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A Class of Implicative Expansions of Belnap-Dunn Logic in which Boolean Negation is Definable

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Abstract
Belnap and Dunn's well-known 4-valued logic FDE is an interesting and useful non-classical logic. FDE is defined by using conjunction, disjunction and negation as the sole propositional connectives. Then the question of expanding FDE with an implication connective is of course of great interest. In this sense, some implicative expansions of FDE have been proposed in the literature, among which Brady's logic BN4 seems to be the preferred option of relevant logicians. The aim of this paper is to define a class of implicative expansions of FDE in whose elements Boolean negation is definable, whence strong logics such as the paraconsistent and paracomplete logic PŁ4 and BN4 itself are definable, in addition to classical propositional logic.

Keywords Belnap-Dunn logic · Implicative expansions of Belnap-Dunn logic · Boolean negation · Two-valued Belnap-Dunn semantics

1 Introduction

The aim of this paper is to define a class of implicative expansions of Belnap-Dunn well-known 4-valued logic in which Boolean negation and, consequently, strong logics such as E4, BN4, PŁ4 and classical propositional logic are definable. We shall focus more on the functional strength of the elements in this class than on the (in many cases) interesting properties sported by the characteristic implication of some or other of said implicative expansions.

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As it is well-known, Belnap and Dunn's *useful four-valued logic* or *first-degree entailment logic* FDE is a "particularly interesting and useful" (cf. [25, p. 1021]) non-classical logic. FDE can be viewed as a 4-valued logic in which wffs (formulas) can be both true and false or neither true nor false, in addition to being only true or only false (cf. [5, 6, 12–14, 25–27]).

FDE is defined in the language $\{\wedge, \vee, \sim\}$ (cf. Definition 2.1) but some implicative expansions of it have been given in the literature (cf. [4, 7, 18, 19, 23, 25–28] and references in the last four items). Among these, Brady's 4-valued logic BN4 (cf. [7]) seems to be regarded in the relevant area as the adequate 4-valued expansion of FDE (cf., e.g., [24, p. 25] or [35, p. 289]).[1]

In this context, the logic E4 is proposed in [28] as the entailment counterpart to BN4 in the sense that E4 is related to BN4 similarly as Anderson and Belnap's E (*Entailment*) is related to their logic R (*Relevance*) (cf. [1] about E and R). Furthermore, in pp. 852-853 of the quoted paper [28], three alternatives to BN4 (along with another three to E4) are summarily discussed together with the question whether one of these options might be preferable to BN4 and/or E4. This question is settled in [33], where its proven that BN4 (resp., E4) and its three alternatives are functionally equivalent logics, whereas BN4 is functionally included in E4, but not conversely. In [33], it is then concluded that E4 is, to some extent, a preferable logic to BN4 as everything that can be done with the latter can be done with the former, which has a greater expressive power, in addition (cf., however, the end of Sections 4 and 7). But be it as it may, let us now enunciate the aims of the present paper.

Let us name $\mathcal{FOUR}$ the matrix determining the logic FDE (cf. Definitions 2.1, 2.2). We define a class MI4^{C} of implicative expansions of $\mathcal{FOUR}$, each element of which is implicative in the sense that the $f_{\rightarrow}$-function defining the connective $\rightarrow$ has the ensuing properties.

1. It is C-extending, that is, is coincides with (the $f_{\rightarrow}$-function for) the classical conditional when restricted to the "classical" truth-values **f** and **t** (cf. Definition 2.2).[2]
2. It satisfies the *modus ponens*.
3. It satisfies the *self-identity* axiom $A \rightarrow A$ (cf. Definition 2.1).

Now, as pointed out above, the purpose of the paper does not center on the characteristic implications of the members of MI4^{C}, but in some aspects of the functional strength of MI4^{C} as a whole. In particular, below it is proved that each element M of MI4^{C} enjoys the two following properties, among others:

4. The logic E4 (and so, BN4) is definable in M.

And, most of all,

[1] A referee of the the Journal of Philosophical Logic remarks: "This is a pretty strong statement. I think that getting *any* kind of consensus from the "relevant area" is wishful thinking, much less agreement that some logic is the "adequate expansion"."

[2] A referee of the Journal of Philosophical Logic remarks: "The "C-extending" property was introduced by Carnielli, Marcos and de Amo in an earlier paper as "hyperclassicality." We suppose the referee refers to [10].

5. Boolean negation is definable in M where by "Boolean negation" we can understand any of the four possibilities considered in [11, p. 833], and, in particular, "the unique classical negation in the four-valued setting" ([11, p. 833]).

Of course, from (5) it follows that M includes classical propositional logic, whence, in its turn, it follows that M includes the logic PŁ4 introduced in [21] and, according to [16], equivalent to De and Omori's logic BD_+, Zaitsev's paraconsistent logic FDEP and Béziau's 4-valued logic PM4N (cf. [16], [32] and references therein). Thus, PŁ4 is a very important 4-valued logic that can be regarded as an implicative expansion of FDE since it is a negation expansion of classical implicative logic $C_\rightarrow$ in which FDE is definable (the negation expanding $C_\rightarrow$ is the characteristic negation of FDE). We note that the logic HSE4 defined in [2] is a definitionally equivalent logic to PŁ4 (it is defined in the language $\{\supset, \wedge, \vee, \neg\}$ where, in said work, $\neg$ represents the characteristic FDE negation).

Independently of those that the characteristic $f_\rightarrow$-function of M can have, we think that the properties M enjoys remarked above make of it a very interesting implicative expansion of $\mathcal{FOUR}$.

The structure of the paper is as follows. In Section 2, the class $MI4^C$ of implicative expansions of $\mathcal{FOUR}$ is defined. It is required that each $f_\rightarrow$-function in $MI4^C$ be such that $f_\rightarrow(\mathbf{n}, \mathbf{n}) = \mathbf{b}$, $f_\rightarrow(\mathbf{b}, \mathbf{b}) = \mathbf{t}$ and $f_\rightarrow(\mathbf{b}, \mathbf{f}) = \mathbf{n}$ or $f_\rightarrow(\mathbf{t}, \mathbf{b}) = \mathbf{n}$, in addition to being a C-extending function verifying the *modus ponens* and the *self-identity* axiom. In Section 3, it is shown that Boolean negation is definable in each member M of $MI4^C$, whence it follows that the material implication is also definable in M. In Section 4, it is proved that the matrices MBN4 and ME4 determining the logics BN4 and E are definable in each M in $MI4^C$. (As noted in Section 1, BN4 is viewed as the correct 4-valued logic in the relevance logic area[3], but it is functionally included in E4, its "entailment counterpart".) In Section 5, it is shown how to give Hilbert-formulations (H-formulations) to the logic LM determined by the matrix M in $MI4^C$. Interestingly, each H-formulation of LM presents it as an expansion of classical propositional logic. The strategy (based on [7] as developed in, e.g., [22, 23, 28]) uses a two-valued Belnap-Dunn semantics equivalent to the matrix semantics definable upon each element in $MI4^C$. In Section 6, the class $MI4^C$ is restricted in order that its members verify the rule (or the axiom, as the case may be) *contraposition* and the rule *transitivity*, since many of the members in $MI4^C$ do not satisfy these rules and/or axiom. And, although as pointed out above, the focus of the paper is on the functional strength of $MI4^C$, it also seemed interesting to select subclasses of $MI4^C$ whose elements would present stronger implication functions. We provide Hilbert-formulations for the logics determined by a couple of matrices resulting from the restrictions referred to above. In Section 7, the paper is ended with some concluding remarks on the results obtained and on the possible future work to be made on the topic.

As it has been indicated above, FDE is defined in the language $\{\wedge, \vee, \sim\}$. Then, the question of expanding it with an implication connective is of course of great interest. In this sense, some implicative expansions of FDE have been proposed in the

[3]Cf. Note 1

literature, the most important of which may be BN4, E4, PŁ4, BD_+, PM4N, FDEP and HSE4. But BN4 is functionally included in E4 (though not conversely) whereas the remaining 5 logics are equivalent, as advertised in the precedent lines. Moreover. it has also been signaled that all the logics just quoted are functionally definable in each M in $MI4^C$. This last fact can suggest that each logic LM is, to certain extent, superior to the referred logics, since everything that can be done with the latter ones can be done with LM, which has in principle greater expressive power. In this regard, it is more than probable that the characteristic implication of some LM logic or other will be useful in some sense or another, besides being capable of defining the strong logics mentioned above (cf. Section 7 below).

To the best of our knowledge, the present paper introduces a class of implicative expansions of Belnap-Dunn logic in which Boolean negation (so, classical propositional logic) and other strong logics are definable, for the first time in the literature, a few specific instances of such type of expansions (all of them definable in each M in $MI4^C$) being at our disposal until now.

2 The Class $MI4^C$ of Implicative Expansions of $\mathcal{FOUR}$

In this section, we define the class $MI4^C$ of matrices (the label $MI4^C$ intends to abbreviate "implicative matrices expanding $\mathcal{FOUR}$ in which a Boolean —classical— negation is definable"). We begin by stating some prior concepts.

