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On Some Weakened Forms of Transitivity in the Logic of Conditional Obligation

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Abstract
This paper examines the logic of conditional obligation, which originates from the works of Hansson, Lewis, and others. Some weakened forms of transitivity of the betterness relation are studied. These are quasi-transitivity, Suzumura consistency, acyclicity and the interval order condition. The first three do not change the logic. The axiomatic system is the same whether or not they are introduced. This holds true under a rule of interpretation in terms of maximality and strong maximality. The interval order condition gives rise to a new axiom. Depending on the rule of interpretation, this one changes. With the rule of maximality, one obtains the principle known as disjunctive rationality. With the rule of strong maximality, one obtains the Spohn axiom (also known as the principle of rational monotony, or Lewis' axiom CV). A completeness theorem further substantiates these observations. For interval order, this yields the finite model property and decidability of the calculus.

Keywords Conditional obligation · Axiomatization · Betterness · Transitivity · Quasi-transitivity · Acyclicity · Suzumura consistency · Interval order · (Strong) maximality

1 Introduction

This paper examines the logic of conditional obligation, which originates from the works of Hansson [35], van Fraassen [80], Lewis [44] and others. Possible worlds are ranked in terms of a preference relation, viewed as a relation of comparative goodness or betterness. In that framework, the truth conditions for the conditional obligation operator are phrased in terms of best antecedent-worlds. The main motivation for such a semantics has to do with the analysis of so-called contrary-to-duty (CTD) obligation sentences [19]. They tell us what comes into force when some other (primary) obli-

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gations are violated. A number of researchers in deontic logic have accepted the idea that an appropriate semantics for contrary-to-duty obligation sentences calls for an ordering on possible worlds, in terms of preference or relative goodness. An evaluation of such a treatment, and of the substantive body of theory and research it spurred, falls outside the scope of this paper.[1]

The paper deals with the meta-theory of such logics. It is part of a larger project, whose aim is to axiomatize the latter logics taking into account two types of consideration. First, the notion of best antecedent-worlds can be understood in different ways. Second, like in traditional modal logic, different properties of the relation in the models give rise to different systems. Only recently have we started to understand what these systems are. Early results [44, 73, 80] were tailored to the case where the betterness relation comes with the full panoply of the standard properties. These are reflexivity, transitivity, totality (which rules out "gaps" in the ranking) and Lewis [44]'s limit assumption (which requires that a set of worlds have a "limit" or best element). These properties were criticized as being too demanding in some contexts. Therefore subsequent research investigated how to extend these results to models equipped with a betterness relation verifying less properties, if any at all [31–34, 55–57]. In this paper, I will look into a problem left open in this previous work. It concerns the role of the property of transitivity of the betterness relation and variant weaker forms discussed in rational choice theory.

Weakenings of Transitivity

Transitivity tells us that, if a is at least as good as b ("$a \succeq b$") and b is at least as good as c ("$b \succeq c$"), then a is at least as good as c ("$a \succeq c$"). Transitivity is often seen as a conceptual truth. Defenders of such a view include Davidson [22] and Broome [13]. However, the intuitive plausibility of such a property has been called into question. This has led some authors to either reject transitivity wholesale or weaken it.

In moral philosophy, it has been proposed to reject the assumption of transitivity to avoid Parfit's so-called repugnant conclusion. It reads: "For any possible population of at least ten billion people, all with a very high quality of life, there must be some much larger imaginable population whose existence, if other things are equal, would be better, even though its members have lives that are barely worth living" [59, p. 331]. The reasoning leading to this conclusion is a continuum (or sorites-style) argument. One starts with a population of ten billion people, all with a very high quality of life. One increases gradually its size, and at the same time decreases the quality of life of its inhabitants. Because the loss in the quality of life is outweighed by a sufficient gain in quantity or whatever makes life worth living, one gets a series of increasingly

[1] For further background on preference-based semantics for deontic logic, see Makinson [48, Section 7], Hilpinen & McNamara [38, p. 112ff.] and references there cited. Preferences may also be used to model the notion of defeasible obligation [1]. They provide a model-theoretic analysis of the notion of supererogation and allied concepts. For an integration with the preference-based treatment of contrary-to-duty obligation into a single framework, the reader may wish to consult McNamara [51]. For a combination with the logic of time, see Åqvist [5]. For a combination with the modal logic of action (STIT) and with epistemic logic, see Kooi & Tamminga [41] and Horty [39, 40]. The application to linguistics is discussed at length in Cariani [15]. A faithful embedding of preference-based dyadic deontic logic into higher-order logic, yielding automation, has been realized by Benzmüller & al. [8].

better worlds. According to [77, 78], if transitivity of "better than" is denied, then the reasoning leading to the repugnant conclusion is blocked.

In rational choice theory, weaker forms of transitivity have been advocated. The following four deserve a special mention: quasi-transitivity [69]; acyclicity [71]; Suzumura consistency [76]; and the interval order condition [24]. Quasi-transitivity demands that the strict part of the betterness relation be transitive. Acyclicity rules out the presence of strict betterness cycles. Each was proposed as a means to avoid Arrow [7]'s impossibility result, which assumes the transitivity of "better than". Suzumura consistency (named after its inventor, following a suggestion made by Bossert [10]) rules out the presence of cycles with at least one instance of strict betterness. One motivation for Suzumura consistency is that a preference relation satisfying this condition is not vulnerable to money pumps.[2]

The interval order condition is usually defined for the strict betterness relation. The present paper focuses on its non-strict counterpart, which requires that $\succeq$ be a reflexive relation that meets the so-called Ferrers condition: $a \succeq b$ and $c \succeq d$ imply $a \succeq d$ or $c \succeq b$.[3] Interval orders were initially introduced to make room for the idea of non-transitive equal goodness (or indifference)[4] relation due to discrimination thresholds. The idea is that agents tend to discriminate between two alternatives, only when their difference exceeds a certain threshold [24, 79].[5] Each world can then be represented by an interval on the real line in such a way that $\succeq$ holds when the interval of the lower-ranking word is completely to the left of the interval of the higher-ranking one, or they overlap.

I will not take a position on the question of whether the assumption of transitivity should be kept in its plain form, weakened, or rejected wholesale. I want to know if or to what extent these different courses of action affect the logic of the conditional obligation operator.

Maximality vs. Strong Maximality

The statement "It ought to be the case that B given A" (in symbolic notation, $\bigcirc(B/A)$) is true if the best A-worlds are all B-worlds. However, the notion of best A-worlds can in turn be characterized in terms of either "optimality" or "maximality".[6] The distinction between the two is well-established in rational choice theory [11, 36, 70, 75]. It has been revived by Rott [64, 65] and Arló-Costa [6] in the context of the study

[2] The "money pump" argument was initially put forth by Davidson, McKinsey and Suppes [23] in support of transitivity. According to this argument, to abandon the assumption of transitivity leaves room for the possibility of an agent being money-pumped: he will accept a series of trade offers that leaves him with the same option he began with, but with less money. According to Bossert [10], Suzumura consistency is "exactly" what is needed to avoid the phenomenon of a money pump.

[3] I use reflexivity where most authors use totality (see, e.g., [60]). The two formulations are equivalent.

[4] In rational choice theory, equal goodness is called indifference.

[5] The classic example used to illustrate this point employs a series of objects so arranged that one cannot distinguish between two adjacent members of the series. But one can differentiate members at a greater distance. For instance, in Quinn [63]'s self-torturer example, a person is repeatedly given the option to increase his torture level by an undetectable increment, in exchange for $10000.

[6] I follow Sen's terminology in [70]. Other names have been used. For instance, Herzberger [36] speaks of "stringent vs. liberal" maximization, and Bossert and Suzumura [11] speak of "greatest element vs. maximal element" rationalizability.

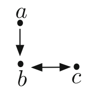

Fig. 1 Violation of IBC. An arrow from a to b means $a \succeq b$. No arrow from b to a means $b \not\succeq a$. (Reflexive loops are omitted)

of belief change theory. Roughly speaking, while optimizing involves choosing an element that is as good as every member of the reference set, maximizing only requires choosing an element that is not worse than any other. The two notions coincide when the betterness relation is total. But the choice between the two matters as long as there are incomparabilities between possible worlds—a situation that cannot be ruled out. Depending on what notion of "best" is used, one gets different truth conditions for the conditional obligation operator, and also different forms of the limit assumption.

Maximality is often seen as more appropriate than optimality when the possibility of incomparability is allowed (see, e.g., [70]). Previous research has shown that the choice between the two has less impact on the logic of the conditional obligation operator than one would have thought [33, 34, 55–57]. In this paper, I will be interested not so much in the contrast between maximality and optimality, but rather in the contrast between the standard notion of maximality and a variant notion called "strong maximality" by Bradley [12]. Roughly speaking, a world a is strongly maximal if no world b is strictly better than any c that is as equally good as a. The role of strong maximality is to ensure that the agent's choice meets the seemingly plausible requirement of (as Bradley calls it) "Indifference based choice" (IBC): two items that are regarded indifferently (or equally good) should always be equally choiceworthy (or best).[7] Here the items are possible worlds. Such a requirement can be violated, if $\succeq$ is no longer transitive. Suppose a is strictly better than b, and b and c are equally good. This is illustrated with Fig. 1. The convention is that a higher vertical position signifies a greater degree of goodness.[8] Both a and c are maximal, and hence chosen. The fact that c is chosen is counter-intuitive. For c is as equally good as b, which is strictly worse than a. One could argue that only a should be chosen. Here is a concrete example.

Example (Trip to Australia) Let b be a world where I go on holiday to Australia, and b^+ a world where I go on holiday to Australia with a bonus of \$100. b^+ is strictly better than b. Perhaps b is also a world where I go to the dentist for a filling. One can envisage a finite sequence of worlds b_1, ... b_n starting with $b = b_1$, where the pain during the dental work gradually increases in intensity, all the way to b_n. The pain increase from one world to the next is too small to be perceptible. So each world is as equally good as its predecessor. The pain in b_n is excruciating. One could argue that b^+ and b_n are too different from each other to make a straightforward comparison possible. So b^+ and b_n are incomparable. With the rule of maximality, the sentence

[7] In rational choice theory, "best" and "choiceworthy" are synonymous.

[8] This diagrammatic convention will be used throughout this paper whenever possible (except in Figs. 6 and 9, showing a cycle of betterness).

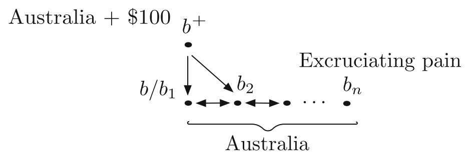

Fig. 2 Trip to Australia. Equal goodness (↔) is not transitive

"It ought to be that I go to Australia with a bonus of $100" is evaluated as false in this model, while with the rule of strong maximality it is evaluated as true. The first outcome may be considered counter-intuitive. This is shown in Fig. 2.

Contribution

This paper will present an axiomatic study of the various classes of models characterized by the presence or absence of the weakened forms of transitivity mentioned above. Reference will be made to five systems of increasing strength, one of them being new.

This paper makes two observations. The first one is that quasi-transitivity, acyclicity and Suzumura consistency make no difference to the logic. The axiomatic system remains the same whether or not these conditions are introduced. This holds under a rule of interpretation in terms of maximality and strong maximality. The second observation is that the interval order condition gives rise to a new axiom, which varies depending on the rule of interpretation. With maximality, one gets the principle known (in the non-monotonic logic literature) as disjunctive rationality [42]. With strong maximality, one gets a stronger principle, called the Spohn axiom [73]. This one is equivalent to the principle of rational monotony [42], or Lewis [44]'s axiom CV. A completeness theorem further substantiates these observations. For interval order, this yields the finite model property and decidability of the associated calculus.[9]

Our results improve the state-of-the-art in dyadic deontic logic, but also in the related areas of non-monotonic logic and the logic of counterfactuals. In particular, they complement those of Makinson, and Kraus, Lehmann & Magidor (KLM) for non-monotonic inference relations [42, 43, 47, 49]. These authors assume a form of the limit assumption called smoothness or stopperedness. Such an assumption has been criticized, notably by Lewis [44]. There is a call for understanding what happens in its absence. We will see that the smoothness condition makes no difference to the issue at hand, at least in the case of transitivity, quasi-transitivity, a-cyclicity and Suzumura consistency. On the other hand, these authors work with a primitive strict or irreflexive relation $\succ$ ("strictly better than") in the models while I will be using a primitive non-strict relation $\succeq$ ("at least as good as"). The advantage of using a primitive non-strict relation is that one can more easily distinguish between worlds that are tied or equally good and worlds that are incomparable. This point is discussed in Section 2.2. One can then provide a finer-grained semantical analysis, and disentangle different notions

[9] The decidability of the other calculi is already known.

of "best" like the above three, and different candidate weakenings of transitivity. Only maximality and quasi-transitivity are considered by these authors.[10]

Our findings also offer a complementary perspective to those presented by Lewis [44] in his work on the logic of counterfactuals. Lewis' prime interest is in strong logics of counterfactuals, for which the similarity relation is transitive and total. Thus he does not address models that might be appropriate for weaker systems. In [44] and also in [45] Lewis directs our attention to the systems that might be suitable for a logic of conditional obligation. He designates his own preferred deontic system as **VN**, providing it with various modelings, with the "preferred" one being formulated in terms of sphere models. Transitivity is inherently embedded within a system of spheres, corresponding to the nesting property among these spheres. It remains an interesting question whether our proposed relaxations of transitivity find analogous counterparts within this framework. In his [44, 46], he proposes two variant evaluation rules for the conditional in terms of betterness, allowing to avoid some of the side effects of letting the limit assumption go. It is not known what happens when transitivity is relaxed.

This paper is organized as follows. Section 2 sets the stage and describes the framework within which the investigation will be conducted. Section 3 deals with the conditions of quasi-transitivity, acyclicity and Suzumura consistency. Section 4 deals with the condition of interval order. Section 5 concludes. The appendix gives the proof of results which would have otherwise cluttered Sections 4 and 5.

2 Setting the Stage

I start by setting the stage, and by describing the framework being used. The language $\mathscr{L}$ is defined by the following BNF:

$$\begin{array}{ll} \text{Atomic formulas:} & p \in \mathbb{P} \\ \text{Formulas:} & A \in \mathscr{L} \end{array}$$
$$A ::= p \mid \neg A \mid A \vee A \mid \Box A \mid \bigcirc(A/A)$$

$\neg A$ is read as "not-A", and $A \vee B$ as "A or B". $\Box A$ is read as "A is settled as true", and $\bigcirc(B/A)$ as "B is obligatory, given A". A is called the antecedent, and B the consequent.

The Boolean connectives other than "$\neg$" and "$\vee$" are defined as usual. $\Diamond A$ is short for $\neg\Box\neg A$. $P(B/A)$ ("B is permitted, given A") is short for $\neg\bigcirc(\neg B/A)$, $\bigcirc A$ ("A is unconditionally obligatory") and PA ("A is unconditionally permitted") are short for $\bigcirc(A/\top)$ and $P(A/\top)$, where $\top$ denotes a tautology.

[10] The interval order condition has been studied within the KLM setting by Booth and Varzinczak [9], but in isolation, without considering other potential weakenings of transitivity, their focus being on strict relations in the models, and on the usual notion of maximality.

