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A Basis for AGM Revision in Bayesian Probability Revision

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Abstract
In standard Bayesian probability revision, the adoption of full beliefs (propositions with probability 1) is irreversible. Once an agent has full belief in a proposition, no subsequent revision can remove that belief. This is an unrealistic feature, and it also makes probability revision incompatible with belief change theory, which focuses on how the set of full beliefs is modified through both additions and retractions. This problem in probability theory can be solved in a model that (i) lets the codomain of the probability function be a hyperreal-valued rather than the real-valued closed interval [0, 1], and (ii) identifies the full beliefs as the propositions whose probability is either 1 or infinitesimally smaller than 1. In this model, changes in the probability function will result in changes in the set of full beliefs (belief set), which constitutes a submodel that can be conceived as the "tip of the iceberg" within the larger model that also contains beliefs on lower levels of probability. The patterns of change in the set of full beliefs in this modified Bayesian model coincides with the corresponding pattern in a slightly modified version of AGM revision, which is commonly conceived as the gold standard of (dichotomous) belief change. The modification only concerns the marginal case of revision by an inconsistent input sentence. These results show that probability revision and dichotomous belief change can be unified in one and the same framework, or – if we so wish – that belief change theory can be subsumed under a modified version of probability revision that allows for iterated change and for the removal of full beliefs.

Keywords Bayesianism · Probability revision · Infinitesimal probabilities · Belief revision · AGM model

1 Introduction

One of the major problems in formal epistemology is the difficulty involved in combining standard probabilistic representations of belief with standard representations

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of full beliefs. The set of sentences to which a probability function assigns the probability 1 is logically closed, and it can therefore be taken as a representation of the set of full beliefs, also called the "belief set". This set is usually assumed to contain both the logically and analytically true statements and those empirical statements that are fully believed. Typically, the empirical statements in which we have full belief are so highly probable that we see no reason to doubt them, but there is still a possibility that reasons to doubt or reject them may arise at some later point in time. Having empirical full beliefs, not only probabilistic ones, reduces the cognitive burden, since full beliefs form the basis for deductive reasoning, which is much less cumbersome than probabilistic reasoning [17, 20, 29].

However, probability theory has difficulties in representing such provisional contingent full beliefs. In the standard Bayesian framework for the revision of probabilities, the adoption of full beliefs is an irreversible process. If we revise a probability function $\mathfrak{p}$ by a sentence a with $\mathfrak{p}(a) > 0$, then the outcome is a new probability function $\mathfrak{p}'$ such that $\mathfrak{p}'(a) = 1$, i.e., a is a full belief, and that for all sentences e, $\mathfrak{p}'(e) = \mathfrak{p}(a\&e)/\mathfrak{p}(a)$. Revision of $\mathfrak{p}'$ by $\neg a$ is not defined, i.e. not possible, since the new probability function $\mathfrak{p}''$ would have to satisfy the condition that $\mathfrak{p}''(e) = \mathfrak{p}'(\neg a\&e)/\mathfrak{p}'(\neg a)$ for all e, thus having a zero-valued denominator. The same applies to revision by any sentence b that logically implies $\neg a$. Therefore, in a series of probability revisions, we continually accumulate more and more beliefs with probability 1, but we cannot ever downgrade any belief from that level to a lower degree of probability. We can call this the *accumulation problem*. It is a highly implausible feature of the standard probabilistic model of the dynamics of beliefs.

Since the 1980s, increasingly sophisticated models of full beliefs have been developed in the tradition of belief change theory (also called belief revision theory). (For an overview, see [8].) Contrary to probabilistic models, these models treat belief as a dichotomous phenomenon rather than one that comes in degrees. Their operations of revision can both add new items to the belief set and remove old items from it. For instance, a belief set that contains a can be revised by $\neg a$, resulting in a new belief set that contains $\neg a$ instead of a. Thus, contrary to probability theory, belief change theory does not suffer from the accumulation problem.

The accumulation problem of probability revision is a problem in its own right, which can be discussed and addressed without comparisons with belief change theory. However, the comparison with belief change theory serves well to highlight the problem, since this theory is largely devoted to representing something that standard probability theory cannot at all represent – namely repeated changes of the set of full beliefs, in which old beliefs are lost and new beliefs replace them. Therefore, it is an appropriate benchmark for a solution of the accumulation problem that it should provide us with *a framework for probability revision, in which the pattern of revisions of the set of full beliefs coincides with a reasonable account of belief change, as propounded in belief change theory*. In such a framework, (dichotomous) belief change will be the tip of an iceberg, i.e., a partial picture of a larger system in which changes of lower degrees of belief are also included.[1] To the extent that such a structure can

[1] Lin and Kelly [29] described this as the problem of finding an operation of belief change that *tracks* Bayesian conditionalization.

be constructed, the two previously competing formal representations of the dynamics of belief can be unified – or, if we so wish, belief change theory, which focuses exclusively on the full beliefs, can be subsumed under a modified version of probability theory, which includes not only full beliefs but also beliefs held to all other possible degrees than the maximal one.

One possible approach to the accumulation problem is to leave the standard framework of probability revision unchanged, but straighten up the interpretation of full beliefs, so that probability 1 is only assigned to sentences that are logically or analytically true. Such a strict interpretation of unit probability was recommended for instance by Richard Jeffrey, who proposed (for other reasons) that a scientist "should refrain from accepting or rejecting hypotheses", and instead provide "a single probability for the hypothesis (whereupon each makes his own decision based on the utilities peculiar to his problem)" [23, p. 245]. From a purely formal point of view, this proposal solves the accumulation problem. If the set $\{e \mid \mathfrak{p}(e) = 1\}$ consists exclusively of beliefs which the agent can never have valid reasons to give up, then the accumulation of such beliefs is essentially problem-free. However, this solution comes at a high price. In our actual epistemic dealings, we commonly believe fully in claims that we consider to be "certain enough", although new unexpected information could potentially make us give them up. The scientific corpus is a collective compilation of such currently undoubted but yet doubtable claims, and in our everyday lives we similarly take much for granted that could in fact be wrong. A belief set that only contains logical and analytical truths would make our reasoning more complicated and burdensome than one that also contains those empirical statements that are deemed to be "certain enough" for our purposes [17]. The literature on belief change provides ample evidence that the patterns of change in a belief set containing currently undoubted (but not undoubtable) contingent sentences are both philosophically interesting and practically important. An accurate model of human belief systems should not exclude this type of beliefs.[2]

Therefore, it makes sense to look for a modified probability framework in which full beliefs can be retracted when new information gives us reason to do so. In [18, 19], it was shown that this can be achieved by (i) letting the codomain of the probability function be a hyperreal-valued rather than the real-valued closed interval [0, 1], and (ii) letting the belief set be the set of all sentences whose probability is either 1 or infinitesimally smaller than 1. Such sets are logically closed. These relatively small changes in the traditional framework are sufficient to ensure that full beliefs can be removed from the belief set, thus solving the accumulation problem.

In [18, 19], these changes were combined with other changes in the standard framework, aimed at transferring the distinction between update and revision (knowledge-adding and change-recording operations) from belief change theory to probability theory, and clarifying how observational data influence the probabilities

[2] See [20] for a more extensive account of the role of provisional full beliefs in human belief systems. – Gärdenfors [10, pp. 23 and 38] refers to the elements of the belief set as being subject to "no doubt", "not a serious possibility", having "probability 1", and being "accepted as certain". These descriptions are not synonymous. The first two of them are in line with practice in the belief revision literature.

assigned to underlying hypotheses or theories about the world. In the present contribution, we put these further changes aside, and focus on the basic task outlined above, namely to combine probability theory and belief change theory in one and the same framework. We are going to show how this can be done in a way that satisfies the benchmark set up above. We will present a model of probability revision containing a belief set. Revisions of probabilities give rise to changes in the belief set, and these changes exhibit plausible patterns that can be expressed in the style of belief change theory. Furthermore, the two theories connected in this way are very close to the standard models, respectively, in probability theory and belief change theory. We will connect standard probability theory, with the two modifications (i) and (ii) presented above, with a variant of the standard AGM theory of belief change [1], differing from the latter only in how it deals with a controversial limiting case.

It should be emphasized that infinitesimals are used here for modelling purposes. It is not suggested that humans assign infinitesimal probabilities to propositions, only that a model using infinitesimals provides an adequate structure that corresponds in useful and interesting ways to our patterns of belief change. It would be possible to build a new structure for that purpose, but the use of a well-known and thoroughly investigated mathematical structure has considerable advantages.

There is a fairly large literature on infinitesimal and hyperreal probabilities. Most commonly, infinitesimal probabilities have been used to ensure that all elements of an infinite domain (event space) receive non-zero probability. For instance, a fair lottery with an infinite number of tickets can be modelled by assigning the same infinitesimal probability to all tickets [4, 38, 38]. For an overview, see [5]. Here, we will instead use propositions with infinitesimal probabilities as "memory tracks" of beliefs that have been given up. This usage was proposed by [37].[3] The identification of the set of full beliefs with the set of propositions whose probability is infinitesimally close to 1 can be found in [38].

Section 2 provides the formal preliminaries needed for this investigation. The new framework, combining probability revision with dichotomous belief change, is presented in Section 3, which also provides an axiomatic characterization of the new, probability-linked belief change operation. In Section 4, this operation is compared both to the original AGM revision and to a slightly modified version of it that never collapses into inconsistency. In Section 5, the new approach is compared to other formal approaches that allow for revision (conditionalization) of a probability function by an input (antecedent) with probability 0. All formal proof are deferred to an appendix.

2 Formal Preliminaries

Sentences, i.e., elements of the language that express propositions, are represented by lowercase letters ($a, b, \ldots$), and sets of sentences by capital letters ($A, B, \ldots$).

[3] Skyrms assumed that Bayesian update would be used. As he admitted, memories of what one believed when holding a to be true would then be "wiped out" in an update by $\neg a$ [37, p. 161].

The object language $\mathfrak{L}$ is formed from a finite number of atomic sentences with the usual truth-functional connectives: negation ($\neg$), conjunction ($\&$), disjunction ($\vee$), implication ($\rightarrow$), and equivalence ($\leftrightarrow$). $\top$ is a tautology and $\bot$ a logically contradictory sentence.

A Tarskian consequence operation Cn expresses the logic. It satisfies the standard conditions: inclusion ($A \subseteq \text{Cn}(A)$), monotony (If $A \subseteq B$, then $\text{Cn}(A) \subseteq \text{Cn}(B)$) and iteration ($\text{Cn}(A) = \text{Cn}(\text{Cn}(A))$). Furthermore, Cn is supraclassical (if a follows from A by classical truth-functional logic, then $a \in \text{Cn}(A)$) and satisfies the deduction property ($b \in \text{Cn}(A \cup \{a\})$ if and only if $a \rightarrow b \in \text{Cn}(A)$). Since $\mathfrak{L}$ is finite, Cn is also compact (if $a \in \text{Cn}(A)$ then there is a finite subset A' of A such that $a \in \text{Cn}(A')$). $\text{Cn}(\varnothing)$ is the set of tautologies. $A \vdash a$ is an alternative notation for $a \in \text{Cn}(A)$ and $\vdash a$ for $a \in \text{Cn}(\varnothing)$.

A set A of sentences is a (consistent) *belief set* if and only if it is consistent and logically closed, i.e., $A = \text{Cn}(A) \neq \text{Cn}(\{\bot\})$. K denotes a belief set. The conjunction of all elements of a finite set A of sentences is denoted $\&A$, and their disjunction is denoted $\bigvee A$. For any finite set A of sentences, $\mathsf{numb}(A)$ is the number of logically non-equivalent elements of A. For all sets A of sentences and all sentences a, the *remainder set* $A \perp a$ is the set of maximal subsets of A not implying a. Thus, $\mathfrak{L} \perp \bot$ is the set of maximal consistent subsets of $\mathfrak{L}$. For any sentence $a \in \mathfrak{L}$, $\|a\| = \{X \in \mathfrak{L} \perp \bot \mid a \in X\}$.