Definition 2.1 (Some preliminary notions) The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ...$, and some or all of the following connectives: $\to$ (conditional or implication[4]), $\wedge$ (conjunction), $\vee$ (disjunction) and $\sim$ (negation). The biconditional ($\leftrightarrow$) and the set of formulas (wffs) are defined in the customary way. A, B, C, etc. are metalinguistic variables. Then the ensuing concepts are understood in a fairly standard sense: logical matrix M, M-interpretation, M-consequence and M-validity. Also, the following notions: functions definable in a matrix, functional inclusion and functional equivalence (cf., e.g., [30, Section 2] or [31]).

As suggested in the introduction, in this paper, logics are primarily viewed as M-determined structures, i.e., as structures of the type $(\mathcal{L}, \vDash_M)$ where $\mathcal{L}$ is a propositional language and $\vDash_M$ is a (consequence) relation defined in $\mathcal{L}$ according to the logical matrix M as follows: for any set of wffs Γ and wff A, $\Gamma \vDash_M A$ iff $I(A) \in D$ whenever $I(\Gamma) \in D$ for all M-interpretations I ($I(\Gamma) \in D$ iff $I(A) \in D$ for all $A \in \Gamma$; D is the set of designated values in M). Thus, from this viewpoint, we can safely travel back and forth from matrices to logics, given the aims of this paper.

[4]We follow Anderson and Belnap's "Grammatical Propaedeutic", Appendix to [1]: "The principal aim of this piece is to convince the reader that it is philosophically respectable to "confuse" implication and entailment with the conditional, and indeed philosophically suspect to harp on the dangers of such "confusion" ([1, p. 473].

Nevertheless, logics are sometimes defined as Hilbert-type axiomatic systems, the notions of "theorem" and "proof from premises" being the usual ones. Furthermore, in a derived or secondary sense, we can regard an M-determined logic as a, say, Hilbert-type system (or a natural deduction system or a Gentzen-type system) L such that $\Gamma \vdash_{\mathrm{L}} A$ iff $\Gamma \vDash_{\mathrm{M}} A$, where $\vDash_{\mathrm{M}}$ is the consequence relation defined above and $\Gamma \vdash_{\mathrm{L}} A$ means "A is provable from Γ in L".

Definition 2.2 (Belnap and Dunn's matrix $\mathcal{FOUR}$) The propositional language consists of the connectives $\wedge$, $\vee$ and $\sim$. Belnap and Dunn's matrix $\mathcal{FOUR}$ is the structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V}$ is $\{\mathbf{f}, \mathbf{n}, \mathbf{b}, \mathbf{t}\}$ and is partially ordered as shown in Lattice 1 (cf. Fig. 1).

(2) $D = \{\mathbf{b}, \mathbf{t}\}$; $\mathtt{F} = \{f_\wedge, f_\vee, f_\sim\}$ where $f_\wedge$ and $f_\vee$ are defined as the glb (or lattice meet) and the lub (or lattice joint), respectively. Finally, $f_\sim$ is an involution with $f_\sim(\mathbf{f}) = \mathbf{t}$, $f_\sim(\mathbf{t}) = \mathbf{f}$, $f_\sim(\mathbf{n}) = \mathbf{n}$, $f_\sim(\mathbf{b}) = \mathbf{b}$ (cf. [5, 6, 12–14]). We display the tables for $\wedge$, $\vee$ and $\sim$:

$\wedge$	**f**	**n**	**b**	**t**
f	**f**	**f**	**f**	**f**
n	**f**	**n**	**f**	**n**
b	**f**	**f**	**b**	**b**
t	**f**	**n**	**b**	**t**

$\vee$	**f**	**n**	**b**	**t**
f	**f**	**n**	**b**	**t**
n	**n**	**n**	**t**	**t**
b	**b**	**t**	**b**	**t**
t	**t**	**t**	**t**	**t**

	$\sim$
f	**t**
n	**n**
b	**b**
t	**f**

Remark 2.3 (On the meaning of the symbols for referring to the four truth-values) The symbols **f**, **n**, **b** and **t** stand for false only, neither true nor false, both true and false and true only, respectively.

Next, we proceed to define the class MI4$^{\mathrm{C}}$. As pointed out in the preceding section, any $f_\rightarrow$-function in MI4$^{\mathrm{C}}$ needs to have at least the ensuing properties: (1) It is a C-extending $f_\rightarrow$-function. (An $f_\rightarrow$-function is C-extending if it coincides with (the $f_\rightarrow$-function for) the classical conditional when restricted to the "classical" values **f** and **t**.) (2) It satisfies the *modus ponens*. (3) It is such that $f_\rightarrow(\mathbf{n}, \mathbf{n}) = \mathbf{b}$ and $f_\rightarrow(\mathbf{b}, \mathbf{b}) = \mathbf{t}$. (4) It is such that $f_\rightarrow(\mathbf{b}, \mathbf{f}) = \mathbf{n}$ or $f_\rightarrow(\mathbf{t}, \mathbf{b}) = \mathbf{n}$. (Notice that the conditions (1) and (3) conjointly taken guarantee that the *self-identity* axiom $A \rightarrow A$ is satisfied by all members in MI4$^{\mathrm{C}}$.)

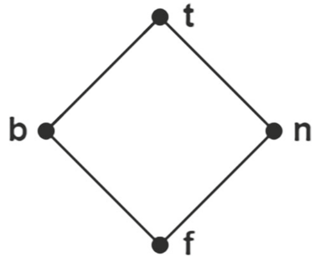

Fig. 1 Lattice 1

Thus, we set:

Definition 2.4 ($f_{\rightarrow}$-functions complying with (1)-(4)) The $4^7 \times 5$ implicative truth-tables describing all $f_{\rightarrow}$-functions fulfilling conditions (1)-(4) enunciated above are contained in the general tables TI-TV displayed below (blank spaces can be filled with no matter which truth-values in $\mathcal{FOUR}$; $b_1, b_2 \in \{\mathbf{f}, \mathbf{n}\}$).[5]

TI

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**			**t**
n		**b**		
b	**f**	b_1	**t**	
t	**f**	b_2	**n**	**t**

TII

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**			**t**
n		**b**		
b	**n**	b_1	**t**	
t	**f**	b_2	**n**	**t**

TIII

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**			**t**
n		**b**		
b	**n**	b_1	**t**	
t	**f**	b_2	**f**	**t**

TIV

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**			**t**
n		**b**		
b	**n**	b_1	**t**	
t	**f**	b_2	**b**	**t**

TV

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**			**t**
n		**b**		
b	**n**	b_1	**t**	
t	**f**	b_2	**t**	**t**

Definition 2.5 (The class $\mathrm{MI4^C}$) The class $\mathrm{MI4^C}$ consists of the implicative expansions of $\mathcal{FOUR}$ defined as follows. Each implicative expansion M is the structure $(\mathcal{V}, D, \mathtt{F})$, where $\mathcal{V}$, D, $f_{\wedge}$, $f_{\vee}$ and $f_{\sim}$ are defined exactly as in $\mathcal{FOUR}$ (Definition 2.2) and $f_{\rightarrow}$ is defined according to one of the $4^7 \times 5$ different implication truth-tables described in the general tables TI, TII, TIII, TIV and TV.

Remark 2.6 ($\mathrm{MI4^C_I}$-$\mathrm{MI4^C_V}$) We shall generally refer by $\mathrm{MI4^C_I}$ (resp., $\mathrm{MI4^C_{II}}$, $\mathrm{MI4^C_{III}}$, $\mathrm{MI4^C_{IV}}$, $\mathrm{MI4^C_V}$) to the members of $\mathrm{MI4^C}$ built from the tables in $\mathrm{MI4^C_I}$ (resp., $\mathrm{MI4^C_{II}}$, $\mathrm{MI4^C_{III}}$, $\mathrm{MI4^C_{IV}}$, $\mathrm{MI4^C_V}$).

3 Definability of Boolean Negation and Material Implication

In this section, Boolean and material implication are defined in each element of $\mathrm{MI4^C}$. Firstly, we note a remark on the proofs to follow. Then, we define four additional negation connectives.

[5] A referee of the Journal of Philosophical Logic remarks: "I'd take care here to prevent the reader from thinking that the functions described by these matrices are partial functions."

Remark 3.1 (Functions and truth-tables. On displaying proofs of definability) Let f_* be a function defined in $\mathcal{V} = \{\mathbf{f}, \mathbf{n}, \mathbf{b}, \mathbf{t}\}$. In this paper, f_* is usually represented by means of a truth-table t_* (or simply $*$), as for instance, it is the case with $\wedge$, $\vee$ and $\sim$ in $\mathcal{FOUR}$ (Definition 2.2). In addition, by k_* (or simply $*$) we refer to the connective defined by t_*. Now, let M be $\mathcal{FOUR}$ or an expansion of it. The proof that a given unary or binary function f_* is definable in M is easily visualized by using the connectives corresponding to the functions in M needed in the proof in question. In general, proofs provided below are simplified as just indicated (A, B refer to any wffs —cf. Definition 2.1) On the other hand, in order to prove that a certain matrix is functionally included in another one, it is clear that it suffices to show that the implication table of the former is definable in the latter, given that we treat only implicative expansions of $\mathcal{FOUR}$. Finally, from now on, by "definable in $\mathrm{MI4}^{\mathrm{C}}$ (resp., $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{II}}$, $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{III}}$, $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{IV}}$, $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{V}}$)", we mean "definable in all members in $\mathrm{MI4}^{\mathrm{C}}$ (resp., $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{II}}$, $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{III}}$, $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{IV}}$, $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{V}}$)". (This convention can also be used w.r.t. other general tables to be introduced in what follows. In case a tester is needed, the one in [15] can be used.)