2.1 Semantics

Definition 1 (Preference model) *A preference (or betterness) model is a structure* $M = (W, \succeq, v)$ *in which:*

(i) $W \neq \emptyset$ *(W is a non-empty set of possible worlds, called the universe);*
(ii) $\succeq \subseteq W \times W$ *(intuitively,* $\succeq$ *is a betterness or comparative goodness relation; "*$a \succeq b$*" can be read as "world a is at least as good as world b");*
(iii) $v : \mathbb{P} \rightarrow \mathscr{P}(W)$ *(v is an assignment, which associates a set of possible worlds to each atomic formula p).*

A model is said to be finite, if its universe has finitely many worlds. $\succ$ is the asymmetric (or strict) part of $\succeq$ defined by $a \succ b$ iff (if and only if) $a \succeq b$ and $b \not\succeq a$. The symmetric part $\approx$ of $\succeq$ is called equal goodness, and is defined by $a \approx b$ iff $a \succeq b$ and $b \succeq a$. The incomparability relation is noted $\|$, and is defined by $a\|b$ iff $a \not\succeq b$ and $b \not\succeq a$. $\succeq$ is said to be reflexive, if for all a, $a \succeq a$. $\succeq$ is said to be transitive, if for all a, b and c, $a \succeq b$ and $b \succeq c$ imply $a \succeq c$. Finally $\succeq$ is said to be total, if for all a and b, either $a \succeq b$ or $b \succeq a$. Note that totality implies reflexivity. Note also that, by definition, $\succ$ is irreflexive (for all a, $a \not\succ a$) and asymmetric (for all a and b, if $a \succ b$ then $b \not\succ a$). The transitive closure of $\succeq$ is written as $tc(\succeq)$ (or, more succinctly, as $\succeq^\star$), and is defined in the usual way. A similar notation is used for the transitive closure of $\succ$.

Fact 2 *(i) If* $\succeq$ *is transitive, then* $\succeq = \succeq^\star$*; (ii) if* $\succ$ *is transitive, then* $\succ = \succ^\star$*; and (iii)* $\succ^\star \subseteq \succeq^\star$.

Proof Straightforward. □

Before defining the truth-conditions for the connectives, I need to introduce an extra notion, that of a world's equal goodness (in Bradley's terminology, indifference) class. As usual, $\|A\|^M$ denotes the truth-set of A in M, i.e., the set of worlds in M at which A holds. I drop reference to M when it is clear what model is intended. The transitive closure of $\approx$ on $\|A\|$ is denoted $\approx^A$, and is defined as follows: for all $a, b \in \|A\|$, $a \approx^A b$ iff (1) $a = b$, or (2) there is a finite sequence of A-worlds a_1, ..., a_n $(n \geq 2)$ with $a_1 = a$, ..., $a_n = b$ and $a_{i-1} \approx a_i$ (for $i = 2, \ldots, n$). The equal goodness class of a in A is $\{b : a \approx^A b\}$.

Fact 3 *The following applies:*

(i) If $a \approx^A b$, $c \in \|A\|$ *and* $b \approx c$*, then* $a \approx^A c$*;*
(ii) $\approx\upharpoonright_A \subseteq \approx^A$*, where* $\approx\upharpoonright_A$ *is the restriction of* $\approx$ *to* $\|A\|$*;*
(iii) If $\succeq$ *is reflexive and transitive, then* $\approx^A \subseteq \approx\upharpoonright_A$.

Proof (i) follows from the definition of $\approx^A$.

For (ii), assume $a \approx b$ with $a, b \in \|A\|$. If $a = b$, then the claim follows by part (1) in the definition of $\approx^A$. If $a \neq b$, then the claim follows by part (2).

For (iii), assume $\succeq$ is reflexive and transitive, and let $a \approx^A b$. Hence $a, b \in \|A\|$. If $a \approx^A b$ holds in virtue of part (1) in the definition of $\approx^A$, then the claim follows from the reflexivity of $\succeq$. If $a \approx^A b$ holds in virtue of part (2) in the definition of $\approx^A$, then there is a finite sequence of A-worlds a_1, ..., a_n $(n \geq 2)$ with $a_1 = a$, $a_n = b$ and $a_{i-1} \approx a_i$ (for $i = 2, \ldots, n$). By transitivity of $\succeq$, it follows at once that $a \approx b$. □

Definition 4 (Satisfaction) *Given a preference model $M = (W, \succeq, v)$ and a world $a \in W$, the satisfaction relation $M, a \vDash A$ (read as "world a satisfies A in model M") is defined as follows:*

$$M, a \vDash p \text{ iff } a \in v(p)$$
$$M, a \vDash \neg A \text{ iff } M, a \nvDash A$$
$$M, a \vDash A \vee B \text{ iff } M, a \vDash A \text{ or } M, a \vDash B$$
$$M, a \vDash \Box A \text{ iff } \forall b \;\; M, b \vDash A$$
$$M, a \vDash \bigcirc(B/A) \text{ iff } \mathrm{best}(\|A\|) \subseteq \|B\|$$

When no confusion can arise, I omit the reference to M and simply write $a \models A$. Intuitively, the evaluation rule for the conditional obligation operator stipulates that $\bigcirc(B/A)$ is true whenever all the best A-worlds are B-worlds. I allow for variation in the way "best" is defined, and distinguish between the following three rules of interpretation (in the rightmost cell, the letter "s" is mnemonic for "strong"):

max rule	**opt rule**	**s-max rule**
$\mathrm{best}(\|A\|) = \max_{\succeq}(\|A\|)$	$\mathrm{best}(\|A\|) = \mathrm{opt}_{\succeq}(\|A\|)$	$\mathrm{best}(\|A\|) = \max^{s}_{\succeq}(\|A\|)$

where

$$a \in \max_{\succeq}(\|A\|) \Leftrightarrow a \models A \;\&\; \neg\exists b\; (b \vDash A \;\&\; b \succ a)$$
$$a \in \mathrm{opt}_{\succeq}(\|A\|) \Leftrightarrow a \models A \;\&\; \forall b\; (b \models A \rightarrow a \succeq b)$$
$$a \in \max^{s}_{\succeq}(\|A\|) \Leftrightarrow a \models A \;\&\; \forall b\; \big((b \models A \;\&\; b \approx^{A} a) \rightarrow \neg\exists c\, (c \models A \;\&\; c \succ b)\big)$$

Intuitively, an A-world a is maximal, if it is not (strictly) worse than any other A-world. It is optimal if it is at least as good as every A-world. It is strongly maximal, if any world b in a's equal goodness class in A is not (strictly) worse than any other A-world.

Proposition 5 *We have:*

(i) $\mathrm{opt}_{\succeq}(\|A\|) = \max_{\succeq}(\|A\|)$ *if $\succeq$ is total;*
(ii) $\max^{s}_{\succeq}(\|A\|) = \max_{\succeq}(\|A\|)$ *if $\succeq$ is reflexive and transitive.*

Proof (i) is straightforward. For (ii), the left-in-right inclusion follows from the fact that $a \approx^{A} a$. For the converse inclusion, assume $\succeq$ is reflexive and transitive. Let a be such that $a \in \max_{\succeq}(\|A\|)$ and $a \notin \max^{s}_{\succeq}(\|A\|)$. From the latter, $\exists b, c$ s.t. $b \models A$, $c \models A$, $b \approx^{A} a$ and $c \succ b$. By Fact 3 (iii), $b \approx a$, and so by transitivity $c \succ a$, a contradiction. □

The notions of semantic consequence, validity and satisfiability are defined as usual.

2.2 Non-strict vs. Strict Betterness

In the introductory section, with reference to the work of Makinson, Kraus, Lehmann and Magidor, I said that the advantage of using a primitive non-strict betterness relation is that one can more easily distinguish between worlds that are equally good (or tied) and worlds that are incomparable. If one uses a primitive non-strict relation $\succeq$, it is a straightforward matter to distinguish between the two. The formal definitions were given in Section 2.1. Two worlds are equally good if $\succeq$ holds between them in each direction. They are incomparable if $\succeq$ holds in no direction. Suppose only the strict relation $\succ$ is available. Because a strict relation is asymmetric (if $a \succ b$, then $b \not\succ a$), one cannot simply replace $\succeq$ with $\succ$ in the previous definition to get equal goodness in terms of $\succ$. Rott [65, p. 158] offers the following variant definition. Two worlds are equally good if they have exactly the same sets of dominating and dominated elements. Two worlds are incomparable if they are not equally good, and none is strictly better than the other. Formally:

$$a \mathrel{\dot{\approx}} b \quad \Leftrightarrow \quad \forall c : c \succ a \Leftrightarrow c \succ b \text{ and } a \succ c \Leftrightarrow b \succ c \tag{tie}$$

$$a \mathrel{\dot{\|}} b \quad \Leftrightarrow \quad a \not\succ b \text{ and } b \not\succ a \text{ and } a \mathrel{\dot{\not\approx}} b \tag{gap}$$

According to Rott, condition (tie) "captures exactly what we want, at least if the relation in question is transitive" [65, p. 158]. He asks: "How could two [worlds] possibly satisfy this condition, and yet be called incomparable *in terms of* [$\succ$]?" [ibid.] A problem immediately arises in Bradley's example in Fig. 1. For $a \mathrel{\dot{\|}} c$ to hold, it must be the case that $a \not\succ c$. For $b \mathrel{\dot{\approx}} c$ to hold, it must be the case that $a \succ c$ (as $a \succ b$). Thus the configuration shown in this example cannot be modeled. The same holds for the example shown in Fig. 2. Small, imperceptible worsenings accumulate, becoming significant and creating an incomparability between b^+ and some b_i $(1 < i \leq n)$. World b_i in the sequence acts as a threshold, where the property of incomparability with b^+ is not only established but also inherited by its successors. For $b^+ \mathrel{\dot{\|}} b_i$ to hold, it must be the case that $b^+ \not\succ b_i$. For $b_{i-1} \mathrel{\dot{\approx}} b_i$ to hold, it must be the case that $b^+ \succ b_i$. I am not aware of any alternative proposal (other than Rott's). The problem disappears if $\succeq$ is taken as primitive.

2.3 Limit Assumption

Lewis [44]'s well-known limit assumption (LA) rules out sets of worlds without a "limit" (viz. a best element). Its exact formulation varies among authors. It exists in (at least) the following two basic forms, where best $\in \{\max_{\succeq}, \text{opt}_{\succeq}, \max^{s}_{\succeq}\}$:

<u>Limitedness</u>

$$\text{If } \exists a \text{ s.t. } a \models A \text{ then best}(\|A\|) \neq \emptyset \tag{LIM}$$

<u>Smoothness</u> (or stopperedness)

$$\text{If } a \models A, \text{ then: either } a \in \text{best}(\|A\|) \text{ or } \exists b \text{ s.t. } b \succ a \;\&\; b \in \text{best}(\|A\|) \tag{SM}$$

A betterness relation $\succeq$ will be called "opt-limited", "max-limited" or "s-max-limited" depending on whether (LIM) holds with respect to $\text{opt}_{\succeq}$, $\max_{\succeq}$ or $\max^{s}_{\succeq}$. Similarly, it will be called "opt-smooth", "max-smooth" or "s-max-smooth" depending on whether (SM) holds with respect to $\text{opt}_{\succeq}$, $\max_{\succeq}$ or $\max^{s}_{\succeq}$. For pointers to literature, and the relationships between the versions of LA cast in terms of optimality and maximality, see [55, 57]. Proposition 6 below clarifies how the versions of LA in terms of maximality and strong maximality relate to one another. For the reader's convenience, these relationships are represented in an implication diagram with the direction of the arrow representing that of implication. This is Fig. 3. The implication relations shown in the picture on the left-hand side hold without restriction, while those shown on the right-hand side hold under the hypothesis that $\succeq$ meets the property (or pair of properties) displayed as labeled.

Proposition 6 *We have:*

(i) s-max-smoothness $\Rightarrow$ max-smoothness; s-max limitedness $\Rightarrow$ max-limitedness; s-max-smoothness $\Rightarrow$ s-max-limitedness; max-smoothness $\Rightarrow$ max-limitedness;
(ii) Given reflexivity and transitivity, max-smoothness $\Rightarrow$ s-max-smoothness, and max-limitedness $\Rightarrow$ s-max-limitedness;
(iii) Given totality, s-max-limitedness $\Rightarrow$ s-max-smoothness, and max-limitedness $\Rightarrow$ max-smoothness.

Proof For (i), assume $\succeq$ is s-max-smooth. Let $a \models A$ and $a \notin \max_{\succeq}(\|A\|)$. By Proposition 5 (ii), $a \notin \max^{s}_{\succeq}(\|A\|)$. So $\exists b \models A$ s.t. $b \succ a$ and $b \in \max^{s}_{\succeq}(\|A\|)$. By Proposition 5 (ii) again, $b \in \max_{\succeq}(\|A\|)$, and the claim is proved. The implication "s-max-limitedness $\Rightarrow$ max-limitedness" is proved similarly. The third and fourth implications follow at once from the definitions involved.

(ii) follows from Proposition 5 (ii).

For (iii), assume $\succeq$ is s-max-limited, and consider some a such that $a \models A$ and $a \notin \max^{s}_{\succeq}(\|A\|)$. Hence $\exists b, c$ s.t. $b \approx^{A} a, c \models A$ and $c \succ b$. By the opening assumption, $\exists d$ s.t. $d \in \max^{s}_{\succeq}(\|A\|)$. Suppose, to reach a contradiction, that $a \succeq d$. Since $d \approx^{A} d$, by definition it follows that $d \succeq a$. But, then, by Fact 3 (i) $b \approx^{A} d$, and $d \notin \max^{s}_{\succeq}(\|A\|)$. So one must conclude that $a \not\succeq d$. By totality, $d \succeq a$, and so $d \succ a$. This shows that $\succeq$ is s-max-smooth. The other implication is proved similarly. $\square$

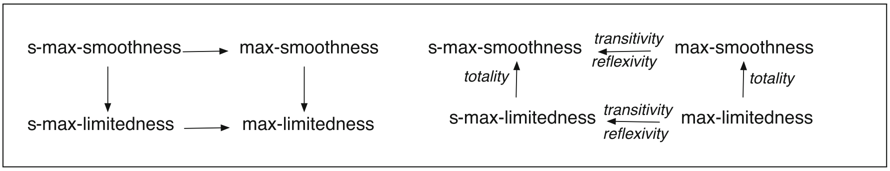

Fig. 3 Limit assumption

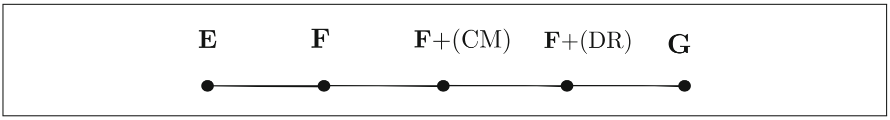

Fig. 4 Systems

2.4 Hilbert Systems

The systems of interest are shown in Fig. 4. They are of increasing strength. A line between two systems indicates that the system to the left is contained in the system to the right. **E**, **F**, **F**+(CM) and **G** have been around for some time now.[11] **F**+(DR) is new.