The letters s, t, u, v, x, y, and z represent hyperreal numbers (which may be real). The letters δ and ϵ represent numbers that are either 0 or infinitesimal.[4] The standard (real) part of a finite hyperreal number s is denoted $\mathsf{st}(s)$, and the following abbreviations are used:

$s \approx t$ if and only if $\mathsf{st}(s) = \mathsf{st}(t)$
$s \not\approx t$ if and only if $\mathsf{st}(s) \neq \mathsf{st}(t)$
$s \ll t$ if and only if $\mathsf{st}(s) < \mathsf{st}(t)$

$\mathfrak{p}$ is a hyperreal-valued probability function on $\mathfrak{L}$. $[\![\mathfrak{p}]\!]$ is the set of sentences whose probability according to $\mathfrak{p}$ is at most infinitesimally smaller than 1, i.e., $[\![\mathfrak{p}]\!] = \{e \in \mathfrak{L} \mid \mathfrak{p}(e) \approx 1\}$. Importantly, $[\![\mathfrak{p}]\!]$ is a belief set, i.e. $[\![\mathfrak{p}]\!] = \text{Cn}([\![\mathfrak{p}]\!])$ [18].

Standard notation in probability theory is rather opaque in its representation of probability revision. The outcome of revising a probability function $\mathfrak{p}$ by an input sentence a is a new probability function, but there is no special notation for the new function that keeps track of its origin in $\mathfrak{p}$ and a. This can be remedied by taking over a notational practice from belief change theory: The new function will be denoted $\mathfrak{p} \star a$. For additional clarity, boldface brackets will be used around composite probability functions. Thus, $\mathbf{(}\mathfrak{p} \star a\mathbf{)}(d)$ denotes the probability assigned to d by the probability function obtained by revising $\mathfrak{p}$ by a.

4 See [25] or [26] for an accessible introduction to hyperreal numbers or [18] p. 1024 for a very brief introduction to finite hyperreal numbers.

3 Hyperreal Bayesian Probability Revision and its Sentential Top

A straightforward approach to the revision of hyperreal probabilities would be to apply the same Bayesian rule as for standard (real) probabilities:

$$(\mathfrak{p} \star a)(d) = \begin{cases} \mathfrak{p}(d) & \text{if } \mathfrak{p}(a) = 0 \\ \mathfrak{p}(d \mid a) = \dfrac{\mathfrak{p}(a\&d)}{\mathfrak{p}(a)} & \text{if } \mathfrak{p}(a) \neq 0 \end{cases} \tag{1}$$

But this would lead to the loss of more and more possibilities when revising, since $(\mathfrak{p} \star a_1 \star a_2 \star \ldots \star a_n)(a_1) = 1$ whenever $\mathfrak{p}(a_1) \neq 0$. In other words, the accumulation problem has not been solved. This can be remedied by applying Jeffrey conditionalization [24, pp. 164-183] and leaving an infinitesimal probability to the beliefs that have been given up:

$$(\mathfrak{p} \star_\delta a)(d) = \begin{cases} \mathfrak{p}(d) & \text{if } \mathfrak{p}(a) = 0 \text{ or } \mathfrak{p}(a) = 1 \\ (1-\delta) \times \dfrac{\mathfrak{p}(a\&d)}{\mathfrak{p}(a)} + \delta \times \dfrac{\mathfrak{p}(\neg a\&d)}{\mathfrak{p}(\neg a)} & \text{if } 0 \neq \mathfrak{p}(a) \neq 1 \end{cases} \tag{2}$$

If we only perform a single revision, then it makes no difference in the resulting belief set if we revise according to Eqs. 1 or 2.

OBSERVATION 1 *Let $\mathfrak{p}$ be a probability function. Let $\star'$ be the operation of revision on $\mathfrak{p}$ defined in Eq. 1, and $\star_\delta$ the operation of revision on $\mathfrak{p}$ defined in Eq. 2, for some δ with $0 \leq \delta \approx 0$. Then it holds for all sentences a in the domain of $\mathfrak{p}$ that $[\![\mathfrak{p} \star' a]\!] = [\![\mathfrak{p} \star_\delta a]\!]$.*

However, in a series of two or more revisions, the difference between the two approaches can be momentous. For instance, in the series $\mathfrak{p} \star' a \star' \neg a$, the second operation has to be performed according to the first clause of Eq. 1, which means that $\neg a$ is not assimilated, and $\mathfrak{p} \star' a \star' \neg a = \mathfrak{p} \star' a$. In contrast, the second operation in the series $\mathfrak{p} \star_{\delta_1} a \star_{\delta_2} \neg a$ follows the second clause of Eq. 2, and if $\delta_1 \neq 0 \neq \delta_2$, then $\neg a$ is assimilated. The infinitesimal probabilities that Eq. 2 assigns to $\neg a$ and to sentences implying $\neg a$ serve as retrievable memories of what it means to believe in $\neg a$. Retaining such memories after adopting the provisional full belief a is essential for solving the accumulation problem. We will therefore use Eq. (2), although iterated revisions will not be further investigated in this article.

DEFINITION 1 *Let $\mathfrak{p}$ be a hyperreal probability function on a language $\mathfrak{L}$ that is closed under truth-functional operations. The* hyperreal Bayesian revision *based on $\mathfrak{p}$ is the operation $\star$ such that for all $a, d \in \mathfrak{L}$ and all δ with $0 \leq \delta \approx 0$:*

$$(\mathfrak{p} \star_\delta a)(d) = \begin{cases} \mathfrak{p}(d) & \textit{if } \mathfrak{p}(a) = 0 \textit{ or } \mathfrak{p}(a) = 1 \\ (1-\delta) \times \dfrac{\mathfrak{p}(a\&d)}{\mathfrak{p}(a)} + \delta \times \dfrac{\mathfrak{p}(\neg a\&d)}{\mathfrak{p}(\neg a)} & \textit{if } 0 \neq \mathfrak{p}(a) \neq 1 \end{cases}$$

Each hyperreal probability function $\mathfrak{p}$ is associated with a belief set, $[\![\mathfrak{p}]\!]$. Changes in $\mathfrak{p}$ give rise to changes in the associated belief set. Therefore, we can derive a sentential revision[5] $*$ on $[\![\mathfrak{p}]\!]$ from the probability revision $\star_\delta$ on $\mathfrak{p}$:

DEFINITION 2 *Let $*$ be a sentential revision on a belief set K in a language $\mathfrak{L}$. Then $*$ is a* hyperreal Bayesian top revision *on K if and only if there is a hyperreal Bayesian revision $\star$, based on a probability function $\mathfrak{p}$ on $\mathfrak{L}$, and some δ with $0 \leq \delta \approx 0$, such that $[\![\mathfrak{p}]\!] = K$ and that $[\![\mathfrak{p} \star_\delta a]\!] = K * a$ for all $a \in \mathfrak{L}$.*

In the axiomatic characterization of the operation introduced in Definition 2, we will have use for *ring systems*, a framework for (dichotomous) belief change that is a variation of the sphere systems commonly used in belief change theory.[6]

DEFINITION 3 *Let $\mathfrak{L}$ be a finite language and K a consistent belief set in $\mathfrak{L}$. A (finite)* ring system *around K is a sequence $\langle \mathfrak{R}_0, \ldots, \mathfrak{R}_n \rangle$ of subsets of $\mathfrak{L}\perp\!\!\perp$, such that $K = \bigcap \mathfrak{R}_0$ and that if $0 \leq k < m \leq n$, then $\mathfrak{R}_k \cap \mathfrak{R}_m = \varnothing$.*

DEFINITION 4 *Let $\langle \mathfrak{R}_0, \ldots, \mathfrak{R}_n \rangle$ be a ring system. Then:*
$\mathfrak{R}(a) = \mathfrak{R}_k$ iff $\bigcap \mathfrak{R}_m \vdash \neg a$ for all $m < k$, and $\bigcap \mathfrak{R}_k \nvdash \neg a$

In other words, $\mathfrak{R}(a)$ is the first element in the sequence $\langle \mathfrak{R}_0, \ldots, \mathfrak{R}_n \rangle$ that has some element containing a.

DEFINITION 5 *Let K be a consistent belief set, and let $\langle \mathfrak{R}_0, \ldots, \mathfrak{R}_n \rangle$ be a ring system such that $K = \bigcap \mathfrak{R}_0$. The* ring-based revision *on K that is based on $\langle \mathfrak{R}_0, \ldots, \mathfrak{R}_n \rangle$ is the sentential operation $*$ such that:*

(1) If $\neg a \in \bigcap(\mathfrak{R}_0 \cup \cdots \cup \mathfrak{R}_n)$, then $K * a = K$, and
(2) otherwise, $K * a = \bigcap(\|a\| \cap \mathfrak{R}(a))$.

As can be seen from clause (1) of Definition 5, the limiting case when the input sentence a is not true in any of the maximal sets represented in the model is treated in different ways in ring systems and sphere systems. In ring systems, $K * a = K$ in this case, whereas in sphere systems, $K * a = \text{Cn}(\{\perp\})$. It should also be noted that since our definition of ring systems does not require that $\mathfrak{R}_0 \cup \cdots \cup \mathfrak{R}_n = \mathfrak{L}\perp\!\!\perp$, the limiting case can cover a larger part of the potential input sentences than in a sphere system.

We are now ready for the main theorem of this contribution, namely an axiomatic characterization of hyperreal Bayesian top revisions:

THEOREM 1 *Let K be a consistent belief set in a finite language $\mathfrak{L}$, and let $*$ be a sentential operation on K. The following three conditions are equivalent:*

(1) $*$ is a hyperreal Bayesian top revision on K.
(2) $*$ is a ring-based revision on K.

[5] A sentential operation on a belief set K is an operation $\circ$ such that for all sentences a, $K \circ a$ is a belief set.

[6] Sphere systems were introduced in [11]. For a brief introduction see [8] pp. 25-31, and for a more in-depth introduction, [15] pp. 220-228 and 294-304. For a previous version of ring systems, see [16] p. 201.

(3) $*$ satisfies the axioms:

$K * a = \mathrm{Cn}(K * a)$ *(closure)*
Either $a \in K * a$ *or* $K * a = K$ *(relative success)*
If $\neg b \notin K * a$, *then* $b \in K * b$ *(strong regularity)*
$K * a \subseteq \mathrm{Cn}(K \cup \{a\})$ *(inclusion)*
$K * a \nvdash \perp$ *(strong consistency)*
If $\vdash a_1 \leftrightarrow a_2$, *then* $K * a_1 = K * a_2$ *(extensionality)*
If $\neg a \notin K$, *then* $\mathrm{Cn}(K \cup \{a\}) \subseteq K * a$ *(vacuity)*
$K * (a \vee b)$ *is either* $K * a$, $K * b$, *or* $(K * a) \cap (K * b)$ *(disjunctive factoring)*

4 A Comparison with AGM and AGM^{C}

Since AGM revision is usually considered to be the gold standard in belief change theory, a comparison of hyperreal Bayesian top revision with AGM revision is of particular interest.[7] AGM revision can be characterized in multiple ways [8, pp. 17-40]. For comparisons with other operations, axiomatic characterizations are particularly useful. A characterization using the following eight axioms was reported already in the original presentation of the AGM model [1]:

$K * a = \mathrm{Cn}(K * a)$ (closure)
$a \in K * a$ (success)
$K * a \subseteq \mathrm{Cn}(K \cup \{a\})$ (inclusion)
If $a \nvdash \perp$, then $K * a \nvdash \perp$ (consistency)
If $\vdash a_1 \leftrightarrow a_2$, then $K * a_1 = K * a_2$ (extensionality)
If $\neg a \notin K$, then $\mathrm{Cn}(K \cup \{a\}) \subseteq K * a$ (vacuity)
$K * (a_1 \& a_2) \subseteq \mathrm{Cn}((K * a_1) \cup \{a_2\})$ (superexpansion)
If $\neg a_2 \notin K * a_1$, then $(K * a_1) \cup \{a_2\} \subseteq K * (a_1 \& a_2)$ (subexpansion)

The first six of these postulates are called “basic” and the remaining two “supplementary”. In the presence of the six basic postulates, the combination of the two supplementary postulates is equivalent with *disjunctive factoring*. (This was shown by Hans Rott, and first reported by Gärdenfors [10]). In what follows, we will use the equivalent characterization in terms of the six basic postulates and *disjunctive factoring*.