Proposition 3.2 (The negation connectives $\overset{\bullet}{\neg}$, $\overset{\circ}{\neg}$, $\overset{\square}{\neg}$ and $\overset{\circledast}{\neg}$) *Consider the negation connectives* $\overset{\bullet}{\neg}$, $\overset{\circ}{\neg}$, $\overset{\square}{\neg}$ *and* $\overset{\circledast}{\neg}$, *given by the truth tables:*

	$\overset{\bullet}{\neg}$	$\overset{\circ}{\neg}$	$\overset{\square}{\neg}$	$\overset{\circledast}{\neg}$
f	**t**	**t**	**t**	**t**
n	**b**	**b**	**t**	**t**
b	**t**	**f**	**b**	**t**
t	**f**	**f**	**f**	**f**

The four connectives are definable in $\mathit{MI4}^{C}$

Proof We set $\overset{\bullet}{\neg}A =_{\mathrm{df}} A \to \sim A$; $\overset{\circ}{\neg}A =_{\mathrm{df}} \sim(\sim A \to A)$; $\overset{\square}{\neg}A =_{\mathrm{df}} \overset{\circ}{\neg}A \vee \sim A$; $\overset{\circledast}{\neg}A =_{\mathrm{df}} \overset{\bullet}{\neg}A \vee \sim A$ ($\to$ is the conditional given by any truth-table in $\mathrm{MI4}^{\mathrm{C}}$). □

Net, we proceed to define Boolean negation in $\mathrm{MI4}^{\mathrm{C}}$.

Proposition 3.3 (Boolean negation in $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{I}}$ and $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{II}}$) *Boolean negation* $\neg$ *as given by the truth table*

	$\neg$
f	**t**
n	**b**
b	**n**
t	**f**

is definable in $\mathit{MI4}^{C}_{I}$ *and* $\mathit{MI4}^{C}_{II}$.

Proof We set $\neg A =_{df} \sim(\overset{\circledast}{\neg} A \rightarrow \overset{\Box}{\neg} A) \vee \overset{\circ}{\neg} A$ ($\rightarrow$ is the conditional given by any of the truth-tables in $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{I}}$ and $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{II}}$). □

Proposition 3.4 (Boolean negation in $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{III}}$, $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{IV}}$ and $\mathrm{MI4}^{\mathrm{C}}_{\mathrm{V}}$) *Boolean negation* $\neg$ *as given by the same truth-table as in the preceding proposition is definable in* $MI4^{C}_{III}$, $MI4^{C}_{IV}$ *and* $MI4^{C}_{V}$.

Proof (1) $f_{\rightarrow}$-functions such that $f_{\rightarrow}(\mathbf{n}, \mathbf{b}) \in \{\mathbf{f}, \mathbf{n}\}$. We set $\neg A =_{df} [\sim(\sim A \rightarrow \overset{\circ}{\neg} A) \vee \overset{\circ}{\neg} A] \wedge \overset{\bullet}{\neg} A$. (2) $f_{\rightarrow}$-functions such that $f_{\rightarrow}(\mathbf{n}, \mathbf{b}) \in \{\mathbf{b}, \mathbf{t}\}$. We set $\neg A =_{df} (\sim A \rightarrow \overset{\circ}{\neg} A) \wedge \overset{\bullet}{\neg} A$. □

Although, as pointed out in the introduction to the paper, De and Omori think that $\neg$ represents "the unique classical negation in the four-valued setting" ([11, p. 833]), they consider three alternatives to it: in addition to $\overset{\circ}{\neg}$, the connectives given by the ensuing truth-tables.

	$\overset{\oplus}{\neg}$	$\overset{\boxplus}{\neg}$
f	**t**	**t**
n	**t**	**t**
b	**f**	**n**
t	**f**	**f**

We have:

Proposition 3.5 ($\overset{\oplus}{\neg}$ and $\overset{\boxplus}{\neg}$ are definable in $\mathrm{MI4}^{\mathrm{C}}$) *The negation connectives* $\overset{\oplus}{\neg}$, $\overset{\boxplus}{\neg}$, *given by the truth-tables displayed above, are definable in* $MI4^{C}$.

Proof (1) Connective $\overset{\oplus}{\neg}$. (1a) $f_{\rightarrow}$-functions such that $f_{\rightarrow}(\mathbf{t}, \mathbf{b}) = \mathbf{n}$. We set $\overset{\oplus}{\neg} A =_{df} (\overset{\circledast}{\neg} A \rightarrow \overset{\Box}{\neg} A) \wedge \overset{\Box}{\neg} A$. (1b) $f_{\rightarrow}$-functions such that $f_{\rightarrow}(\mathbf{b}, \mathbf{f}) = \mathbf{n}$. We set $\overset{\oplus}{\neg} A =_{df} [[\overset{\bullet}{\neg} A \rightarrow (\sim A \wedge \neg A)] \wedge \overset{\Box}{\neg} A] \vee \overset{\circ}{\neg} A$. (2) Connective $\overset{\boxplus}{\neg}$. We set $\overset{\boxplus}{\neg} A =_{df} \overset{\oplus}{\neg} A \vee \neg A$. □

Turning to material implication, we recall that Omori and Wansing [25, p. 1036] note two versions of it given by the tables displayed below.

$\overset{b}{\rightarrow}$	**f**	**n**	**b**	**t**
f	**t**	**t**	**t**	**t**
n	**b**	**t**	**b**	**t**
b	**n**	**n**	**t**	**t**
t	**f**	**n**	**b**	**t**

$\overset{e}{\rightarrow}$	**f**	**n**	**b**	**t**
f	**t**	**t**	**t**	**t**
n	**t**	**t**	**t**	**t**
b	**f**	**n**	**b**	**t**
t	**f**	**n**	**b**	**t**

where b (resp., e) abbreviates "Boolean" (resp., "exclusion"), Boolean and exclusion being negations given by the connectives $\neg$ and $\overset{\oplus}{\neg}$ defined above, respectively. Of course, they mean that $\overset{b}{\to}$ (resp., $\overset{e}{\to}$) can be defined by disjunction and Boolean (resp., exclusion) negation.

Proposition 3.6 ($\overset{b}{\to}$ and $\overset{e}{\to}$ are definable in MI4^{C}) *Material implication, as given by $\overset{b}{\to}$ or $\overset{e}{\to}$, is definable in MI4^{C}.*

Proof We set $A \overset{b}{\to} B =_{df} \neg A \vee B$; $A \overset{e}{\to} B =_{df} \overset{\oplus}{\neg} A \vee B$. □

The connectives $\overset{b}{\to}$ and $\overset{e}{\to}$ represent material implication in the sense that the respective truth-tables defining them verify classical implicative propositional logic (as, e.g., firstly defined by Łukasiewicz and Tarski [20]: $A \supset (B \supset A)$, $[A \supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)]$, $[(A \supset B) \supset A] \supset A$ and *modus ponens*), on the one hand, and on the other hand, that said tables do not verify any invalid classical implicative wff as the functions $f_{\overset{b}{\to}}$ and $f_{\overset{e}{\to}}$ are C-extending $f_{\to}$-functions (cf. the introduction to the paper). Concerning this question, let us remark that it is possible to give definitions of material implication by using the connectives $\overset{\circ}{\neg}$ and $\overset{\boxplus}{\neg}$.

Proposition 3.7 (Alternative definition of $\supset$) *Consider the following implicative tables.*

$\overset{\overset{\circ}{\neg}}{\to}$	f	n	b	t
f	t	t	t	t
n	b	t	b	t
b	f	n	b	t
t	f	n	b	t

$\overset{\overset{\boxplus}{\neg}}{\to}$	f	n	b	t
f	t	t	t	t
n	t	t	t	t
b	n	n	t	t
t	f	n	b	t

These tables are definable in MI4^{C} by putting $A \overset{\overset{\circ}{\neg}}{\to} B =_{df} \overset{\circ}{\neg} A \vee B$; $A \overset{\overset{\boxplus}{\neg}}{\to} B =_{df} \overset{\boxplus}{\neg} A \vee B$. Now, $\overset{\overset{\circ}{\neg}}{\to}$ and $\overset{\overset{\boxplus}{\neg}}{\to}$ represent material implication in the same sense as $\overset{b}{\to}$ and $\overset{e}{\to}$.

4 Definability of E4 and BN4

The logics BN4 and E4 briefly discussed in the introduction to the paper are determined by the implicative expansions of $\mathcal{FOUR}$ defined when adding to it the $f_{\to}$-functions described by the ensuing truth-tables.

$\overset{BN4}{\to}$	f	n	b	t
f	t	t	t	t
n	n	t	n	t
b	f	n	b	t
t	f	n	f	t

$\overset{E4}{\to}$	f	n	b	t
f	t	t	t	t
n	f	b	f	t
b	f	f	b	t
t	f	f	f	t

As recalled in Section 1, BN4 is regarded as the adequate 4-valued implicative logic, in the relevant area. Nevertheless, in [33], it is shown that BN4 is definable in E4, though not conversely. Below, it is shown that E4 (and so, BN4) is definable in MI4^{C}.

