P and s is the sole modal state that exactly bans P. In this normal m-model, therefore, s is the exact verifier for $\Box P$, i.e., the proposition that it is necessary that water is composed uniquely of H_2O molecules. If we in addition let Q stand for the proposition that water is composed of XYZ molecules, and let t and s respectively be an exact verifier and falsifier for Q. Then t would be an exact verifier for $\Box Q$, i.e., the proposition that it is necessary water is composed of XYZ molecules.

Example 2 Note that the modal boundaries in Σ_1 were modally sound and complete. Here we give another example of a normal m-model which is based on the same but whose modal boundaries are not modally complete. This time, we define an m-space Σ_2 by modifying μ thus:

$$\begin{aligned}
&\alpha(\bot) = \{\bot\}, \quad \beta(\bot) = \emptyset;\\
&\alpha(s) = \{s\}, \quad \beta(s) = \emptyset;\\
&\alpha(t) = \{t\}, \quad \beta(t) = \emptyset;\\
&\alpha(\top) = \emptyset, \quad \beta(\top) = \mathcal{S}.
\end{aligned}$$

Σ_2 can be diagrammatically represented as in Fig. 2.

We check that Σ_2 is a normal m-space. Notice that the modally sound states of Σ_2 are s, t, and $\bot$. Since the only proper extension of s and t, namely $\top$, is modally unsound, we can see that the modal boundaries of Σ_2 are s and t. So, $\mathcal{S}^\Diamond = \{\bot, s, t\}$. So (**N1**) is met. (**N2**) and (**N3**) are clearly met also; for instance, $\overline{\alpha}(s) = \alpha(s) \cup \alpha(\bot) = \{s, \bot\}$ and $\overline{\beta}(s) = \beta(s) \cup \beta(\bot) = \emptyset$. To check that (**N4**) is satisfied, it suffices to note that $\overline{\alpha}(s) = \{\bot, s\}$ and $\overline{\alpha}(t) = \{\bot, t\}$. (**N5i**) is satisfied because every $x \in \mathcal{S}$ is hit by some arrow (i.e., either exactly allowed or banned by some state). Finally (**N5ii**) is vacuously satisfied, because $\overline{\beta}(s) = \overline{\beta}(t) = \emptyset$ and because for any $x \in \mathcal{S}^\Diamond$ and for any $U \subseteq \mathcal{S}$, $\{x\} \sqcup (\mathcal{A}(U) \cup \mathcal{B}(U))$ is nonempty by (**N5i**).

Let $\Gamma = \{P \wedge Q\}$. We define a normal m-model $\mathcal{M}_2 = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ for Γ by setting $[s]^+ = \{P\}$, $[t]^+ = \{Q\}$, and $[s]^- = [t]^- = \emptyset$; and for the other $x \in \mathcal{S}$, $[x]^+ = [x]^- = \emptyset$. So, $|P|^+ \cup |P|^- = \{s\}$ and $|Q|^+ \cup |Q|^- = \{t\}$. (**V1**) is clearly met. Again, (**V2**) is vacuously satisfied because $\overline{\beta}(s) = \overline{\beta}(t) = \emptyset$.

It is worth noting here that no states behave like possible worlds in $\mathcal{M}_2$. Recall that a possible world are considered a state that is both modally and atomically sound and complete; and there is no such state in $\mathcal{S}$. $\top$ is modally unsound. s is neither modally nor atomically complete; and similarly for t and $\bot$.

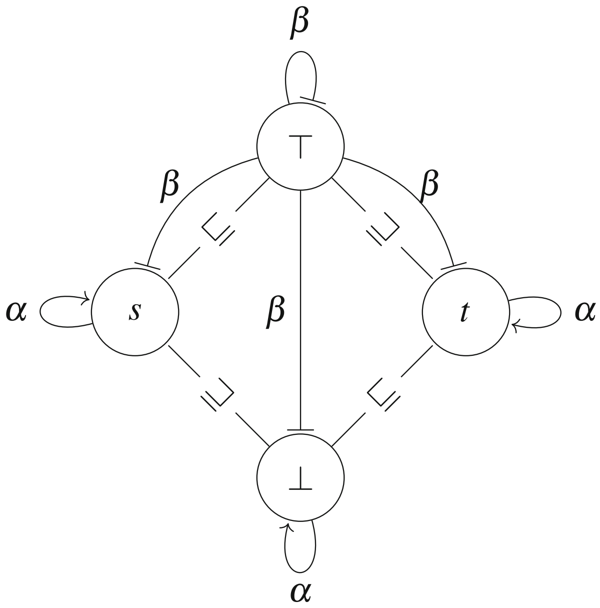

Fig. 2 Diagrammatic representation of Σ_2

Example 3 For a bit more complex example, let:

$$\mathcal{S} = \{\bot, s, t, w, w', u, \top\};$$
$$\sqsubseteq = \{\langle \bot, x\rangle : x \in \mathcal{S}\} \cup \{\langle x, \top\rangle : x \in \mathcal{S}\} \cup \{\langle x, u\rangle : x \in \mathcal{S} \setminus \{\top\}\} \cup \{\langle s, w'\rangle, \langle t, w'\rangle\}$$
$$\cup \{\langle x, x\rangle : x \in \mathcal{S}\}.$$

$\langle \mathcal{S}, \sqsubseteq\rangle$ clearly is a state space. Then we let μ be a function with domain $\{w, u\}$ such that

$$\alpha(w) = \{w', s, t, \bot\}, \quad \beta(w) = \{\top\};$$
$$\alpha(u) = \{w\}, \quad \beta(u) = \{w, u\}.$$

Then $\Sigma_3 = \langle \mathcal{S}, \sqsubseteq, \mu\rangle$ can be represented as in Fig. 3 below.

Σ_3 clearly satisfies (**N1**)-(**N3**). Since u is modally unsound, the modal boundaries of Σ_3 are w and w' and $\mathcal{S}^{\Diamond} = \{w, w', s, t, \bot\}$. (**N4**) is also met because everything in $\overline{\alpha}(w)$ is part of w'. Notice also that every state is hit by some arrow. So, (**N5i**) is satisfied. To verify that (**N5ii**) is satisfied, we only need to consider $\overline{\beta}(w) = \{\top\}$ because $\overline{\beta}(w') = \emptyset$. We check that for each $y \in \mathcal{S}^{\Diamond}$, if $y \notin \overline{\beta}(w)$ then there is no set U of states such that

$$\{y\} \sqcup (\mathcal{A}(U) \cup \mathcal{B}(U)) \subseteq \overline{\beta}(w) \tag{*}$$

Let $y = w'$. Notice that if there is a set U satisfying (*) then $\mathcal{A}(U) \cup \mathcal{B}(U) = \{\top\}$. Since $\top$ is not a modal state, there is no $U \subseteq \mathcal{S}$ such that $\top \in \mathcal{A}(U)$. For all $U \subseteq \mathcal{S}$, moreover, either $\mathcal{B}(U) = \{\bot\}$ (if $U = \emptyset$), or $\mathcal{B}(U) \subseteq \{w, u\}$ (otherwise). This is