As can be seen from the postulate *strong consistency* in Theorem 1 and the *success* postulate of AGM, hyperreal Bayesian top revision is not an AGM revision. However, this difference refers to the limiting case of revision by a logical contradiction. It follows from *success* and *closure* that in AGM revision, $K * \perp = \mathrm{Cn}(\{\perp\})$. This is a problematic feature of AGM revision, since the operation is intended to represent a rational pattern of belief revision, and acquiring beliefs in all propositions that are

[7] Relations between AGM and probabilities have been subject to several studies. [9] and [30] investigated connections between AGM postulates and changes in the top of a (real-valued) probability function under Bayesian update. [29] studied the relationship between AGM revision and changes in the set of beliefs supported by the Lockean thesis that result from Bayesian updates. They reported an impossibility theorem that puts limits to that relationship. Further results relating to the Lockean thesis have been reported in [28, pp. 159–229], [34, 36].

expressible in the language is certainly not a rational epistemic behaviour. It would seem much more rational for an epistemic agent just to reject a logically inconsistent input. Given the purpose of belief revision, it is therefore justified to investigate a minimally modified version of AGM revision, differing from the original operation only in how it treats this limiting case. We will call this system "consistent AGM" and denote it AGM^C.

DEFINITION 6 *Let $*'$ be a sentential operation on a consistent belief set K. Then $*'$ is an* AGM^C revision *(consistent AGM revision) on K if and only if there is an AGM operation $*$ on K such that:*
(a) $K *' a = K * a$ if $a \nvdash \bot$, and
(b) $K *' a = K$ if $a \vdash \bot$.

OBSERVATION 2 *A sentential operation $*$ on a consistent belief set K is an AGM^C revision if and only if it satisfies:*

$K * a = \mathrm{Cn}(K * a)$ *(closure)*
If $a \nvdash \bot$, then $a \in K * a$ (consistent success)
Either $a \in K * a$ or $K * a = K$ (relative success)
$K * a \subseteq \mathrm{Cn}(K \cup \{a\})$ *(inclusion)*
$K * a \nvdash \bot$ *(strong consistency)*
If $\vdash a_1 \leftrightarrow a_2$, then $K * a_1 = K * a_2$ (extensionality)
If $\neg a \notin K$, then $\mathrm{Cn}(K \cup \{a\}) \subseteq K * a$ (vacuity)
$K * (a \vee b)$ is either $K * a$, $K * b$, or $K * a \cap K * b$ (disjunctive factoring)

As compared to the AGM postulates (in the variant with *disjunctive factoring* instead of *superexpansion* and *subexpansion*), to characterize AGM^C we have strengthened *consistency* to *strong consistency*, replaced *success* by two of its weakenings, *consistent success* and *relative success*, and left the other postulates unchanged.

AGM^C is a special case of hyperreal Bayesian top revision, which we characterized in Theorem 1:

OBSERVATION 3 *Let $*$ be a sentential operation on a consistent belief set K. The following three conditions are equivalent:*

(1) $*$ is a hyperreal Bayesian top revision on K that satisfies consistent success.
(2) $*$ is a hyperreal Bayesian top revision on K, based on a probability function $\mathfrak{p}$ such that if $a \nvdash \bot$, then $\mathfrak{p}(a) \neq 0$.
(3) $*$ is an AGM^C revision.

Clause (2) of Observation 3 requires that $\mathfrak{p}$ assigns probability zero only to logical contradictions, and consequently, unit probability only to logical truths. This means that no contingent sentence can be irretrievably lost or irreversibly included in the belief set. This property of the belief system is also encoded in the postulates of *consistent success* and *strong consistency*, which together ensure that if a is not a logical truth, then it can be removed from K through revision by $\neg a$. (If $\nvdash a$, then $\neg a \nvdash \bot$, thus $\neg a \in K * \neg a$, and since $K * \neg a \nvdash \bot$, we have $a \notin K * \neg a$.) Thus, this operation does not have the accumulation problem discussed in Section 2.

In the Introduction, we set forth the task of combining probability revision and (dichotomous) belief revision in one and the same framework, such that the pattern of changes in full beliefs resulting from probability revision (the changes on the tip of the changing iceberg) coincides with a reasonable account of changes of full beliefs expressible in the style of belief change theory. Observation 3 is a proof of concept, showing that this can be done, and that it can be done with fairly small modifications of the two theories that have been reconciled. In probability theory, we have extended the codomain of the probability function to finite hyperreal numbers, and included sentences with probabilities infinitesimally close to 1 in the belief set. In belief change theory, we have only adjusted the standard AGM theory in the limiting case of revision by an inconsistent sentence. This adjustment can be justified independently of our present endeavour, since this limiting case is a weak point of AGM theory.

5 Discussion

Two other approaches than infinitesimal probabilities have been proposed to avoid the accumulation of permanent full beliefs that results from Bayesian updates of probabilities: primitive dyadic probabilities and lexicographic probabilities.

A primitive dyadic ("conditional") probability function is a function $\ddot{\mathfrak{p}}$ with the real-valued codomain [0, 1], interpretated such that $\ddot{\mathfrak{p}}(b, a)$ is the probability of b, conditional on a. It is a generalization of a monadic real-valued probability function $\mathfrak{p}$, such that $\mathfrak{p}(b) = \ddot{\mathfrak{p}}(b, \top)$ for all b. Revision of a dyadic probability function $\ddot{\mathfrak{p}}$ by a proposition a gives rise to a new monadic probability function $\mathfrak{p} \star a$ such that $(\mathfrak{p} \star a)(b) = \ddot{\mathfrak{p}}(b, a)$ for all b. The associated set of full beliefs is $\{b \mid \ddot{\mathfrak{p}}(b, a) = 1\}$. This works even if $\mathfrak{p}(a) = \ddot{\mathfrak{p}}(a, \top) = 0$. Such dyadic functions have often been called "Popper functions".[8] For an overview, see [32].

A major problem with this approach is that although revision of $\ddot{\mathfrak{p}}$ by a results in a new monadic probability function $\mathfrak{p} \star a$ such that $(\mathfrak{p} \star a)(b) = \ddot{\mathfrak{p}}(b, a)$ for all b, it does not provide a new dyadic probability function with which additional updates can be made ([2] p. 585; cf. [30] p. 98).[9] McGee ([33] pp. 181-183) showed that the formula $\ddot{\mathfrak{p}}(b, a) = \mathrm{st}(\mathfrak{p}(a\&b)/\mathfrak{p}(a))$ constitutes a bridge between a primitive dyadic, real-valued probability function $\ddot{\mathfrak{p}}$ and a monadic hyperreal probabilty function $\mathfrak{p}$. Through this formula, every hyperreal probability function $\mathfrak{p}$ gives rise to a dyadic function $\ddot{\mathfrak{p}}$ that satisfies standard axioms for such functions, and conversely. This result provides a connection between dyadic and hyperreal probability functions, but it does not provide a mechanism for iterated change of primitive dyadic probabilities.

Lexicographic probabilities are obtained with an n-tuple of real-valued probability functions, $\langle \mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_n \rangle$ with $n > 1$. When we revise (conditionalize) by a, the outcome is obtained by conditionalizing the first probability function $\mathfrak{p}_k$ in

8 Arló-Costa and Parikh [3] p. 117 criticized this terminology.

9 One possibility would be to let $(\ddot{\mathfrak{p}} * a * b)(c, \top) = \ddot{\mathfrak{p}}(c, a\&b)$ for all c ([30] p. 98 and [32] p. 130). However, this construction has implausible properties, such as that that $(\ddot{\mathfrak{p}} * a * \neg a)(c, \top) = (\ddot{\mathfrak{p}} * \neg a * a)(c, \top)$ for all a and c.

$\langle \mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_n \rangle$ that assigns a non-zero probability to a. This construction was introduced by Blume, Brandenburger, and Dekel [6, 7].

There is a close correspondence between lexicographic sequences and hyperreal probabilities [13]. This can be seen by comparing the sequence $\langle \mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_n \rangle$ with the sum $\varepsilon^0 \times \mathfrak{p}_0(\cdot) + \varepsilon^1 \times \mathfrak{p}_1(\cdot) + \cdots + \varepsilon^n \times \mathfrak{p}_n(\cdot)$. Provided that $\mathfrak{p}_k(a) \neq 0$, the term $\varepsilon^k \times \mathfrak{p}_k(a)$ is infinitely larger than all its successors, which amounts to a lexicographic priority.[10] However, just like primitive dyadic probabilities, lexicographic probabilities have a problem with repeated change. After revising by a, we obtain a new monadic probability function, but not a plausible new lexicographic sequence that can be used for a new revision. The standard proposal for a new lexicographic sequence after revision by a is obtained by removing all elements from the original sequence that gave a probability zero ([12] p. 115; [13] p. 158). This removes any information (such as memories) that the epistemic agent may have of what it means to believe $\neg a$.

In contrast to these two proposals, primitive dyadic probabilities and lexicographic probabilities, the model with hyperreal probabilities presented here has no problem with repeated change. The outcome of a revision of a hyperreal probability function $\mathfrak{p}$ by a is a new hyperreal probability function $\mathfrak{p} \star_\delta a$ that can again be revised. As shown elsewhere, the tip-of-the-iceberg part of such repeated revision satisfies the standard axioms for iterated change that have been proposed in the belief change literature [21]. The use of infinitesimals to model an epistemic agent's memories of discarded but retrievable beliefs is a promising approach that should be further investigated.

Appendix: Proofs

DEFINITION 7 *[35] pp.88-89; [14] pp. 46-47 Let $\bar{\varepsilon}$ be a hyperreal number such that $0 < n\bar{\varepsilon} < 1$ for all positive integers n. $\mathfrak{F}$ is the set of fractions of the form*

$$\frac{s_0 \times \bar{\varepsilon}^0 + s_1 \times \bar{\varepsilon}^1 + s_2 \times \bar{\varepsilon}^2 + \cdots + s_k \times \bar{\varepsilon}^k}{t_0 \times \bar{\varepsilon}^0 + t_1 \times \bar{\varepsilon}^1 + t_2 \times \bar{\varepsilon}^2 + \cdots + t_n \times \bar{\varepsilon}^n}$$

within the closed hyperreal interval $[0, 1]$*, such that $s_0, \ldots, s_k$ and $t_0, \ldots, t_n$ are finite series of real numbers and at least one of $t_0, \ldots, t_n$ is non-zero.*

POSTULATE 1 *Probability functions have the codomain $\mathfrak{F}$.*

DEFINITION 8 *[19] A hyperreal number $y \in \mathfrak{F}$ is an infinitesimal of the* first order *(*in *$\mathfrak{F}$) if and only if $0 \neq y \approx 0$ but there is no $z \in \mathfrak{F}$ such that $0 \neq z \approx 0$ and $y/z \approx 0$.*

An infinitesimal $y \in \mathfrak{F}$ is an infinitesimal of the n^{th} order*, for some $n > 1$, if and only if:*

(1) There is a series $z_1, \ldots, z_{n-1}$ of non-zero elements of $\mathfrak{F}$, such that $z_1 \approx 0$, $z_k/z_{k-1} \approx 0$ whenever $1 < k \leq n-1$ and $y/z_{n-1} \approx 0$, and
(2) There is no series $z'_1, \ldots, z'_n$ of non-zero elements of $\mathfrak{F}$, such that $z'_1 \approx 0$, $z'_k/z'_{k-1} \approx 0$ whenever $1 < k \leq n$ and $y/z'_n \approx 0$.

An infinitesimal is finite-ordered *if and only if it is of the n^{th} order for some positive integer n.*

[10] More precisely, if $k < m$, then either $\varepsilon^m \times \mathfrak{p}_m(a) = 0$ or $\varepsilon^m \times \mathfrak{p}_m(a)/\varepsilon^k \times \mathfrak{p}_k(a)$ is infinitesimal.