Proposition 4.1 (E4 is definable is MI4^{C}) *The logic E4 (so, also BN4) is definable in* MI4^{C}.

Proof It suffices to show that the characteristic implicative table of E4 is definable in MI4^{C}. Consider then the unary connective ∇ and the implicative connectives $\overset{1}{\to}$ and $\overset{2}{\to}$ as given by the truth-tables below.

	∇
f	**f**
n	**f**
b	**b**
t	**f**

$\overset{1}{\to}$	**f**	**n**	**b**	**t**
f	**t**	**t**	**t**	**t**
n	**f**	**t**	**f**	**t**
b	**f**	**f**	**b**	**t**
t	**f**	**f**	**f**	**t**

$\overset{2}{\to}$	**f**	**n**	**b**	**t**
f	**t**	**t**	**t**	**t**
n	**f**	**b**	**n**	**t**
b	**f**	**b**	**n**	**t**
t	**f**	**b**	**n**	**t**

These connectives are definable in MI4^{C} as follows. $\nabla A =_{\text{df}} \sim(\overset{\circ}{\neg} A \vee \sim A) \wedge \overset{\square}{\neg} A$; $A \overset{1}{\to} B =_{\text{df}} [(\neg A \vee B) \wedge (\sim B \overset{\text{b}}{\to} \sim A)] \wedge (\overset{\circ}{\neg} A \vee B)$; $A \overset{2}{\to} B =_{\text{df}} (\neg A \wedge \sim A) \vee \sim\neg B$. Then we have $A \overset{\text{E4}}{\to} B =_{\text{df}} \nabla(A \overset{1}{\to} B) \vee [(A \overset{2}{\to} B) \wedge (\overset{\circ}{\neg} A \vee B)]$. □

Thus, we see, the logic, let us name it L, built upon an arbitrary matrix in MI4^{C} contains, among other logics, classical propositional logic, Cp, E4 and BN4. So L is, to some extent, superior to the logics it contains in the sense that anything that can be done with these logics can also be done with L, which has more expressive power, in addition. Nevertheless, of course, it does not mean that it is advisable or convenient to drop Cp, E4 or BN4 in favor of L: one of these logics may have properties we require for some purpose or another. For example, BN4 is a strong 4-valued extension of contractionless relevant logic R, while E4 is a strong 4-valued extension of reductioless entailment logic E, which can be axiomatized by using *modus ponens* (for the E4-conditional) and adjunction as the sole rules of inference (cf. [7, 28]).

Perhaps, we can elaborate on the question by taking an example from 3-valued logic. As it is known, Kleene's strong 3-valued matrix with two designated values, MK3_{II}, can be defined as follows.

Definition 4.2 (The matrix MK3_{II}) The propositional language is the same as in $\mathcal{FOUR}$. The matrix MK3_{II} is the structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V} = \{\mathbf{f}, \mathbf{b}, \mathbf{t}\}$ and it is ordered as shown in Lattice 2 (cf. Fig. 2).

(2) $D = \{\mathbf{b}, \mathbf{t}\}$; (3) $\mathtt{F} = \{f_\wedge, f_\vee, f_\sim\}$, where $f_\wedge$ and $f_\vee$ are defined as in $\mathcal{FOUR}$ and $f_\sim$ is an involution with $f_\sim(\mathbf{t}) = \mathbf{f}$, $f_\sim(\mathbf{f}) = \mathbf{t}$ and $f_\sim(\mathbf{b}) = \mathbf{b}$.

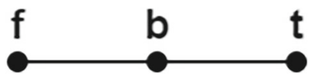

Fig. 2 Lattice 2

Then the logic Pac ("paraconsistency") is the logic determined by the implicative expansion of $\mathrm{MK3_{II}}$ built up by adding the $f_{\rightarrow}$-function described by the following table:

$\rightarrow$	**f**	**b**	**t**
f	**t**	**t**	**t**
b	**f**	**b**	**t**
t	**f**	**b**	**t**

Pac, maybe the most important paraconsistent 3-valued logic, has been defined independently by many authors (cf. [17] and references therein). It can be axiomatized by adding to classical positive propositional logic the following axioms: $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$, $\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$, $\sim(A \rightarrow B) \leftrightarrow (A \wedge \sim B)$, $A \leftrightarrow \sim\sim A$ and $A \vee \sim A$ (cf. [29]).

Now, in [30], it is shown that Pac together with other 26 natural implicative expansions of $\mathrm{MK3_{II}}$, one of them being the quasi-relevant logic RM3, can be defined from 27 additional implicative expansions of $\mathrm{MK3_{II}}$, among which Łukasiewicz's Ł3 or the paraconsistent logic $\mathrm{G3_{Ł}}$ are to be found, but not conversely. One of these expansions is the logic, let us name it L′, (also axiomatized in [29]) determined by the implicative expansion of $\mathrm{MK3_{II}}$ built by adding the $f_{\rightarrow}$-function described by the following table:

$\rightarrow$	**f**	**b**	**t**
f	**t**	**f**	**t**
b	**f**	**t**	**f**
t	**f**	**f**	**t**

But it is clear that it does not seem wise to drop Pac in favor of L′, a logic in which wffs such as $(A \wedge B) \rightarrow A$, $(A \wedge B) \rightarrow B$, $A \rightarrow (A \vee B)$ or $B \rightarrow (A \vee B)$ are not provable.

5 On Defining Hilbert-formulations of the L-logics

Let us generally refer by L-logics to the class of logics determined by the matrices in $\mathrm{MI4^C}$. In this section, we show how to use the fact that classical propositional logic is definable in each L-logic, in order to give easy Hilbert-style formulations (H-formulations) of the L-logics. The H-formulations we define present the L-logics as expansions of classical propositional logic.

As a way of an example, we take the logic, let us provisionally name it L1, determined by the matrix in MI4^{C}, let us provisionally name it M1, defined by using the $f_{\rightarrow}$-function described in the ensuing table:

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**	**t**	**t**	**t**
n	**f**	**b**	**n**	**t**
b	**n**	**n**	**t**	**t**
t	**f**	**f**	**t**	**t**

In order to give an H-formulation for L1, we rely upon a strategy based upon Belnap-Dunn two-valued semantics introduced by Brady in [7] (cf. also [8, 9, 34]), as illustrated in some papers such as [21–23] or [28].

As it is well-known, Belnap-Dunn two-valued semantics is characterized by the possibility of assigning T, F, both T and F or neither T nor F to the formulas of a given language (cf. [5, 6, 12–14]; T represents truth and F represents falsity).

Given M an implicative expansion of $\mathcal{FOUR}$ (cf. Definition 2.2), the idea for defining a BD-semantics, M$'$, equivalent to the matrix semantics based upon M is simple: a wff A is assigned T and F in M$'$ iff it is assigned **b** in M; A is assigned neither T nor F in M$'$ iff it is assigned **n** in M; finally A is assigned T but not F (resp., F but not T) in M$'$ iff it is assigned **t** (resp., **f**) in M.

Then below a BD-semantics for L1 is introduced by defining the notion of an L1-model and the accompanying notions of L1-consequence and L1-validity.

Definition 5.1 (L1-models) An L1-model is a structure (K, I) where (i) $K = \{\{T\}, \{F\}, \{T, F\}, \emptyset\}$, and (ii) I is an L1-interpretation from the set of all wffs to K, this notion being defined according to the following conditions ('clauses') for each propositional variable p and wffs A, B:

$$
\begin{aligned}
1.\;\; & I(p) \in K \\
2a.\;\; & T \in I(\sim A) \text{ iff } F \in I(A) \\
2b.\;\; & F \in I(\sim A) \text{ iff } T \in I(A) \\
3a.\;\; & T \in I(A \wedge B) \text{ iff } T \in I(A) \;\;\&\;\; T \in I(B) \\
3b.\;\; & F \in I(A \wedge B) \text{ iff } F \in I(A) \text{ or } F \in I(B) \\
4a.\;\; & T \in I(A \vee B) \text{ iff } T \in I(A) \text{ or } T \in I(B) \\
4b.\;\; & F \in I(A \vee B) \text{ iff } F \in I(A) \;\;\&\;\; F \in I(B) \\
5a.\;\; & T \in I(A \rightarrow B) \text{ iff } [T \notin I(A) \;\;\&\;\; F \in I(A)] \text{ or } \\
& [T \in I(B) \;\;\&\;\; F \notin I(B)] \text{ or } [T \notin I(A) \;\;\&\;\; F \notin I(B)] \text{ or } \\
& [F \in I(A) \;\;\&\;\; T \in I(B)] \text{ or } [T \in I(A) \;\;\&\;\; T \in I(B)] \\
5b.\;\; & F \in I(A \rightarrow B) \text{ iff } F \notin I(A) \;\;\&\;\; T \notin I(B).
\end{aligned}
$$

Definition 5.2 (L1-consequence, L1-validity) Let M be an L1-model. For any set of wffs Γ and wff A:

1. $\Gamma \vDash_M A$ (A is a consequence of Γ in M) iff $T \in I(A)$ whenever $T \in I(\Gamma)$. ($T \in I(\Gamma)$ iff $\forall A \in \Gamma(T \in I(A))$; $F \in I(\Gamma)$ iff $\exists A \in \Gamma(F \in I(A))$.)
2. $\Gamma \vDash_{L1} A$ (A is a consequence of Γ in L1-semantics) iff $\Gamma \vDash_M A$ for each L1-model M.
3. In particular, $\vDash_{L1} A$ (A is valid in L1-semantics) iff $\vDash_M A$ for each L_1-model M (i.e., iff $T \in I(A)$ for each L1-model M).