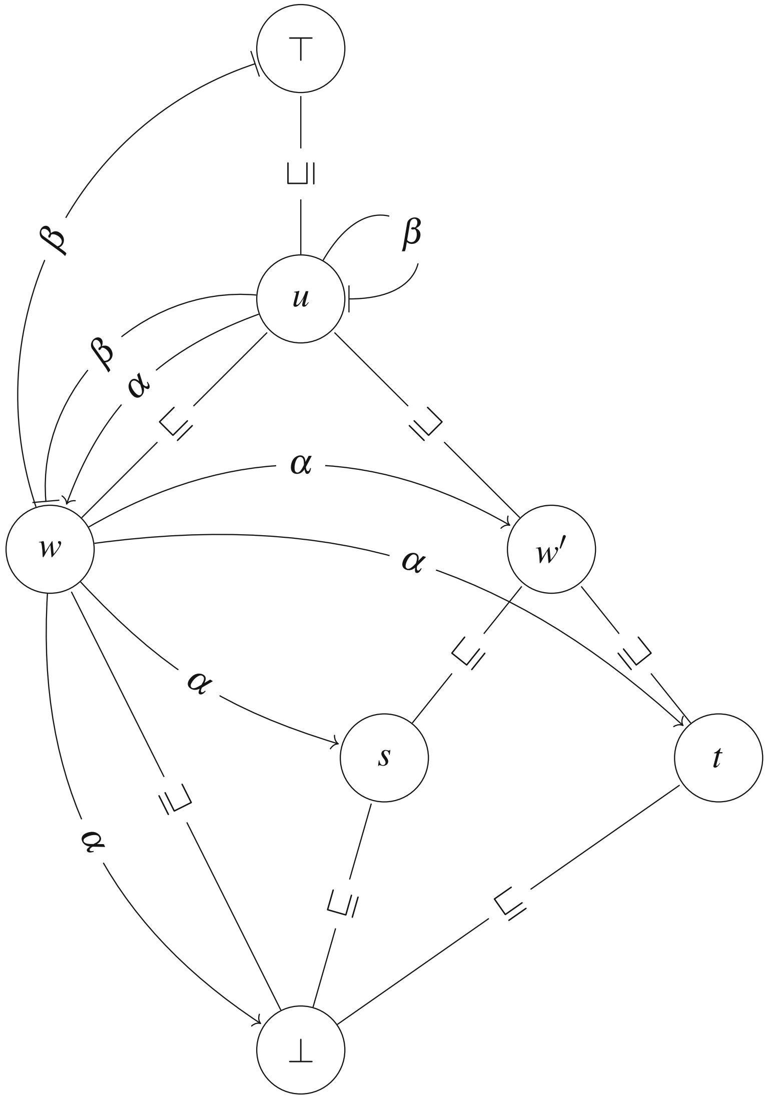

Fig. 3 Diagrammatic representation of Σ_3

because w and u are the only states that exactly bans a state and because $w \sqsubseteq u$. Hence there is no $U \subseteq \mathcal{S}$ such that $\mathcal{A}(U) \cup \mathcal{B}(U) = \{\top\}$. Therefore, no $U \subseteq \mathcal{S}$ satisfies (*) when $y = w'$. The same consideration applies to all the other states in $\mathcal{S}^{\Diamond}$. Hence (**N5ii**) is satisfied.

Let $\Gamma = \{P \wedge Q\}$, and define a normal m-model $\mathcal{M}_3$ based on Σ_3 by setting $[s]^+ = \{P\}$ and $[t]^+ = \{Q\}$, and, for the other $x \in \mathcal{S}$, $[x]^+ = [x]^- = \emptyset$. So, $|P|^+ \cup |P|^- = \{s\}$ and $|Q|^+ \cup |Q|^- = \{t\}$. (**V1**) is clearly met. To verify that (**V2**) is also met, note first that $\overline{\beta}(w') = \emptyset$. So we only need to consider $\overline{\beta}(w)$. It suffices to observe that $\{x\} \sqcup (|P|^+ \cup |P|^-) \subseteq \overline{\beta}(w)$ only if $x = \top$; and $\top \in \overline{\beta}(w)$. $|Q|^+ \cup |Q|^-$ can be treated similarly.

It is easy to see that no states in $\mathcal{M}_3$ behave like a possible world. $\top$ and u are modally unsound. w is neither modally nor atomically complete. w' is only atomically, but not modally, complete.

Finally, we provide the definition of consequence. For any set Γ of formulas and any formula A, we let $\Gamma; A$ denote $\Gamma \cup \{A\}$.

Definition 4 For any set Γ of formulas and a formula A, A is a *consequence of* Γ *in* a normal m-model $\mathcal{M}$ for $\Gamma; A$ iff for all $s \in \mathcal{S}^\Diamond$ such that $s \rhd^+ B$ for all $B \in \Gamma$, $s \not\rhd^- A$. A is a *consequence of* Γ *on* a normal m-space Σ iff A is consequence of Γ in all normal m-models based on Σ. A is *a consequence of* Γ *with respect to* a class $\mathfrak{S}$ of normal m-spaces iff A is a consequence of Γ on all normal m-spaces in $\mathfrak{S}$. In each case, A is said to be *valid* in the corresponding sense if $\Gamma = \emptyset$.

It should be obvious that this definition is a straightforward formalization of (MC2) from Section 5.

7 Basic Results

Here we establish some basic results. The upshots will be, first, that every modally sound state is consistent (Theorem 4), and second that the axiom K is valid in every normal m-model, i.e., valid with respect to the class of normal m-spaces (Theorem 5).

Proposition 1 *Let* Γ *be a set of formulas and* $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ *be a normal m-model of* Γ. *For each* $A \in Fml(\Gamma)$, $|A|^+ \cup |A|^- \neq \emptyset$.

Proof By induction on A. The base case holds because we require that for every propositional atom $P \in At(\Gamma)$, $|P|^+ \cup |P|^- \neq \emptyset$. For the induction step, assume that $|B|^+ \cup |B|^- \neq \emptyset$ and that $|C|^+ \cup |C|^- \neq \emptyset$. Let $A = \neg B$. Then

$$\begin{aligned} |A|^+ \cup |A|^- &= |\neg B|^+ \cup |\neg B|^- \\ &= |B|^- \cup |B|^+, \end{aligned}$$

which is nonempty by the I.H. Let $A = B \wedge C$. Notice:

$$|A|^+ \cup |A|^- = (|B|^+ \sqcup |C|^+) \cup (|B|^- \cup |C|^-).$$

So, it suffices to show that if $|B|^- = |C|^- = \emptyset$ then $|B|^+ \sqcup |C|^+ \neq \emptyset$. Suppose that $|B|^- = |C|^- = \emptyset$. Then, by the I.H., both $|B|^+$ and $|C|^+$ are nonempty. Hence $|B|^+ \sqcup |C|^+ \neq \emptyset$. The case where $A = B \vee C$ can be treated dually. Let $A = \Box B$. $|A|^+ \cup |A|^- = \mathcal{A}(|B|^-) \cup \mathcal{B}(|B|^-)$, which is nonempty by (**N5i**). □

Notice that Proposition 1 shows that every formula in $Fml(\Gamma)$ has either an exact verifier or falsifier in $\mathcal{M}$, which one would naturally want. This is why we have (**N5i**) in the formal definition of normal m-model.

Lemma 2 *Let $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ be a normal m-model of Γ. Let s be a modally sound state and t be a state. For any formula $A \in Fml(\Gamma)$, if $\{t\} \sqcup (|A|^+ \cup |A|^-) \subseteq \overline{\beta}(s)$ then $t \in \overline{\beta}(s)$.*

Proof Induction on A. The base case holds by (**V2**). Assume as the induction hypothesis that the property holds for B and C.

Let $A = \neg B$. Suppose that $\{t\} \sqcup (|A|^+ \cup |A|^-) \subseteq \overline{\beta}(s)$. Then, since $|A|^+ = |\neg B|^+ = |B|^-$ and $|A|^- = |\neg B|^- = |B|^+$, it follows that $\{t\} \sqcup (|B|^+ \cup |B|^-) \subseteq \overline{\beta}(s)$. By the I.H., then $t \in \overline{\beta}(s)$.

Let $A = B \wedge C$. Assume that $\{t\} \sqcup (|A|^+ \cup |A|^-) \subseteq \overline{\beta}(s)$. Since $|A|^+ = |B \wedge C|^+ = |B|^+ \sqcup |C|^+$ and $|A|^- = |B \wedge C|^- = |B|^- \cup |C|^-$, we have:

$$\{t\} \sqcup ((|B|^+ \sqcup |C|^+) \cup (|B|^- \cup |C|^-)) \subseteq \overline{\beta}(s),$$

which is equivalent to:

$$(\{t\} \sqcup (|B|^+ \sqcup |C|^+)) \cup (\{t\} \sqcup |B|^-) \cup (\{t\} \sqcup |C|^-) \subseteq \overline{\beta}(s).$$

Now, suppose that $|B|^+ \sqcup |C|^+ = \emptyset$. Then either $|B|^+ = \emptyset$ or $|C|^+ = \emptyset$. In the former case, it follows from Proposition 1 that $|B|^-$ is not empty; and $\{t\} \sqcup (|B|^+ \cup |B|^-) = \{t\} \sqcup |B|^- \subseteq \overline{\beta}(s)$. In the latter case, similarly, $\{t\} \sqcup (|C|^+ \cup |C|^-) \subseteq \overline{\beta}(s)$. Either way, it follows from the I.H. that $t \in \overline{\beta}(s)$. Suppose, on the other hand, that $|B|^+ \sqcup |C|^+ \neq \emptyset$. Let $b \in |B|^+$. Then,

$$\{t \sqcup b\} \sqcup |C|^+ \subseteq \overline{\beta}(s),$$

because $\{t \sqcup b\} \sqcup |C|^+ \subseteq \{t\} \sqcup (|B|^+ \sqcup |C|^+)$. Notice also that

$$\{t \sqcup b\} \sqcup |C|^- \subseteq \overline{\beta}(s),$$

because $\{t\} \sqcup |C|^- \subseteq \overline{\beta}(s)$ and $\overline{\beta}(s)$ is upward closed. By the I.H., then, $t \sqcup b \in \overline{\beta}(s)$. Since b was an arbitrary element of $|B|^+$, it follows that $\{t\} \sqcup |B|^+ \subseteq \overline{\beta}(s)$. Then, by the I.H., $t \in \overline{\beta}(s)$. The case where $A = B \vee C$ can be treated dually.