LEMMA 1 *$\bar{\varepsilon}$ is a first-order infinitesimal.*

PROOF OF LEMMA 1: Clearly, $0 \neq \bar{\varepsilon} \approx 0$. It remains to show that there is no $z \in \mathfrak{F}$ with $0 \neq z \approx 0$ and $\bar{\varepsilon}/z \approx 0$. Suppose to the contrary that there is such a z. According to Definition 7, there is some positive real number s and some positive integer k such that $(s \times \bar{\varepsilon}^k)/z \approx 1$, thus $(\bar{\varepsilon}/z) \times s \times \bar{\varepsilon}^{k-1} \approx 1$, contrary to $\bar{\varepsilon}/z \approx 0$. This contradiction concludes the proof. □

LEMMA 2 *If $x \in \mathfrak{F}$ and $0 \neq x \approx 0$, then x is finite-ordered.*

PROOF OF LEMMA 2: Let s_u be the first non-zero coefficient in the numerator of x and t_v the first non-zero coefficient in its denominator. It follows from $0 \approx x$ that $v < u$. Dividing both numerator and denominator by $\bar{\varepsilon}^v$, we obtain:

$$x = \frac{s_u \times \bar{\varepsilon}^{u-v} + s_{u+1} \times \bar{\varepsilon}^{u-v+1} + \cdots + s_k \times \bar{\varepsilon}^k}{t_v \times \bar{\varepsilon}^0 + t_{v+1} \times \bar{\varepsilon}^1 + \cdots + t_n \times \bar{\varepsilon}^n} \approx \frac{s_u}{t_v} \times \bar{\varepsilon}^{u-v}$$

If $u - v = 1$, then x is a first-order infinitesimal. If $u - v > 1$, then x is an infinitesimal of order $u - v$. □

LEMMA 3 *If y and y' are both positive n^{th} order infinitesimals, then $0 \ll y/y'$.*

PROOF OF LEMMA 3: Suppose that this is not the case. Then $y/y' \approx 0$, and we have a series:

$z_1 \approx 0$, $z_2/z_1 \approx 0, \ldots y'/z_{n-1} \approx 0$, $y/y' \approx 0$

so that y is of at least $(n+1)^{\text{th}}$ order, contrary to the assumption. □

LEMMA 4 *If $\mathfrak{L}$ is finite and $\mathfrak{p}(a) \neq 0$, then*
$\mathfrak{p}(a) \approx \sum\{\mathfrak{p}(\&X) \mid a \in X \in \mathfrak{L}{\perp\!\!\perp}\bot$ *and* $0 \ll \mathfrak{p}(\&X)/\mathfrak{p}(a)\}$.

PROOF OF LEMMA 4: Let $\mathfrak{p}(a) \neq 0$. Then:

$\mathfrak{p}(a) = \sum\{\mathfrak{p}(\&X) \mid a \in X \in \mathfrak{L}{\perp\!\!\perp}\bot\}$

$1 = \sum\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid a \in X \in \mathfrak{L}{\perp\!\!\perp}\bot\}$ since $\mathfrak{p}(a) \neq 0$

$1 \approx \sum\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid a \in X \in \mathfrak{L}{\perp\!\!\perp}\bot$ and $0 \ll \mathfrak{p}(\&X)/\mathfrak{p}(a)\}$

a finite number of infinitesimal or zero terms removed

$\mathfrak{p}(a) \approx \sum\{\mathfrak{p}(\&X) \mid a \in X \in \mathfrak{L}{\perp\!\!\perp}\bot$ and $0 \ll \mathfrak{p}(\&X)/\mathfrak{p}(a)\}$ □

LEMMA 5 *Let K be a consistent belief set and $*$ a sentential operation on K. If $*$ satisfies* relative success *and* vacuity, *then it satisfies:*

If $K \not\subseteq K * a$, then $K \cup (K * a) \vdash \bot$ (consistent expansion, [22] p. 1583)

PROOF OF LEMMA 5: Let $K \not\subseteq K * a$. First suppose that $a \notin K * a$. Then due to *relative success*, $K = K * a$, contrary to $K \not\subseteq K * a$. Thus $a \in K * a$. It follows from *vacuity* and $K \not\subseteq K * a$ that $\neg a \in K$. We can conclude from $\neg a \in K$ and $a \in K * a$ that $K \cup (K * a) \vdash \bot$. □

LEMMA 6 *Let K be a consistent belief set and $*$ a sentential operation on K. If $*$ satisfies* relative success, inclusion, *and* vacuity, *then it satisfies:*

$K * \top = K$

PROOF OF LEMMA 6: It follows from *inclusion* that $K * \top \subseteq \mathrm{Cn}(K)$, and the logical closure of K yields $K * \top \subseteq K$.

Suppose that $K \nsubseteq K * \top$. *Consistent expansion* (Lemma 5) yields $K \cup (K * \top) \vdash \bot$. It then follows from $K * \top \subseteq K$, which we have just proved, that $K \vdash \bot$, contrary to our assumption. This contradiction concludes the proof that $K * \top = K$. □

LEMMA 7 *Let $*$ be a sentential operation on the consistent belief set K in a finite language $\mathfrak{L}$, and let $*$ satisfy* closure *and* strong consistency. *If $X \in \mathfrak{L}\perp\perp$ and $\&X \in K * \&X$, then $K * \&X = X$.*

PROOF OF LEMMA 7: It follows from $\&X \in K * \&X$ and *closure* that $X \subseteq K * \&X$. Since all proper supersets of X are inconsistent, *strong consistency* yields $K * \&X = X$. □

LEMMA 8 *Let $*$ be a sentential operation on the consistent belief set K in a finite language $\mathfrak{L}$, and let $*$ satisfy* closure, strong consistency, strong regularity, extensionality *and* disjunctive factoring. *Furthermore, let $\{X_1, \dots, X_n\} \subseteq \mathfrak{L}\perp\perp$, let $\&X_k \in K * \&X_k$ for all $X_k \in \{X_1, \dots, X_n\}$, and let it hold for all elements X_k and X_m of $\{X_1, \dots, X_n\}$ that $K * (\&X_k \vee \&X_m) \subseteq K * \&X_k$. Then:*

$K * (\&X_1 \vee \dots \vee \&X_n) = (K * \&X_1) \cap \dots \cap (K * \&X_n)$

PROOF OF LEMMA 8: The proof will be inductive. In the base case, $n = 2$, consider the set $\{X_1, X_2\} \in \mathfrak{L}\perp\perp$. Due to the conditions of the lemma, $K * (\&X_1 \vee \&X_2) \subseteq (K * \&X_1) \cap (K * \&X_2)$. For the other direction, *strong consistency* yields $\neg(\&X_1 \vee \&X_2) \notin K * \&X_1$, and due to *strong regularity*, $(\&X_1 \vee \&X_2) \in K * (\&X_1 \vee \&X_2)$. Since $\mathrm{Cn}(\{\&X_1 \vee \&X_2\}) = X_1 \cap X_2$, it follows from *closure* that $X_1 \cap X_2 \subseteq K * (\&X_1 \vee \&X_2)$, and Lemma 7 yields $(K * \&X_1) \cap (K * \&X_2) \subseteq K * (\&X_1 \vee \&X_2)$.

In the induction step, $n > 2$, we use *extensionality*, *disjunctive factoring*, the induction hypothesis, and Lemma 7 twice. First, we use them to conclude that one of the following three conditions holds:

(1) $K * (\&X_1 \vee \dots \vee \&X_{n+1}) = K * (\&X_1 \vee \dots \vee \&X_n) = (K * \&X_1) \cap \dots \cap (K * \&X_n) = X_1 \cap \dots \cap X_n$
(2) $K * (\&X_1 \vee \dots \vee \&X_{n+1}) = K * \&X_{n+1} = X_{n+1}$
(3) $K * (\&X_1 \vee \dots \vee \&X_{n+1}) = (K * (\&X_1 \vee \dots \vee \&X_n)) \cap (K * \&X_{n+1}) = X_1 \cap \dots \cap X_{n+1}$

Next we conclude in the same way that one of the following three conditions holds:

(4) $K * (\&X_1 \vee \dots \vee \&X_{n+1}) = K * \&X_1 = X_1$
(5) $K * (\&X_1 \vee \dots \vee \&X_{n+1}) = K * (\&X_2 \vee \dots \vee \&X_{n+1}) = (K * \&X_2) \cap \dots \cap (K * \&X_{n+1}) = X_2 \cap \dots \cap X_{n+1}$
(6) $K * (\&X_1 \vee \dots \vee \&X_{n+1}) = (K * \&X_1) \cap (K * (\&X_2 \vee \dots \vee \&X_{n+1})) = X_1 \cap \dots \cap X_{n+1}$

It holds for all $X_k, X_m \in \{X_1, \dots, X_{n+1}\}$ that if $X_k \neq X_m$, then $X_k \nvdash \neg\&X_m$. Therefore, the only way in which it can be true both that (1), (2) or (3) holds, and also that (4), (5) or (6) holds, is that (3) and (6) hold. Thus, $K * (\&X_1 \vee \dots \vee \&X_{n+1}) = (K * \&X_1) \cap \dots \cap (K * \&X_{n+1})$. □

PROOF OF OBSERVATION 1: Equation (1) is equivalent with the following:

$$(\mathfrak{p} \star a')(d) = \begin{cases} \mathfrak{p}(d) & \text{if } \mathfrak{p}(a) = 0 \text{ or } \mathfrak{p}(a) = 1 \\ \mathfrak{p}(d \mid a) = \dfrac{\mathfrak{p}(a\&d)}{\mathfrak{p}(a)} & \text{if } 0 \neq \mathfrak{p}(a) \neq 1 \end{cases} \quad (1')$$

Since the first clause of equation (1') coincides with the first clause of equation (2), we only have to prove the case represented by the second clause of the two equations:

$d \in [\![\mathfrak{p} \star {}_{\delta}a]\!]$ iff $(1-\delta) \times \dfrac{\mathfrak{p}(a\&d)}{\mathfrak{p}(a)} + \delta \times \dfrac{\mathfrak{p}(\neg a\&d)}{\mathfrak{p}(\neg a)} \approx 1$ Equation (2)

iff $\dfrac{\mathfrak{p}(a\&d)}{\mathfrak{p}(a)} \approx 1$ Two infinitesimal terms removed

iff $d \in [\![\mathfrak{p} \star {}'a]\!]$ Equation (1')□

PROOF OF THEOREM 1: The proof of the theorem consists of four parts. Part I takes us from a hyperreal Bayesian top revision on a given belief set K to a ring-based revision on K. Part II takes us from a ring-based revision on K to a hyperreal Bayesian top revision on K. Part III takes us from a ring-based revision to the axioms, and Part IV from the axioms to a ring-based revision.

PART I, FROM A HYPERREAL BAYESIAN TOP REVISION TO A RING-BASED REVISION

Construction of the ring system: Let $\mathfrak{p}$ be a hyperreal probability function and $*$ the hyperreal Bayesian top revision on the belief set K that it gives rise to according to Definition 2. Then $K = [\![\mathfrak{p}]\!]$.

Let $\mathfrak{R} = \{X \in \mathfrak{L}\!\perp\!\!\perp_{\perp} \mid \mathfrak{p}(\&X) \neq 0\}$. It follows from Postulate 1 and Lemma 2 that for all $X \in \mathfrak{L}\!\perp\!\!\perp_{\perp}$, $\mathfrak{p}(\&X)$, either $0 \ll \mathfrak{p}(\&X)$ or $\mathfrak{p}(\&X)$ is a finite-ordered infinitesimal. We can therefore construct a sequence $\langle \mathfrak{R}_0, \dots, \mathfrak{R}_n \rangle$ such that $\mathfrak{R}_0$ consists of the elements X of $\mathfrak{R}$ such that $0 \ll \mathfrak{p}(\&X)$, and each $\mathfrak{R}_k$ with $0 < k$ consists of the elements of X whose conjunctions have a probability that is an infinitesimal of the k^{th} order.

Let $\bar{*}$ be the operation of revision that is based on $\langle \mathfrak{R}_0, \dots, \mathfrak{R}_n \rangle$ according to Definition 5. We need to verify that $\langle \mathfrak{R}_0, \dots, \mathfrak{R}_n \rangle$ is a ring system around K and that $*$ and $\bar{*}$ coincide.