By $\vDash_{L1}$ we shall refer to the relation just defined.

Now, given Definition 2.5 together with the adjoined notions of M1-interpretation and M1-validity (cf. Definition 2.1) and Definitions 5.1 and 5.2. We easily prove:

Proposition 5.3 (Coextensiveness of $\vDash_{M1}$ and $\vDash_{L1}$) *For any set of wffs Γ and a wff A, $\Gamma \vDash_{M1} A$ iff $\Gamma \vDash_{L1} A$. In particular, $\vDash_{M1} A$ iff $\vDash_{L1} A$.*

Proof See the proof of Theorem 8 in [7] or Proposition 4.4 in [22] where the simple proof procedure is exemplified in the cases of the logics BN4 and Sm4, respectively. □

Proposition 5.3 simply formalizes the intuitive translation (explained above) of the matrix semantics based upon M1 into Belnap and Dunn's two-valued type L1-semantics. Nevertheless, it is a useful proposition, since it gives us the possibility of proving soundness of L1 w.r.t. $\vDash_{M1}$ while proving completeness w.r.t. $\vDash_{L1}$ by using a canonical model construction. But let us now define the H-system HL1. We use $\supset$ (as interpreted by table $\overset{b}{\to}$ —cf. Proposition 3.6), $\to$ and $\sim$ as primitive connectives.

Definition 5.4 (The system HL1) The system HL1 can be formulated as follows ($A_1, ...A_n \Rightarrow B$ means "if A_1, ..., A_n, then B").

Axioms:

$$A1.\quad A \supset (B \supset A)$$
$$A2.\quad A \supset (B \supset C) \supset [(A \supset B) \supset (A \supset C)]$$
$$A3.\quad (\neg A \supset \neg B) \supset (B \supset A)$$
$$A4.\quad (A \to B) \supset (A \supset B)$$
$$A5.\quad A \leftrightarrow \sim\sim A$$
$$A6.\quad \sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$$
$$A7.\quad \sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$$
$$A8.\quad \sim A \to [A \vee (A \to B)]$$
$$A9.\quad B \to [\sim B \vee (A \to B)]$$
$$A10.\quad (A \vee \sim B) \vee (A \to B)$$
$$A11.\quad [(A \vee \sim A) \wedge B] \supset (A \to B)$$
$$A12.\quad [(A \to B) \wedge \sim B] \supset [\sim A \vee (A \wedge B)]$$

$$A13. \quad (\sim A \vee B) \vee \sim(A \to B)$$
$$A14. \quad [\sim(A \to B) \wedge (\sim A \vee B)] \supset C$$

Rules:

$$\text{MP}_{\supset}.\ A \supset B, A \Rightarrow B$$

Definitions:

$$A \wedge B =_{\text{df}} \quad \neg(A \supset \neg B)$$
$$A \vee B =_{\text{df}} \quad \neg A \supset B$$
$$A \leftrightarrow B =_{\text{df}} \quad (A \to B) \wedge (B \to A)$$
$$\overset{\bullet}{\neg} A =_{\text{df}} \quad A \to \sim A$$
$$\overset{\circ}{\neg} A =_{\text{df}} \quad \sim(\sim A \to A)$$
$$\neg A =_{\text{df}} \quad A \supset \sim(\overset{\circ}{\neg} A \to \overset{\bullet}{\neg} A)$$

We note that the tables resulting from the definitions of $\wedge$, $\vee$ and $\neg$ are exactly those for $\wedge$ and $\vee$ in $\mathcal{FOUR}$ (cf. Definition 2.2) and the one for Boolean negation (cf. Proposition 3.3), respectively.

Below, we remark some properties of HL1.

Proposition 5.5 (Some rules of HL1) *The following are provable in HL1:*

1. *Deduction theorem (DT). If* $\Gamma, A \vdash_{HL1} B$, *then* $\Gamma \vdash_{HL1} A \supset B$.
2. *If A is a classical propositional tautology, then* $\vdash_{HL1} A$.
3. *Modus ponens for* $\to$ *(*$MP_{\to}$*).* $A \to B, A \Rightarrow B$.
4. *Adjunction.* $A, B \Rightarrow A \wedge B$.

Proof They are immediate. (1) By A1 and A2, since $\text{MP}_{\supset}$ is the sole rule of inference. (2) By A1, A2, A3 and $\text{MP}_{\supset}$, since, as known, the three axioms and the rule axiomatize classical propositional logic. (3) By A4 and $\text{MP}_{\supset}$. (4) By $A \supset [B \supset (A \wedge B)]$ and $\text{MP}_{\supset}$. □

In what follows, we proceed to the proofs of soundness and completeness.

Theorem 5.6 (Soundness of HL1) *For any set of wffs* Γ *and a wff A, if* $\Gamma \vdash_{HL1} A$ *then (1)* $\Gamma \vDash_{M1} A$ *and (2)* $\Gamma \vDash_{L1} A$.

Proof (1) It is immediate: the axioms of HL1 are M1-valid and $\text{MP}_{\supset}$ preserves M1-validity. (Recall that the material conditional is understood according to table $\overset{\text{b}}{\to}$; cf. Proposition 3.6; in case a tester is needed, the one in [15] can be used.) (On the other hand, the definitions can be understood as mere abbreviations. Nevertheless, they provide the right table for the respective connective, as pointed out before.) □

Concerning completeness, it is proved by a canonical model construction, as suggested above. Let us see how this proof proceeds. We begin by stating a couple of definitions.

Definition 5.7 (HL1-theories) An HL1-theory is a set of wffs containing all HL1-theorems and closed under $\mathrm{MP}_{\supset}$. An HL1-theory t is prime if whenever $A \vee B \in t$, then $A \in t$ or $B \in t$; and t is non-trivial if it does not contain all wffs.

Definition 5.8 (Canonical HL1-models) Let $\mathcal{T}$ be a non-trivial prime L1-theory. A canonical HL1-model is the structure $(K, I_{\mathcal{T}})$ where (i) K is defined as in Definition 5.1 and (ii) $I_{\mathcal{T}}$ is a function from the set of all wffs to K defined as follows: For each wff A, $T \in I_{\mathcal{T}}(A)$ iff $A \in \mathcal{T}$ and $F \in I_{\mathcal{T}}(A)$ iff $\sim A \in \mathcal{T}$.

Then, in order to prove completeness, we have to prove the ensuing two facts:

1. An HL1-theory without a given wff can be extended to a prime HL1-theory without the same wff.
2. Let $\mathcal{T}$ be a non-trivial prime HL1-theory. Then $I_{\mathcal{T}}$ (as defined in Definition 5.8) fulfills clauses (2a), (2b), (3a), (3b) (4a), (4b), (5a) and (5b) (it is immediate that $I_{\mathcal{T}}$ fulfills clause (1)). That is, we have to prove that the canonical translations of clauses (1) through (5b) are provable in $\mathcal{T}$.

We proceed to the proofs of facts 1 and 2.

Lemma 5.9 (Primeness) *Let A be a wff and t an HL1-theory such that $A \notin t$. Then there is a prime HL1-theory $\mathcal{T}$ such that $t \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$.*

Proof We extend t to a maximal HL1-theory $\mathcal{T}$ such that $A \notin \mathcal{T}$. If $\mathcal{T}$ is not prime, there are wffs B, C such that $B \vee C \in \mathcal{T}$ but $B \notin \mathcal{T}$ and $C \notin \mathcal{T}$. We then define the sets $[\mathcal{T}, B]$, $[\mathcal{T}, C]$ as follows: $[\mathcal{T}, B] = \{D \mid \exists E \in \mathcal{T}[\vdash_{\mathrm{HL1}} (B \wedge E) \supset D]\}$; $[\mathcal{T}, C]$ is defined similarly.

We show (I) $[\mathcal{T}, B]$ is closed under $\mathrm{MP}_{\supset}$ (the proof for $[\mathcal{T}, C]$ is similar). Suppose then (1) $\vdash_{\mathrm{HL1}} (B \wedge E) \supset (D \supset G)$ and (2) $\vdash_{\mathrm{HL1}} (B \wedge E') \supset D$ for some wffs D, G and $E, E' \in \mathcal{T}$. Obviously, we get (3) $\vdash_{\mathrm{HL1}} [B \wedge (E \wedge E')] \supset [(D \supset G) \wedge D]$, whence (4) $\vdash_{\mathrm{HL1}} [B \wedge (E \wedge E') \supset G$ follows, i.e., $G \in [\mathcal{T}, B]$ since $E \wedge E' \in \mathcal{T}$ as $\mathcal{T}$ is closed under Adj.