All the systems contain the classical propositional calculus and the modal system S5.[12] Then they add the following axiom schemata:

- For **E** (the naming follows [55]):

$$\bigcirc(B \to C/A) \to (\bigcirc(B/A) \to \bigcirc(C/A)) \quad \text{(COK)}$$
$$\bigcirc(B/A) \to \Box\bigcirc(B/A) \quad \text{(Abs)}$$
$$\Box A \to \bigcirc(A/B) \quad \text{(O-nec)}$$
$$\Box(A \leftrightarrow B) \to (\bigcirc(C/A) \leftrightarrow \bigcirc(C/B)) \quad \text{(Ext)}$$
$$\bigcirc(A/A) \quad \text{(Id)}$$
$$\bigcirc(C/A \wedge B) \to \bigcirc(B \to C/A) \quad \text{(Sh)}$$

- For **F**: axioms of **E** plus

$$\Diamond A \to \neg(\bigcirc(B/A) \wedge \bigcirc(\neg B/A)) \quad (\text{D}^\star)$$

- For **F**+(CM): axioms of **F** plus

$$(\bigcirc(B/A) \wedge \bigcirc(C/A)) \to \bigcirc(C/A \wedge B) \quad \text{(CM)}$$

- For **F**+(DR): axioms of **F** plus

$$\bigcirc(C/A \vee B) \to (\bigcirc(C/A) \vee \bigcirc(C/B)) \quad \text{(DR)}$$

- For **G**: axioms of **F** plus

$$(P(B/A) \wedge \bigcirc(B \to C/A)) \to \bigcirc(C/A \wedge B) \quad \text{(Sp)}$$

[11] **E**, **F** and **G** are Åqvist [2, 4]'s. **F**+(CM) is from the study [55]. Their Hanssonian counterparts may be found in Goble [33]. **F**+(CM) is to **G** what the KLM system P is to their system R [42, 43]. Note the so-called principle of consistency preservation—the analog of (D*), with consistency used as a surrogate of possibility—is not part of the latter two.

[12] S5 is characterized by the rule of necessitation ("If $\vdash A$, then $\vdash \Box A$"), and the K, T and 5 axioms (5 is $\Diamond A \to \Diamond\Box A$).

(COK) is the conditional analogue of the familiar distribution axiom K. (Abs) is the absoluteness axiom of Lewis [44], and reflects the fact that the ranking is not world-relative. (O-nec) is the deontic counterpart of the familiar necessitation rule. (Ext) permits the replacement of necessarily equivalent sentences in the antecedent of deontic conditionals. (Id) is the deontic analogue of the identity principle. (Sh) is named after Shoham [72, p. 77], who seems to have been the first to discuss it. The question of whether (Id) is a reasonable law for deontic conditionals has been much debated. A defense of (Id) can be found in Hansson [35], Prakken and Sergot [61] and Parent [54]. Intuitively, (D*) rules out the possibility of conflicting obligations, for a possible antecedent. It entails the following version of the Kantian principle "*ought* implies *can*":

$$\bigcirc(B/A) \rightarrow (\Diamond A \rightarrow \Diamond(A \wedge B)) \tag{Ought2Can}$$

(CM) is the principle of cautious monotony from [42]. It says that fulfilling an obligation in a given context does not modify our other obligations in the same context. (DR) is the principle of disjunctive rationality. It says that if a disjunction of states of affairs triggers an obligation, then at least one disjunct taken alone triggers the obligation in question. (Sp) is named after Spohn [73]. In **E**, it is equivalent to the well-known principle of rational monotony [42] and Lewis [44]'s axiom CV. It says that realizing a permission does not affect our other obligations arising in the same context:[13]

$$(P(B/A) \wedge \bigcirc(C/A)) \rightarrow \bigcirc(C/A \wedge B) \tag{RM}$$

For future reference, I note the following (the label "RW" is borrowed from the non-monotonic logic literature, and is mnemonic for "Right weakening"; the label "VCM" is short for "Very cautious monotony", and is taken from Goble [31]):

Proposition 7 *We have:*

$$\textit{If } \vdash_{\mathbf{E}} B \rightarrow C \textit{ then } \vdash_{\mathbf{E}} \bigcirc(B/A) \rightarrow \bigcirc(C/A) \tag{RW}$$

$$\vdash_{\mathbf{E}} (\bigcirc(B/A) \wedge \bigcirc(C/A)) \rightarrow \bigcirc(B \wedge C/A) \tag{AND}$$

$$\vdash_{\mathbf{F}+(\mathrm{CM})} \bigcirc(B \wedge C/A) \rightarrow \bigcirc(C/A \wedge B) \tag{VCM}$$

$$\vdash_{\mathbf{G}} (\bigcirc(B/A) \wedge \bigcirc(C/A)) \rightarrow \bigcirc(C/A \wedge B) \tag{CM}$$

Proof For (RW) and (AND), see [57, Theorem 3.1]. For (VCM), assume $\bigcirc(B \wedge C/A)$. By (RW), $\bigcirc(B/A)$. By (CM), $\bigcirc(B \wedge C/A \wedge B)$. By (RW) again, $\bigcirc(C/A \wedge B)$. For (CM), see [57, Theorem 3.3 (v)]. □

The notions of syntactical consequence, theoremhood, consistency and weak (*resp.* strong) completeness are defined as usual. Unless stated otherwise completeness is understood in its strong sense.

Theorem 8 is established in [55, 56] for the max rule and the opt rule. In part (ii) of the theorem, it is understood that max-limitedness is used when deontic formulas are

[13] For a proof of their equivalence, see [57, Theorem 3.3 (iv)].

interpreted using the max rule, and opt-limitedness is used when they are interpreted using the opt rule. A similar remark applies to parts (iii) and (iv).

Theorem 8 *(i)* **E** *is sound and complete with respect to the class of all preference models; (ii)* **F** *is sound and complete with respect to the class of preference models in which* $\succeq$ *is max-limited* (resp. *opt-limited); (iii)* **F**+*(CM) is sound and complete with respect to the class of preference models in which* $\succeq$ *is max-smooth* (resp. *opt-smooth); and (iv)* **G** *is sound and complete with respect to the class of preference models in which* $\succeq$ *is max-smooth* (resp. *opt-smooth), transitive and total.*

Proof See [55, 56]. □

I mentioned that **E**, **F**, **F**+(CM), **F**+(DR) and **G** form a series of systems of increasing strength. The inclusions **E** $\subset$ **F** $\subset$ **F**+(CM)$\subset$ **G** are known. I show **F**+(CM) $\subset$ **F**+(DR) $\subset$ **G**.

Theorem 9 *(i) (DR) is a theorem of* **G***; (ii) (CM) is a theorem of* **F**+*(DR); (iii) (DR) is not a theorem of* **F**+*(CM); (iv) (Sp) is not a theorem of* **F**+*(DR); and (v) (D***) is not a theorem of* **E**+*(DR).*

Proof For (i), the argument draws on Makinson [49, p. 94]. Assume $\bigcirc(C/A \vee B)$ and $\neg\bigcirc(C/A)$. To show: $\bigcirc(C/B)$. If $\neg\bigcirc(\neg A/A \vee B)$, then $\bigcirc(C/(A \vee B) \wedge A)$ by (RM). This yields $\bigcirc(C/A)$ by (Ext), in contradiction with the second opening assumption. So $\bigcirc(\neg A/A \vee B)$. If $\neg\bigcirc(\neg B/A \vee B)$, then $\bigcirc(C/(A \vee B) \wedge B)$ by (RM). This yields the desired conclusion $\bigcirc(C/B)$ by (Ext). So assume $\bigcirc(\neg B/A \vee B)$. By (AND), $\bigcirc(\neg A \wedge \neg B/A \vee B)$. By (Id), (AND) and (RW), $\bigcirc(B/A \vee B)$. From this and the first opening assumption one gets $\bigcirc(C/B)$ by (CM) and (Ext). This concludes the proof.

For (ii), assume $\bigcirc(B/A)$ and $\bigcirc(C/A)$. By (AND), $\bigcirc(B \wedge C/A)$. By (Ext), $\bigcirc(B \wedge C/(A \wedge B) \vee (A \wedge \neg B))$. By (DR), either (a) $\bigcirc(B \wedge C/A \wedge B)$ or (b) $\bigcirc(B \wedge C/A \wedge \neg B)$. In case (a) (RW) yields $\bigcirc(C/A \wedge B)$ as required. So assume (b) holds. By (Id), $\bigcirc(A \wedge \neg B/A \wedge \neg B)$. From this and (b) one gets $\bigcirc(B \wedge \neg B/A \wedge \neg B)$ using (AND) and (RW). By (RW) and propositional logic, $\bigcirc(B/A \wedge \neg B) \wedge \bigcirc(\neg B/A \wedge \neg B)$. (D*) then yields $\neg\Diamond(A \wedge \neg B)$. By S5 $\Box(A \leftrightarrow (A \wedge B))$, and so by (Ext) the second opening assumption yields $\bigcirc(C/A \wedge B)$ as required.

For (iii), (iv) and (v), see Corollary 26. □

2.5 Weakenings of Transitivity

I list below the weakened forms of transitivity discussed in this paper.

Definition 10 *Let* $\succeq$ *be a given relation.*

- $\succeq$ *is quasi-transitive, if* $\succ$ *is transitive;*
- $\succeq$ *is acyclic, if* $a \succ^\star b$ *implies* $b \not\succ a$*;*
- $\succeq$ *is Suzumura consistent, if* $a \succeq^\star b$ *implies* $b \not\succ a$*;*
- $\succeq$ *is an interval order, if* $\succeq$ *is reflexive and Ferrers (*$a \succeq b$ *and* $c \succeq d$ *imply* $a \succeq d$ *or* $c \succeq b$*).*

Fact 11 *If $\succeq$ is an interval order, then:*

(i) *$\succeq$ is "negatively" transitive (i.e., $a \not\succeq b$ and $b \not\succeq c$ imply $a \not\succeq c$);*
(ii) *$\succeq$ is total (i.e., $a \succeq b$ or $b \succeq a$).*

Proof For part (i), let $a \not\succeq b$ and $b \not\succeq c$. By Ferrers, $a \not\succeq c$ or $b \not\succeq b$. By reflexivity, the latter is impossible, and so only the former applies.

For part (ii), let a and b be in W. By reflexivity, $a \succeq a$ and $b \succeq b$. By Ferrers, $a \succeq b$ or $b \succeq a$. □

The relationships between these conditions can be depicted as in Fig. 5. The arrow symbol is used to represent implication. Transitivity and interval order are independent. Transitivity implies quasi-transitivity and Suzumura consistency, but not the other way around. Each of quasi-transitivity and Suzumura consistency implies acyclicity, but the converses fail. Quasi-transitivity and Suzumura consistency are independent, as are interval order and Suzumura consistency.

I state these relationships in full.

Proposition 12 *The following applies:*

(i) *Interval order $\Rightarrow$ quasi-transitivity;*
(ii) *Transitivity $\Rightarrow$ quasi-transitivity and Suzumura consistency;*
(iii) *Quasi-transitivity or Suzumura consistency $\Rightarrow$ acyclicity.*

Proof For (i), assume $\succeq$ is an interval order. Suppose $a \succ b$ and $b \succ c$. By definition of $\succ$, $a \succeq b$, $b \succeq c$, $b \not\succeq a$, and $c \not\succeq b$. By negative transitivity (see Fact 11 (i)) $c \not\succeq a$. By totality (see Fact 11 (ii)), $a \succeq c$. By definition of $\succ$, $a \succ c$.

For (ii), assume $\succeq$ is transitive. First, I consider the case of quasi-transitivity. Assume $a \succ b$ and $b \succ c$. By definition, $a \succeq b$ and $b \succeq c$. By transitivity of $\succeq$, $a \succeq c$. By hypothesis, $b \not\succeq a$. By transitivity of $\succeq$ again, $c \not\succeq a$. Hence $a \succ c$ as required.

Next, I consider the case of Suzumura consistency. Assume, per absurdum, that $\succeq$ is not Suzumura consistent. Hence, for some a and b, $a \succeq^\star b$ but $b \succ a$. The latter implies that $a \not\succeq b$, by definition, while the former implies that $a \succeq b$, by Fact 2 (i). Contradiction.

For the first half of (iii), assume that $\succeq$ is quasi-transitive, but not acyclic. Hence, for some a and b, $a \succ^\star b$ but $b \succ a$. Given quasi-transitivity, the former implies $a \succ b$,

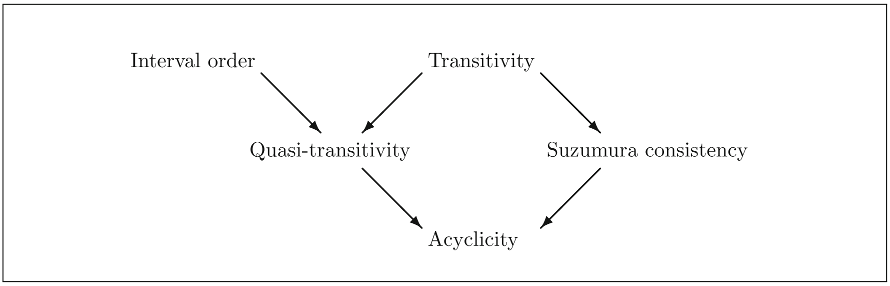

Fig. 5 Implication relations

by Fact 2 (ii). Combined with the latter, one gets $a \succ a$, by quasi-transitivity again. This contradicts the irreflexivity of $\succ$.

For the second half of (iii), assume $\succeq$ is Suzumura consistent. To see acyclicity, let $a \succ^\star b$. By Fact 2 (iii), $a \succeq^\star b$, and so $b \not\succ a$ by Suzumura consistency. □

Proposition 13 *The following applies:*

(i) Interval order or quasi-transitivity or Suzumura consistency $\not\Rightarrow$ transitivity;
(ii) Interval order $\not\Leftrightarrow$ Suzumura consistency;
(iii) Acyclicity $\not\Rightarrow$ quasi-transitivity or Suzumura consistency;
(iv) Quasi-transitivity $\not\Leftrightarrow$ Suzumura consistency;
(v) Transitivity $\not\Rightarrow$ interval order.

Proof Let $\succeq = \{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b), (a, c)\}$. $\succeq$ is an interval order and is quasi-transitive (and hence acyclic), but not Suzumura consistent (and hence not transitive).

Let $\succeq = \{(a, b), (b, c)\}$. $\succeq$ is Suzumura consistent (and hence acyclic), but not an interval order nor quasi-transitive (and hence it is not transitive).

Let $\succeq = \{(a, a), (b, b)\}$. $\succeq$ is transitive, but not Ferrers, and hence not an interval order. □

Proposition 14 considers the case of a finite preference model, and tells us which weakening of transitivity "enforces" which form of the limit assumption. I give the weakest weakening of transitivity having such an effect. It should not come as a surprise that smoothness requires a stronger condition than limitedness to be "enforced" in the model.

Proposition 14 *Suppose W is finite.*

(i) If $\succeq$ is quasi-transitive, then $\succeq$ is max-smooth (but not necessarily s-max-smooth);
(ii) If $\succeq$ is acyclic, then $\succeq$ is max-limited (but not necessarily s-max-limited);
(iii) If $\succeq$ is Suzumura-consistent, then $\succeq$ is s-max-limited.

Proof For the first claim in part (i), see [55, Proposition 4]. Figure 6 establishes the second claim. $\succeq$ is (vacuously) quasi-transitive, but not s-max-smooth. We have $\max^s_\succeq(\|p\|) = \emptyset$.