Let $A = \Box B$. Suppose that $\{t\} \sqcup (|\Box B|^+ \cup |\Box B|^-) \subseteq \overline{\beta}(s)$. This means:

$$\{t\} \sqcup (\mathcal{A}(|B|^-) \cup \mathcal{B}(|B|^-)) \subseteq \overline{\beta}(s).$$

By (**N5ii**), $t \in \overline{\beta}(s)$. □

Lemma 3 *Let $\mathcal{M}$ be a normal m-model of Γ and A be any formula. For all states s,*

$$s \rhd^+ \Box A \quad \Leftrightarrow \quad |A|^- \subseteq \overline{\beta}(s).$$

Proof The left-to-right direction is obvious. To see the other direction, suppose that $|A|^- \subseteq \overline{\beta}(s)$. Then, for each $t \in |A|^-$, there is $s_t \sqsubseteq s$ such that $t \in \beta(s_t)$. Then let f be a function that maps each $t \in |A|^-$ to s_t. ran(f) is an exact ban on $|A|^-$ and $\bigsqcup ran(f) \sqsubseteq s$. □

It is worth noting that this proof requires the axiom of choice.

Theorem 4 *Let $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ be a normal m-model of Γ. For all $A \in Fml(\Gamma)$ and for all $s \in \mathcal{S}^{\Diamond}$, either $s \not\triangleright^{+} A$ or $s \not\triangleright^{-} A$.*

Proof By induction on A. It suffices to check that if s is both atomically and modally sound, then either $s \not\triangleright^{+} A$ or $s \not\triangleright^{-} A$. So, let s be a atomically and modally sound state.

The base case holds because s is atomically sound.

Let $A = \neg B$. By the I.H., either $s \not\triangleright^{+} B$ or $s \not\triangleright^{-} B$. In the former case, $s \not\triangleright^{-} \neg B$, and in the latter case $s \not\triangleright^{+} \neg B$.

Let $A = B \wedge C$. If $s \triangleright^{+} B$ and $s \triangleright^{+} C$, then it follows from the I.H. that $s \not\triangleright^{-} B$ and $s \not\triangleright^{-} C$ and hence that $s \not\triangleright^{-} B \wedge C$. Otherwise, $s \not\triangleright^{+} B \wedge C$. The case where $A = B \vee C$ can be treated dually.

Finally, $A = \Box B$. If $s \triangleright^{+} \Box B$ and $s \triangleright^{-} \Box B$, then it would follow from Lemma 3 that $|B|^{-} \subseteq \overline{\beta}(s)$. Also, $|B|^{-} \cap \overline{\alpha}(s) \neq \emptyset$; but this is impossible because s is modally sound. □

Theorem 5 *Let Γ be a set of formulas and $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ be a normal m-model of Γ. For any $s \in \mathcal{S}^{\Diamond}$, $s \not\triangleright^{-} \Box(A \supset B) \supset (\Box A \supset \Box B)$ (where $A, B \in Fml(\Gamma)$).*

Proof Pick any $s \in \mathcal{S}^{\Diamond}$, and let t be a modal boundary containing s. It suffices to verify that $t \not\triangleright^{-} \Box(A \supset B) \supset (\Box A \supset \Box B)$. Assume, for contradiction, that $t \triangleright^{-} \Box(A \supset B) \supset (\Box A \supset \Box B)$. Then $t \triangleright^{+} \Box(A \supset B)$, $t \triangleright^{+} \Box A$, and $t \triangleright^{-} \Box B$. By Lemma 3, this means:

$$|A|^{+} \sqcup |B|^{-} \subseteq \overline{\beta}(t), \quad |A|^{-} \subseteq \overline{\beta}(t), \quad \text{and } |B|^{-} \cap \overline{\alpha}(t) \neq \emptyset.$$

Let $u \in |B|^{-} \cap \overline{\alpha}(t)$. Then $\{u\} \sqcup (|A|^{+} \cup |A|^{-}) \subseteq \overline{\beta}(s)$. By Lemma 2, then, $u \in \overline{\beta}(t)$. But then $u \in \overline{\alpha}(t) \cap \overline{\beta}(t)$, contradicting the modal soundness of t. □

8 World Model

In Section 6, we have seen a couple of examples of normal m-models in which there are no possible worlds, i.e., modally and atomically complete states. These examples show that normal m-models do not necessarily presuppose the existence of possible worlds. In this section, we shall introduce a special subclass of normal m-models with states that behave just like possible worlds.

Let $\Sigma = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$ be an m-space. We say that a subset $\mathcal{R}$ of $\mathcal{S}$ is *worldly* if and only if $\mathcal{R}$ is nonempty and every $w \in \mathcal{R}$ is modally sound and complete.

Definition 5 An m-space $\Sigma = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$ is said to be a *world space* (or, simply, *w-space*) if and only if it satisfies (**N2**), (**N3**), (**N4**), and

(**W1**) the modal boundaries are worldly.

It should be clear that the definition is a straightforward formalization of the informal analysis of possible worlds in Section 4 that possible worlds are modally sound and complete states that are robust about the possibilities.

Proposition 6 *W-spaces are normal m-spaces.*

Proof It suffices to check that a w-space satisfies (**N1**), (**N5i**), and (**N5ii**). It is clear that (**N1**) is satisfied. (**N5i**) is also met because the modal boundaries are modally complete. To verify (**N5ii**), let w be a modal boundary and $t \in \mathcal{S}^{\Diamond}$. Suppose that there is a set U of states such that

$$\{t\} \sqcup (\mathcal{A}(U) \cup \mathcal{B}(U)) \subseteq \overline{\beta}(w).$$

We need to show that $t \in \overline{\beta}(w)$. Since w is modally complete, it suffices to show that $t \notin \overline{\alpha}(w)$. Assume, for contradiction, that $t \in \overline{\alpha}(w)$. Then it would follow from (**N4**) that there is a modal boundary w' such $t \sqsubseteq w'$ and $w' \in \overline{\alpha}(w)$. Since w' should also be modally complete, w' should extend some $u \in \mathcal{A}(U) \cup \mathcal{B}(U)$. Now, consider $t \sqcup u$. Notice that $t \sqcup u \sqsubseteq w'$ and $t \sqcup u \in \overline{\beta}(w)$. By (**N3**), then, $w' \in \overline{\beta}(w)$. Hence $w' \in \overline{\alpha}(w) \cap \overline{\beta}(w)$, contradicting the modal soundness of w. Therefore, $t \notin \overline{\alpha}(w)$. □

Definition 6 Let Γ be a set of formulas. We define a *world model* (in short, *w-model*) $\mathcal{M}$ to be $= \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$, where $\langle \mathcal{S}, \sqsubseteq, \mu \rangle$ is a w-space and v is a valuation satisfying:

(**W2**) every modal boundary is atomically sound and complete.

In a world model, each modal boundary either verifies or falsifies every propositional atom in the inexact sense.

Proposition 7 *W-models are normal m-models.*

Proof Let $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ be a w-model. It is obvious that (**W2**) implies (**V1**). For, if every modal boundary is atomically sound, then everything below it must also be atomically sound. To check that $\mathcal{M}$ satisfies (**V2**), let w be a modal boundary and $t \in \mathcal{S}^{\Diamond}$. Assume that there is a propositional atom $P \in At(\Gamma)$ such that

$$\{t\} \sqcup (|P|^{+} \cup |P|^{-}) \subseteq \overline{\beta}(w).$$

We need to show that $t \in \overline{\beta}(w)$. Again, it suffices to show that $t \notin \overline{\alpha}(w)$. Suppose, for contradiction, that $t \in \overline{\alpha}(w)$. By (**N4**), there would be a modal boundary w' such that $t \sqsubseteq w'$ and $w' \in \overline{\alpha}(w)$. By (**W2**), then, w' would be atomically complete. Hence some part u of w' would belong to $(|P|^{+} \cup |P|^{-})$. Then it would follow from the assumption that $t \sqcup u \in \overline{\beta}(w)$. Since $t \sqcup u \sqsubseteq w'$, it would follow by (**N3**) that $w' \in \overline{\beta}(w)$, contradicting the modal soundness of w. □

Theorem 8 *Let* $\mathcal{M}$ *be a world model of* Γ *and* w *be a modal boundary. Then for all formulas* A*, either* $w \rhd^{+} A$ *or* $w \rhd^{-} A$*, but not both.*

Proof By induction on A. The base case holds because of (**W2**). The cases for the Boolean connectives are straightforward, so we omit them here. Let $A = \Box B$. We argue by cases. Suppose first that $|B|^{-} \subseteq \overline{\beta}(w)$. By Lemma 3, $w \rhd^{+} \Box B$. Since w is modally sound, moreover, $|B|^{-} \cap \overline{\alpha}(w) = \emptyset$. So, $w \not\rhd^{-} \Box B$. Now suppose that $|B|^{-} \nsubseteq \overline{\beta}(w)$. By Lemma 3, then, $w \not\rhd^{+} \Box B$. Since w is modally complete, $|B|^{-} \cap \overline{\alpha}(w) \neq \emptyset$; so, $w \rhd^{-} \Box B$. □

This theorem shows that the modal boundaries in a w-model consistently determine the truth value of every formula and hence are plausibly considered possible worlds.