(II) $\mathcal{T} \subset [\mathcal{T}, B], [\mathcal{T}, C]$. Immediate by the HL1-theorems $(D \wedge E) \supset D$, $(D \wedge E) \supset E$ and the supposition that $B \notin \mathcal{T}$, $C \notin \mathcal{T}$.

It follows from (I) and (II) that $[\mathcal{T}, B]$ and $[\mathcal{T}, C]$ are HL1-theories in which $\mathcal{T}$ is strictly included. By the maximality of $\mathcal{T}$, we have (1) $\vdash_{\mathrm{HL1}} (B \wedge E) \supset A$ and (2) $\vdash_{\mathrm{HL1}} (C \wedge E') \supset A$ for some $E, E' \in \mathcal{T}$, that is (3) $\vdash_{\mathrm{HL1}} [(B \wedge E) \vee (C \wedge E')] \supset A$, whence by the distributive properties between $\wedge$ and $\vee$ we get (4) $[(B \vee C) \wedge (E \wedge E')] \supset A$ and, finally, $A \in \mathcal{T}$, as $(B \vee C) \wedge (E \wedge E') \in \mathcal{T}$, which is impossible. Therefore, $\mathcal{T}$ is a prime HL1-theory such that $A \notin \mathcal{T}$. □

Lemma 5.10 (Canonical HL1-models are HL1-models) *Let M_c be a canonical HL1-model. Then M_c is indeed an HL1-model.*

Proof Let $\mathcal{T}$ be a non-trivial prime HL1-theory and $\mathrm{M_c}$ be the canonical HL1-model built upon it as indicated in Definition 5.8. In order to prove that $\mathrm{M_c}$ is indeed an HL1-model it suffices to prove that $I_{\mathcal{T}}$ fulfills clauses (2a) through (5b). We have:

- Clause (2a). It is trivial.
- Clause (2b). By using A5 $A \leftrightarrow \sim\sim A$.
- Clause (3a). By the HL1-theorems $(A \wedge B) \supset A$, $(A \wedge B) \supset B$, $A \supset [B \supset (A \wedge B)]$.
- Clause (3b). By A7 $\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$.
- Clause (4a). By primeness of $\mathcal{T}$ and the HL1-theorems $A \supset (A \vee B)$, $B \supset (A \vee B)$.
- Clause (4b). By A6 $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$.
- Clause (5a). ($\Rightarrow$) Suppose $A \to B \in \mathcal{T}$. We have to show that at least one of the following alternative obtains: $[A \notin \mathcal{T}\ \ \&\ \ \sim A \in \mathcal{T}]$ or $[B \in \mathcal{T}\ \ \&\ \ \sim B \notin \mathcal{T}]$ or $[A \notin \mathcal{T}\ \ \&\ \ \sim B \notin \mathcal{T}]$ or $[\sim A \in \mathcal{T}\ \ \&\ \ B \in \mathcal{T}]$ or $[A \in \mathcal{T}\ \ \&\ \ B \in \mathcal{T}]$. For *reductio*, suppose that there are wffs A, B such that (1) $A \in \mathcal{T}$ or $\sim A \notin \mathcal{T}$ and (2) $B \notin \mathcal{T}$ or $\sim B \in \mathcal{T}$ and (3) $A \in \mathcal{T}$ or $\sim B \in \mathcal{T}$ and (4) $\sim A \notin \mathcal{T}$ or $B \notin \mathcal{T}$ and (5) $A \notin \mathcal{T}$ or $B \notin \mathcal{T}$. We have 32 possibilities to consider, but each one of them contains either (a) or (b), (a) being $A \in \mathcal{T}\ \ \&\ \ B \notin \mathcal{T}$ and (b) being $\sim B \in \mathcal{T}$ and either $\sim A \notin \mathcal{T}$ or one of $A \notin \mathcal{T}$, $B \notin \mathcal{T}$. But (a) and (b) are impossible: (a) since $\mathcal{T}$ is closed under MP, and (b) by A12, $[(A \to B) \wedge \sim B] \supset [\sim A \vee (A \wedge B)]$.

  ($\Leftarrow$) Suppose $[A \notin \mathcal{T}\ \ \&\ \ \sim A \in \mathcal{T}]$ or $[B \in \mathcal{T}\ \ \&\ \ \sim B \notin \mathcal{T}]$ or $[A \notin \mathcal{T}\ \ \&\ \ \sim B \notin \mathcal{T}]$ or $[\sim A \in \mathcal{T}\ \ \&\ \ B \in \mathcal{T}]$ or $[A \in \mathcal{T}\ \ \&\ \ B \in \mathcal{T}]$. We have to prove that $A \to B \in \mathcal{T}$ follows from each one of these five alternatives. Now, this is immediate by using A8, A9, A10, A11 and A11, respectively.
- Clause (5b). ($\Rightarrow$) Suppose $\sim(A \to B) \in \mathcal{T}$. We have to prove $\sim A \notin \mathcal{T}\ \ \&\ \ B \notin \mathcal{T}$, which is immediate by A14: if either $\sim A \in \mathcal{T}$ or $B \in \mathcal{T}$, then $C \in \mathcal{T}$ for any wff C, contradicting the non-triviality of $\mathcal{T}$.

  ($\Leftarrow$) Suppose $\sim A \notin \mathcal{T}$ and $B \notin \mathcal{T}$. We have to prove $\sim(A \to B) \in \mathcal{T}$, which is immediate by A13 and primeness of $\mathcal{T}$. □

Once Lemmas 5.9 and 5.10 proved, completeness is at hand.

Theorem 5.11 (Completeness of HL1) *For any set of wffs Γ and wff A, (1) if $\Gamma \vDash_{M1} A$, then $\Gamma \vdash_{HL1} A$; (2) if $\Gamma \vDash_{L1} A$, then $\Gamma \vdash_{HL1} A$.*

Proof Firstly, case (2) is proved. (2) Suppose $\Gamma \nvdash_{\mathrm{HL1}} A$, i.e., that A is not included in the set of consequences derivable in HL1 from Γ (in symbols, $A \notin \mathrm{Cn}\Gamma[\mathrm{HL1}]$). Then, $\mathrm{Cn}\Gamma[\mathrm{HL1}]$ is extended to a prime HL1-theory $\mathcal{T}$ such that $A \notin \mathcal{T}$. Next, the canonical HL1-model $\mathrm{M_c} = (K, I_{\mathcal{T}})$ based upon $\mathcal{T}$ is defined, and we have $\Gamma \nvDash_{\mathrm{M_c}} A$, since $T \in I_{\mathcal{T}}(\Gamma)$ (as $T \in I_{\mathcal{T}}(\mathrm{Cn}\Gamma[\mathrm{HL1}])$ but $T \notin I_{\mathcal{T}}(A)$), whence $\Gamma \nvDash_{\mathrm{L1}} A$ (by Definitions 5.1 and 5.2), as was to be proved.

(1) It is immediate by (2) and Proposition 5.3. □

6 Restricting Tables TI-TV in Order to Verify the Contraposition and Transitivity Rules

As discussed at the end of Section 4, the fact that a given logic L is functionally included in another one L′ does not mean that L has to be forgotten in favor of L′:

L may have properties desirable from some perspective or another, whence it would follow the convenience of maintaining the independence status of L. In this regard, it has to be remarked that the logics determined by all matrices MIV and MV lack the rule contraposition (con): let M be any of such matrices and I be an M-interpretation assigning **t** and **b** to different propositional variables p and q, respectively. Then $I(p \to q) = \mathbf{t}$ but $I(\sim q \to \sim p) = \mathbf{n}$. Consequently, the L-logics built upon MIV and MV may lack the replacement theorem, as it is the case with L1, where the rule $A \leftrightarrow B \Rightarrow \sim B \leftrightarrow \sim A$ fails (of course, L1 has a replacement theorem for a number of conditionals definable in it such as $\overset{b}{\to}$, $\overset{E4}{\to}$ or $\overset{BN4}{\to}$). In addition, it also has to be remarked that the transitivity rule (trans), $A \to B, B \to C \Rightarrow A \to C$, fails in all L-logics built upon MIV or MV, the $f_\to$-function of which has $f_\to(\mathbf{f}, \mathbf{b}) \in \{\mathbf{f}, \mathbf{n}\}$ or both $f_\to(\mathbf{n}, \mathbf{t}) = \mathbf{t}$ and $f_\to(\mathbf{n}, \mathbf{b}) \in \{\mathbf{f}, \mathbf{n}\}$. Finally, as it is shown below, many of the L-logics definable upon TI-TIII also lack the rule con.

Therefore, it seemed interesting to restrict tables TI-TV in order to the rules con and trans to be verified thus giving us a class of L-logics whose characteristic implication has stronger properties than the ones presented by the modest ones definable upon MI-MV in their present form.

Then in what follows, we operate the restrictions commented upon above and next provide H-formulations for a couple of L-logics built upon the resulting general tables. We shall limit ourselves to supply the BD-semantics for said logics and the H-formulations definable from them, being the soundness and completeness theorems entirely similar to those for L1 developed in the previous section.