For the first claim in part (ii), the argument is virtually the same as for part (i) except that the weaker condition of acyclicity is invoked. Let $\succeq$ be acyclic, and suppose $a_1 \models A$. Assume to reach a contradiction that (a) $\forall a$ if $a \models A$ then $\exists b$ s.t. $b \models A$ and $b \succ a$.

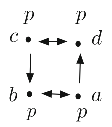

Fig. 6 Failure of s-max-smoothness. (Reflexive loops are omitted)

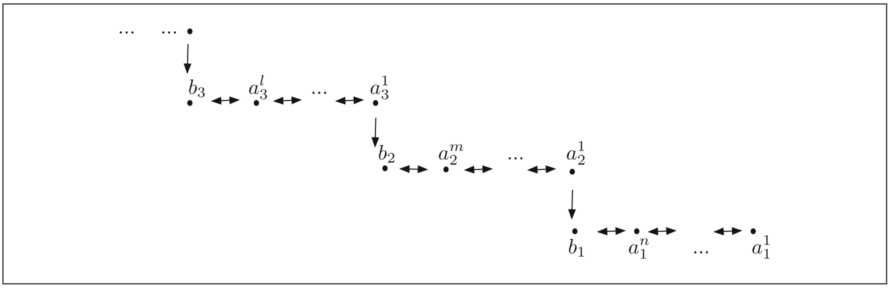

Fig. 7 Failure of s-max-limitedness (cont'd). It is understood that each world satisfies A

Successive applications of (a) yields an infinite sequence of increasingly better A-worlds $a_1 \prec a_2 \prec ... \prec a_n \prec ....$ Acyclicity guarantees that they are all distinct. This contradicts the assumption of finiteness of W.

Figure 6 also shows the second claim in part (ii). $\succeq$ is acyclic, but not s-max-limited.

For part (iii), the argument is similar. Suppose $a_1^1 \models A$. Assume to reach a contradiction that $\succeq$ is not s-max-limited. The obtained sequence of A-worlds is organized in "floors" as in Fig 7. To avoid a proliferation of symbols, the labels "A" are not shown. For all $i \leq n$, $a_i^1 \notin \max^s_{\succeq}(\|A\|)$, and a_{i+1}^1 is the world (to be called a "witness") which makes this so. This is because, for all $i \leq n$, $a_{i+1}^1 \succ b_i$. In the presence of Suzumura consistency, one gets an infinite sequence of witnesses (all distinct). To see why, define $\downarrow b_i$ $(1 \leq i \leq n)$ as $\{x : b_i \succeq^\star x\}$. When generated, each witness a_{i+1}^1 is "new", in the sense that, for all $i \leq n$, $a_{i+1}^1 \notin \downarrow b_i \cup \{b_i\}$. Suppose not. So, for some $i \leq n$, $a_{i+1}^1 \in \downarrow b_i \cup \{b_i\}$. We already have $a_{i+1}^1 \succ b_i$. By irreflexivity of $\succ$, $b_i \neq a_{i+1}^1$, so $b_i \succeq^\star a_{i+1}^1$. By Suzumura consistency, $a_{i+1}^1 \not\succ b_i$. Contradiction. □

2.6 Selection Functions

This section provides some background information on so-called selection function semantics, through which the proof of completeness will make a detour. The argument will be in two steps. The first one consists in establishing the result for models equipped with a selection function. The second one consists in deriving from this the analog result for models with a betterness relation. Perhaps a direct argument is possible, but I leave it open.

Stemming from Stalnaker [74] and generalized by Chellas [16], such a semantics was adapted to the present setting by Åqvist [4]. I call these new structures "selection models", to distinguish them from those described above. In models of this sort, the betterness relation $\succeq$ is replaced with a so-called selection function $\mathfrak{f}$ from formulas[14] such that, for all A in $\mathscr{L}$, $\mathfrak{f}(A) \subseteq W$. Intuitively, $\mathfrak{f}(A)$ outputs all the best worlds

[14] I adopt the definition proposed by Stalnaker, Lewis [44, p. 77] and Åqvist, for whom $\mathfrak{f}$ takes a formula (rather than a set of possible worlds) as input. In contrast, many scholars tend to follow Chellas [16, p. 134], who defines the domain of $\mathfrak{f}$ as the set of all the subsets of W. A selection function of the second type is sometimes called "propositional" (as opposed to a "sentential" one). A problem with the second approach is that there is no guarantee that all $X \subseteq W$ is the truth-set of a formula, and thus expresses a proposition.

satisfying A. The evaluation rule for the dyadic obligation operator is phrased thus:

$$M, a \vDash \bigcirc(B/A) \text{ iff } \mathfrak{f}(A) \subseteq \|B\|^M$$

As before I will drop the reference to M when it is clear what model is intended. The relevant constraints for $\mathfrak{f}$ are:

$$\text{If } \|A\| = \|B\| \text{ then } \mathfrak{f}(A) = \mathfrak{f}(B) \qquad \text{(syntax-independence)}$$
$$\mathfrak{f}(A) \subseteq \|A\| \qquad \text{(inclusion)}$$
$$\mathfrak{f}(A) \cap \|B\| \subseteq \mathfrak{f}(A \wedge B) \qquad \text{(Chernoff)}$$
$$\text{If } \|A\| \neq \emptyset \text{ then } \mathfrak{f}(A) \neq \emptyset \qquad \text{(consistency-preservation)}$$
$$\text{If } \mathfrak{f}(A \vee B) \subseteq \|C\| \text{ then } \mathfrak{f}(A) \subseteq \|C\| \text{ or } \mathfrak{f}(B) \subseteq \|C\| \qquad \text{(s-drat)}$$
$$\text{If } \mathfrak{f}(A) \cap \|B\| \neq \emptyset \text{ then } \mathfrak{f}(A \wedge B) \subseteq \mathfrak{f}(A) \cap \|B\| \qquad \text{(Arrow)}$$

The names used for the first four and the sixth constraints are from [57]. All these constraints have known counterparts within the framework of rational choice theory (for an overview, see Moulin [52]). The third constraint is identical to so-called Chernoff [18]'s condition also known as Sen's condition α. The label (s-drat) is from Goble [32].

For future reference I note the following:

Proposition 15 *(syntax-independence), (inclusion), (consistency-preservation) and (Arrow) imply (s-drat).*

Proof Assume $\mathfrak{f}(A \vee B) \subseteq \|C\|$. Suppose $\mathfrak{f}(A \vee B) \cap \|A\| \neq \emptyset$. Using both (syntax-independence) and (Arrow), we get the desired result:

$$\mathfrak{f}(A) = \mathfrak{f}((A \vee B) \wedge A) \subseteq \mathfrak{f}(A \vee B) \cap \|A\| \subseteq \mathfrak{f}(A \vee B) \subseteq \|C\|$$

Suppose $\mathfrak{f}(A \vee B) \subseteq \|\neg A\|$. If $\mathfrak{f}(A \vee B) \cap \|B\| \neq \emptyset$, then (Arrow) and (syntax-independence) immediately yields the desired result:

$$\mathfrak{f}(B) = \mathfrak{f}((A \vee B) \wedge B) \subseteq \mathfrak{f}(A \vee B) \cap \|B\| \subseteq \mathfrak{f}(A \vee B) \subseteq \|C\|$$

So assume $\mathfrak{f}(A \vee B) \subseteq \|\neg B\|$. Hence $\mathfrak{f}(A \vee B) \subseteq \|\neg A \wedge \neg B\|$. By (inclusion), $\mathfrak{f}(A \vee B) = \emptyset$. By (consistency-preservation), $\|A \vee B\| = \emptyset = \|B\|$. By (inclusion) $\mathfrak{f}(B) = \emptyset \subseteq \|C\|$ as required. □

The following result will be used later on.

This creates a problem in the proof of completeness when verifying that in the canonical model $\mathfrak{f}$ verifies the required constraints, like (Chernoff) and (consistency-preservation). The use of a sentential selection function allows us to bypass such a problem. An alternative strategy is to modify the standard formulation of the constraints put on $\mathfrak{f}$ and make them relative to an extra parameter, the set P of all propositions (see McNamara [50] for an example of implementation of this strategy in another context).

Theorem 16 *(i)* **E** *is sound and complete with respect to the class of selection models in which $\mathfrak{f}$ meets (syntax-independence), (inclusion), (Chernoff); (ii)* **F** *is sound and complete with respect to the class of selection models in which $\mathfrak{f}$ also meets (consistency-preservation); (iii)* **F**+*(DR) is sound and complete with respect to the class of selection models in which $\mathfrak{f}$ meets in addition (s-drat); and (iv)* **G** *is sound and complete with respect to the class of selection models in which $\mathfrak{f}$ meets (syntax-independence), (inclusion), (Chernoff), (consistency-preservation) and (Arrow).*

Proof Soundness is straightforward. Proofs of completeness are given by Åqvist [4] for **E**, **F** and **G**. To extend the argument to **F**+(DR), one just needs to verify that the proposed canonical model satisfies (s-drat). To be precise, one works with the canonical model generated from a given maximal consistent set (MCS) w. For some MCS a to be in its universe, it must be the case that $\{A : \Box A \in w\} \subseteq a$. And $a \in \mathfrak{f}(B)$ iff $\{C : \bigcirc(C/B) \in w\} \subseteq a$.

The argument for (s-drat) appeals to the truth-lemma, stating that the truth-conditions of formulas in a world coincide with the set-membership relation between formulas and maximal consistent sets. Let $\mathfrak{f}(A) \nsubseteq \|C\|$ and $\mathfrak{f}(B) \nsubseteq \|C\|$. Hence $\exists a, b$ such that $a \in \mathfrak{f}(A)$, $b \in \mathfrak{f}(B)$, $a \not\models C$ and $b \not\models C$. By the truth-lemma, $C \notin a$ and $C \notin b$. Hence $\bigcirc(C/A) \notin w$ and $\bigcirc(C/B) \notin w$. By (DR), it follows that $\bigcirc(C/A \vee B) \notin w$. A standard argument establishes that $\{D : \bigcirc(D/A \vee B) \in w\} \cup \{\neg C\}$ is consistent, and can be extended to a MCS, call it c. By (O-nec), c is in the universe of the canonical model. By construction $c \in \mathfrak{f}(A \vee B)$ and $C \notin c$ so that $c \notin \|C\|$, by the truth-lemma. Hence $\mathfrak{f}(A \vee B) \nsubseteq \|C\|$ as required. □

3 Quasi-transitivity, Acyclicity and Suzumura Consistency

I start with quasi-transitivity, acyclicity and Suzumura consistency. First, together with Theorem 8, Theorem 17 tells us that under the max rule the assumption of transitivity has no impact in the presence or absence of the limit assumption: imposing transitivity adds no new theorems to the logic.

Theorem 17 *(i) Under the max rule,* **E** *is sound and complete with respect to the class of preference models in which $\succeq$ is transitive; (ii) under the max rule,* **F** *is sound and complete with respect to the class of preference models in which $\succeq$ is max-limited and transitive; and (iii) under the max rule,* **F**+*(CM) is sound and complete with respect to the class of preference models in which $\succeq$ is max-smooth and transitive.*

For **E** and **F**, the crux of the argument consists in showing that, starting with a given preference model, one can transform it into one in which the betterness relation is transitive in such a way that it may be guaranteed to satisfy exactly the same formulas. This is what the adjective "equivalent" refers to in the statement of the theorem below. The precise definition of this notion is best explained once the construction has been given.

Theorem 18 *[Goble [33]] For every model $M = (W, \succeq, v)$, there is a model $M' = (W', \succeq', v')$ in which $\succeq'$ is transitive and such that under the max rule M and M' are equivalent. Furthermore, if $\succeq$ is max-limited, then $\succeq'$ is also max-limited.*

Proof I recall the construction.[15] Let $M = (W, \succeq, v)$. Define $M' = (W', \succeq', v')$ as follows:

- $W' = \{\langle a, b, n\rangle \mid a, b \in W, n \in \omega\}$
- $\langle a, b, n\rangle \succeq' \langle c, d, m\rangle$ iff (1) $\langle a, b, n\rangle = \langle c, d, m\rangle$ or

$$(2) \begin{cases} \text{(a)}\ \ b = d \ \&\ n \geq m \\ \text{and} \\ (\text{b}_1)\ c \neq d \ \&\ a = c \text{ or } (\text{b}_2)\ c = d \ \&\ a \succ c \end{cases}$$

- $v'(p) = \{\langle a, b, n\rangle \mid a \in v(p)\}$

The verification that the construction actually does the desired job proceeds via a series of lemmas, for which we refer the reader to [33, p. 44 *sqq*] or [57, Theorem C1]. The two models are equivalent in the following sense: for all formulas A, all $a, b \in W$ and all $n \in \omega$, $a \models A$ iff $\langle a, b, n\rangle \models A$. □

For **F**+(CM), the proof is different. The reader is referred to [32, Theorem 56] or [57, Theorem 4.2] for more details.

While a number of theorems have been presented in the literature on the iddleness of transitivity under a rule of interpretation in terms of maximality (see also Grossi & al. [34]),[16] the following extremely simple corollary seems to have been overlooked:

Theorem 19 *(i) Under the max rule,* **E** *is sound and complete with respect to the class of preference models whose relation* $\succeq$ *is quasi-transitive, Suzumura consistent or acyclic; (ii) under the max rule,* **F** *is sound and complete with respect to the class of preference models whose relation* $\succeq$ *is max-limited and either quasi-transitive, Suzumura consistent or acyclic; and (iii) under the max rule,* **F**+*(CM) is sound and complete with respect to the class of preference models whose relation* $\succeq$ *is max-smooth and either quasi-transitive, Suzumura consistent or acyclic.*

Proof Soundness follows from the fact that none of the axioms and rules appeal to the conditions of quasi-transitivity, Suzumura consistency and acyclicity. Completeness follows from Theorem 17 and Proposition 12 above. Where Γ is a set of formulas, suppose $\Gamma \nvdash A$ in, e.g., **E**. According to Theorem 17 (i), $\Gamma \cup \{\neg A\}$ is falsifiable in a preference model in which $\succeq$ is transitive. By Proposition 12, $\Gamma \cup \{\neg A\}$ is falsifiable in a preference model in which $\succeq$ is quasi-transitive, Suzumura consistent or acyclic. Hence $\Gamma \not\models A$ w.r.t. the class of preference models in which $\succeq$ is quasi-transitive, Suzumura consistent or acyclic. For **F** and **F**+(CM), the argument is similar. □

These results extend to the classes of models as specified in the statement of Theorem 19 whose relation $\succeq$ is in addition reflexive. They also carry over to the rule of interpretation in terms of strong maximality:

[15] A similar one may be found in Schlechta [68, p. 77].

[16] Their system **MOU** is equivalent to **E**. **MOU** has (Triv) $\bigcirc(\top/A)$ in place of (O-nec), and the extra axiom (L-Ext) $\Box(B \leftrightarrow C) \rightarrow (\bigcirc(B/A) \leftrightarrow \bigcirc(C/A))$. (Triv) is derivable from (O-nec), since this one gives $\Box\top \rightarrow \bigcirc(\top/A)$ and $\Box\top$ is a theorem. Suppose $\Box(B \leftrightarrow C)$, and hence $\Box(B \rightarrow C)$. (O-nec) and (COK) give $\bigcirc(B/A) \rightarrow \bigcirc(C/A)$, and likewise for the converse implication. Thus, both (Triv) and (L-Ext) are theorems of **E**. Suppose $\Box A$ holds. So $\Box(A \leftrightarrow \top)$ holds. By (Triv) $\bigcirc(\top/B)$, and so $\bigcirc(A/B)$ by (L-Ext). Hence, (O-nec) is a theorem of **MOU**.