Verification that $\langle \mathfrak{R}_0, \dots, \mathfrak{R}_n \rangle$ *is a ring system around* K: We need to prove that $\bigcap \mathfrak{R}_0 = K$, which is done as follows:

$a \in K$ iff $a \in [\![\mathfrak{p}]\!]$

iff $\mathfrak{p}(a) \approx 1$

iff $\sum\{\mathfrak{p}(\&X) \mid X \in \|a\|\} \approx 1$

iff $\sum\{\mathfrak{p}(\&X) \mid X \in \|a\| \cap \mathfrak{R}_0\} \approx 1$

A finite number of infinitesimal terms excluded, namely all $\mathfrak{p}(\&X)$ with $X \in \|a\| \setminus \mathfrak{R}_0$

iff $\sum\{\mathfrak{p}(\&X) \mid X \in \|a\| \cap \mathfrak{R}_0\} \approx \sum\{\mathfrak{p}(\&X) \mid X \in \mathfrak{R}\}$

iff $\sum\{\mathfrak{p}(\&X) \mid X \in \|a\| \cap \mathfrak{R}_0\} \approx \sum\{\mathfrak{p}(\&X) \mid X \in \mathfrak{R}_0\}$

$\mathfrak{R} \setminus \mathfrak{R}_0$ is finite and the conjunctions of its elements all have infinitesimal probabilities

iff $\sum\{\mathfrak{p}(\&X) \mid X \in \mathfrak{R}_0 \setminus \|a\|\} \approx 0$

iff $\mathfrak{R}_0 \setminus \|a\| = \varnothing$ $0 \ll \mathfrak{p}(\&X)$ for all $X \in \mathfrak{R}_0$

iff $\mathfrak{R}_0 \subseteq \|a\|$
iff $a \in \bigcap \mathfrak{R}_0$
Verification that $*$ *and* $\bar{*}$ *coincide*: If $\mathfrak{p}(a) = 0$, then:
$d \in K * a$
iff $d \in [\![\mathfrak{p} \star_{\delta} a]\!]$ Definition 2
iff $d \in [\![\mathfrak{p}]\!]$ Definition 1
iff $d \in K$
iff $d \in K \bar{*} a$ Definition 5, clause 1, $\neg a \in \bigcap(\mathfrak{R}_0 \cup \dots \cup \mathfrak{R}_n)$ since $\mathfrak{p}(a) = 0$

If $\mathfrak{p}(a) = 1$, then we obtain $K * a = K$ in the same way as in the previous case. It follows from $\mathfrak{p}(a) = 1$ that $\mathfrak{R} \subseteq \|a\|$, thus $\|a\| \cap \mathfrak{R}(a) = \|a\| \cap \mathfrak{R}_0 = \mathfrak{R}_0$, and it follows from Definition 5, clause 2, that $K \bar{*} a = \bigcap \mathfrak{R}_0$, thus $K \bar{*} a = K$. Hence, $K * a = K \bar{*} a$.

In the main case, $0 \neq \mathfrak{p}(a) \neq 1$, we proceed as follows:
$d \in K * a$
iff $d \in [\![\mathfrak{p} \star_{\delta} a]\!]$ Definition 2
iff $(\mathfrak{p} \star_{\delta} a)(d) \approx 1$
iff $\dfrac{\mathfrak{p}(a\&d)}{\mathfrak{p}(a)} \approx 1$ Definition 1, excluding two infinitesimal terms
iff $\dfrac{\sum\{\mathfrak{p}(\&X) \mid X \in \|a\&d\|\}}{\sum\{\mathfrak{p}(\&X) \mid X \in \|a\|\}} \approx 1$
iff $\dfrac{\sum\{\mathfrak{p}(\&X) \mid X \in \|a\&d\| \cap \mathfrak{R}(a)\} + \sum\{\mathfrak{p}(\&X) \mid X \in \|a\&d\| \setminus \mathfrak{R}(a)\}}{\sum\{\mathfrak{p}(\&X) \mid X \in \|a\| \cap \mathfrak{R}(a)\} + \sum\{\mathfrak{p}(\&X) \mid X \in \|a\| \setminus \mathfrak{R}(a)\}} \approx 1$
iff $\dfrac{\sum\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid X \in \|a\&d\| \cap \mathfrak{R}(a)\} + \sum\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid X \in \|a\&d\| \setminus \mathfrak{R}(a)\}}{\sum\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid X \in \|a\| \cap \mathfrak{R}(a)\} + \sum\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid X \in \|a\| \setminus \mathfrak{R}(a)\}} \approx 1$

Due to the construction of $\langle \mathfrak{R}_0, \dots, \mathfrak{R}_n \rangle$ for this proof, $0 \ll \sum\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid X \in \|a\| \cap \mathfrak{R}(a)\}$, whereas $\sum\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid X \in \|a\| \setminus \mathfrak{R}(a)\} \approx 0$ and consequently $\sum\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid X \in \|a\&d\| \setminus \mathfrak{R}(a)\} \approx 0$. We can therefore delete the latter two parts of the expression, without affecting the standard part of the ratio:
iff $\dfrac{\sum\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid X \in \|a\&d\| \cap \mathfrak{R}(a)\}}{\sum\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid X \in \|a\| \cap \mathfrak{R}(a)\}} \approx 1$ Equation (3)

$\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid X \in \|a\&d\| \cap \mathfrak{R}(a)\}$ is a subset of $\{\mathfrak{p}(\&X)/\mathfrak{p}(a) \mid X \in \|a\| \cap \mathfrak{R}(a)\}$, and all the elements of the latter set are positive numbers with a non-zero standard part. Thus Equation 3 holds if and only if:
iff $\|a\&d\| \cap \mathfrak{R}(a) = \|a\| \cap \mathfrak{R}(a)$
iff $d \in \bigcap(\|a\| \cap \mathfrak{R}(a))$
iff $d \in K \bar{*} a$ Definition 5, clause 2

PART II: FROM RING-BASED REVISION TO HYPERREAL BAYESIAN TOP REVISION

Construction: Let $\bar{*}$ be the ring-based revision on K that is based on the ring system $\langle \mathfrak{R}_0, \dots, \mathfrak{R}_n \rangle$. (Then $K = \bigcap \mathfrak{R}_0$.) For each k with $0 < k \leq n$, let ε_k be an infinitesimal of the k^{th} order. Let $\mathfrak{p}$ be a probability function such that for each $X \in \mathfrak{L}\bot\!\!\bot$:

(1) If $X \notin \mathfrak{R}_0 \cup \dots \cup \mathfrak{R}_n$, then $\mathfrak{p}(\&X) = 0$
(2) If $X \in \mathfrak{R}_k$ and $0 < k$, then $\mathfrak{p}(\&X) = \dfrac{\varepsilon_k}{\mathsf{numb}(\mathfrak{R}_k)}$

(3) If $X \in \mathfrak{R}_0$, then $\mathfrak{p}(\&X) = \dfrac{1 - \sum\{\varepsilon_k \mid 0 < k \leq n\}}{\mathsf{numb}(\mathfrak{R}_0)}$

Let $\star$ be the hyperreal Bayesian revision that is based on $\mathfrak{p}$ according to Definition 1, and let $*$ be the hyperreal Bayesian top revision on K that is based on $\star$ according to Definition 2. We are going to show that $*$ coincides with $\bar{*}$. There are four cases.

Verification for $\mathfrak{p}(a) = 0$:

$d \in K * a$ iff $d \in [\![\mathfrak{p} \star {}_\delta a]\!]$ — Definition 2
iff $(\mathfrak{p} \star {}_\delta a)(d) \approx 1$
iff $\mathfrak{p}(d) \approx 1$ — Definition 1, case 1
iff $d \in [\![\mathfrak{p}]\!]$
iff $d \in K$ — Definition 2
iff $d \in K \bar{*} a$ — Definition 5, clause 1

Verification for $\mathfrak{p}(a) = 1$: We obtain $K * a = K$ in the same way as in the previous case. It follows from $\mathfrak{p}(a) = 1$ and our construction of $\mathfrak{p}$ that $\mathfrak{R}_0 \cup \cdots \cup \mathfrak{R}_n \subseteq \|a\|$, thus $\|a\| \cap \mathfrak{R}(a) = \mathfrak{R}_0$. It follows from Definition 5, clause 2, that $K \bar{*} a = \bigcap \mathfrak{R}_0 = K$, thus $K * a = K \bar{*} a$.

Verification for $0 \ll \mathfrak{p}(a) < 1$:

$d \in K * a$ iff $d \in [\![\mathfrak{p} \star {}_\delta a]\!]$ — Definition 2
iff $(\mathfrak{p} \star {}_\delta a)(d) \approx 1$
iff $\dfrac{\mathfrak{p}(a\&d)}{\mathfrak{p}(a)} \approx 1$ — Definition 1, two infinitesimal terms removed
iff $\mathfrak{p}(a\&d) \approx \mathfrak{p}(a)$ — Since $0 \ll \mathfrak{p}(a)$
iff $\|a\&d\| \cap \mathfrak{R}_0 = \|a\| \cap \mathfrak{R}_0$ — Construction of $\mathfrak{p}$
iff $d \in \bigcap(\|a\| \cap \mathfrak{R}_0)$
iff $d \in K \bar{*} a$ — Definition 5, clause 2

Verification for $0 < \mathfrak{p}(a) \approx 0$: We assume that $\mathfrak{R}(a) = \mathfrak{R}_k$. Then ε_k is the infinitesimal of the k^{th} order introduced in the construction for this part of the proof.

$d \in K * a$ iff $\dfrac{\mathfrak{p}(a\&d)}{\mathfrak{p}(a)} \approx 1$ — As in the previous case

$$\text{iff } \frac{\mathfrak{p}(a\&d)}{\sum\{\mathfrak{p}(\&X) \mid X \in \|a\| \cap \mathfrak{R}(a)\} + \sum\{\mathfrak{p}(\&X) \mid X \in \|a\| \setminus \mathfrak{R}(a)\}} \approx 1$$

$$\text{iff } \frac{\mathfrak{p}(a\&d)}{\dfrac{\varepsilon_k \times \mathsf{numb}(\|a\| \cap \mathfrak{R}(a))}{\mathsf{numb}(\mathfrak{R}(a))} + \sum\{\mathfrak{p}(\&X) \mid X \in \|a\| \setminus \mathfrak{R}(a)\}} \approx 1$$

$$\text{iff } \frac{\mathfrak{p}(a\&d)/\varepsilon_k}{\dfrac{\mathsf{numb}(\|a\| \cap \mathfrak{R}(a))}{\mathsf{numb}(\mathfrak{R}(a))} + \dfrac{\sum\{\mathfrak{p}(\&X) \mid X \in \|a\| \setminus \mathfrak{R}(a)\}}{\varepsilon_k}} \approx 1$$

$$\text{iff } \frac{\mathfrak{p}(a\&d)/\varepsilon_k}{\dfrac{\mathsf{numb}(\|a\| \cap \mathfrak{R}(a))}{\mathsf{numb}(\mathfrak{R}(a))}} \approx 1$$ — Infinitesimal term after positive real-valued term deleted

$$\text{iff } \frac{\dfrac{\sum\{\mathfrak{p}(\&X) \mid X \in \|a\&d\| \cap \mathfrak{R}(a)\}}{\varepsilon_k} + \dfrac{\sum\{\mathfrak{p}(\&X) \mid X \in \|a\&d\| \setminus \mathfrak{R}(a)\}}{\varepsilon_k}}{\dfrac{\mathsf{numb}(\|a\| \cap \mathfrak{R}(a))}{\mathsf{numb}(\mathfrak{R}(a))}} \approx 1$$

$$\text{iff } \frac{\frac{1}{\varepsilon_k} \times \sum\{\mathfrak{p}(\&X) \mid X \in \|a\&d\| \cap \mathfrak{R}(a)\}}{\frac{\mathsf{numb}(\|a\| \cap \mathfrak{R}(a))}{\mathsf{numb}(\mathfrak{R}(a))}} \approx 1$$

Infinitesimal term removed, the denominator is positive real-valued

$$\text{iff } \frac{\frac{1}{\varepsilon_k} \times \frac{\varepsilon_k \times \mathsf{numb}(\|a\&d\| \cap \mathfrak{R}(a))}{\mathsf{numb}(\mathfrak{R}(a))}}{\frac{\mathsf{numb}(\|a\| \cap \mathfrak{R}(a))}{\mathsf{numb}(\mathfrak{R}(a))}} \approx 1$$

iff $\mathsf{numb}(\|a\&d\| \cap \mathfrak{R}(a)) = \mathsf{numb}(\|a\| \cap \mathfrak{R}(a))$

iff $d \in \bigcap(\|a\| \cap \mathfrak{R}(a))$

iff $d \in K \bar{*} a$ — Definition 5, clause 2

PART III: FROM RING-BASED REVISION TO AXIOMS

Let K be a belief set and $\langle \mathfrak{R}_0, \dots, \mathfrak{R}_n \rangle$ a ring system according to Definition 3 with $K = \bigcap \mathfrak{R}_0$, and let $*$ be the revision on K based on that ring system according to Definition 5.

Closure, $K * a = \text{Cn}(K * a)$: Directly from Definition 5.

Relative success, Either $a \in K * a$ *or* $K * a = K$: Directly from Definition 5.

Strong regularity, If $\neg b \notin K * a$*, then* $b \in K * b$: If $\neg b \notin K * a$, then there is some $X \in \mathfrak{R}_0 \cup \dots \cup \mathfrak{R}_n$ such that $\neg b \notin X$. It follows from Definition 5 that $b \in K * b$.