Proposition 6.1 ($f_\to$-functions in TI-TV falsifying con) 1. *Any $f_\to$-function in tables TIV and TV falsifies con.*

2. *If an $f_\to$-function in tables TI-TIII satisfies one of the conditions (a)-(f) below falsifies con:*

(a) $f_\to(\mathbf{n}, \boldsymbol{f}) \in \{\mathbf{b}, \mathbf{t}\}$
(b) $f_\to(\mathbf{n}, \boldsymbol{b}) \in \{\mathbf{b}, \mathbf{t}\}$
(c) $f_\to(\mathbf{n}, \mathbf{t}) \in \{\mathbf{b}, \mathbf{t}\}$ & $f_\to(\mathbf{f}, \mathbf{n}) \in \{\mathbf{f}, \mathbf{n}\}$
(d) $f_\to(\mathbf{f}, \mathbf{n}) \in \{\mathbf{b}, \mathbf{t}\}$ & $f_\to(\mathbf{n}, \mathbf{t}) \in \{\mathbf{f}, \mathbf{n}\}$
(e) $f_\to(\mathbf{f}, \mathbf{b}) \in \{\mathbf{b}, \mathbf{t}\}$ & $f_\to(\mathbf{b}, \mathbf{t}) \in \{\mathbf{f}, \mathbf{n}\}$
(f) $f_\to(\mathbf{b}, \mathbf{t}) \in \{\mathbf{b}, \mathbf{t}\}$ & $f_\to(\mathbf{f}, \mathbf{b}) \in \{\mathbf{f}, \mathbf{n}\}$

Proof (1) As discussed above, it suffices to note that $f_\to(\mathbf{t}, \mathbf{b}) \in \{\mathbf{b}, \mathbf{t}\}$ in all tables in TIV and TV. (2) As summarily shown in the diagram below (cf. Definition 2.4).

	p	$\to$	q	$\sim q$	$\to$	$\sim p$
(a)	**n**	**b/t**	**f**	**t**	b_2	**n**
(b)	**n**	**b/t**	**b**	**b**	b_1	**n**
(c)	**n**	**b/t**	**t**	**f**	**f/n**	**n**
(d)	**f**	**b/t**	**n**	**n**	**f/n**	**t**
(e)	**f**	**b/t**	**b**	**b**	**f/n**	**t**
(f)	**b**	**b/t**	**t**	**f**	**f/n**	**b**

□

A corollary of Proposition 6.1 is the ensuing proposition:

Proposition 6.2 (Tables in TI-TV verifying con) *Consider the following general truth-tables TVI-TIX*[6]*:*

TVI

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**	a_1	a_2	**t**
n	b_1	**b**	b_2	a_3
b		b_3	**t**	a_4
t	**f**	b_4		**t**

TVII

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**	a_1	b_1	**t**
n	b_2	**b**	b_3	a_2
b		b_4	**t**	b_5
t	**f**	b_6		**t**

TVIII

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**	b_1	a_1	**t**
n	b_2	**b**	b_3	b_4
b		b_5	**t**	a_2
t	**f**	b_6		**t**

TIX

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**	b_1	b_2	**t**
n	b_3	**b**	b_4	b_5
b		b_6	**t**	b_7
t	**f**	b_8		**t**

where a_i $(1 \leq i \leq 4) \in \{\boldsymbol{b}, \boldsymbol{t}\}$, b_i $(1 \leq i \leq 8) \in \{\boldsymbol{f}, \boldsymbol{n}\}$ *and there are three possibilities for filling the blank spaces: (1)* $f_{\rightarrow}(\mathbf{b}, \mathbf{f}) = f_{\rightarrow}(\mathbf{t}, \mathbf{b}) = \mathbf{n}$*; (2)* $f_{\rightarrow}(\mathbf{b}, \mathbf{f}) = \mathbf{n}$ & $f_{\rightarrow}(\mathbf{t}, \mathbf{b}) = \mathbf{f}$*; (3)* $f_{\rightarrow}(\mathbf{b}, \mathbf{f}) = \mathbf{f}$ & $f_{\rightarrow}(\boldsymbol{t}, \mathbf{b}) = \mathbf{n}$. *The* $2^{10} \times 3$ *matrices definable upon TVI through TIX are the only ones in tables TI-TIII verifying the rule con.*

Proof It is easy to show that all matrices in TVI-TIX verify con: the cases where $f_{\rightarrow}(\mathbf{f}, \mathbf{f}) = f_{\rightarrow}(\mathbf{f}, \mathbf{t}) = f_{\rightarrow}(\mathbf{b}, \mathbf{b}) = f_{\rightarrow}(\mathbf{t}, \mathbf{t}) = \mathbf{t}$ and $f_{\rightarrow}(\mathbf{n}, \mathbf{n}) = \mathbf{b}$ are trivial, while it is immediate to check that the rest of the cases of interest verify con. Consider TVI: it is obvious that if $A \rightarrow B$ is assigned any of the pairs $\langle \mathbf{f}, \mathbf{n} \rangle$, $\langle \mathbf{f}, \mathbf{b} \rangle$, $\langle \mathbf{n}, \mathbf{t} \rangle$ or $\langle \mathbf{b}, \mathbf{t} \rangle$ then $\sim B \rightarrow \sim A$ is assigned a designated value. Therefore, from Proposition 6.1, it follows that tables TVI-TIX are the only ones in TI-TV verifying con: any other particular table in TI-TV not in TVI-TIX satisfies one of the conditions (a)-(f) in Proposition 6.1, thus falsifying con. □

In addition to verifying con, tables TVI-TIX also verify trans.

Proposition 6.3 (TVI-TIX verify trans) *Matrices definable upon the general tables TVI-TIX verify the rule trans, i.e.,* $A \rightarrow B, B \rightarrow C \Rightarrow A \rightarrow C$.

Proof The easy proof is left to the reader: if $A \rightarrow C$ takes a non-designated value, then either $A \rightarrow B$ or $B \rightarrow C$ also takes a non-designated value. □

The following proposition proposes the last restriction of tables TI-TV carried out in the present paper, this time in order to verify the contraposition axiom, $(A \rightarrow B) \rightarrow (\sim B \rightarrow \sim A)$.

[6]Cf. Note 5

Proposition 6.4 (Tables in TVI-TIX verifying the contraposition axioms) *Consider the following general table TX*

TX

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**	a	d	**t**
n	b	**b**	c	a
b	e	c	**t**	d
t	**f**	b	e	**t**

where $a, d \in \{\mathbf{f}, \mathbf{n}, \mathbf{b}, \mathbf{t}\}$ *and* $b, c, e \in \{\mathbf{f}, \mathbf{n}\}$. *The* 2^7 *matrices definable upon TX are the only ones in tables TVI-TIX verifying the contraposition axiom* $(A \rightarrow B) \rightarrow (\sim B \rightarrow \sim A)$.

Proof It is easy. As a means to verify the contraposition axiom, we need the following equations: $f_\rightarrow(\mathbf{f}, \mathbf{n}) = f_\rightarrow(\mathbf{n}, \mathbf{t})$; $f_\rightarrow(\mathbf{n}, \mathbf{f}) = f_\rightarrow(\mathbf{t}, \mathbf{n})$; $f_\rightarrow(\mathbf{n}, \mathbf{b}) = f_\rightarrow(\mathbf{b}, \mathbf{n})$; $f_\rightarrow(\mathbf{b}, \mathbf{f}) = f_\rightarrow(\mathbf{t}, \mathbf{b})$; $f_\rightarrow(\mathbf{f}, \mathbf{b}) = f_\rightarrow(\mathbf{b}, \mathbf{t})$. □

In Section 5, we have considered a matrix in TV falsifying both the con and trans rules. Next, we briefly treat a matrix in TIX verifying both said rules and another one in TX verifying the contraposition axiom in addition to the rule trans.

Definition 6.5 (The logics L2 and L3) The logics L2 and L3 are those determined by the matrices M2 and M3 built up by adding to $\mathcal{FOUR}$ the $f_\rightarrow$-functions described by the tables t2 and t3, respectively:

*t*2

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**	**n**	**n**	**t**
n	**f**	**b**	**n**	**n**
b	**f**	**f**	**t**	**n**
t	**f**	**f**	**n**	**t**

*t*3

$\rightarrow$	**f**	**n**	**b**	**t**
f	**t**	**f**	**f**	**t**
n	**f**	**b**	**n**	**f**
b	**n**	**n**	**t**	**f**
t	**f**	**f**	**n**	**t**

(Notice that t2 is one of the tables in TIX, while t3 belongs to TX.)

Next, BD-semantics for L2 and L3 are provided by defining the concepts of an L2-model and an L3-model (the notions of L2- (L3-) consequence and L2- (L3-) validity are defined similarly as the corresponding ones for the logic L1 —cf. Definition 5.2).