Theorem 20 *(i) Under the s-max rule, $\mathbf{E}$ is sound and complete with respect to the class of all preference models, and with respect to the class of those whose relation $\succeq$ is transitive, quasi-transitive, Suzumura consistent or acyclic; (ii) under the s-max rule, $\mathbf{F}$ is sound and complete with respect to the class of preference models whose relation $\succeq$ is s-max-limited, and with respect to the class of those in which in addition $\succeq$ is transitive, quasi-transitive, Suzumura consistent or acyclic; and (iii) under the s-max rule, $\mathbf{F}$+(CM) is sound and complete with respect to the class of preference models whose relation $\succeq$ is s-max-smooth, and with respect to the class of those in which in addition $\succeq$ is transitive, quasi-transitive, Suzumura consistent or acyclic.*

Proof For the soundness half, it is enough to verify that all the axioms and rules are valid under the s-max rule. I show (Sh) and (CM).

- For (Sh), suppose that $a \models \bigcirc(C/A \wedge B)$. Let b be such that $b \in \max^s_{\succeq}(\|A\|)$ and $b \models B$. Suppose, to reach a contradiction, that $b \notin \max^s_{\succeq}(\|A \wedge B\|)$. Hence there are c and d with $b \approx^{A \wedge B} c$, $d \models A \wedge B$ and $d \succ c$. We have $b \approx^A c$ and $d \models A$, so that $b \notin \max^s_{\succeq}(\|A\|)$, contrary to the opening assumption. Hence $b \in \max^s_{\succeq}(\|A \wedge B\|)$, and so $b \models C$ as required.
- For (CM), assume $\succeq$ is s-max-smooth, and let a be such that $a \models \bigcirc(B/A)$ and $a \models \bigcirc(C/A)$. Consider also some b such that $b \in \max^s_{\succeq}(\|A \wedge B\|)$. Suppose, to reach a contradiction, that $b \notin \max^s_{\succeq}(\|A\|)$. We have $b \models A$. By s-max-smoothness, it follows that there is some $c \succ b$ such that $c \in \max^s_{\succeq}(\|A\|)$. Clearly, $c \models A \wedge B$. Since $b \approx^{A \wedge B} b$, one gets that $b \notin \max^s_{\succeq}(\|A \wedge B\|)$, a contradiction. So $b \in \max^s_{\succeq}(\|A\|)$, and hence $b \models C$, which suffices for $a \models \bigcirc(C/A \wedge B)$.

The completeness half follows from Theorem 17 (extended to models whose relation $\succeq$ is reflexive), Proposition 5 (ii) and Proposition 12. Suppose $\Gamma \nvdash A$ in, e.g., $\mathbf{E}$. According to Theorem 17 (i) and Proposition 12, under the max rule $\Gamma \cup \{\neg A\}$ is falsifiable in a preference model in which $\succeq$ is reflexive and transitive, and hence quasi-transitive, Suzumura consistent and acyclic. By Proposition 5 (ii), under the s-max rule $\Gamma \cup \{\neg A\}$ is falsified in the same model. Hence under the s-max rule $\Gamma \not\models A$ w.r.t. the class of all preference models, and the class of those in which $\succeq$ meets the appropriate conditions. For $\mathbf{F}$ and $\mathbf{F}$+(CM), the argument is similar. □

The result also holds for each class of models mentioned in the statement of Theorem 20 whose relation $\succeq$ is also reflexive.

Proposition 21 below clarifies the status of the law:

$$(\bigcirc(A_2/A_1) \wedge \ldots \wedge \bigcirc(A_1/A_n)) \rightarrow \bigcirc(A_n/A_1) \qquad \text{(loop)}$$

(loop) captures a restricted form of transitivity for the conditional obligation operator. Kraus, Lehmann and Magidor [42, p. 187] observe that in their setting, where a state is identified with a set of valuations or possible worlds, the principle (loop) is the syntactical counterpart of quasi-transitivity. Their observation does not carry over to the present setting. By itself, quasi-transitivity does not give us (loop) under either rule of interpretation.

Proposition 21 (loop) *Under the max rule (resp. s-max rule), it holds that:*

(i) *((loop)) is not valid in the class of preference models whose relation $\succeq$ is quasi-transitive;*
(ii) *((loop)) is derivable in* $\mathbf{F}+(CM)$*, and hence (by Theorems 8(iii) and 20(iii)) valid in the class of preference models whose relation $\succeq$ is max-smooth (*resp. *s-max-smooth).*

Proof For (i), consider the instance of (loop) where $n = 3$, and put $W = \{a_i : i \in \omega\}$, $a_i \succeq a_j$ iff $i \geq j$, and $V(p) = \{a_1\}$, $V(q) = W$, and $V(r) = \emptyset$ for all other atomic formulas $r \in \mathbb{P}$. $\succeq$ is quasi-transitive since $\geq$ is. Under the max rule, it holds that:

$a_1 \models \bigcirc(q/p)$
$a_1 \models \bigcirc(r/q)$
$a_1 \models \bigcirc(p/r)$
$a_1 \not\models \bigcirc(r/p)$ (witness: a_1)

$\succeq$ is reflexive and transitive, since $\geq$ is. Therefore, the above point also holds under the s-max rule, by Proposition 5 (ii).

For (ii), assume $\bigcirc(A_2/A_1)$, ... and $\bigcirc(A_1/A_n)$. First, we show that ($\star$) for all i, j s.t. $\bigcirc(A_j/A_i)$ is in this loop, $\bigcirc(A_i \to A_j/\bigvee_{k=1}^n A_k)$. Let $\bigcirc(A_j/A_i)$. By (RW), $\bigcirc(A_i \to A_j/A_i)$. By (Ext), $\bigcirc(A_i \to A_j/(\bigvee_{k=1}^n A_k) \wedge A_i)$. By (Sh), $\bigcirc(A_i \to (A_i \to A_j)/\bigvee_{k=1}^n A_k)$. By (RW), $\bigcirc(A_i \to A_j/\bigvee_{k=1}^n A_k)$. This shows ($\star$). Now, by ($\star$) and (AND), it follows that $\bigcirc(\bigwedge_{i=1}^{n-1}(A_i \to A_{i+1}) \wedge (A_n \to A_1)/\bigvee_{k=1}^n A_k)$. By (Id), (AND) and (RW), $\bigcirc(\bigwedge_{k=1}^n A_k/\bigvee_{k=1}^n A_k)$. By (VCM), $\bigcirc(\bigwedge_{k=2}^n A_k/(\bigvee_{k=1}^n A_k) \wedge A_1)$. By (Ext), $\bigcirc(\bigwedge_{k=2}^n A_k/A_1)$. By (RW), $\bigcirc(A_n/A_1)$. □

The next section deals with our fourth candidate weakening of transitivity: the interval order condition.

4 Interval Order

I start with the max and opt rules. They can be handled simultaneously, because given the assumption of totality the two come to the same thing.[17] Intuitively the interval order condition may seem to be at odds with the s-max rule. For instance, the condition is not met in the motivating example shown in Fig. 1. For the sake of completeness, I will nevertheless consider such a rule as well.

I will assume that models are finite. The rationale for this is purely technical. As we will soon see, the main (completeness) proof relies on such an assumption. Consequently, the result is formulated as a weak completeness theorem with respect to the class of finite models. It establishes a match between theorems and validities only. This point is further discussed after the proof of Theorem 28, on p. 24 f., and after Corollary 41, on p. 28 f.

Theorem 22 (Soundness, opt and max rules) *Under the max and opt rules,* $\mathbf{F}$+*(DR) is (weakly) sound with respect to the class of finite preference models whose relation $\succeq$ is an interval order.*

[17] A similar result (but in a slightly different, KLM-style setting) is reported by Booth and Varzinczak [9]. Their proof is different and has not been published yet. I have drawn on Rott [66] whose setting remains very different.

Proof The axioms other than (D*) and (DR) are valid regardless of the properties of $\succeq$. (D*), the distinctive postulate of **F**, calls for max-limitedness (*resp.* opt-limitedness). It holds, given the assumption of finiteness of the models and the interval order condition, by Proposition 5 (i), Proposition 12 (i) and (iii) and Proposition 14 (ii). It suffices to show that (DR) is validated, given the interval order condition. I give the argument for the max rule only. It applies *mutatis mutandis* to the opt rule.

Suppose (i) $\max_{\succeq}(\|A \vee B\|) \subseteq \|C\|$ but (ii) $\max_{\succeq}(\|A\|) \nsubseteq \|C\|$ and (iii) $\max_{\succeq}(\|B\|) \nsubseteq \|C\|$. From (ii), $\exists a$ s.t. $a \in \max_{\succeq}(\|A\|)$ and $a \not\models C$. By (iii), $\exists b$ s.t. $b \in \max_{\succeq}(\|B\|)$ and $b \not\models C$. By (i), $a \notin \max_{\succeq}(\|A \vee B\|)$ and $b \notin \max_{\succeq}(\|A \vee B\|)$. So $\exists c \succ a$ s.t. $c \models A \vee B$ and $\exists d \succ b$ s.t. $d \models A \vee B$. Clearly, $c \models B$ and $d \models A$. This is illustrated with Fig. 8. By Ferrers, $a \succeq d$ and $b \succeq c$ would imply $a \succeq c$ or $b \succeq d$. But $a \not\succeq c$ and $b \not\succeq d$. So $a \not\succeq d$ or $b \not\succeq c$. Suppose the first applies. By totality, $d \succeq a$, so that $d \succ a$, contradicting the fact that $a \in \max_{\succeq}(A)$. Now suppose $b \not\succeq c$. By totality again $c \succeq b$, so that $c \succ b$, contradicting the fact that $b \in \max_{\succeq}(\|B\|)$. This completes the proof. □

Theorem 23 (Soundness, s-max) *Under the s-max rule,* **G** *is (weakly) sound with respect to the class of finite preference models whose relation $\succeq$ is an interval order and s-max-limited.*

Proof (D*), the distinctive axiom of **F**, is validated given s-max-limitedness.

The proof of validity of (Sp) appeals to the assumption of totality only. Suppose (i) $\max^s_{\succeq}(\|A\|) \subseteq \|B \to C\|$, (ii) $\max^s_{\succeq}(\|A\|) \cap \|B\| \neq \emptyset$ and (iii) $\max^s_{\succeq}(\|A \wedge B\|) \cap \|\neg C\| \neq \emptyset$. From (iii), $\exists a \in \max^s_{\succeq}(\|A \wedge B\|)$ such that $a \not\models C$. By (i), $a \notin \max^s_{\succeq}(\|A\|)$. Hence $\exists b, c$ s.t. $b \approx^A a$, $c \models A$, and $c \succ b$. From (ii), $\exists d \in \max^s_{\succeq}(\|A\|)$ such that $d \models B$. By totality, either (α) $d \succeq a$ or (β) $a \succeq d$. Suppose (β) applies. By definition, $d \approx^A d$. But $d \in \max^s_{\succeq}(\|A\|)$, so $a \models A$ and $a \succeq d$ imply $d \succeq a$. Suppose (α) applies. By definition $a \approx^{A \wedge B} a$. But $a \in \max^s_{\succeq}(\|A \wedge B\|)$. So $d \models A \wedge B$ and $d \succeq a$ imply $a \succeq d$. Either way $a \approx d$. From this and $b \approx^A a$, one gets $b \approx^A d$, by Fact 3 (i). It then follows that $d \notin \max^s_{\succeq}(\|A\|)$, a contradiction. One concludes that $\max^s_{\succeq}(\|A \wedge B\|) \subseteq \|C\|$. □

Remark 24 *Unlike with Theorem 22, the form of the limited assumption needed to validate (D*), s-max-limitedness, must appear in the statement of the soundness theorem, Theorem 23. This is because in a finite model the interval order condition does not imply s-max-limitedness. Consider, e.g., $M = (W, \succeq, v)$, where*

- $W = \{a, b, c\}$

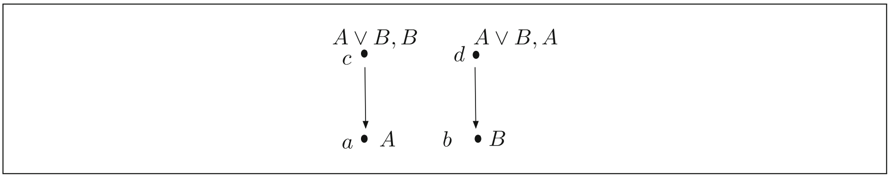

Fig. 8 Disjunctive rationality

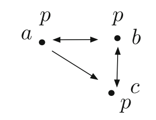

Fig. 9 Failure of s-max-limitedness in a model whose relation $\succeq$ is an interval order. (Reflexive loops are omitted)

- $\succeq$ = *the reflexive closure of* $\{(a, b), (b, a), (b, c), (c, b), (a, c)\}$
- $V(p) = W$ *for all atomic formulas* $p \in \mathbb{P}$

This is shown in Fig. 9. In this model, the interval order condition is met, but not s*-max-limitedness. For all* p, $\|p\| \neq \emptyset$, *but* $a \approx^p c$ *while* $a \succ c$, *so* $\max^s_{\succeq}(\|p\|) = \emptyset$.

Before addressing completeness, I justify the claims of independence appearing in the statement of Theorem 9.

Proposition 25 *(i) There is a preference model whose relation* $\succeq$ *is max-smooth, in which (DR) is falsified under the max rule; (ii) there is a preference model whose relation* $\succeq$ *is an interval order and max-limited, in which (Sp) is falsified under the max rule; and (iii) there is a selection model in which* $\mathfrak{f}$ *meets (syntax-independence), (inclusion), (Chernoff) and (s-drat), in which (D***) is falsified.*

Proof For (i), put $M = (W, \succeq, v)$ with $W = \{a, b, c, d\}$, $\succeq = \{(c, a), (d, b)\}$, $v(p) = \{a, d\}$, $v(q) = \{b, c\}$, $v(r) = \{c, d\}$ and $v(s) = W$ for all other atomic formulas $s \in \mathbb{P}$. $\succeq$ is quasi-transitive. By Proposition 14 (i), $\succeq$ is max-smooth, since W is finite. $\bigcirc(r/p \vee q)$ holds under the max rule, but not $\bigcirc(r/p)$ nor $\bigcirc(r/q)$. Hence (DR) is falsified.

For (ii), put $M = (W, \succeq, v)$ with W and $\succeq$ as in Remark 24, $v(p) = W$, $v(q) = \{b, c\}$, $v(r) = \{a, b\}$ and $v(s) = W$ for all other atomic formulas $s \in \mathbb{P}$. $\succeq$ is an interval order, and hence acyclic, by Proposition 12 (i) and (iii). Since W is finite, $\succeq$ is max-limited, by Proposition 14 (ii). $\bigcirc(q \to r/p)$ and $P(q/p)$ both hold under the max rule, but not $\bigcirc(r/p \wedge q)$. For $\max_{\succeq}(\|p\|) = \{a, b\}$ while $\max_{\succeq}(\|p \wedge q\|) = \{b, c\}$. Hence (Sp) is falsified.