Inclusion, $K * a \subseteq \text{Cn}(K \cup \{a\})$:

Case 1, $a \notin K * a$: Clause (1) of Definition 5 yields $K * a = K \subseteq \text{Cn}(K \cup \{a\})$.

Case 2, $a \in K * a$ and $K \nvdash \neg a$: Then $\mathfrak{R}(a) = \mathfrak{R}_0$. Thus:

$K * a = \bigcap(\|a\| \cap \mathfrak{R}_0)$ — Definition 5, clause 2

$= \bigcap\{X \in \mathfrak{L}\bot\!\!\!\bot_\perp \mid a \in X \text{ and } K \subseteq X\}$ — $K = \bigcap \mathfrak{R}_0$

$= \bigcap\{X \in \mathfrak{L}\bot\!\!\!\bot_\perp \mid \text{Cn}(K \cup \{a\}) \subseteq X\}$

$= \text{Cn}(K \cup \{a\})$

Case 3, $a \in K * a$ and $K \vdash \neg a$: Then $\text{Cn}(K \cup \{a\}) = \text{Cn}(\{\perp\})$, thus $K * a \subseteq \text{Cn}(K \cup \{a\})$.

Strong consistency, $K * a \nvdash \perp$: Directly from Definition 5.

Extensionality: If $\vdash a_1 \leftrightarrow a_2$*, then* $K * a_1 = K * a_2$: Directly from Definition 5.

Vacuity, If $\neg a \notin K$*, then* $\text{Cn}(K \cup \{a\}) \subseteq K * a$:

$\neg a \notin K$

$\|a\| \cap \mathfrak{R}_0 \neq \varnothing$ — $K = \bigcap \mathfrak{R}_0$

$K * a = \bigcap(\|a\| \cap \mathfrak{R}_0)$ — Definition 5, clause 2

$K * a = \bigcap\{X \in \mathfrak{L}\bot\!\!\!\bot_\perp \mid K \cup \{a\} \subseteq X\}$ — $K = \bigcap \mathfrak{R}_0$

$K * a = \text{Cn}(K \cup \{a\})$

Disjunctive factoring, $K * (a \vee b)$ *is either* $K * a$*,* $K * b$*, or* $(K * a) \cap (K * b)$: Let $\mathfrak{R} = \mathfrak{R}_0 \cup \dots \cup \mathfrak{R}_n$.

Case 1, $\neg a \in \bigcap \mathfrak{R}$ and $\neg b \in \bigcap \mathfrak{R}$: Then $\neg(a \vee b) \in \bigcap \mathfrak{R}$, and if follows from clause (1) of Definition 5 that $K * (a \vee b) = K * a = K * b = K$.

Case 2, $\neg a \in \bigcap \mathfrak{R}$ and $\neg b \notin \bigcap \mathfrak{R}$: Then it holds for all $X \in \mathfrak{R}$ that $(a \vee b) \in X$ if and only if $b \in X$, and it follows from Definition 5 that $K * (a \vee b) = K * b$.

Case 3, $\neg a \notin \bigcap \mathfrak{R}$, $\neg b \notin \bigcap \mathfrak{R}$, and $\mathfrak{R}(a)$ precedes $\mathfrak{R}(b)$ in the sequence: Then $\mathfrak{R}(a) = \mathfrak{R}(a \vee b)$, and since $\neg b \in \bigcap \mathfrak{R}(a)$ we have: $K * a = \bigcap(\|a\| \cap \mathfrak{R}(a)) = \bigcap(\|a \vee b\| \cap \mathfrak{R}(a)) = \bigcap(\|a \vee b\| \cap \mathfrak{R}(a \vee b)) = K * (a \vee b)$.

Case 4, $\neg a \notin \bigcap \mathfrak{R}$, $\neg b \notin \bigcap \mathfrak{R}$, and $\mathfrak{R}(a) = \mathfrak{R}(b)$: Then $\mathfrak{R}(a \vee b) = \mathfrak{R}(a) = \mathfrak{R}(b)$, and:

$K * (a \vee b) = \bigcap(\|a \vee b\| \cap \mathfrak{R}(a \vee b))$
$= \bigcap((\|a\| \cup \|b\|) \cap \mathfrak{R}(a \vee b))$
$= \bigcap((\|a\| \cap \mathfrak{R}(a \vee b)) \cup (\|b\| \cap \mathfrak{R}(a \vee b)))$
$= \bigcap(\|a\| \cap \mathfrak{R}(a \vee b)) \cap \bigcap(\|b\| \cap \mathfrak{R}(a \vee b))$
$= \bigcap(\|a\| \cap \mathfrak{R}(a)) \cap \bigcap(\|b\| \cap \mathfrak{R}(b))$
$= (K * a) \cap (K * b)$

PART IV: FROM AXIOMS TO RING-BASED REVISION

Part IV.1, construction:

Let $*$ be a sentential operation on the consistent belief set K that satisfies the axioms. Let

$\mathfrak{R} = \{X \in \mathfrak{L}\perp\perp \mid \&X \in K * \&X\}$

Let $\sqsubseteq$ (with the strict part $\sqsubset$) be the relation on $\mathfrak{R}$ such that:

$X_1 \sqsubseteq X_2$ if and only if $K * (\&X_1 \vee \&X_2) \subseteq K * X_1$

Part IV.2, proof that $\sqsubseteq$ is complete and transitive:

That $\sqsubseteq$ is complete follows directly from *disjunctive factoring*. We proceed to show that it is transitive. Let $X_1 \sqsubseteq X_2$ and $X_2 \sqsubseteq X_3$. Then $K * (\&X_1 \vee \&X_2) \subseteq K * \&X_1$ and $K * (\&X_2 \vee \&X_3) \subseteq K * \&X_2$. It follows from *disjunctive factoring* that either $K*(\&X_1 \vee \&X_2 \vee \&X_3) \subseteq K*\&X_1$ or $K*(\&X_1 \vee \&X_2 \vee \&X_3) \subseteq K*(\&X_2 \vee \&X_3)$.[11] Due to our assumption $K * (\&X_2 \vee \&X_3) \subseteq K * \&X_2$, $K * (\&X_1 \vee \&X_2 \vee \&X_3)$ is either a subset of $K * \&X_1$ or a subset of $K * \&X_2$. Due to *closure*, *strong consistency* and Lemma 7, we have $K * \&X_1 = X_1$, $K * \&X_2 = X_2$ and $K * \&X_3 = X_3$, and since $X_1, X_2, X_3 \in \mathfrak{L}\perp\perp$, we can conclude that $K * (\&X_1 \vee \&X_2 \vee \&X_3) \neq K * \&X_3$.

It follows from *disjunctive factoring* and $K * (\&X_1 \vee \&X_2 \vee \&X_3) \neq K * \&X_3$ that $K * (\&X_1 \vee \&X_2 \vee \&X_3) \subseteq K * (\&X_1 \vee \&X_2)$. Due to our assumption $K * (\&X_1 \vee \&X_2) \subseteq K * \&X_1$, we can conclude that $K * (\&X_1 \vee \&X_2 \vee \&X_3) \subseteq K * \&X_1$.

Next we note that due to *disjunctive factoring* and *extensionality*, one of the following three conditions must hold:

(1) $K * (\&X_1 \vee \&X_2 \vee \&X_3) = K * \&X_2$
(2) $K * (\&X_1 \vee \&X_2 \vee \&X_3) = K * (\&X_1 \vee \&X_3)$
(3) $K * (\&X_1 \vee \&X_2 \vee \&X_3) = (K * (\&X_1 \vee \&X_3)) \cap (K * \&X_2)$

Case (1): Since $K * (\&X_1 \vee \&X_2 \vee \&X_3) \subseteq K * \&X_1 = X_1$, $K * \&X_2 = X_2$ and $X_1, X_2 \in \mathfrak{L}\perp\perp$, this case is impossible.

Case (2): Since $K*(\&X_1 \vee \&X_2 \vee \&X_3) \subseteq K*\&X_1$, we obtain $K*(\&X_1 \vee \&X_3) \subseteq K * \&X_1$, i.e. $X_1 \sqsubseteq X_3$.

Case (3): Suppose for reductio that $K * (\&X_1 \vee \&X_3) = K * \&X_3$. Then:

[11] Depending on whether $K*(\&X_1 \vee \&X_2 \vee \&X_3)$ is constructed as an abbreviation of $K*(\&X_1 \vee (\&X_2 \vee \&X_3))$ or of $K*((\&X_1 \vee \&X_2) \vee \&X_3)$, *extensionality* is involved either when we use *disjunctive factoring* to conclude that $K*(\&X_1 \vee \&X_2 \vee \&X_3)$ is either $K*\&X_1$, $K*(\&X_2 \vee \&X_3)$ or $(K*\&X_1) \cap K*(\&X_2 \vee \&X_3)$ or when we use it to conclude that $K * (\&X_1 \vee \&X_2 \vee \&X_3)$ is either $K * (\&X_1 \vee \&X_2)$, $K * \&X_3$, or $K * (\&X_1 \vee \&X_2) \cap (K * \&X_3)$.

$K * (\&X_1 \vee \&X_2 \vee \&X_3) = (K * \&X_3) \cap (K * \&X_2)$
$(K * \&X_3) \cap (K * \&X_2) \subseteq K * \&X_1$ Since $K * (\&X_1 \vee \&X_2 \vee \&X_3) \subseteq K * \&X_1$
$X_3 \cap X_2 \subseteq X_1$ *Closure*, *strong consistency* and Lemma 7
$\neg\&X_1 \in X_1$, Since $\neg\&X_1 \in X_3$ and $\neg\&X_1 \in X_2$
which is impossible. Thus $K * (\&X_1 \vee \&X_3) \neq K * \&X_3$, and it follows from *disjunctive factoring* that $K * (\&X_1 \vee \&X_3) \subseteq K * \&X_1$, i.e. $X_1 \sqsubseteq X_3$.

This concludes the proof that $\sqsubseteq$ is transitive.

Part IV.3, the construction continued:
Since $\sqsubseteq$ is transitive and complete, we can divide $\mathfrak{R}$ into $\sqsubseteq$-equivalence classes, arranged in a sequence $\langle \mathfrak{R}_0, \ldots, \mathfrak{R}_v \rangle$, such that if $0 \leq k \leq v$ and $0 \leq m \leq v$ then (1) $\mathfrak{R}_k \cap \mathfrak{R}_m = \varnothing$ if $\mathfrak{R}_k \neq \mathfrak{R}_m$, and (2) $\mathfrak{R}_k \sqsubseteq \mathfrak{R}_m$ if and only if $k \leq m$. Let $\hat{*}$ be the ring-based operation based on $\langle \mathfrak{R}_0, \ldots, \mathfrak{R}_v \rangle$ according to Definition 5. To complete the proof we need to show that this is a ring system and that $\hat{*}$ coincides with $*$, or more precisely that $K = \bigcap \mathfrak{R}_0$, and that $K \hat{*} a = K * a$ for all $a \in \mathfrak{L}$.

Part IV.4, proof that $K = \bigcap \mathfrak{R}_0$:
Step 1, proof that if $K \subseteq X \in \mathfrak{L} \perp\!\!\!\perp_{\perp}$, then $X \in \mathfrak{R}$: It follows from $K \subseteq X \in \mathfrak{L} \perp\!\!\!\perp_{\perp}$ that $\neg\&X \notin K$. *Relative success*, *inclusion*, *vacuity*, and Lemma 6 yield $\neg\&X \notin K * \top$, *strong regularity* yields $\&X \in K * \&X$, and our construction of $\mathfrak{R}$ in part IV.1 of this proof yields $X \in \mathfrak{R}$.

Step 2, proof that if $X, Y \in \mathfrak{R}$ and $K \subseteq X$, then $X \sqsubseteq Y$:
$K * (\&X \vee \&Y) \subseteq \mathrm{Cn}(K \cup \{\&X \vee \&Y\})$ *Inclusion*
$K * (\&X \vee \&Y) \subseteq \mathrm{Cn}(K \cup \{\&X\})$
$K * (\&X \vee \&Y) \subseteq X$ Since $K \subseteq X$
$K * (\&X \vee \&Y) \subseteq K * \&X$ *Closure*, *strong consistency*, and Lemma 7
$X \sqsubseteq Y$.