Definition 6.6 (L2-models) An L2-model is a structure (K, I) where (i) $K = \{\{T\}, \{F\}, \{T, F\}, \emptyset\}$ and (ii) I is an L2-interpretation from the set of all wffs to K fulfilling the same conditions (1), (2a), (2b), (3a), (3b), (4a) and (4b) as in L1-models, while conditions (5a) and (5b) are as follows:

(5a) $T \in I(A \rightarrow B)$ iff $[T \notin I(A)\ \&\ F \in I(A)\ \&\ T \notin I(B)\ \&\ F \in I(B)]$ or $[T \notin I(A)\ \&\ F \notin I(A)\ \&\ T \notin I(B)\ \&\ F \notin I(B)]$ or

$$[T \in I(A)\ \ \&\ \ F \in I(A)\ \ \&\ \ T \in I(B)\ \ \&\ \ F \in I(B)] \text{ or}$$
$$[T \in I(A)\ \ \&\ \ F \notin I(A)\ \ \&\ \ T \in I(B)\ \ \&\ \ F \notin I(B)] \text{ or}$$
$$[T \notin I(A)\ \ \&\ \ F \in I(A)\ \ \&\ \ T \in I(B)\ \ \&\ \ F \notin I(B)]$$

(5b) $F \in I(A \to B)$ iff $[T \in I(A)\ \ \&\ \ T \notin I(B)]$ or $[F \notin I(A)\ \ \&\ \ T \notin I(B)]$

Definition 6.7 (L3-models) An L3-model is a structure (K, I) where K is defined as in L2-models and an (L3-interpretation) I is defined exactly as in L2-models except for the clause (5b), which now reads as follows:

(5b) $F \in I(A \to B)$ iff $[F \notin I(A)\ \ \&\ \ T \notin I(B)]$ or
$[T \notin I(A)\ \ \&\ \ T \notin I(B)\ \ \&\ \ F \notin I(B)]$ or
$[T \notin I(A)\ \ \&\ \ F \notin I(A)\ \ \&\ \ F \notin I(B)]$ or
$[T \notin I(A)\ \ \&\ \ F \in I(A)\ \ \&\ \ T \in I(B)\ \ \&\ \ F \in I(B)]$ or
$[T \in I(A)\ \ \&\ \ F \in I(A)\ \ \&\ \ T \in I(B)\ \ \&\ \ F \notin I(B)]$

Next, H-formulations of L2 and L3 are provided. The base is the same as that for L1: $\supset$, $\to$ and $\sim$ as primitive connectives; $\overset{\bullet}{\neg}$, $\overset{\circ}{\neg}$, $\neg$, $\wedge$, $\vee$ and $\leftrightarrow$ are defined connectives with $\text{MP}_{\supset}$ as the sole rule of inference. The axiomatization, especially in the case of L3, is a bit more complicated than that of L1, since the clauses (5a) and (5b) in L2- and L3-models are more involved than those of L1-models. But, anyway, the resulting H-formulations are not more complex than, say, those for strong 3-valued logics (see, e.g., [3] and references therein).

Definition 6.8 (The system HL2) The system HL2 can be formulated as follows.
Axioms: A1-A7 of HL1 and:

$A8.$ $[(A \to B) \wedge (B \to C)] \supset (A \to C)$
$A9.$ $(A \to B) \supset (\sim B \to \sim A)$
$A10.$ $(A \wedge B) \supset [(\sim A \vee \sim B) \vee (A \to B)$
$A11.$ $(\sim A \wedge B) \supset [(A \vee \sim B) \vee (A \to B)]$
$A12.$ $[(A \wedge \sim A) \wedge (B \wedge \sim B)] \supset (A \to B)$
$A13.$ $[(A \vee \sim A) \vee (B \vee \sim B)] \vee (A \to B)$
$A14.$ $[(A \to B) \wedge \sim A] \supset (B \vee \sim B)$
$A15.$ $[(A \to B) \wedge (A \wedge \sim A)] \supset \sim B$
$A16.$ $(\sim A \vee B) \vee \sim (A \supset B)$
$A17.$ $A \supset [B \vee \sim (A \to B)]$
$A18.$ $[\sim (A \to B) \wedge (\sim A \vee B)] \supset A$
$A19.$ $[\sim (A \to B) \wedge B] \supset C$

Rules:

$$\text{MP}_{\supset}: \ A \supset B, A \Rightarrow B$$

Definitions: As in HL1.

Definition 6.9 (The system HL3) The system HL3 can be formulated as follows.

Axioms: A1-A8 and A10-A16 of HL2 with A9′ $(A \to B) \leftrightarrow (\sim B \to \sim A)$ instead of A9 and:

$$A17. \quad [(A \vee \sim A) \vee \sim B] \vee \sim (A \to B)$$
$$A18. \quad [(A \wedge \sim A) \wedge B] \supset [\sim B \vee \sim (A \to B)$$
$$A19. \quad [\sim (A \to B) \wedge (A \wedge \sim A)] \supset B$$
$$A20. \quad [\sim (A \to B) \wedge (A \wedge B)] \supset \sim A$$
$$A21. \quad [\sim (A \to B) \wedge (\sim A \wedge B)] \supset (A \vee \sim B)$$
$$A22. \quad [\sim (A \to B) \wedge [(A \wedge \sim A) \wedge \sim B)]] \supset C$$

Rules:

$$\text{MP}_{\supset}: \ A \supset B, A \Rightarrow B$$

Definitions: As in HL1.

Proposition 6.10 (Some theorems of HL2 and HL3) *The ensuing wffs are theorems of HL2:*

$$t1. \quad (\sim A \wedge \sim B) \supset [(A \vee B) \vee (A \to B)]$$
$$t2. \quad (\sim A \to B) \supset (\sim B \to A)$$
$$t3. \quad (A \to \sim B) \supset (B \to \sim A)$$
$$t4. \quad (\sim A \to \sim B) \supset (B \to A)$$
$$t5. \quad [(A \to B) \wedge (B \wedge \sim B)] \supset A$$
$$t6. \quad [(A \to B) \wedge B] \supset (A \vee \sim A)$$

In addition, the rule modus tollens (MT) $A \to B, \sim B \Rightarrow \sim A$, also holds in HL2. Then, in addition to t1-t6 and MT, the following are provable in HL3:

$$t7. \quad \sim (A \to B) \leftrightarrow \sim (\sim B \to \sim A)$$
$$t8. \quad [(B \vee \sim B) \vee A] \vee \sim (A \to B)$$
$$t9. \quad [(B \wedge \sim B) \wedge \sim A] \supset [A \vee \sim (A \to B)]$$
$$t10. \quad [\sim (A \to B) \wedge (B \wedge \sim B)] \supset \sim A$$
$$t11. \quad [\sim (A \to B) \wedge (\sim A \wedge \sim B)] \supset B$$
$$t12. \quad [\sim (A \to B) \wedge [(B \wedge \sim B) \wedge A]] \supset C$$

Proof It is easy and is left to the reader □

By following the pattern set up in the case of HL1 in Section 5, it is easy to prove that HL2 and HL3 are sound and complete in the same sense as HL1 is.

7 Concluding Remarks

In this paper, the class of matrices $\mathrm{MI4^C}$ is defined. $\mathrm{MI4^C}$ is composed of a wealth of implicative expansions of the matrix $\mathcal{FOUR}$ characterizing the well-known Belnap-Dunn 4-valued logic FDE. Each matrix M in $\mathrm{MI4^C}$ determines the logic LM in the usual way: LM = $(\mathcal{L}, \vDash_{\mathrm{M}})$ where $\mathcal{L}$ is the implicative expansion of the language upon which $\mathcal{FOUR}$ is built and $\vDash_{\mathrm{M}}$ is the consequence relation defined in M, as it is customary. As in Section 5, we shall refer by L-logics to the class of logics determined by the matrices in $\mathrm{MI4^C}$.

Boolean negation is definable in each member M of $\mathrm{MI4^C}$, whence it follows that matrices determining strong logics such as E4, BN4, PŁ4 and classical propositional logic are also definable. But in addition to initiate the study of the functional strength of $\mathrm{MI4^C}$, it also seemed interesting to begin to explore the characteristic properties of the implication functions in $\mathrm{MI4^C}$. In this sense, it has to be noted that many of them lack the rule *transitivity* (trans), whereas the rule *contraposition* (con) does not hold in more than the seventy per cent of the L-logics. Consequently, in order to obtain L-logics with a stronger implication, we investigated two restrictions in $\mathrm{MI4^C}$. The first one, results in a subset, say S, of $\mathrm{MI4^C}$ validating con. Next, it develops that trans is also validated in S. The second restriction gives us a subset of S, say S′, validating the contraposition axiom (we note that only 128 matrices in $\mathrm{MI4^C}$ validate the contraposition axiom; cf. Proposition 6.4). Of course, still stricter restrictions are possible. For instance, are there elements in S′ validating the *suffixing* and *prefixing* axioms or at least the corresponding rules? In this way, we obtain L-logics with a stronger characteristic implication, which can be useful in some way or another, in addition to being capable of defining such strong logics as BN4 and PŁ4.

It has been shown how to give Hilbert-formulations to the L-logics by leaning upon a two-valued Belnap-Dunn semantics equivalent to the matrix semantics definable upon the elements in $\mathrm{MI4^C}$.

As far as we know, this paper is the first item in the literature presenting a class of implicative expansions of Belnap-Dunn logic in which Boolean negation is definable. Until now, only a few specific instances of such type of expansions could be found in it, all of them definable in $\mathrm{MI4^C}$.

There is a number of ways in which the investigation carried out in the present paper could be pursued. We remark on three of them.

1. To continue the study of the functional strength of the L-logics.
2. To investigate the functional relations the L-logics maintain to each other.
3. To study whether there are L-logics with an interesting characteristic implication among those verifying con or the contraposition axiom.

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