9 Completeness Result

We now establish the soundness and completeness results for the minimal system K of normal modal propositional logic with respect to the proposed truthmaker semantics. This result will be extended to stronger systems in the next section. Below we shall assume that the present readership is already familiar with the standard Kripke semantics.

To establish the soundness result, we show that every normal m-model can be "completed" to a w-model, which can in turn be translated to an equivalent Kripke model. This shows that no theorem of K has a countermodel in the proposed truthmaker semantics. To prove the completeness result, on the other hand, we show that every Kripke model can be translated to an equivalent w-model, *a fortiori*, to an equivalent normal m-model. This shows that every formula, if not a theorem of K, has a countermodel in the proposed truthmaker semantics.

Translation from normal m-models to Kripke models

We first consider how to translate normal m-models to Kripke models. Let $\Sigma = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$ be a normal m-space. A *completion* of Σ is a *w-space* $\Sigma' = \langle \mathcal{S}, \sqsubseteq, \mu' \rangle$ such that $dom(\mu) \subseteq dom(\mu')$ and such that, for each $s \in dom(\mu)$, $\alpha(s) \subseteq \alpha'(s)$ and $\beta(s) \subseteq \beta'(s)$, where $\mu(s) = \langle \alpha(s), \beta(s) \rangle$ and $\mu'(s) = \langle \alpha'(s), \beta'(s) \rangle$. Intuitively, a completion Σ' is obtained from Σ by extending μ so as to make the modal boundaries modally complete.

One natural way of obtaining a completion is by "closing off," so to speak, the modal boundaries; that is, by letting each modal boundary be a modal state that bans every state that it does not allow (in the inexact sense) under the original μ. To make this precise, let $\Sigma = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$ be a normal m-space. We extend μ to μ^* as follows: $dom(\mu^*) = dom(\mu) \cup \{w : w \text{ is a modal boundary of } \Sigma\}$ and μ^* agrees with μ on all $s \in dom(\mu)$ except that for all modal boundaries w of Σ, $\beta^*(w) = \mathcal{S} \setminus \overline{\alpha}(w)$.[18] Then define the *close-off* Σ^* of Σ to be $\langle \mathcal{S}, \sqsubseteq, \mu^* \rangle$.

Proposition 9 *For every normal m-space Σ, Σ^* is a completion of Σ.*

Proof Let $\Sigma = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$ be a normal m-space. Let $\Sigma^* = \langle \mathcal{S}, \sqsubseteq, \mu^* \rangle$ be as just defined. It is obvious from the definition of μ^* that, for all $s \in dom(\mu)$, $\alpha(s) \subseteq \alpha^*(s)$ and $\beta(s) \subseteq \beta^*(s)$. So, it only remains to show that $\Sigma^* = \langle \mathcal{S}, \sqsubseteq, \mu^* \rangle$ is a w-space.

Since it is clear from the construction that the modal boundaries of Σ^* are modally sound and complete, we only need to check that Σ^* satisfies (**N2**), (**N3**), and (**N4**). To verify that Σ^* satisfies (**N2**), it is sufficient to observe that, for all states $s \in \mathcal{S}$, $\overline{\alpha^*}(s) = \overline{\alpha}(s)$ and $\overline{\alpha}(s)$ is downward closed because Σ is a normal m-space.

We now show that Σ^* satisfies (**N3**), i.e., that for all $s \in \mathcal{S}$, $\overline{\beta^*}(s)$ is upward closed. We argue by cases:

Case 1: s is an absolute possibility that is not a modal boundary of Σ. In this case, $\overline{\beta^*}(s) = \overline{\beta}(s)$. Hence $\overline{\beta^*}(s)$ is upward closed.

[18] In case $w \in dom(\mu)$, we let $\alpha^*(w) = \alpha(w)$. Otherwise, $\alpha^*(w) = \emptyset$.

Case 2: s is a modal boundary of Σ. In this case, $\overline{\beta^*}(s) = \mathcal{S} \setminus \overline{\alpha}(s)$, which is upward closed because $\overline{\alpha}(s)$ is downward closed.

Case 3: s is not an absolute possibility of Σ. Suppose that $t \in \overline{\beta^*}(s)$. Pick any $t' \in \mathcal{S}$ with $t \sqsubseteq t'$. Now, either $t \in \overline{\beta}(s)$ or $t \in \overline{\beta^*}(w)$ for some modal boundary $w \sqsubseteq s$. Either way, $t' \in \overline{\beta^*}(s)$; for both $\overline{\beta}(s)$ and $\overline{\beta^*}(w)$ are upward closed.

Before turning to (**N4**), observe that Σ and Σ^* have exactly the same modal boundaries because the modally sound states of Σ remain modally sound in Σ^*, and similarly for the modally unsound states of Σ. Now we show that Σ^* satisfies (**N4**). Let w be a modal boundary of Σ^* and $t \in \mathcal{S}^{\Diamond}$. Suppose that $t \in \overline{\alpha^*}(w)$. Recall that $\overline{\alpha^*}(w) = \overline{\alpha}(w)$. So, $t \in \overline{\alpha}(w)$. Since Σ satisfies (**N4**), it follows that there is a modal boundary w' of Σ such that $t \sqsubseteq w'$ and $w' \in \overline{\alpha}(w)$. Since Σ and Σ^* have exactly the same modal boundary and $\overline{\alpha}(w) = \overline{\alpha^*}(w)$, it follows that $t \sqsubseteq w'$ and $w' \in \overline{\alpha^*}(w)$ in Σ^*. □

Now let $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ be a normal m-model. We define a *completion* of $\mathcal{M}$ to be a w-model $\langle \mathcal{S}, \sqsubseteq, \mu', v' \rangle$ such that $\langle \mathcal{S}, \sqsubseteq, \mu' \rangle$ is a completion of $\langle \mathcal{S}, \sqsubseteq, \mu \rangle$ and v' is such that for each s, $[s]^+ \subseteq [s]'^+$ and $[s]^- \subseteq [s]'^-$, and such that v' satisfies (**C**). Given any normal m-model $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$, let v^* be a valuation such that v^* agrees with v for all $s \in \mathcal{S}$ except that for all modal boundaries w of Σ^*, $[w]^{*-} = \{P \in At(\Gamma) : w \not\trianglerighteq^+ P\}$; that is, v^* is obtained from v by letting each modal boundary exactly falsify all the propositional atoms that it does not verify (in the inexact sense) under the original v. Then define the *close-off* $\mathcal{M}^*$ of $\mathcal{M}$ to be $\langle \mathcal{S}, \sqsubseteq, \mu^*, v^* \rangle$, where μ^* is as defined above. Then it is immediate from the definition that $\mathcal{M}^*$ satisfies (**W2**). By Proposition 9, we thus have:

Proposition 10 *For every normal m-model $\mathcal{M}$, $\mathcal{M}^*$ is a completion of $\mathcal{M}$.*

Lemma 11 *Let $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ be a normal m-model of Γ and let $\mathcal{M}' = \langle \mathcal{S}, \sqsubseteq, \mu', v' \rangle$ be a completion of $\mathcal{M}$. Then, for all $A \in Fml(\Gamma)$:*

$$\mathcal{M}, s \trianglerighteq^+ A \quad \Rightarrow \quad \mathcal{M}', s \trianglerighteq^+ A;$$
$$\mathcal{M}, s \trianglerighteq^- A \quad \Rightarrow \quad \mathcal{M}', s \trianglerighteq^- A.$$

Proof By straightforward induction on A. □

Now we show that every w-model has an equivalent Kripke model. Let Γ be a set of formulas and $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ be a w-model of Γ. Define the corresponding Kripke model $K(\mathcal{M}) = \langle W^{\mathcal{M}}, R^{\mathcal{M}}, \Phi^{\mathcal{M}} \rangle$ of Γ as follows:

- $W^{\mathcal{M}}$ = the modal boundaries of $\mathcal{M}$;
- $R^{\mathcal{M}} = \{\langle w, w' \rangle \in W_{\mathcal{M}} \times W_{\mathcal{M}} : w' \in \overline{\alpha}(w)\}$;
- $\Phi^{\mathcal{M}}(w, P) = \begin{cases} T, & w \trianglerighteq^+ P; \\ F, & \text{otherwise.} \end{cases}$

This clearly defines a Kripke model of Γ. The notion of a formula A's *being true at* a world w in a Kripke model (in symbol, $w \models A$) is defined in the usual way.