For (iii), define $M = (W, \mathfrak{f}, v)$ as follows: $W = \{a\}$; $\mathfrak{f}(A) = \emptyset$ for all $A \in \mathscr{L}$; and $v(p) = \emptyset$ for all atomic formulas $p \in \mathbb{P}$. $\mathfrak{f}$ meets (syntax-independence), (inclusion), (Chernoff) and (s-drat). The formulas $\Diamond\neg p$, $\bigcirc(q/\neg p)$ and $\bigcirc(\neg q/\neg p)$ hold at a. Hence (D*) is falsified. $\square$

Corollary 26 (Independence) *(i) (DR) is not a theorem of* **F**+*(CM); (ii) (Sp) is not a theorem of* **F**+*(DR); and (iii) (D***) is not a theorem of* **E**+*(DR).*

Proof For (i), note that under the max rule **F**+(CM) is sound with respect to the class of models in which $\succeq$ is max-smooth. So if (DR) was a theorem of **F**+(CM), (DR) would be valid in the class of preference models in which $\succeq$ is max-smooth. This contradicts Proposition 25 (i).

For (ii), the argument is similar. Under the max rule **F**+(DR) is sound with respect to the class of preference models in which $\succeq$ is max-limited and an interval order. If (Sp) was a theorem of **F**+(DR), (Sp) would be valid in this class of models, contradicting Proposition 25 (ii).

For (iii), **E**+(DR) is sound with respect to the class of selection models in which $\mathfrak{f}$ meets (syntax-independence), (inclusion), (Chernoff) and (s-drat). If (D*) was a theorem of **E**+(DR), (D*) would be valid in this class of models, contradicting Proposition 25 (iii). □

Corollary 26 (iii) explains why (DR) and (D*) are considered separately. It also gives us the following:

Proposition 27 (Incompleteness) **E**+*(DR) is not (weakly) complete with respect to the class of finite models whose relation* $\succeq$ *is an interval order.*

Proof By Proposition 12 (i), Proposition 14 (i) and Proposition 6 (i), (D*) is valid in the class of finite models in which $\succeq$ is an interval order, and hence max-limited. By Corollary 26 (iii), (D*) is not a theorem in **E**+(DR). Hence, there exists a formula that is valid in the class of finite models in which $\succeq$ is an interval order, but not derivable in **E**+(DR). □

The final part of the paper is devoted to establishing the following result.

Theorem 28 (Completeness, opt and max) *Under the opt and max rules,* **F**+*(DR) is (weakly) complete with respect to the class of finite preference models whose relation* $\succeq$ *is an interval order.*

The proof takes a detour through the modeling in terms of selection functions. It will be helpful to draw in the concept of expressibility described in Goble [30]. Briefly, a model M is said to meet the expressibility constraint when any set of worlds in M is expressible by some formula A:

$$\forall X \subseteq W \ \exists A \in \mathscr{L} \text{ s.t. } X = \|A\| \tag{expr}$$

An immediate corollary is that, in a selection model M meeting (expr), $\mathfrak{f}$ is (in the terminology of Schlechta [68]) "definability-preserving":

$$\forall A \in \mathscr{L} \ \exists B \in \mathscr{L} \text{ s.t. } \mathfrak{f}(A) = \|B\| \tag{dp}$$

Theorem 28 follows from Theorem 16 (iii) combined with Theorems 29 and 31:

Theorem 29 (F.m.p., selection functions) **F**+*(DR) has the finite model property (f.m.p.) with respect to selection functions. That is, if A is satisfiable in a selection model $M = (W, \mathfrak{f}, v)$ in which $\mathfrak{f}$ meets (syntax-independence), (inclusion), (Chernoff), (consistency-preservation) and (s-drat), then A is satisfiable in a finite such model $M^\star = (W^\star, \mathfrak{f}^\star, v^\star)$ meeting condition (expr).*

Proof The proof extends that for **F** in [57]. One must perform two extra verifications. First, one must show that, if $\mathfrak{f}$ meets (s-drat), then so does $\mathfrak{f}^\star$. Second, one must establish that (expr) is fulfilled. Details are given in Appendix A. □

The following property (I give it Rott [66]'s name) is closely related to (s-drat):

$$\text{Either } \mathfrak{f}(A) \subseteq \mathfrak{f}(A \vee B) \text{ or } \mathfrak{f}(B) \subseteq \mathfrak{f}(A \vee B) \tag{II$^+$}$$

Our completeness result relies on the following fact, made possible by (expr) and its corollary (dp):

Fact 30 *(i)* (II$^+$) *implies (s-drat); and (ii) given (expr), (s-drat) implies* (II$^+$).

Proof For (i), suppose (II$^+$) holds, and assume $\mathfrak{f}(A \vee B) \subseteq \|C\|$ for a given C. It follows at once that either $\mathfrak{f}(A) \subseteq \|C\|$ or $\mathfrak{f}(B) \subseteq \|C\|$ as required.

For (ii), assume (expr) and (s-drat) both hold. By construction, $\mathfrak{f}(A \vee B) \subseteq W$. By (dp), $\exists C$ s.t. $\mathfrak{f}(A \vee B) = \|C\|$. Trivially, $\mathfrak{f}(A \vee B) \subseteq \mathfrak{f}(A \vee B) = \|C\|$. By (s-drat), it follows that $\mathfrak{f}(A) \subseteq \mathfrak{f}(A \vee B)$ or $\mathfrak{f}(B) \subseteq \mathfrak{f}(A \vee B)$. □

Now for the crux of the argument:

Theorem 31 *For every finite selection model $M = (W, \mathfrak{f}, v)$ in which $\mathfrak{f}$ meets (syntax-independence), (inclusion), (Chernoff), (consistency-preservation), (s-drat) and (expr), there is a preference model $M' = (W, \succeq, v)$ (with W and v unchanged) in which $\succeq$ is an interval order, such that under the max or opt rules M' is equivalent with M.*

Proof Starting with $M = (W, \mathfrak{f}, v)$, define M' by putting $a \succeq b$ iff $\exists A$ s.t. $\|A\|^M = \{a, b\}$ and $a \in \mathfrak{f}(A)$. □

Lemma 32 *$\succeq$ is total (and hence reflexive).*

Proof of Lemma 32 This follows at once from (expr), (consistency-preservation) and (inclusion). □

Lemma 33 *$\succeq$ is Ferrers.*

Proof of Lemma 33 Assume $a \succeq b$ and $c \succeq d$. By definition $\exists A$ s.t. $a \in \mathfrak{f}(A)$ and $\|A\| = \{a, b\}$ and $\exists B$ s.t. $c \in \mathfrak{f}(B)$ and $\|B\| = \{c, d\}$. Hence $\|A \vee B\| = \{a, b, c, d\}$. By (II$^+$), $a \in \mathfrak{f}(A \vee B)$ or $c \in \mathfrak{f}(A \vee B)$. Assume $a \in \mathfrak{f}(A \vee B)$. By (expr), $\exists C$ s.t. $\|C\| = \{a, d\}$. By (Chernoff), $\mathfrak{f}(A \vee B) \cap \|C\| \subseteq \mathfrak{f}((A \vee B) \wedge C)$. So $a \in \mathfrak{f}((A \vee B) \wedge C)$. Clearly, $\|(A \vee B) \wedge C\| = \{a, d\}$. So by definition of $\succeq$, $a \succeq d$ as required. If $c \in \mathfrak{f}(A \vee B)$, then a similar argument yields $c \succeq b$. □

Lemma 34 establishes equivalence between models.

Lemma 34 *For all a, $M, a \models A \Leftrightarrow M', a \models A$.*

Proof of Lemma 34 By induction on A. The only case of interest is when $A := \bigcirc(C/B)$. It will help to note that, under the inductive hypothesis, □

Sub-lemma 35 $a \in \mathfrak{f}(B) \Leftrightarrow a \in \text{opt}_{\succeq}(\|B\|) \Leftrightarrow a \in \max_{\succeq}(\|B\|)$.

Proof of Sub-lemma 35 The equivalence of the right pair is immediate, given totality of $\succeq$. Therefore, I focus on the equivalence of the left pair.

($\Rightarrow$) Assume $a \in \mathfrak{f}(B)$. By (inclusion), $M, a \models B$. By the inductive hypothesis, $M', a \models B$. Let b be s.t. $M', b \models B$. By (expr), $\exists C$ s.t. $\|C\|^M = \{a, b\}$. By (Chernoff), $\mathfrak{f}(B) \cap \|C\| \subseteq \mathfrak{f}(B \wedge C)$. So $a \in \mathfrak{f}(B \wedge C)$. By the inductive hypothesis, $M, b \models B$. So $\|B \wedge C\|^M = \|C\|^M$. By (syntax-independence), $a \in \mathfrak{f}(C)$. So $a \succeq b$, which suffices for $a \in opt_{\succeq}(\|B\|)$.

($\Leftarrow$) Let $a \in opt_{\succeq}(\|B\|)$. $\|B\|^M$ is finite, since W is finite. Let a_1, ..., a_n be an enumeration of the elements of $\|B\|^M$. By the inductive hypothesis, $\|B\|^{M'} = \{a_1, ...a_n\}$. a is one of these elements. By the opening assumption, $a \succeq a_i$, for all $i \in \{1, ..., n\}$. By definition of $\succeq$, $\forall a_i$ $(i \in \{1, ..., n\})$ $\exists A_i$ s.t. $\|A_i\|^M = \{a, a_i\}$ and $a \in \mathfrak{f}(A_i)$. For instance, $a \in \mathfrak{f}(A_1)$ and $a \in \mathfrak{f}(A_2)$. By (II$^+$), it follows that $a \in \mathfrak{f}(A_1 \vee A_2)$. The number of A_i's is finite. Reiterating this argument n times one reaches the conclusion that $a \in \mathfrak{f}(A_1 \vee A_2 \vee ... \vee A_n)$. But $\|B\|^M = \|A_1 \vee A_2 \vee ... \vee A_n\|^M$. By (syntax-independence), $a \in \mathfrak{f}(B)$ as required. □

With Sub-lemma 35 in hand, it is a straightforward matter to show that the two models are equivalent. Details are omitted.

The proof of Theorem 28 can now begin. I show the contrapositive, i.e., if A is not derivable in **F**+(DR), then A is not valid in the class of finite preference models whose relation $\succeq$ is an interval order. Suppose A is not derivable in **F**+(DR). By Theorem 16 (iii), A is false in some world a in some selection model $M_1 = (W_1, \mathfrak{f}_1, v_1)$ whose selection function $\mathfrak{f}_1$ meets (syntax-independence), (inclusion), (Chernoff), (consistency-preservation) and (s-drat). By Theorem 29, A is false in some world b in some finite selection model $M_2 = (W_2, \mathfrak{f}_2, v_2)$ in which $\mathfrak{f}_2$ meets (syntax-independence), (inclusion), (Chernoff), (consistency-preservation), (s-drat) and (expr). By Theorem 31, there is a finite preference model $M_3 = (W_2, \succeq, v_2)$ whose relation $\succeq$ is an interval order, in which A is false at b. Hence A is not valid in the class of finite preference models whose relation $\succeq$ is an interval order.

Theorem 28 is stated as a weak completeness theorem with respect to the class of finite models. Theorem 28 can be strengthened into a weak completeness theorem irrespective of the cardinality of the models as long as limitedness is made explicit in the statement of the theorem. In other words, Theorem 28 also holds with respect to the class of models whose relation $\succeq$ is an interval order and max-limited (*resp.* opt-limited). Nevertheless, Theorem 28 falls short of establishing strong completeness. In other words, it does not guarantee a correspondence between the syntactic and semantic consequence relation while also accommodating a potentially infinite set of assumptions. That is due to the need to take the path through finite models.

To elaborate further, the proof of Theorem 31 specifically relies on the input model being finite and fulfilling condition (expr). These two properties are "enforced" through the application of the filtration method (see Appendix A). Let us try to re-run the proof of Theorem 28 and make it an argument for strong completeness, starting with the assumption that A is not provable from a set Γ of assumptions. A problem arises in the proof of the finite model property with respect to selection models, Theorem 29. For weak completeness, the "filter" is the set of $\neg A$'s sub-formulas, where A is the formula appearing in the statement of the completeness theorem. Such a set is finite, and hence the filtrated model $M^\star$ is also finite, while meeting (expr). For strong completeness, the filter contains $\Gamma \cup \{\neg A\}$. Because such a set is potentially infinite, there is no guarantee that the filtrated model $M^\star$ is finite, and meets (expr). I leave open the possibility of establishing strong completeness of **F**+(DR) by other means.

The added value of the finitary method proposed in this paper is that it gives us the finite model property with respect to preference models, and also decidability of the associated calculus.

Corollary 36 (F.m.p., preferences) *If under the max rule (*resp. *opt rule) A is satisfiable in a preference model $M = (W, \succeq, v)$ whose relation $\succeq$ is an interval order and max-limited (*resp. *opt-limited), then under the max rule (*resp. *opt rule) A is satisfiable in a finite such model.*

Proof The argument is straightforward, using Theorems 29, 31 and the following observation: if under the max rule (*resp.* opt rule) A is satisfiable in a preference model $M = (W, \succeq, v)$ whose relation $\succeq$ is an interval order and max-limited (*resp.* opt-limited), then A is satisfiable in a selection model $M' = (W, \mathfrak{f}, v)$ (with W and v unchanged) in which $\mathfrak{f}$ meets (syntax-independence), (inclusion), (Chernoff), (consistency-preservation) and (s-drat). Define $\mathfrak{f}$ by putting, for all $A \in \mathscr{L}$, $\mathfrak{f}(A) = \max_{\succeq}(\|A\|)$ (*resp.* $\mathrm{opt}_{\succeq}(\|A\|)$). Note that, in the proof of Theorem 31, the relation $\succeq$ derived from $\mathfrak{f}$ is max-limited (*resp.* opt-limited). □

The theoremhood problem ("Is A a theorem?") in **E**, **F**, **F**+(CM) and **G** is decidable.[18] As a spin-off result of Theorem 28, one also gets:

Corollary 37 *The theoremhood problem in* **F**+*(DR) is decidable.*

Proof The argument is standard, and is omitted. (See, e.g., [17].) □

I turn to completeness under the s-max rule.

Lemma 38 *Given (Arrow), the relation $\succeq$ as defined in the proof of Theorem 31 is transitive.*

Proof of Lemma 38 Assume $a \succeq b$ and $b \succeq c$. So $\exists A, B$ s.t. $\{a, b\} = \|A\|$, $a \in \mathfrak{f}(A)$, $\{b, c\} = \|B\|$ and $b \in \mathfrak{f}(B)$. By (expr), $\exists C$ s.t. $\{a, c\} = \|C\|$. Clearly $\{a, b, c\} = \|A \vee B \vee C\|$. If one can show that $a \in \mathfrak{f}(A \vee B \vee C)$, then we are done, because by (Chernoff) one gets $a \in \mathfrak{f}((A \vee B \vee C) \wedge C)$, and then by (syntax-independence) $a \in \mathfrak{f}(C)$, which suffices for $a \succeq c$.