Step 3, proof that $\bigcap \mathfrak{R}_0 \subseteq K$:
$\{X \in \mathfrak{R} \mid K \subseteq X\} \subseteq \mathfrak{R}_0$ Step 2
$\bigcap \mathfrak{R}_0 \subseteq \bigcap \{X \in \mathfrak{R} \mid K \subseteq X\}$
$\bigcap \mathfrak{R}_0 \subseteq \bigcap \{X \in \mathfrak{L} \perp\!\!\!\perp_{\perp} \mid K \subseteq X\}$ Step 1
$\bigcap \mathfrak{R}_0 \subseteq K$ $K = \bigcap \{X \in \mathfrak{L} \perp\!\!\!\perp_{\perp} \mid K \subseteq X\}$

Step 4, proof that $K \subseteq \bigcap \mathfrak{R}_0$:
We know from Steps 1 and 2 that there is a set $\{X_1, \ldots, X_n\} \subseteq \mathfrak{R}_0$ such that $K = X_1 \cap \cdots \cap X_n$. Due to *closure*, *strong consistency*, and Lemma 7, $K = (K * \&X_1) \cap \cdots \cap (K * \&X_n)$. It follows from *closure*, *strong consistency*, *strong regularity*, *extensionality*, *disjunctive factoring* and Lemma 8 that $K = K * (\&X_1 \vee \cdots \vee \&X_n)$. Suppose for contradiction that there is some $Y \in \mathfrak{R}_0 \setminus \{X_1, \ldots, X_n\}$. It follows from Lemmas 7 and 8 that $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y) = X_1 \cap \cdots \cap X_n \cap Y$. Thus, $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y)$ is a subset of K. Since it does not contain $\neg\&Y$, which is an element of K, we also have $K \nsubseteq K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y)$. Due to *relative success*, *vacuity* and Lemma 5, *consistent expansion* holds, and therefore $K \cup (K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y)) \vdash \perp$, which is impossible since K is consistent and $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y)$ is a subset of K. We can conclude from this contradiction that $\mathfrak{R}_0 \setminus \{X_1, \ldots, X_n\}$ is empty, thus:
$\mathfrak{R}_0 \subseteq \{X_1, \ldots, X_n\}$
$\bigcap \{X_1, \ldots, X_n\} \subseteq \bigcap \mathfrak{R}_0$

$K \subseteq \bigcap \mathfrak{R}_0$

Part IV.5, proof of the identity of $\hat{*}$ and $*$:

In the limiting case when $a \notin K * a$, it follows from *relative success* that $K * a = K$. Suppose that there is some $X \in \mathfrak{R}$ with $a \in X$. Due to the construction of $\mathfrak{R}$ in Part IV.1, $\&X \in K * \&X$. Due to *closure*, *strong consistency* and Lemma 7, $X = K * \&X$. Thus $\neg a \notin K * \&X$, and *strong regularity* yields $a \in K * a$, contrary to our assumption for this case. Thus there is no $X \in \mathfrak{R}$ with $a \in X$, thus $\neg a \in \bigcap(\mathfrak{R}_0 \cup \cdots \cup \mathfrak{R}_v)$. It follows from clause 1 of Definition 5 that $K \hat{*} a = K$, i.e. $K \hat{*} a = K * a$.

The proof of the main case, $a \in K * a$, is divided into three steps.

Part IV.5.1, first step:

It follows from *closure* and *strong consistency* that $K * a = \bigcap\{X \in \mathfrak{L} \perp \perp \mid K * a \subseteq X\}$. It holds for each element X of $\{X \in \mathfrak{L} \perp a \mid K * a \subseteq X\}$ that $\neg \&X \notin K * a$, and it follows from *strong regularity* that $\&X \in K * \&X$, thus $X \in \mathfrak{R}$. We can conclude that $K * a = \bigcap\{X \in \mathfrak{R} \mid K * a \subseteq X\}$. Let $\{X \in \mathfrak{R} \mid K * a \subseteq X\} = \{X_1, \ldots, X_n\}$.

Due to $a \in K * a$ and *strong consistency*, a is a consistent element of $\mathfrak{L}$. There is therefore a subset $\{Z_1, \ldots, Z_m\}$ of $\mathfrak{L} \perp \perp$, such that $\vdash a \leftrightarrow \&(Z_1 \cap \cdots \cap Z_m)$, or equivalently, $\vdash a \leftrightarrow (\&Z_1 \vee \cdots \vee \&Z_m)$. Due to *extensionality*, $K * a = K * (\&Z_1 \vee \cdots \vee \&Z_m)$.

Next, suppose that $\{X_1, \ldots, X_n\} \nsubseteq \{Z_1, \ldots, Z_m\}$. More specifically, let $X_n \notin \{Z_1, \ldots Z_m\}$. It follows from $a \in K * a$, *extensionality* and *closure* that $(\&Z_1 \vee \cdots \vee \&Z_m) \in K * (\&Z_1 \vee \cdots \vee \&Z_m)$. Since each element of $\{\&Z_1, \ldots, \&Z_m\}$ implies $\neg \&X_n$, so does $\&Z_1 \vee \cdots \vee \&Z_m$, and due to *closure*, $\neg \&X_n \in K * (\&Z_1 \vee \cdots \vee \&Z_m)$. However, $\neg \&X_n \notin X_1 \cap \cdots \cap X_n$. We can conclude from this contradiction that $\{X_1, \ldots, X_n\} \subseteq \{Z_1, \ldots, Z_m\}$.

In summary, we have found that there are $Z_1, \ldots, Z_m \in \mathfrak{L} \perp \perp$ and $X_1, \ldots, X_n \in \mathfrak{R}$ such that $\vdash a \leftrightarrow (\&Z_1 \vee \cdots \vee \&Z_m)$, $\{X_1, \ldots, X_n\} \subseteq \{Z_1, \ldots, Z_m\}$, and $K * a = K * (\&Z_1 \vee \cdots \vee \&Z_m) = X_1 \cap \cdots \cap X_n$.

Part IV.5.2, second step:

In this step, we are going to show that all elements of $\{X_1, \ldots, X_n\}$ belong to the same $\sqsubseteq$-equivalence class. The proof is divided into two cases, depending on whether $\{X_1, \ldots, X_n\}$ is identical to, or a proper subset of, $\{Z_1, \ldots, Z_m\}$.

Part IV.5.2.1, first case of the second step, $\{X_1, \ldots, X_n\} = \{Z_1, \ldots, Z_m\}$**:**

In this case, $K * (\&X_1 \vee \cdots \vee \&X_n) = X_1 \cap \cdots \cap X_n$. If $n = 1$, we are finished. If $n = 2$, we use *closure*, *strong consistency* and Lemma 7 to obtain $X_1 = K * \&X_1$ and $X_2 = K * \&X_2$. It follows that $K * (\&X_1 \vee \&X_2) \subseteq K * \&X_1$ and $K * (\&X_1 \vee \&X_2) \subseteq K * \&X_2$, i.e. $X_1 \sqsubseteq X_2 \sqsubseteq X_1$.

For $n > 2$, suppose for contradiction that $X_2 \not\sqsubseteq X_1$. Then $K * (\&X_1 \vee \&X_2) \nsubseteq K * \&X_2$, and *conjunctive factoring* yields $K * (\&X_1 \vee \&X_2) = K * \&X_1$. Another application of *conjunctive factoring* shows that one of the following three conditions holds:

(1) $K * (\&X_1 \vee \cdots \vee \&X_n) = K * (\&X_1 \vee \&X_2) = K * \&X_1$

(2) $K * (\&X_1 \vee \cdots \vee \&X_n) = K * (\&X_3 \vee \cdots \vee \&X_n)$

(3) $K * (\&X_1 \vee \cdots \vee \&X_n) = (K * (\&X_1 \vee \&X_2)) \cap (K * (\&X_3 \vee \cdots \vee \&X_n)) = (K \& X_1) \cap (K * (\&X_3 \vee \cdots \vee \&X_n))$

Due to *closure*, *strong consistency*, and Lemma 7, $\neg\&X_2 \in K * \&X_1$. It follows from $\&X_3 \in K * \&X_3$ and *strong consistency* that $\neg(\&X_3 \vee \cdots \vee \&X_n) \notin K * \&X_3$. *Strong regularity* yields $(\&X_3 \vee \cdots \vee \&X_n) \in K * (\&X_3 \vee \cdots \vee \&X_n)$, thus due to *closure*, $\neg\&X_2 \in K * (\&X_3 \vee \cdots \vee \&X_n)$. It follows from $\neg\&X_2 \in K * \&X_1$ and $\neg\&X_2 \in K * (\&X_3 \vee \cdots \vee \&X_n)$ that in all three cases (1), (2), and (3), $\neg\&X_2 \in K*(\&X_1 \vee \cdots \vee \&X_n)$. However, it follows from $K*(\&X_1 \vee \cdots \vee \&X_n) = X_1 \cap \cdots \cap X_n$ that $\neg\&X_2 \notin K * (\&X_1 \vee \cdots \vee \&X_n)$. We can conclude from this contradiction that $X_2 \sqsubseteq X_1$.

Part IV.5.2.2, second case of the second step, $\{X_1, \ldots, X_n\} \subset \{Z_1, \ldots, Z_m\}$**:** Let $\{Z_1, \ldots, Z_m\} \setminus \{X_1, \ldots, X_n\} = \{Y_1, \ldots, Y_k\}$. Leaving out the trivial subcase $n = 1$, we have two subcases to treat.

Subcase 1, $n = 2$: We have $K * (\&X_1 \vee \&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k) = X_1 \cap X_2$. Suppose for contradiction that $X_1 \not\sqsubseteq X_2$, i.e. $K*(\&X_1 \vee \&X_2) \not\subseteq K*\&X_1$. *Disjunctive factoring* yields $K * (\&X_1 \vee \&X_2) = K * \&X_2$.

Noting that due to *extensionality*, $K * (\&X_1 \vee \&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k) = K * ((\&X_1 \vee \&X_2) \vee (\&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k))$, we can use *extensionality*, *disjunctive factoring*, and Lemma 7 to conclude that one of the following three conditions holds:

(1) $K * (\&X_1 \vee \&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k) = K * (\&X_1 \vee \&X_2) = K * \&X_2 = X_2$
(2) $K * (\&X_1 \vee \&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k) = K * (\&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k)$
(3) $K * (\&X_1 \vee \&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k) = (K * (\&X_1 \vee \&X_2)) \cap (K * (\&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k)) = X_2 \cap (K * (\&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k))$

It follows from $\&X_2 \in K * \&X_2$ and *strong consistency* that $\neg(\&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k) \notin K * \&X_2$, thus due to *strong regularity*, $(\&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k) \in K*(\&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k)$. Due to *closure* and $(\&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k) \vdash \neg\&X_1$, it follows that $\neg\&X_1 \in K * (\&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k)$.

In each of the three cases (1), (2), and (3), it follows from $\neg\&X_1 \in X_2$ and $\neg\&X_1 \in K * (\&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k)$ that $\neg\&X_1 \in K * (\&X_1 \vee \&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k)$, contrary to $K * (\&X_1 \vee \&X_2 \vee \&Y_1 \vee \cdots \vee \&Y_k) = X_1 \cap X_2$. We can conclude from this contradiction that $X_1 \sqsubseteq X_2$.

Subcase 2, $n > 2$: We have $K*(\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = X_1 \cap \cdots \cap X_n$. Suppose for contradiction that $X_1 \not\sqsubseteq X_2$. Just as in the previous case, it follows that $K * (\&X_1 \vee \&X_2) = K * \&X_2$. Due to *disjunctive factoring* and Lemma 7, one of the following three conditions holds:

(1) $K*(\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = K*(\&X_1 \vee \&X_2) = K*\&X_2 = X_2$
(2) $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = K * (\&X_3 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k)$
(3) $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = K * (\&X_1 \vee \&X_2) \cap K * (\&X_3 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = X_2 \cap K * (\&X_3 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k)$

It follows from $\&X_3 \in K * \&X_3$ and *strong consistency* that $\neg(\&X_3 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) \notin K * \&X_3$, and *strong regularity* yields $(\&X_3 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) \in K * (\&X_3 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k)$. Due to *closure* and $(\&X_3 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) \vdash \neg\&X_1$, it follows that $\neg\&X_1 \in K * (\&X_3 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k)$.

In each of the three cases (1), (2), and (3), it follows from $\neg\&X_1 \in X_2$ and $\neg\&X_1 \in K*(\&X_3 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k)$ that $\neg\&X_1 \in K*(\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee$

$\cdots \vee \&Y_k$), contrary to $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = X_1 \cap \cdots \cap X_n$. We can conclude from this contradiction that $X_1 \sqsubseteq X_2$.

