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Monstrous Content and the Bounds of Discourse

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Abstract

Bounds consequence provides an interpretation of a multiple-conclusion consequence relation in which the derivability of a sequent $[\Gamma \succ \Delta]$ is understood as the claim that it is conversationally out-of-bounds to take a position in which each member of $\boldsymbol{\Gamma}$ is asserted while each member of $\boldsymbol{\Delta}$ is denied. Two of the foremost champions of bounds consequence—Greg Restall and David Ripley—have independently indicated that the shape of the bounds in question is determined by conversational practice. In this paper, I suggest that the standard treatments of bounds consequence have focused heavily on the matter of veridicality at the expense of ignoring other features by which conversational bounds are set, prime among them being the matter of content or subject-matter. Furthermore, I argue that the semantic behavior of propositions containing "monstrous" content—content whose introduction is inappropriate to a context independently of veridical considerations—leads to a weak Kleene account of bounds consequence.

Keywords Bounds consequence · Weak Kleene logic · Strict-tolerant logic · Subject-matter

1 Introduction

Sequents—structures $[\Gamma \succ \Delta]$, where Γ, Δ are multisets of formulae—serve as the foundational objects of study in sequent calculus formulations of proof theory. Traditionally, sequents receive an *operational* interpretation, in which the sequent separator $\succ$ is understood as an entailment relation. Sam Buss's [1], for example,

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interprets a sequent $[\varphi_1, ..., \varphi_k \succ \xi_1, ..., \xi_l]$ as follows (with slight modifications to notation):

> The intuitive meaning of the sequent is that the conjunction of the φ_i's implies the disjunction of the ξ_j's. Thus, a sequent is equivalent in meaning to the formula $\bigwedge_{i=1}^{k} \varphi_i \supset \bigvee \xi_{j=1}^{l}$ [1, p. 10].

This operational reading is indeed intuitive—which goes some way to explain its entrenched status. But it is not inevitable.

In [2], Greg Restall offers an alternative interpretation of sequents $[\Gamma \succ \Delta]$ as *states* or *positions* available to an individual. With notation altered slightly, he writes:

> A state might be used to represent the *outlook* of an agent which we take to *accept* each statement in Γ and *reject* each statement in Δ. We might also use a state to represent the *context* in some dialogue or discourse at which each statement Γ is *asserted* and each statement in Δ is *denied* [2].

By placing equal emphasis on assertions and denials, this reading positions it among *bilateral* approaches in logic, including the work in Blasio, Caleiro, and Marcos' [3], which offers a closely related interpretation of sequents. Their remark that their interpretation "[thinks] about assertions and denials both as first-class citizens" [3, p. 5482] is equally applicable in the present case.

Understanding sequents as positions motivates an interpretation identifying derivability of $[\Gamma \succ \Delta]$ with the condition that taking the position lands individuals "out-of-bounds" by flouting some discursive norm. To illustrate, the derivability of $[\varphi \succ \varphi]$ corresponds to a proscription against "blowing hot and cold" with respect to arbitrary formulae φ, i.e., for someone to both assert and deny φ. (The pejorative character of this idiom constitutes social evidence that taking the position $[\varphi \succ \varphi]$ is indeed a *discursive transgression*.) On this reading, the contours of correct derivability closely follow the contours of the *bounds of discourse*, whence the corresponding interpretation may be described as *bounds consequence*.

But, even granted this interpretation, there remains room for dispute about the shape of such transgressions; indeed, many philosophical disagreements concerning the correct account of logical validity can be straightforwardly translated into disputes about the demarcation of conversational bounds. In the same way that philosophical logicians may argue about the semantic properties over which logical consequence, operationally construed, is determined, a central question becomes: What is it that determines these bounds?

If the shape of the bounds directly impact the shape of logical validity, the choice of which categories of conversational features are logically relevant is critical. The complexity and subtlety of social practices ensure a wide range of dimensions from which to draw. One can confirm this breadth by surveying the variety of faults for which a position may be deemed out-of-bounds in informal practice.

As cultural norms are prone to idiosyncrasy and irrationality, there are many bounds-determining conversational features of negligible logical importance. In a political debate, for example, a position may be judged out-of-bounds for reasons independent of its content or truth-value, such as whether its phrasing sufficiently demonstrates an affinity with a particular constituency. Thus, we must take care to avoid mistaking *ad hoc* features of conversational bounds for regularities. But there remain many features that are arguably logically significant, that is, enjoy sufficient importance and regularity to influence the measurement of the adequacy of a particular sequent calculus.