Theorem 12 *Let $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v\rangle$ be a w-model of Γ and $K(\mathcal{M}) = \langle W^{\mathcal{M}}, R^{\mathcal{M}}, \Phi^{\mathcal{M}}\rangle$ be the corresponding Kripke model. For all formulas A and for all $w \in W^{\mathcal{M}}$,*

$$\begin{aligned} \mathcal{M}, w \rhd^{+} A &\quad\Leftrightarrow\quad w \models A \text{ in } K(\mathcal{M}); \\ \mathcal{M}, w \rhd^{-} A &\quad\Leftrightarrow\quad w \not\models A \text{ in } K(\mathcal{M}). \end{aligned}$$

Proof By Theorem 8, it suffices to verify the first equivalence. By induction on A. Let P be a propositional atom. For all $w \in W^{\mathcal{M}}$:

$$\begin{aligned} w \rhd^{+} P &\quad\Leftrightarrow\quad \Phi^{\mathcal{M}}(w, P) = T; \\ &\quad\Leftrightarrow\quad w \models P. \end{aligned}$$

Let $A = \neg B$, for some B. Then

$$\begin{aligned} w \rhd^{+} \neg B &\quad\Leftrightarrow\quad w \rhd^{-} B; && \text{then, by the I.H.,} \\ &\quad\Leftrightarrow\quad w \not\models B; \\ &\quad\Leftrightarrow\quad w \models \neg B. \end{aligned}$$

Let $A = B \wedge C$, for some B and C. Then

$$\begin{aligned} w \rhd^{+} B \wedge C &\quad\Leftrightarrow\quad w \rhd^{+} B \text{ and } w \rhd^{+} C; && \text{then, by the I.H.,} \\ &\quad\Leftrightarrow\quad w \models B \text{ and } w \models C; \\ &\quad\Leftrightarrow\quad w \models B \wedge C; \end{aligned}$$

The case where $A = B \vee C$ can be proved dually.

Finally, let $A = \Box B$ for some B.

$$\begin{aligned} w \rhd^{+} \Box B &\Leftrightarrow |B|^{-} \subseteq \overline{\beta}(w) && (1) \\ &\Leftrightarrow (w' \in W^{\mathcal{M}})(w' \rhd^{-} B \Rightarrow w' \in \overline{\beta}(w)); && (2) \\ &\Leftrightarrow (w' \in W^{\mathcal{M}})(w' \notin \overline{\beta}(w) \Rightarrow w' \not\rhd^{-} B); && \text{then, since } w \text{ is modally sound and complete,} \\ &\Leftrightarrow (w' \in W^{\mathcal{M}})(w' \in \overline{\alpha}(w) \Rightarrow w' \not\rhd^{-} B); && \text{then, by Theorem 8,} \\ &\Leftrightarrow (w' \in W^{\mathcal{M}})(w' \in \overline{\alpha}(w) \Rightarrow w' \rhd^{+} B); && \text{then, by the definition of } R^{\mathcal{M}}, \\ &\Leftrightarrow (w' \in W^{\mathcal{M}})(R^{\mathcal{M}}(w, w') \Rightarrow w' \rhd^{+} B); && \text{then, by the I.H.,} \\ &\Leftrightarrow (w' \in W^{\mathcal{M}})(R^{\mathcal{M}}(w, w') \Rightarrow w' \models B); \\ &\Leftrightarrow w \models \Box B. \end{aligned}$$

We check the equivalence between (1) and (2). Assume (1). Let $w' \in W_{\mathcal{M}}$. Suppose that $w' \rhd^{-} B$, i.e., that w' extends some $t' \in |B|^{-}$. Since $t' \in \overline{\beta}(w)$, $w' \in \overline{\beta}(w)$. Conversely, assume (2). Let $t \in |B|^{-}$. Assume, for contradiction, that $t \notin \overline{\beta}(w)$. Since w is modally complete, $t \in \overline{\alpha}(w)$. So there is $w' \in W^{\mathcal{M}}$ such that $t \sqsubseteq w'$ and $w' \in \overline{\alpha}(w)$. But then (2) implies that $w' \in \overline{\beta}(w)$. So $w' \in \overline{\alpha}(w) \cap \overline{\beta}(w)$, which contradicts the modal soundness of w. So, $|B|^{-} \subseteq \overline{\beta}(w)$. □

Translation from Kripke models to normal m-models

We now consider the converse embedding of Kripke models into normal m-models. There is a sense in which the Kripke models form a special subclass of w-models. Let $K = \langle W, R, \Phi \rangle$ be a Kripke model of Γ, where W is a nonempty set, R is a binary relation on W, and Φ is a valuation that assigns a truth-value to every $P \in At(\Gamma)$ at each $w \in W$. For each $w \in W$, let $R_w = \{w' \in W : R(w, w')\}$.

Define the corresponding w-space $\Sigma(K) = \langle \mathcal{S}^K, \sqsubseteq^K, \mu^K \rangle$ as follows:

- $\mathcal{S}^K = W \cup \{\bot, \top\}$.
- $\sqsubseteq^K = \{\langle x, x \rangle : x \in \mathcal{S}\} \cup \{\langle \bot, x \rangle : x \in \mathcal{S}\} \cup \{\langle x, \top \rangle : x \in \mathcal{S}\}$
- μ^K is a function with domain $W \cup \{\top\}$ such that for all $w \in W$,

$$\alpha(w) = \emptyset \text{ and } \beta(w) = \mathcal{S}, \qquad \text{if } R_w = \emptyset;$$
$$\alpha(w) = R_w \cup \{\bot\} \text{ and } \beta(w) = \mathcal{S} \setminus \alpha(w), \qquad \text{if } R_w \neq \emptyset.$$

and such that $\alpha(\top) = \beta(\top) = \{\top\}$.

That is, $\Sigma(K)$ is obtained from K by adding the bottom and top states, placing in between the members of W as mutually incomparable states, and then setting up the allowing and banning relations in the obvious way.

Example 4 Define a Kripke frame $K_1 = \langle W, R \rangle$ as follows:

- $W = \{w_1, w_2\}$
- $R = \{\langle w_1, w_2 \rangle\}$

We may represent K with the following diagram, where thick triangle arrows are used to indicate the accessibility relation R in Fig. 4 below.

To obtain the corresponding w-space $\Sigma(K_1)$, we first let $\mathcal{S}^{K_1} = W \cup \{\bot, \top\} = \{w_1, w_2, \bot, \top\}$. Then we let w_1 and w_2 be mutually incomparable states between $\bot$ and $\top$. To set up μ^{K_1}, we first let $dom(\mu^{K_1}) = W \cup \{\top\}$. Consider w_1. Since $R_{w_1} = \{w_2\}$, we let w_1 allow $\alpha(w_1) = \{\bot, w_2\}$ and $\beta(w_1) = \{\top\}$; and since $R_{w_2} = \emptyset$, let $\alpha(w_2) = \emptyset$ and $\beta(w_2) = \{w_1, w_2, \bot, \top\}$. Finally, we set $\alpha(\top) = \beta(\top) = \{\top\}$. Then $\Sigma(K_1)$ can be diagrammatically represented as in Fig. 5.

We now define the corresponding w-model $\mathcal{M}(K)$ by adjoining to $\Sigma(K)$ a valuation=v^K defined as follows: for all $w \in W$ and $P \in At(\Gamma)$,

$$P \in [w]^+ \quad \Leftrightarrow \quad \Phi(w, P) = T;$$
$$P \in [w]^- \quad \Leftrightarrow \quad \Phi(w, P) = F;$$

In a sense, the Kripke models can be thought of as constituting a special subclass of w-models in which each world consists, as it were, of "one great fact."