By (consistency-preservation) and (inclusion), at least one of a, b and c is in $\mathfrak{f}(A \vee B \vee C)$. Suppose it is b. Then, $\mathfrak{f}(A \vee B \vee C) \cap \|A\| \neq \emptyset$. By (Arrow), $\mathfrak{f}((A \vee B \vee C) \wedge A) \subseteq \mathfrak{f}(A \vee B \vee C) \cap \|A\|$. By (syntax-independence), $\mathfrak{f}((A \vee B \vee C) \wedge A) = \mathfrak{f}(A)$, so that $a \in \mathfrak{f}(A \vee B \vee C)$. Now suppose $c \in \mathfrak{f}(A \vee B \vee C)$. A similar argument yields $b \in \mathfrak{f}(A \vee B \vee C)$, from which $a \in \mathfrak{f}(A \vee B \vee C)$ follows. Either way, $a \in \mathfrak{f}(A \vee B \vee C)$ as required. □

[18] See, e.g., [3, 57].

Theorem 39 (F.m.p., selection functions, cont'd) **G** *has the finite model property (f.m.p.) with respect to selection functions. That is, if A is satisfiable in a selection model* $M = (W, \mathfrak{f}, v)$ *in which* $\mathfrak{f}$ *meets (syntax-independence), (inclusion), (Chernoff), (consistency-preservation) and (Arrow), then A is satisfiable in a finite such model meeting condition (expr).*

Proof See [3, p. 34] and [57, Theorem 5.10]. □

With this in hand, one shows:

Theorem 40 (Completeness, s-max) *Under the* s*-max rule,* **G** *is (weakly) complete with respect to the class of finite preference models whose relation* $\succeq$ *is an interval order and* s*-max-limited.*

Proof To establish this result, we need only make a few adjustments to the proof of the corresponding completeness theorem for **F**+(DR), Theorem 28. Suppose A is not derivable in **G**. By Theorem 16 (iv), A is false in some world a in some selection model $M_1 = (W_1, \mathfrak{f}_1, v_1)$ in which $\mathfrak{f}_1$ meets (syntax-independence), (inclusion), (Chernoff), (consistency-preservation) and (Arrow). By Theorem 39, A is false in some world b in some finite selection model $M_2 = (W_2, \mathfrak{f}_2, v_2)$ in which $\mathfrak{f}_2$ meets (syntax-independence), (inclusion), (Chernoff), (consistency-preservation), (Arrow) and (expr). In this model, $\mathfrak{f}_2$ meets (s-drat), by Proposition 15. As before, by Theorem 31, there is a finite preference model $M_3 = (W_2, \succeq, v_2)$ whose relation $\succeq$ is an interval order, in which under the max rule A is false at b. In this model, $\succeq$ is reflexive. Because $\mathfrak{f}_2$ meets (Arrow), $\succeq$ is also transitive, by Lemma 38. Using Proposition 5 (ii) it follows that in M_3 the s-max rule and the max rule coincide. Also s-max-limitedness follows by Proposition 12 (ii) and Proposition 14 (iii). Thus, the analog of Theorem 31 allows us to conclude that under the s-max rule A is falsified in a finite preference model whose relation $\succeq$ is an interval order and s-max-limited. □

It is worth mentioning that, in Theorems 23 and 40, s-max-limitedness can be replaced with Suzumura consistency.

Corollary 41 *Under the* s*-max rule,* **G** *is (weakly) sound and complete with respect to the class of finite preference models whose relation* $\succeq$ *is an interval order and Suzumura consistent.*

Proof For soundness, it suffices to invoke Proposition 14 (iii). For completeness, it is enough to remark that (as defined in the proof of Theorem 40) the relation $\succeq$ is Suzumura consistent, since it is transitive, Proposition 12 (ii). □

Unlike Theorem 28, Theorem 40 can be strengthened into a strong completeness result. This follows at once from [55, Theorem 9], which establishes strong completeness of **G** under the max rule with respect to the class of models whose relation $\succeq$ is transitive, total, and max-limited.[19] Let Γ be the consistent set of formulas whose satisfiability must be established. Consider the canonical model used in the proof of [55, Theorem 9]. The relation $\succeq$ in this model is max-limited, transitive and total (hence

[19] I owe this remark to an anonymous referee.

reflexive).[20] Γ is satisfiable under the max rule. By Proposition 5 (ii), Γ is satisfiable under the s-max rule. By Proposition 6 (ii), $\succeq$ is s-max-limited. A total and transitive relation is Ferrers, and hence an interval order.

What is striking with the s-max rule is not so much completeness, but soundness. The interval order condition boosts the logic from **F**+(CM) to **G**. This is in sharp contrast with the rule of maximality, where the interval order condition is not sufficient for the validity of (Sp), but one needs plain transitivity (in addition to totality).[21] Completeness has an interest in its own right. It makes it clear that no new axiom apart from (Sp) must be added.

It is also worth stressing that under the s-max rule (Sp) requires only totality (cf. Theorem 23). Furthermore, given totality, s-max-limitedness and s-max-smoothness are equivalent, by Proposition 6 (i) and (iii). It follows that, under the latter rule, **G** is also complete with respect to the class of models whose relation $\succeq$ is total and s-max-limited (or s-max-smooth). Hence, given totality and s-max-limitedness (or s-max-smoothness), Ferrers is idle.

Note, finally, that an analog of Corollary 36 is available:

Corollary 42 *If under the s-max rule A is satisfiable in a preference model $M = (W, \succeq, v)$ whose relation $\succeq$ is an interval order and s-max-limited (or s-max-smooth), then under the s-max rule A is satisfiable in a finite such model.*

Proof This conclusion is derived from Theorem 39 and the counterpart of Theorem 31. It is also based on a similar extra observation as employed in the proof of Theorem 36. □

5 Conclusion

A number of completeness theorems were reported, which enhance our understanding of the role of transitivity and some candidate weakenings of it: quasi-transitivity, Suzumura consistency, acyclicity and the interval order condition. Reference was made to five systems of increasing strength: **E**, **F**, **F**+(CM), **F**+(DR) and **G**.

Our first finding is that, under the max and strong max rules, the conditions of transitivity, quasi-transitivity, acyclicity and Suzumura consistency make no difference on the logic. The same system (**E**, **F** or **F**+(CM)) is sound and complete with respect to their matching classes of models irrespective of whether the condition is fulfilled. Our second finding is that the interval order condition corresponds to a new axiom, which varies depending on the choice of the rule of interpretation. Under the max rule, the latter condition validates the principle of disjunctive rationality, and hence it boosts the logic from **F**+(CM) to **F**+(DR). Under the s-max rule, the condition validates the Spohn axiom, corresponding to the principle of rational monotony, or Lewis's axiom CV. The condition boosts the logic from **F**+(CM) to the stronger system **G**. These last two points were substantiated further through the establishment of a weak completeness theorem with respect to the class of finite models, yielding the finite model property and decidability.

[20] Note the argument for transitivity appeals to (Sp), the distinctive axiom of **G**.

[21] See Table 2.

Table 1 Soundness and completeness results

Property of $\succeq$	max	s-max
–		
+ acyclicity		
+ Suzumura consistency	**E**	**E**
+ quasi-transitivity		
+ transitivity		
limitedness		
+ acyclicity		
+ Suzumura consistency	**F**	**F**
+ quasi-transitivity		
+ transitivity		
smoothness		
+ acyclicity		
+ Suzumura consistency	**F**+(CM)	**F**+(CM)
+ quasi-transitivity		
+ transitivity		
interval order[1]	**F**+(DR)[2]	**G**[3]

[1] With limitedness
[2] Weak completeness w.r.t. finite models and weak completeness *tout court*
[3] Weak completeness w.r.t. finite models and strong completeness

Table 1 recapitulates the above points. The left-most column shows the condition on the betterness relation. The other columns show the corresponding systems under each rule of interpretation. It is understood that in each column limitedness and smoothness are cast in terms of the appropriate notion of best.

These findings are interesting, but they do not give us the full story yet. I indicate two obvious directions for future research. This will give me the opportunity to nuance the above points.

First, the outcome is likely to be different, if another notion of best is used like maximality-in-the-limit or variations thereof, where there are no best worlds, but (non-empty) sequences of ever-better ones, which approximate the ideal (see, e.g., [14, 44, 62, 68]). The role of quasi-transitivity, Suzumura consistency and acyclicity under the opt rule remains to be understood too. Table 2 provides a concise overview of additional significant findings in dyadic deontic logic, putting them side-by-side with the results stated in Theorem 8.[22] This table should be self-explanatory. I only briefly comment on the new axiom (transit). It captures a principle of transitivity for a notion of weak preference given by $A \geq B =_{def} P(A/A \vee B)$:[23]

[22] The interested reader can find in [32, 34, 57] supplementary information. Analog results for Hansson's original family of DSDL systems may be found in [33]. Kratzer's semantics is axiomatized in [31]. Results for multiplex semantics are given in [28, 29]. Analytic sequent calculi for **E** and **F** may be found in [20, 21].

[23] Cf. [44, p. 54].

Table 2 Main existing results

Property of $\succeq$	max	opt	Reference
−			
+ reflexivity	**E**	**E**	[34, 56]
+ totality			
transitivity	**E**	**E**+(Sp)+(transit)	[33, 34, 57]
+ reflexivity			[33, 34, 57]
+ totality	**E**+(Sp)+(transit)+(DR)	as for max	[34]
limitedness			
+ reflexivity	**F**	**F**	[32, 56]
+ totality			
smoothness			
+ reflexivity	**F**+(CM)	**F**+(CM)	[32, 55]
+ totality			
smoothness			
+ reflexivity	**F**+(CM)	**G**	[32, 53, 55, 57]
+ transitivity			
+ transitivity/totality	**G**		

$$P(A/A \vee B) \wedge P(B/B \vee C) \to P(A/A \vee C) \tag{transit}$$

In [32, p. 48] it was conjectured that, under the opt rule, **E**+(Sp)+(transit) is complete with respect to the class of models whose relation $\succeq$ is transitive.[24] This point is echoed in [57, Section 4.3]. Goble's conjecture was settled in the positive in [34, Theorem 70], where the pair {(Sp), (transit)} is replaced by an axiom called "Disjunctive monotony" (DM), also known as "γ^+" or "strong expansion":[25]

$$\bigcirc(B/A \vee C) \wedge P(C/A) \to \bigcirc(B/C) \tag{DM}$$

The fact that (DM) is equivalent with {(Sp), (transit)} is shown in Appendix B. It is interesting to remark that a syntactical counterpart of the transitivity of betterness is obtained, but only at the cost of using a stronger notion of best, which (as mentioned) is often considered less appropriate when the possibility of incomparability is kept

[24] Goble does not refer to **E**, but to its counterpart for a semantics in which the ranking is made relative to worlds, and a distinction is drawn between assessable and non-assessable worlds. (Sp) is replaced with (RM) and the reflexivity of the betterness relation is mentioned. The system is called DDL-0.RT.

[25] See [27, 67]. For the completeness of **E**+(Sp)+(transit), take the canonical model given for **G** in [53] and [57, Definition 4.10], and delete condition (a) in the definition of the betterness relation. Re-run the proof for **G** in [53] with appropriate editing, using Lemma 57 in [34] for the right-to-left direction of the induction for $\bigcirc(-/-)$. Let w^B be a shorthand for $\{C : \bigcirc(C/B) \in w\}$, where w is the MCS from which the canonical model is generated. The lemma reads: where $b \in \text{opt}_{\succeq}(\|B\|)$ and $\Delta = \{D : w^D \subseteq b\}$, if w^B is inconsistent, then $\{B\} \cup \{\neg D : D \in \Delta\}$ is consistent. This allows to by-pass the fact that (D*) is no longer available.

open. A similar remark applies to the max rule under totality. I have not looked into the question of whether quasi-transitivity, acyclicity and Suzumura consistency have a counterpart under the opt rule.

Second, the restriction to finite models creates some discrepancy between the first three weakened forms of transitivity as shown in Table 3. These differences are due to the fact that, when a model is finite, a given condition may or may not enforce a given form of the limit assumption.

Under the max rule, the choice between one of "acyclicity/Suzumura consistency" and one of "quasi-transitivity/transitivity" makes a difference. First, Suzumura consistency entails acyclicity. This one in turn entails max-limitedness, which validates (D*). Hence under the max rule **F** is (weakly) sound with respect to the class of finite models in which $\succeq$ is acyclic, and with respect to the class of those in which $\succeq$ is Suzumura consistent. It would be interesting to know if **F** is also complete with respect to these two classes of models. As shown by Fact 43, (CM) is falsifiable in each of them. Fact 43 also considers strong maximality.

Fact 43 *There exists a finite preference model $M = (W, \succeq, v)$ whose relation $\succeq$ is Suzumura consistent (and hence acyclic), but not max-smooth (and hence not s-max-smooth), in which (CM) is falsified under the max and s-max rules.*

Proof Consider $M = (W, \succeq, v)$ where $W = \{a, b, c\}$, $\succeq$ is the reflexive closure of $\{(a, b), (b, c)\}$ and $v(p) = W$, $v(q) = \{a, c\}$, $v(r) = \{a\}$ and $v(s) = \emptyset$ for all other $s \in \mathbb{P}$. $\succeq$ is Suzumura consistent, and hence acyclic. $\succeq$ is not max-smooth, since $c \models p$, $\max_{\succeq}(\|p\|) = \{a\}$ and $a \not\succ c$. Hence $\succeq$ is not s-max-smooth either, by Proposition 6 (i). Under each rule, $a \models \bigcirc(q/p)$, $a \models \bigcirc(r/p)$ and $a \not\models \bigcirc(r/p \wedge q)$ (witness: c). □

Transitivity implies quasi-transitivity, which in turn entails max-smoothness, and hence (CM) is validated. Therefore under the max rule **F**+(CM) is sound with respect to the class of finite models in which $\succeq$ is either transitive or quasi-transitive.

With strong maximality, the picture is slightly different. To facilitate comparison, I assume reflexivity of $\succeq$. First, acyclicity and quasi-transitivity do not validate (D*) nor (CM). Indeed, Fig. 9 tells us that s-max-limitedness, and hence s-max-smoothness, may fail in a finite model whose relation $\succeq$ is an interval order, and hence quasi-transitive or acyclic. Second, Suzumura consistency entails s-max-limitedness, hence validating (D*). Fact 43 adds to this that (CM) remains falsifiable. By contrast, in the presence of reflexivity, transitivity entails s-max-smoothness, hence validating (CM).

Table 3 Correspondences (finite models)

Property of $\succeq$	max	s-max
acyclicity	(D*)	–
Suzumura consistency	(D*)	(D*)
quasi-transitivity	(CM)	–
transitivity	(CM)	(CM)[1]

[1] With reflexivity of $\succeq$

Therefore, Suzumura consistency boosts the logic from **E** to **F**, and transitivity from **F** to **F**+(CM). Other things being equal, acyclicity and quasi-transitivity do not have these effects.

Appendix A: Proof of Theorem 29

I restate the theorem to be established:

Theorem 29 (F.m.p., selection functions) **F**+*(DR) has the finite model property (f.m.p.) with respect to selection functions. That is, if A is satisfiable in a selection model* $M = (W, \mathfrak{f}, v)$ *in which* $\mathfrak{f}$ *meets (syntax-independence), (inclusion), (Chernoff), (consistency-preservation) and (s-drat), then A is satisfiable in a finite such model* $M^\star = (W^\star, \mathfrak{f}^\star, v^\star)$ *meeting condition (expr).*

Proof The proof given here extends the one given for **E**, **F** and **G** in [3, 57]. I recall the definition of the notion of filtration, and the main steps of the proof. I only carry out in detail the verification of two new claims.