Part IV.5.3, third and final step:

We have shown that for each sentence $a \in \mathfrak{L}$, there are $X_1, \ldots, X_n \in \mathfrak{R}_t$ for some $\sqsubseteq$-equivalence class $\mathfrak{R}_t$, such that either (1) $\vdash a \leftrightarrow (\&X_1 \vee \cdots \vee \&X_n)$ and $K * a = K * (\&X_1 \vee \cdots \vee \&X_n) = X_1 \cap \cdots \cap X_n$, or (2) there are $Y_1, \ldots, Y_k \in \mathfrak{L}\bot\bot$ such that $\vdash a \leftrightarrow (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k)$, and $K * a = K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = X_1 \cap \cdots \cap X_n$. We are now going to show that in the latter case, it holds for all $X' \in \{X_1, \ldots, X_n\}$ and $Y' \in \{Y_1, \ldots, Y_k\}$ $Y' \not\sqsubseteq X'$. There are three cases.

Case 1, $n = k = 1$: Then $K * (\&X_1 \vee \&Y_1) = X_1$. Suppose that $Y_1 \sqsubseteq X_1$. Since $\sqsubseteq$ is a relation on $\mathfrak{R}$ (cf. Part IV.1 above), it follows that $\&Y_1 \in K * \&Y_1$. Due to *closure*, *strong consistency* and Lemma 7, $K * \&Y_1 = Y_1$. It therefore follows from $Y_1 \sqsubseteq X_1$, i.e. $K * (\&X_1 \vee \&Y_1) \subseteq K * \&Y_1$, that $X_1 \subseteq Y_1$, which is impossible since $X_1 \neq Y_1$ and $X_1, Y_1 \in \mathfrak{L}\bot\bot$. We can conclude that $Y_1 \not\sqsubseteq X_1$.

Case 2, $n = 1$ and $k > 1$: Then $K * (\&X_1 \vee \&Y_1 \vee \cdots \vee \&Y_k) = X_1$. First let $K * (\&X_1 \vee \&Y_1 \vee \cdots \vee \&Y_k) \subseteq K * (\&Y_2 \vee \cdots \vee \&Y_k)$. Due to *relative success*, either $(\&Y_2 \vee \cdots \vee \&Y_k) \in K * (\&Y_2 \vee \cdots \vee \&Y_k)$ or $K * (\&Y_2 \vee \cdots \vee \&Y_k) = K$. In the former case, it follows from *closure* and $\&Y_2 \vee \cdots \vee \&Y_k \vdash \neg\&X_1$ that $\neg\&X_1 \in K * (\&Y_2 \vee \cdots \vee \&Y_k)$, contrary to *strong consistency* and $X_1 = K * (\&X_1 \vee \&Y_1 \vee \cdots \vee \&Y_k) \subseteq K * (\&Y_2 \vee \cdots \vee \&Y_k)$. Thus, $K * (\&Y_2 \vee \cdots \vee \&Y_k) = K$, thus $X_1 \subseteq K$, thus $\mathfrak{R}_0 = \{X_1\}$ and $Y_1 \not\sqsubseteq X_1$.

Next, let $K * (\&X_1 \vee \&Y_1 \vee \cdots \vee \&Y_k) \not\subseteq K * (\&Y_2 \vee \cdots \vee \&Y_k)$. *Disjunctive factoring* yields $K * (\&X_1 \vee \&Y_1 \vee \cdots \vee \&Y_k) = K * (\&X_1 \vee \&Y_1)$, thus $K * (\&X_1 \vee \&Y_1) = X_1$, thus due to Lemma 7, $K * (\&X_1 \vee \&Y_1) \not\subseteq K * \&Y_1$, thus $Y_1 \not\sqsubseteq X_1$.

Case 3, $n > 1$: Then $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = X_1 \cap \cdots \cap X_n$. Due to *extensionality* and *disjunctive factoring*, one of the following conditions holds:

(1) $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = K * (\&X_1 \vee \&Y_1)$
(2) $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = K * (\&X_2 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k)$
(3) $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = (K * (\&X_1 \vee \&Y_1)) \cap (K * (\&X_2 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k))$

In (1) we note that $\neg(\&X_1 \vee \&Y_1) \notin K * \&X_1$ due to *strong consistency* and $\&X_1 \in K * \&X_1$. It follows from *strong regularity* that $\&X_1 \vee \&Y_1 \in K * (\&X_1 \vee \&Y_1)$. Due to *closure* and $\&X_1 \vee \&Y_1 \vdash \neg\&X_2$ we obtain $\neg\&X_2 \in K * (\&X_1 \vee \&Y_1)$, which is impossible since $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = X_1 \cap \cdots \cap X_n$.

In (2), we similarly obtain $\neg\&X_1 \in K * (\&X_2 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k)$, which is impossible since $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = X_1 \cap \cdots \cap X_n$. Thus this case must also be excluded.

In the remaining case (3), suppose for contradiction that $Y_1 \sqsubseteq X_1$. Since $\sqsubseteq$ is a relation on $\mathfrak{R}$, it follows that $Y_1 \in \mathfrak{R}$ and thus $\&Y_1 \in K * \&Y_1$. Due to *disjunctive factoring* and $Y_1 \sqsubseteq X_1$, $K * (\&X_1 \vee \&Y_1)$ is equal to either $(K * \&X_1) \cap (K * \&Y_1)$ or $K * \&Y_1$. In both cases, it follows from (3) that $\neg\&Y_1 \notin K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k)$, contrary to $K * (\&X_1 \vee \cdots \vee \&X_n \vee \&Y_1 \vee \cdots \vee \&Y_k) = X_1 \cap \cdots \cap X_n$. It follows from this contradiction that $Y_1 \not\sqsubseteq X_1$. □

PROOF OF OBSERVATION 2: *Part 1, From construction to postulates.* Let $*$ satisfy the AGM postulates, and let $*'$ be the AGM^C operation derived from it according to Definition 6.

Closure of $*'$ follows from the closure property of $*$ when $a \nvdash \perp$ and from the logical closure of K when $a \vdash \perp$.

Consistent success of $*'$ follows from the success property of $*$.

Relative success of $*'$ follows from the success property of $*$.

Inclusion of $*'$ follows from the inclusion property of $*$ when $a \nvdash \perp$. For $a \vdash \perp$, we have $K *' a = K \subseteq \text{Cn}(\{\perp\}) = \text{Cn}(K \cup \{a\})$.

Strong consistency of $*'$: If $a \nvdash \perp$, then $K *' a \nvdash \perp$ follows from the consistency property of $*$. Otherwise, it follows from the consistency of K.

Extensionality of $*'$: Let $\vdash a_1 \leftrightarrow a_2$. If $a_1 \nvdash \perp$, then $a_2 \nvdash \perp$, and *extensionality* of $*'$ follows from the extensionality property of $*$. If If $a_1 \vdash \perp$, then $a_2 \vdash \perp$, and $K *' a_1 = K = K *' a_2$.

Vacuity of $*'$: Let $\neg a \notin K$. Since K is logically closed, $a \nvdash \perp$, thus $K *' a = K * a$, and *vacuity* of $*'$ follows from the corresponding property of $*$.

Disjunctive factoring of $*'$: Case 1, $a \vdash \perp$ and $b \vdash \perp$: Then $a \vee b \vdash \perp$, and $K *' (a \vee b) = K *' a = K *' b = K$.

Case 2, $a \vdash \perp$ and $b \nvdash \perp$: Then $\vdash a \vee b \leftrightarrow b$, and *extensionality* of $*'$, which was proved above, yields $K *' (a \vee b) = K *' b$.

Case 3, $a \nvdash \perp$ and $b \nvdash \perp$: Then $a \vee b \nvdash \perp$, and *disjunctive factoring* of $*'$ follows from the corresponding property of $*$.

Part 2, from postulates to construction: Let $*$ be an operation on K that satisfies the AGM^C postulates. Let $\bar{*}$ be the operation such that:

(a) If $a \nvdash \perp$, then $K \bar{*} a = K * a$, and

(b) If $a \vdash \perp$, then $K \bar{*} a = \text{Cn}(\{\perp\})$,

It follows directly that $*$ is derivable from $\bar{*}$ in the manner of Definition 6. It remains to show that $\bar{*}$ satisfies the AGM postulates.

Closure of $\bar{*}$ follows from closure of $*$ if $a \nvdash \perp$ and directly from clause (b) if $a \vdash \perp$.

Success of $\bar{*}$ follows from *consistent success* of $*$ if $a \nvdash \perp$ and directly from clause (b) if $a \vdash \perp$.

Inclusion of $\bar{*}$ follows from *inclusion* of $*$ if $a \nvdash \perp$. If $a \vdash \perp$, then $K \bar{*} a = \text{Cn}(\{\perp\}) = \text{Cn}(K \cup \{a\})$.

Vacuity: Let $\neg a \notin K$. Since K is consistent, $a \nvdash \perp$, and *vacuity* of $\bar{*}$ follows from the *vacuity* of $*$.

Consistency: Let $a \nvdash \perp$. Then $K \bar{*} a = K * a$. *Strong consistency* of $*$ yields $K * a \nvdash \perp$, thus $K \bar{*} a \nvdash \perp$.

Extensionality: Let $\vdash a_1 \leftrightarrow a_2$. If $a_1 \nvdash \perp$, then $a_2 \nvdash \perp$, thus $K \bar{*} a_1 = K * a_1$ and $K \bar{*} a_2 = K * a_2$. The extensionality of $*$ yields $K * a_1 = K * a_2$, thus $K \bar{*} a_1 = K \bar{*} a_2$. If $a_1 \vdash \perp$, then $a_2 \vdash \perp$, thus $K \bar{*} a_1 = \text{Cn}(\{\perp\}) = K \bar{*} a_2$.

Conjunctive factoring: Case 1, $a_1 \vdash \perp$ and $a_2 \vdash \perp$. Then $a_1 \vee a_2 \vdash \perp$, and $K \bar{*} a_1 = K \bar{*} a_2 = K \bar{*} (a_1 \vee a_2) = \text{Cn}(\{\perp\})$.

Case 2, $a_1 \vdash \perp$ and $a_2 \nvdash \perp$. Then $\vdash a_1 \vee a_2 \leftrightarrow a_2$. *Extensionality* of $*$ yields $K * a_2 = K * (a_1 \vee a_2)$, thus $K \bar{*} a_2 = K \bar{*} (a_1 \vee a_2)$

Case 3, $a_1 \nvdash \perp$ and $a_2 \nvdash \perp$. Then $a_1 \vee a_2 \nvdash \perp$, thus $K \bar{*} a_1 = K * a_1$, $K \bar{*} a_2 = K * a_2$ and $K \bar{*}(a_1 \vee a_2) = K * (a_1 \vee a_2)$. *Conjunctive factoring* for $\bar{*}$ follows from *conjunctive factoring* for $*$. $\square$

PROOF OF OBSERVATION 3: *From (1) to (2)*: Let $a \nvdash \perp$. *Consistent success* yields $a \in K * a$, thus according to Definition 2, $(\mathfrak{p} \star {}_\delta a)(a) \approx 1$, thus according to Definition 1, $\mathfrak{p}(a) \neq 0$.

From (2) to (1): Let $a \nvdash \perp$. Then $\mathfrak{p}(a) \neq 0$. If $\mathfrak{p}(a) = 1$, then it follows from the first clause of Definition 1 that $(\mathfrak{p} \star {}_\delta a)(a) = 1$, If $0 \neq \mathfrak{p}(a) \neq 1$, then it follows from the second clause of the same definition that $(\mathfrak{p} \star {}_\delta a)(a) \approx 1$. In both cases, it follows from $(\mathfrak{p} \star {}_\delta a)(a) \approx 1$ and Definition 2 that $a \in K * a$.

From (1) to (3): With the exception of *consistent success*, all the axioms characterizing an AGM^C according to Observation 2 were shown in Theorem 1 to be satisfied by hyperreal Bayesian top revisions.

From (3) to (1): According to Observation 2, AGM^C revision satisfies *consistent success* and seven of the eight axioms characterizing hyperreal Bayesian top revisions according to Theorem 1. It remains to show that it satisfies the remaining of these axioms, namely *strong regularity*. Let $*$ be an AGM^C revision and let $\neg b \notin K * a$. Due to *closure*, $b \nvdash \perp$. It follows from *consistent success* that $b \in K * b$. $\square$

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