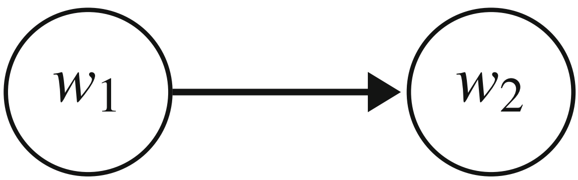

Fig. 4 Diagrammatic representation of K_1

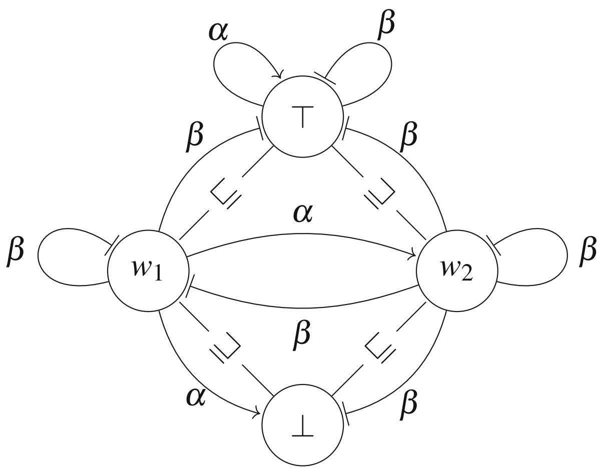

Fig. 5 Diagrammatic representation of $\Sigma(K_1)$

Proposition 13 *Let $K = \langle W, R, \Phi\rangle$ be a Kripke model. Then $\Sigma(K) = \langle \mathcal{S}^K, \sqsubseteq^K, \mu^K\rangle$ is a w-space, and hence a normal m-space.*

Proof Let $K = \langle W, R, \Phi\rangle$ be a Kripke model. Let $\Sigma(K) = \langle \mathcal{S}^K, \sqsubseteq^K, \mu^K\rangle$ be defined as above. Clearly, $\langle \mathcal{S}^K, \sqsubseteq^K\rangle$ is a state space. It is also clear from the definition that every $w \in W$ is modally sound and complete. It is also immediate from the definition of μ^K that $\Sigma(K)$ satisfies (**N2**) and (**N3**). To check that $\Sigma(K)$ satisfies (**N4**), notice first that W forms the modal boundaries of $\Sigma(K)$. And it is clear from the definition that every $w \in W$ is modally sound and complete. Now let $w \in W$ and $s \in (\mathcal{S}^K)^\Diamond$. Suppose that $s \in \overline{\alpha}(w)$. We need to check that there is a modal boundary $w' \in \mathcal{S}^K$ such that $s \sqsubseteq w'$ and $w' \in \overline{\alpha}(w)$. Notice that when $s \in \overline{\alpha}(w)$, either $s = \bot$ or $s = w^*$ for some $w^* \in R_w$. In the latter case, we can let w^* be the desired w' itself. In the former case, notice first that, by construction, $R_w \neq \emptyset$. Picking any $w' \in R_w$, therefore, we have: $\bot \sqsubseteq w'$ and $w' \in \overline{\alpha}(w)$. Hence $\Sigma(K)$ satisfies (**N4**).

Lemma 14 *Let $K = \langle W, R, \Phi\rangle$ be a Kripke model. Then $\mathcal{M}(K) = \langle \mathcal{S}^K, \sqsubseteq^K, \mu^K, v^K\rangle$ is a w-model, and hence a normal m-model.*

Proof It suffices to show that $\mathcal{M}(K)$ satisfies (**W2**). Observe that in the original Kripke model, either $w \models P$, in which case $w \rhd^+ P$, or $w \not\models P$, in which case $w \rhd^- P$, but never both. □

Theorem 15 *Let $K = \langle W, R, \Phi\rangle$ be a Kripke model of Γ and $\mathcal{M}(K) = \langle \mathcal{S}^K, \sqsubseteq^K, \mu^K, v^K\rangle$ be the corresponding w-model. For any $w \in W$ and any $A \in Fml(\Gamma)$,*

$$w \models A \text{ in } K \quad \Leftrightarrow \quad w \rhd^+ A \text{ in } \mathcal{M}(K);$$
$$w \not\models A \text{ in } K \quad \Leftrightarrow \quad w \rhd^- A \text{ in } \mathcal{M}(K).$$

Proof By Theorem 8, it suffices to verify the first equivalence. By induction on A. The base case is immediate from the construction, and the cases for the Boolean connectives are straightforward; so we omit them. Let $A = \Box B$. To verify the first equivalence, suppose first that $w \models \Box B$ in K. By Lemma 3, it suffices to show that $|B|^- \subseteq \overline{\beta}(w)$ in $\mathcal{M}(K)$. Assume, for contradiction, that $|B|^- \not\subseteq \overline{\beta}(w)$. Since w is modally complete, $|B|^- \cap \overline{\alpha}(w) \neq \emptyset$. By (**N4**), then, there would be a $w' \in W$ such that $w' \in \overline{\alpha}(w)$ and $w' \rhd^- B$. By the I.H., then, $w' \in R_w$ and $w' \not\models B$ in K. But this contradicts the assumption that $w \models \Box B$. Conversely, suppose that $w \rhd^+ \Box B$ in $\mathcal{M}(K)$. We show that for all $w' \in W$, if $w' \not\models B$ in K then $w' \notin R_w$. Pick an arbitrary w' and assume that $w' \not\models B$ in K. By the I.H., $w' \rhd^- B$ in $\mathcal{M}(K)$. Since $w \rhd^+ \Box B$, it follows that $w' \in \overline{\beta}(w)$; therefore, $w' \notin R_w$. □

We now turn to consequence. For any set Γ formulas and a formula A, we shall write $\Gamma \models A$ to mean that, for all Kripke models $K = \langle W, R, \Phi \rangle$ of $\Gamma; A$ and for all worlds w in W, if $w \models B$ for all $B \in \Gamma$ then $w \models A$.[19] We say that $\Gamma \mid\models A$ if and only if A is a consequence of Γ with respect to the class of normal m-spaces. In other words, $\Gamma \mid\models A$ if and only if for all normal m-models $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ of $\Gamma; A$ and for all $s \in \mathcal{S}^\Diamond$, if $s \rhd^+ B$ for all $B \in \Gamma$ then $s \not\rhd^- A$. It is immediate from Definition 4 that A is valid with respect to the class of normal m-models just in case $\emptyset \mid\models A$.

Theorem 16 *For all sets Γ of sentences and a sentence A,*

$$\Gamma \models A \quad \Leftrightarrow \quad \Gamma \mid\models A.$$

Proof Suppose that $\Gamma \mid\not\models A$. For some normal m-model $\mathcal{M}$ and for some state $s \in \mathcal{S}^\Diamond$, $s \rhd^+ B$ for all $B \in \Gamma$ and $s \rhd^- A$. Let $\mathcal{M}'$ be a completion of $\mathcal{M}$, and consider the corresponding Kripke model $K(\mathcal{M}')$. By Lemma 11 and Theorem 12, it follows that there is a world $w \in W^{\mathcal{M}}$ such that $w \models B$ for all $B \in \Gamma$ and $w \not\models A$ in $K(\mathcal{M})$; that is, $\Gamma \not\models A$.

Conversely, suppose that $\Gamma \not\models A$. For some Kripke model $K = \langle W, R, \Phi \rangle$ and some world $w \in W$, $w \models B$ for all $B \in \Gamma$ and $w \not\models A$. Let $\mathcal{M}(K) = \langle \mathcal{S}^K, \sqsubseteq^K, \mu^K, v^K \rangle$ be the corresponding w-model as defined above. It follows from Theorem 15 that in $\mathcal{M}(K)$, $w \rhd^+ B$ for all $B \in \Gamma$ and $w \rhd^- A$. Since $w \in (\mathcal{S}^K)^\Diamond$, $\Gamma \mid\not\models A$. □

Let's write $\Gamma \vdash_{\mathsf{K}} A$ to mean that A is derivable from Γ in the system K. Given the soundness and completeness results of K with respect to the Kripke semantics, we have:

Corollary 17 *For all sets Γ of sentences and a sentence A,*

$$\Gamma \vdash_{\mathsf{K}} A \quad \Leftrightarrow \quad \Gamma \mid\models A.$$

10 Truthmaker Semantical Analysis of Modal Axioms

One of the main advantages of the Kripke semantics is that it offers the "reduction" of various modal axioms to conditions on frames. The result is well-known as the correspondence theorem, summarized in Table 1 below (adopted from [4]). The current

[19] This is sometimes called the *local* consequence relation in the context of the Kripke semantics.

Table 1 Some well-known modal axioms and the corresponding frame conditions

Modal Axiom	Scheme	Frame Condition
D	$\Box A \supset \Diamond A$	Serial
T	$\Box A \supset A$	Reflexive
4	$\Box A \supset \Box\Box A$	Transitive
B	$A \supset \Box\Diamond A$	Symmetric
5	$\Diamond A \supset \Box\Diamond A$	Euclidean

approach analyzes the accessibility relation between possible worlds in terms of the allowing and banning relations on states. So it is natural to seek a truthmaker semantical analogue of the correspondence theorem.

Let's say that a Kripke model $K = \langle W, R, \Phi \rangle$ is *based on* a Kripke frame $\langle W, R \rangle$. Given a class $\mathfrak{F}$ of Kripke frames, we also say that K is *in* $\mathfrak{F}$ if and only if K is based on a frame in $\mathfrak{F}$. Table 2 provides the standard names of some well-known classes of frames.