Γ denotes a non-empty and finite set of sentences closed under sub-formulas. § stands for a designated atomic formula in Γ. Put $\top = \S \rightarrow \S$ and $\bot = \neg\top$. For any selection model $M = (W, \mathfrak{f}, v)$, the equivalence relation $\equiv_\Gamma$ on W is defined by setting

$$a \equiv_\Gamma b \text{ iff for every } A \text{ in } \Gamma : a \vDash A \text{ iff } b \vDash A$$

Given $a \in W$, $[a]$ will be the equivalence class of a under $\equiv_\Gamma$. Given some Γ, we define the translation function τ, transforming every formula into a formula whose atomic formulas are all in Γ. Function τ is defined as follows:

$$\tau(p) = \begin{cases} p & \text{if } p \in \Gamma \\ \S & \text{if } p \notin \Gamma \end{cases}$$

$$\tau(\neg A) = \neg\tau(A) \qquad \tau(A \vee B) = \tau(A) \vee \tau(B)$$
$$\tau(\Box A) = \Box\tau(A) \qquad \tau(\bigcirc(B/A)) = \bigcirc(\tau(B)/\tau(A))$$

Since neither $\top$ nor $\bot$ are primitive symbols, and Γ is non-empty, there is always one such atomic formula § in Γ. The filtration of $M = (W, \mathfrak{f}, v)$ through Γ is the model $M^\star = (W^\star, \mathfrak{f}^\star, v^\star)$ where:

(i) $W^\star = \{[a] : a \in W\}$;
(ii) $\mathfrak{f}^\star(A) = \{[a] : \exists b \in [a] \ \& \ b \in \mathfrak{f}(\tau(A))\}$;
(iii) $v^\star(p) = \{[a] : a \in v(\tau(p))\}$ for all $p \in \mathbb{P}$.

We have:

Fact 44 *Let* Γ*,* τ *and* M *be as above. Then, for all* $A \in \mathscr{L}$ *and all* $a, b \in W$*, if* $a \equiv_\Gamma b$*, then* $a \vDash \tau(A)$ *iff* $b \vDash \tau(A)$*.*

The so-called filtration theorem comes in two versions:

Fact 45 (Filtration theorem)

(i) For all $A \in \Gamma$ and all $a \in W$, $M^\star, [a] \vDash A$ iff $M, a \vDash A$;
(ii) For all $A \in \mathscr{L}$ and all $a \in W$, $M^\star, [a] \vDash A$ iff $M, a \vDash \tau(A)$.

The first claim to be verified concerns (s-drat). One must show the following:

Lemma 46 *In a filtered model $M^\star$, $\mathfrak{f}^\star$ meets (s-drat), if $\mathfrak{f}$ does.*

Proof of Lemma 46 Suppose $\mathfrak{f}^\star(A \vee B) \subseteq \|C\|^{M^\star}$. First, we show that $\mathfrak{f}(\tau(A \vee B)) \subseteq \|\tau(C)\|^M$. Let $a \in \mathfrak{f}(\tau(A \vee B))$. Since $a \in [a]$, $[a] \in \mathfrak{f}^\star(A \vee B)$ by definition of $\mathfrak{f}^\star$. From the opening assumption, $[a] \vDash C$. By Fact 45 (ii), $a \vDash \tau(C)$. Hence, $\mathfrak{f}(\tau(A \vee B)) \subseteq \|\tau(C)\|^M$. By definition of τ, $\mathfrak{f}(\tau(A) \vee \tau(B)) \subseteq \|\tau(C)\|^M$. By (s-drat) for $\mathfrak{f}$, $\mathfrak{f}(\tau(A)) \subseteq \|\tau(C)\|^M$ or $\mathfrak{f}(\tau(B)) \subseteq \|\tau(C)\|^M$. Now, suppose for a reductio that $\mathfrak{f}^\star(A) \nsubseteq \|C\|^{M^\star}$ and $\mathfrak{f}^\star(B) \nsubseteq \|C\|^{M^\star}$. Hence $\exists[a]$ s.t. $[a] \in \mathfrak{f}^\star(A)$ and $[a] \nvDash C$. By Fact 45 (ii), $a \nvDash \tau(C)$. Also $\exists[b]$ s.t. $[b] \in \mathfrak{f}^\star(B)$ and $[b] \nvDash C$. By Fact 45 (ii), $b \nvDash \tau(C)$. By definition, $\exists c$ s.t $c \in [a]$ and $c \in \mathfrak{f}(\tau(A))$, and $\exists d$ s.t $d \in [b]$ and $d \in \mathfrak{f}(\tau(B))$. By Fact 44, $c \nvDash \tau(C)$ and $d \nvDash \tau(C)$. Hence $\mathfrak{f}(\tau(A)) \nsubseteq \|\tau(C)\|^M$ and $\mathfrak{f}(\tau(B)) \nsubseteq \|\tau(C)\|^M$. Contradiction. □

The second claim to verify concerns (expr). One must show that $M^\star$ fulfills this condition. For the reader's convenience, I outline the main steps of the argument as given by Goble [30], who uses a different filtration method, called "thin" (or, in Gabbay's terminology, "selective").

Let $\mathscr{B}(\Gamma)$ denote the Boolean closure of Γ, i.e., $\mathscr{B}(\Gamma)$ is the smallest set of formulas such that $\Gamma \subseteq \mathscr{B}(\Gamma)$, and if $A, B \in \Gamma$, then $A \vee B \in \mathscr{B}(\Gamma)$ and $\neg A \in \mathscr{B}(\Gamma)$. It is a straightforward matter to show that the filtration theorem, Fact 45, generalizes to $\mathscr{B}(\Gamma)$:

Lemma 47 *Let M be a selection model and $M^\star$ be its filtration through Γ. Then for all $A \in \mathscr{B}(\Gamma)$ and all $a \in W$:*

$$M^\star, [a] \vDash A \text{ iff } M, a \vDash A.$$

Proof of Lemma 47 Proof by induction on A. If $A = p$ or $\Box B$ or $\bigcirc(C/B)$, then $A \in \Gamma$, and the claim follows from Fact 45 (i). If $A = B \vee C$ or $A = \neg B$, the result follows directly from the inductive hypothesis. □

Lemma 48 *Let M be a selection model and $M^\star$ its filtration through Γ. $M^\star$ is distinguishable in the following sense: for all $[a], [b] \in W^\star$, if $[a] \neq [b]$, then there is some $A \in \mathscr{B}(\Gamma)$ such that $M^\star, [a] \vDash A$ but $M^\star, [b] \nvDash A$.*

Proof of Lemma 48 Suppose $[a] \neq [b]$. So either (i) $\exists c$ s.t. $c \in [a]$ and $c \notin [b]$, or (ii) $\exists d$ s.t. $d \notin [a]$ and $d \in [b]$. Suppose (i) applies. From $c \notin [b]$, one gets (i.a) $\exists A \in \Gamma$ s.t. $b \vDash A$ and $c \nvDash A$, or (i.b) $\exists A' \in \Gamma$ s.t. $b \nvDash A'$ and $c \vDash A'$. In case (i.a), the claim is verified for $\neg A$. Indeed, $[b] \vDash A$, by Fact 45 (i), and so $[b] \nvDash \neg A$. On the other hand, $a \nvDash A$, since $a \sim_\Gamma c$. So by Fact 45 (i) $[a] \nvDash A$, hence $[a] \vDash \neg A$. In case (i.b), the claim is verified for A'. Case (ii) is handled similarly. □

Lemma 49 *For all* $[a] \in W^\star$*, there is* $A \in \mathscr{B}(\Gamma)$ *such that* $\|A\|^{M^\star} = \{[a]\}$.

Proof of Lemma 49 This is [30, Lemma 2]. □

Lemma 50 *For all* $X \subseteq W^\star$*, there is* $A \in \mathscr{B}(\Gamma)$ *such that* $\|A\|^{M^\star} = X$.

Proof of Lemma 50 This is [30, Lemma 3]. □

This completes the proof of Theorem 29.

Appendix B

Proposition 51 shows the relationship between (DM) and the pair {(Sp)+(transit)}, aka {(RM)+(transit)}.

Proposition 51 *(i) Each of (Sp) and (transit) is a theorem of* **E**+*(DM); and (ii) (DM) is a theorem of* **E**+*(Sp)+(transit).*

Proof For (i), I begin with (Sp). Assume $\bigcirc(B \to C/A)$ and $P(B/A)$. By (Ext), $\bigcirc(B \to C/A \vee (A \wedge B))$. Assume, to reach a contradiction, that $\bigcirc(\neg(A \wedge B)/A)$. By (RW), $\bigcirc(A \to \neg B/A)$. By (Id), (AND) and (RW), $\bigcirc(\neg B/A)$, contradicting the second hypothesis. So $\neg\bigcirc(\neg(A \wedge B)/A)$, i.e., $P(A \wedge B/A)$. By (DM), $\bigcirc(B \to C/A \wedge B)$. By (Ext), $\bigcirc(B \to C/A \wedge B \wedge B)$. By (Sh), $\bigcirc(B \to (B \to C)/A \wedge B)$. By (Id), (AND) and (RW), $\bigcirc(C/A \wedge B)$ as required.

For (transit), suppose $P(A/A \vee B)$ and $P(B/B \vee C)$. Assume, to reach a contradiction, that $\bigcirc(\neg A/A \vee C)$. By (Ext), $\bigcirc(\neg A/(A \vee B \vee C) \wedge (A \vee C))$. By (Sh) and (RW), $\bigcirc(\neg A/A \vee B \vee C)$. By (Ext), $\bigcirc(\neg A/(A \vee B \vee C) \vee ((A \vee B) \wedge (A \vee B \vee C))$. Suppose, to reach a contradiction, that $\bigcirc(\neg(A \vee B)/A \vee B \vee C)$. By (Ext), $\bigcirc(\neg(A \vee B)/B \vee C \vee A \vee B)$. By (DM), either $\bigcirc(\neg(A \vee B)/B \vee C)$ or $\bigcirc(\neg(A \vee B)/A \vee B)$. In the first case, by (RW), $\bigcirc(\neg B/B \vee C)$, contradicting the second assumption. In the second case, by (RW), $\bigcirc(\neg A/A \vee B)$, contradicting the first assumption. So $\neg\bigcirc(\neg(A \vee B)/A \vee B \vee C)$. By (RW), $\neg\bigcirc(\neg((A \vee B) \wedge (A \vee B \vee C))/A \vee B \vee C)$, i.e., $P((A \vee B) \wedge (A \vee B \vee C)/A \vee B \vee C)$. By (DM), $\bigcirc(\neg A/(A \vee B) \wedge (A \vee B \vee C))$. By (Ext), $\bigcirc(\neg A/A \vee B)$, in contradiction with the first assumption. Hence one must conclude that $\neg\bigcirc(\neg A/A \vee C)$, i.e., $P(A/A \vee C)$.

For (ii), assume $\bigcirc(B/A \vee C)$ and $P(C/A)$. I break the argument into cases:
Case 1: $\bigcirc(\neg C/A \vee C)$. By (transit), either (i) $\bigcirc(\neg C/(A \wedge C) \vee C)$ or (ii) $\bigcirc(\neg(A \wedge C)/(A \wedge C) \vee A)$. Case (ii) is not possible. By (Ext), (Id), (AND) and (RW), $\bigcirc(\neg C/A)$, contradicting the second opening assumption. So case (i) must hold. By (Ext), $\bigcirc(\neg C/C)$. By (Id), (AND) and (RW), $\bigcirc(B/C)$.
Case 2: $\neg\bigcirc(\neg C/A \vee C)$, i.e., $P(C/A \vee C)$. From the first opening assumption, $\bigcirc(C \to B/A \vee C)$, by (RW). By (Sp) and (Ext), $\bigcirc(B/C)$.
Either way, $\bigcirc(B/C)$ as required. □

Proposition 52 clarifies the relationship between **E**+(Sp)+(transit) and the systems studied in this paper.

Proposition 52 *The following applies:*

(i) $\mathbf{E} \subset \mathbf{E}$+*(Sp)*+*(transit)*;
(ii) $\mathbf{E}$+*(Sp)*+*(transit)* $\subset \mathbf{G}$;
(iii) $\mathbf{E}$+*(Sp)*+*(transit) and* $\mathbf{F}$ *(*resp. $\mathbf{F}$+*(CM)*, $\mathbf{F}$+*(DR)) are distinct, in the sense that none is contained in the other.*

Proof For (i), that $\mathbf{E} \subseteq \mathbf{E}$+(Sp)+(transit) is obvious. To show the containment is proper, it suffices to remark that under, e.g., the max rule each of (Sp) and (transit) is falsifiable in the class of all preference models (see [57, Observations 2.10 and 2.12]), and so none is derivable in $\mathbf{E}$, by Theorem 8 (i).

For (ii), note, first, that (transit) is derivable in $\mathbf{G}$ (see [33, Lemma 5 (2)]). This shows that $\mathbf{E}$+(Sp)+(transit) $\subseteq \mathbf{G}$. That the containment is proper follows from the fact that under the opt rule ($D^\star$) is falsifiable in the class of models whose relation $\succeq$ is transitive, and the fact that under the opt rule $\mathbf{E}$+(Sp)+(transit) is sound with respect to this class of models (see, e.g., [57, Observations 2.9 and 2.11]).

For (iii), the previous point about ($D^\star$) establishes that none of $\mathbf{F}$, $\mathbf{F}$+(CM) and $\mathbf{F}$+(DR) is contained in $\mathbf{E}$+(Sp)+(transit). To show that $\mathbf{E}$+(Sp)+(transit) is not contained in $\mathbf{F}$, $\mathbf{F}$+(CM) or $\mathbf{F}$+(DR), it suffices to invoke Theorem 9 (iv) and the fact that $\mathbf{F} \subset \mathbf{F}$+(CM) $\subset \mathbf{F}$+(DR). □

Note that, in $\mathbf{E}$, (Sp) does not derive (transit) (although it does in $\mathbf{G}$, as just observed).

Proposition 53 *(i) Under the s-max rule,* $\mathbf{E}$+*(Sp) is sound with respect to the class of preference models whose relation* $\succeq$ *is total; and (ii) there is a preference model whose relation* $\succeq$ *is total in which (transit) is falsified under the s-max rule.*

Proof (i) was already observed in the proof of Theorem 23. For (ii), put $M = W, \succeq, v)$ with

- $W = \{a, b, c, d\}$
- $\succeq$= the reflexive closure of $\{(a, b), (c, b), (a, c), (c, a), (d, c), (d, a), (b, d)\}$
- $v(p) = \{a, c\}$, $v(q) = \{b\}$ $v(r) = \{d\}$ and $v(s) = W$ for all other $s \in \mathbb{P}$.

$\succeq$ is total. We have $\max^s_\succeq(\|p \vee q\|) = \{a, c\}$, $\max^s_\succeq(\|q \vee r\|) = \{b\}$ and $\max^s_\succeq(\|p \vee r\|) = \{d\}$. Hence, $P(p/p \vee q)$ and $P(q/q \vee r)$ hold, but not $P(p/p \vee r)$. □

Corollary 54 *(transit) is not a theorem of* $\mathbf{E}$+*(Sp).*

Proof This can be inferred from Proposition 53 using the same line of reasoning as before. □

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