We shall show that for each of these classes of Kripke frames, there is a class of normal m-spaces that is equivalent to it. Let's say that a normal m-model $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ is *based on* a normal m-space $\Sigma = \langle \mathcal{S}, \sqsubseteq, \mu \rangle$. A normal m-model $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ *is in* a class $\mathfrak{S}$ of normal m-spaces if and only if $\mathcal{M}$ is based on a normal m-space in $\mathfrak{S}$. We say that a class $\mathfrak{F}$ of Kripke frames is *equivalent* to a class $\mathfrak{S}$ of normal m-spaces if and only if (1) for each Kripke model K in $\mathfrak{F}$, the corresponding normal m-model $\mathcal{M}(K)$ is in $\mathfrak{S}$, and (2) for each normal m-model $\mathcal{M}$ in $\mathfrak{S}$, the corresponding Kripke model $K(\mathcal{M}^*)$ is in $\mathfrak{F}$, where $\mathcal{M}^*$ is the close-off of $\mathcal{M}$ as defined above. Notice here that we pick a particular way of completing the modal boundaries of a normal m-model for reasons that will become clearer later on (see the comment to the proof of Theorem 20 below).
Now we list five conditions on normal m-spaces:

(TM-D) For all modal boundaries w, $\overline{\alpha}(w) \neq \emptyset$.
(TM-T) For all modal boundaries w, $w \in \overline{\alpha}(w)$.
(TM-4) For all modal boundaries w and w' and for any state t, if $t \notin \overline{\alpha}(w)$ and $t \in \overline{\alpha}(w')$, then $w' \in \overline{\beta}(w)$.
(TM-B) For all modal boundaries w and w', if $w \notin \overline{\alpha}(w')$, then $w' \in \overline{\beta}(w)$.

Table 2 Some well-known classes of Kripke frames

Class	Frame Conditions
D	Serial
T	Reflexive
4	Transitive
KB	Symmetric
5	Euclidean
S4	Reflexive, Transitive
S5	Reflexive, Transitive, Symmetric

(TM-5) For all modal boundaries w and w' and for any state t, if $t \in \overline{\alpha}(w)$ and $t \notin \overline{\alpha}(w')$, then $w' \in \overline{\beta}(w)$.

The conditions on normal m-spaces may seem quite complex at first glance. However, it is not so difficult to see that each condition offers an analysis of the corresponding frame condition in terms of the allowing and banning relations. (TM-T), for example, says that every modal boundary w allows itself. From the way we translated normal m-models to Kripke models, we can easily see that any normal m-model satisfying (TM-T) translates to a Kripke model in **T**. Using these conditions, we define the classes of normal m-models that correspond to those of Kripke frames as given Table 3:
Then we can show that each of these classes is equivalent to the corresponding class of Kripke frames.

Theorem 18 $\mathfrak{S}(\mathbf{D})$ *is equivalent to* **D**.

Proof Let a normal m-model $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ be in $\mathfrak{S}(\mathbf{D})$. Let $\mathcal{M}^* = \langle \mathcal{S}, \sqsubseteq, \mu^* \rangle$ be the close-off of $\mathcal{M}$. Then consider the corresponding Kripke model $K(\mathcal{M}^*) = \langle W^{\mathcal{M}^*}, R^{\mathcal{M}^*}, \Phi^{\mathcal{M}^*} \rangle$. It should be obvious from the construction that, for all $w \in W^{\mathcal{M}}$, $R_w \neq \emptyset$; hence $K(\mathcal{M})$ is in **D**. Conversely, suppose that a Kripke model $K = \langle W, R, \Phi \rangle$ is in **D**. Then the corresponding normal m-model $\mathcal{M}(K) = \langle \mathcal{S}^K, \sqsubseteq^K, \mu^K, v^K \rangle$ clearly satisfies (TM-D). □

Similarly, it is also easy to check (so we omit the proof):

Theorem 19 $\mathfrak{S}(\mathbf{T})$ *is equivalent to* **T**.

Theorem 20 $\mathfrak{S}(\mathbf{4})$ *is equivalent to* **4**.

Proof Let $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ be a normal m-model in $\mathfrak{S}(\mathbf{4})$. Let $\mathcal{M}^* = \langle \mathcal{S}, \sqsubseteq, \mu^*, v^* \rangle$ be the close-off of $\mathcal{M}$. We show that $\mathcal{M}^*$ satisfies (TM-4). Pick any modal boundary w and w', and t be any state. Suppose that $t \notin \overline{\alpha}^*(w)$ and $t \in \overline{\alpha}^*(w')$. Since μ^* agrees with μ concerning what's allowed by each state, it follows that $t \notin \overline{\alpha}(w)$ and $t \in \overline{\alpha}(w')$. Since $\mathcal{M}$ is assumed to satisfy (TM-4), $w' \in \overline{\beta}(w)$. Therefore, $w' \in \overline{\beta}^*(w)$. Now consider the corresponding Kripke model $K(\mathcal{M}^*) = \langle W^{\mathcal{M}^*}, R^{\mathcal{M}^*}, \Phi^{\mathcal{M}^*} \rangle$. Suppose that $R^{\mathcal{M}^*}(w_1, w_2)$ and $R^{\mathcal{M}^*}(w_2, w_3)$. In $\mathcal{M}^*$, $w_2 \in \overline{\alpha}^*(w_1)$ and $w_3 \in \overline{\alpha}^*(w_2)$. So, $w_2 \notin \overline{\beta}^*(w_1)$. By (TM-4), either $w_3 \in \overline{\alpha}^*(w_1)$ or $w_3 \notin \overline{\alpha}^*(w_2)$. Since $w_3 \in \overline{\alpha}^*(w_2)$, it follows that $w_3 \in \overline{\alpha}^*(w_1)$. Therefore, $R^{\mathcal{M}^*}(w_1, w_3)$.

Table 3 Some classes of normal m-spaces and their defining conditions

Class	Conditions on normal m-spaces
$\mathfrak{S}(\mathbf{D})$	(TM-D)
$\mathfrak{S}(\mathbf{T})$	(TM-T)
$\mathfrak{S}(\mathbf{4})$	(TM-4)
$\mathfrak{S}(\mathbf{KB})$	(TM-B)
$\mathfrak{S}(\mathbf{5})$	(TM-5)
$\mathfrak{S}(\mathbf{S4})$	(TM-T), (TM-4)
$\mathfrak{S}(\mathbf{S5})$	(TM-T), (TM-4), (TM-B)

Conversely, suppose that a Kripke model $K = \langle W, R, \Phi \rangle$ is in **4**. Let $\mathcal{M}(K) = \langle \mathcal{S}^K, \sqsubseteq^K, \mu^K, v^K \rangle$ be the corresponding normal m-model. To show that $\mathcal{M}(K)$ satisfies (TM-4), let w_1 and w_2 be modal boundaries and t be any state. Suppose that $w_2 \notin \overline{\beta}(w_1)$ and that $t \in \overline{\alpha}(w_2)$. We need to show that $t \in \overline{\alpha}(w_1)$. Since $\mathcal{M}(K)$ is a w-model, $w' \in \overline{\alpha}(w_2)$. Since we also assumed that $t \in \overline{\alpha}(w_2)$, there is also a modal boundary w_3 such that $t \sqsubseteq w_3$ and $w_3 \in \overline{\alpha}(w_2)$. In the original Kripke model K, then $R(w_1, w_2)$ and $R(w_2, w_3)$. Since K is assumed to be in **4**, it follows that $R(w_1, w_3)$. In $\mathcal{M}(K)$, therefore, $w_3 \in \overline{\alpha}(w_1)$. Hence $t \in \overline{\alpha}(w)$. □

Theorem 21 $\mathfrak{S}(\mathbf{KB})$ *is equivalent to* **KB**.

Proof Let $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ be a normal m-model in $\mathfrak{S}(\mathbf{KB})$. Let $\mathcal{M}^* = \langle \mathcal{S}, \sqsubseteq, \mu^*, v^* \rangle$ be the close-off of $\mathcal{M}$. We first show that $\mathcal{M}^*$ is also in $\mathfrak{S}(\mathbf{KB})$. Pick any modal boundaries w and w'. Suppose that $w \notin \overline{\alpha}^*(w')$. Then, by construction, $w \notin \overline{\alpha}(w')$. So, $w \in \overline{\beta}(w')$. Therefore, $w \in \overline{\beta}^*(w')$. Now, consider the corresponding Kripke model $K(\mathcal{M}^*) = \langle W^{\mathcal{M}^*}, R^{\mathcal{M}^*}, \Phi^{\mathcal{M}^*} \rangle$. We need to show that $K(\mathcal{M}^*)$ is in **KB**. Pick any two worlds w_1 and w_2. Suppose that $R^{\mathcal{M}^*}(w_1, w_2)$. Assume, for contradiction, that $R^{\mathcal{M}^*}(w_2, w_1)$ fails. In $\mathcal{M}^*$, then, $w_2 \in \overline{\alpha}^*(w_1)$ and $w_1 \notin \overline{\alpha}^*(w_2)$. By (TM-B), then, $w_2 \in \overline{\beta}^*(w_1)$. But this contradicts the modal soundness of w_1. Hence $R^{\mathcal{M}^*}(w_2, w_1)$ holds. So, $K(\mathcal{M}^*)$ is in **KB**.

Conversely, suppose that a Kripke model $K = \langle W, R, \Phi \rangle$ is in **KB**. Let $\mathcal{M}(K) = \langle \mathcal{S}^K, \sqsubseteq^K, \mu^K, v^K \rangle$ be the corresponding m-space. Let w and w' be modal boundaries. Suppose that $w \notin \overline{\alpha}(w')$. Then $R(w', w)$ does not hold in K. Since K is assumed to be in **KB**, $R(w, w')$ also fails in K. In $\mathcal{M}(K)$, therefore, $w' \notin \overline{\alpha}(w)$. Since $\Sigma(K)$ is a w-space, $w' \in \overline{\beta}(w)$. Therefore, $\Sigma(K)$ is in **KB**. □

Theorem 22 $\mathfrak{S}(\mathbf{5})$ *is equivalent to* **5**.

Proof Let $\mathcal{M} = \langle \mathcal{S}, \sqsubseteq, \mu, v \rangle$ be a normal m-model in $\mathfrak{S}(\mathbf{5})$. Let $\mathcal{M}^* = \langle \mathcal{S}, \sqsubseteq, \mu^* \rangle$ be the close-off. We first show that $\mathcal{M}^*$ is also in $\mathfrak{S}(\mathbf{5})$. Pick modal boundaries w and w'. Suppose that $t \in \overline{\alpha}^*(w)$ and $t \notin \overline{\alpha}^*(w')$. In $\mathcal{M}$, then, $t \in \overline{\alpha}(w)$ and $t \notin \overline{\alpha}(w')$. Since $\mathcal{M}$ is assumed to be in $\mathfrak{S}(\mathbf{5})$, it follows that $w' \in \overline{\beta}(w)$ in $\mathcal{M}$. Therefore, $w' \in \overline{\beta}^*(w)$. Now, consider the corresponding Kripke model $K(\mathcal{M}^*) = \langle W^{\mathcal{M}^*}, R^{\mathcal{M}^*}, \Phi^{\mathcal{M}^*} \rangle$. Suppose that $R^{\mathcal{M}^*}(w_1, w_2)$ and $R^{\mathcal{M}^*}(w_1, w_3)$. We need to show that $R^{\mathcal{M}^*}(w_2, w_3)$. In $\mathcal{M}^*$, then, $w_2 \in \overline{\alpha}^*(w_1)$ and $w_3 \in \overline{\alpha}^*(w_1)$. From the former, we have: $w_2 \notin \overline{\beta}^*(w_1)$. Since $\mathcal{M}^*$ in in $\mathfrak{S}(\mathbf{5})$, either $w_3 \notin \overline{\alpha}^*(w_1)$ or $w_3 \in \overline{\alpha}^*(w_2)$. Since $w_3 \in \overline{\alpha}^*(w_1)$, it follows that $w_3 \in \overline{\alpha}^*(w_2)$. In $K(\mathcal{M}^*)$, therefore, $R^{\mathcal{M}^*}(w_2, w_3)$. Hence $K(\mathcal{M}^*)$ is in **5**.

Conversely, suppose that a Kripke model $K = \langle W, R, \Phi \rangle$ is in **5**. Let $\mathcal{M}(K) = \langle \mathcal{S}^K, \sqsubseteq^K, \mu^K \rangle$ be the corresponding normal m-model. Let w_1 and w_2 be modal boundaries and t be any state. Suppose that $t \in \overline{\alpha}(w_1)$ and $t \notin \overline{\alpha}(w_2)$. We verify that $w_2 \in \overline{\beta}(w_1)$. Since $\mathcal{M}(K)$ is a w-model, $t \in \overline{\beta}(w_2)$. Now let w_3 be a world such that $t \sqsubseteq w_3$ and $w_3 \in \overline{\alpha}(w)$. Since $t \in \overline{\beta}(w_2)$ and $t \sqsubseteq w_3$, $w_3 \in \overline{\beta}(w_2)$. In the original Kripke model K, therefore, $R(w_1, w_3)$ holds and $R(w_2, w_3)$ fails. Since K is assumed to be in **5**, it follows that $R(w_1, w_2)$ does not hold. So, $w_2 \in \overline{\beta}(w_1)$. □

Corollary 23 $\mathfrak{S}(\mathbf{S4})$ *and* $\mathfrak{S}(\mathbf{S5})$ *are respectively equivalent to* **S4** *and* **S5**.

Comment: It should be clear from the inspection that the proofs of Theorems 20,21 and 22 make use of the definition of μ^*. Hence we may say that $\mathfrak{S}(\mathbf{4})$, $\mathfrak{S}(\mathbf{KB})$, $\mathfrak{S}(\mathbf{5})$ are equivalent, respectively, to **4**, **KB**, **5** *modulo the "closing off" construction*. This suggests that the conditions on normal m-models need to be fine-tuned depending on how they are to be completed. In contrast, observe that the proofs of Theorems 18 and 19 do not depend on the definition of μ^*. So it seems that for modal formulas of degree 2 (or higher), we need to take into account how normal m-models are to be completed. These considerations naturally lead to questions of some technical interest, such as whether there are conditions on normal m-models that translate to transitivity, symmetry, and Euclidean condition on Kripke frames "categorically," i.e., independently of how to complete normal m-models, and, if there are no such categorical conditions, whether it holds for all modal axioms of degree 2 or higher.

With these results, we can easily establish the soundness and completeness results for a well-known family of systems of normal propositional modal logic. Let D, T, KB, K4, S4, and S5 be the systems of normal propositional modal logic that are characterized by (i.e., sound and complete with respect to) **D**, **T**, **4**, **KB**, **K4**, **S4**, and **S5**, respectively. Let's say that a system S is *sound* with respect to a class $\mathfrak{S}$ of normal m-models if and only if every theorem of S is valid with respect to $\mathfrak{S}$. S is said to be *complete* with respect to $\mathfrak{S}$ if and only if every formula that is valid with respect to $\mathfrak{S}$ is a theorem of S. Then we have:

Corollary 24 *The systems* D, T, KB, K4, S4, *and* S5 *are sound and complete with respect to* $\mathfrak{S}(\boldsymbol{D})$, $\mathfrak{S}(\mathbf{T})$, $\mathfrak{S}(\mathbf{4})$, $\mathfrak{S}(\mathbf{KB})$, $\mathfrak{S}(\mathbf{S4})$, *and* $\mathfrak{S}(\mathbf{S5})$, *respectively.*

Proof We shall consider the system D; the other cases are similar. We first check that D is sound with respect to $\mathfrak{S}(\mathbf{D})$. Assume, for contradiction, that some theorem A of D is not valid with respect to $\mathfrak{S}(\mathbf{D})$. Let $\mathcal{M}$ be a normal m-model in $\mathfrak{S}(\mathbf{D})$ in which A is exactly falsified by some absolute possibility. Then it would follow from Theorems 12 and 18 that there is a Kripke model in **D**—namely $K(\mathcal{M}^*)$—in which A is not true at some world. But this contradicts the standard soundness result for D with respect to the class **D** of Kripke frames.

Conversely, we now check that D is complete with respect to $\mathfrak{S}(\mathbf{D})$. Let A be a formula that is not a theorem of D. Then it follows from the standard completeness result for D with respect to **D** that there is a Kripke model K in which A is not true at some world. Then it follows from Theorems 15 and 18 that there is a normal m-model in $\mathfrak{S}(\mathbf{D})$—namely $\mathcal{M}(K)$—in which A is exactly falsified by some absolute possibility. Hence A is not valid with respect to $\mathfrak{S}(\mathbf{D})$. □

11 Concluding Remarks

In this paper, we have developed a bilateralist truthmaker semantics for normal modal propositional logic. The main proposal is that an exact verifier for $\Box P$ is the fusion of an exact ban on the exact falsifiers for P, and that an exact verifier for $\Diamond P$ is a state that exactly allows an exact verifier for P. These clauses are shown to be equivalent to the corresponding clauses in the standard Kripke semantics under the natural analysis of the accessibility relation between possible worlds in terms of the allowing and banning relations between states (given some plausible assumptions about the modal behaviors of possible worlds and their constituent parts). In this way, the current proposal can be considered as an exactification of the Kripke semantics. On the basis of this proposal, a formal semantics was developed. The soundness and completeness results for a well-known family of the systems of normal modal propositional logic were established.

There remain many philosophical and technical questions of some interest. The logical inquiry leaves unanswered the substantive philosophical question of what modal states really are. This question relates to what is often called *the problem of the source of necessity* in the philosophical literature [17]: what makes a necessary truth necessarily true? It is hoped that a full truthmaker account of modality can be developed on the basis of the current logical framework that can address this issue. From a formal point of view, it is of some interest how the current framework can be adopted to give a semantics for modal logics based on multi-valued logics (such as FDE, strong and weak Kleene's three-valued logic, and LP; see footnote 8). I hope that this paper inspires further work on these and other philosophical and technical issues.

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