Among these, the logical significance of the veridical dimension of bounds-setting is easily most recognizable. Indeed, it feels natural to appeal to *truth* and *falsity* in a justification of the out-of-boundedness of an unsatisfiable position. E.g., most individuals accept that the *truth* of formula $\varphi \wedge \psi$ is incompatible with the *falsity* of φ and would, moreover, consider this a cogent explanation of the out-of-boundedness of the position $[\varphi \wedge \psi \succ\!\!- \varphi]$. Despite the centrality these veridical dimensions enjoy, there remain many *extra-veridical* bounds-determining considerations that exhibit logical significance.

As an example, an evidentiary dimension often factors into setting conversational bounds—a position $[\Gamma \succ\!\!- \Delta]$ may be judged as out-of-bounds when an agent takes a stand on $[\Gamma \succ\!\!- \Delta]$ without appropriate justification. Indeed, in some contexts, justificatory factors are prior to veridical factors; promulgating the conclusions of a falsified experiment is worthy of censure even if the conclusions are correct. Just as evidentiary considerations play a strong role in discussions of standard readings of logical consequence, they are just as intuitive in the bounds-consequence setting. For example, Restall suggests that bounds consequence "is compatible with an anti-realist account of intuitionistic logic" [2]; such an account would place evidentiary dimensions—like constructivity—on an equal footing with veridical dimensions.[1]

What I aim to achieve in this paper is threefold. First, I will argue that the bounds of discourse are not determined by veridical grounds alone, but rest also on *content-theoretic* concerns, e.g., features of positions' subject-matter. Consequently, I wish to argue that the veridical approaches exemplified by e.g., [4] overlook critical features of conversational practice. Second, I will argue that such content-theoretic constraints on bounds accommodate "monstrous" content—content whose introduction into a discourse is out-of-bounds for its very mention. Finally, I will argue that the regularities characterizing a content-responsive account are captured by giving a strict-tolerant reading (in the style of [5]) to the *weak* Kleene matrices and that the logic of content-theoretic bounds is characterized by simple alterations to the strict-tolerant scheme.

2 Bounds Consequence

As I have suggested, bounds consequence gives an interpretation of the familiar sequent reading a position $[\Gamma \succ\!\!- \Delta]$ as a representation of an abstract situation in which one strictly asserts each element of Γ and strictly denies each element of Δ. This may be refined in many ways; a position can be a representation of one's

[1] Stuart Shapiro, in conversation, has independently suggested that a bounds consequence reading of intuitionistic logic can be considered in this way.

publicly-facing utterances in a discourse or the commitments an agent has made mentally. Although the operational interpretation of the derivability of position $[\Gamma \succ \Delta]$ is prescriptive in nature—e.g., that an agent accepting Γ ought to accept some member of Δ—bounds consequence in contrast gives proscriptive reading. This coheres with the model-theoretic interpretation that validity of $[\Gamma \succ \Delta]$ rules out models in which the formulae of Γ are true and those in Δ are false.

E.g., David Ripley offers the following (with notation slightly altered):

> Let a *position* be some pair $[\Gamma \succ \Delta]$ of assertions and denials. These might be... the assertions and denials actually made by one party to a particular debate. The position $[\Gamma \succ \Delta]$ asserts each thing in Γ and denies each thing in Δ [4, p. 141].

Thus, when some statement in Γ cannot be asserted truthfully, alethic considerations provide grounds to proscribe taking the position $[\Gamma \succ \Delta]$.

Such proscriptions are material ones; what distinguishes the innocent from the transgressive hinges on the features of conversational practices. Thus, our use of language—possibly circumscribed by semantic or metaphysical features—gives meaning to our logical vocabulary. The particular conversational principles governing the bounds, for example, have immediate consequences for the meaning of notions of validity or negation in that setting.

Reading consequence in terms of conversational bounds is surprisingly successful in the way in which new subtleties are teased out of familiar concepts. The value of this analysis is apparent in how it enables us to reframe familiar notions in new terms and reappraise them.

2.1 The Bounds Consequence Interpretation of Cut

Notably, the bounds consequence reading has a fascinating impact on the Cut rule, which is often encountered like so:

$$\frac{[\Gamma, \varphi \succ \Delta] \qquad [\Gamma \succ \varphi, \Delta]}{[\Gamma \succ \Delta]}\ \text{[Cut]}$$

Traditional, operational readings—as in [1]—view Cut as a generalized modus ponens. On the standard semantical interpretation, whenever Γ and φ jointly entail the truth of Δ—and Γ independently entails the truth of either φ or Δ—Γ alone entails the truth of Δ. Any contribution that φ may have made as guarantor of Δ is therefore redundant.

Bounds consequence permits a radical recasting of the Cut rule, however, licensing Restall's characterization of Cut—in contraposed form—as a condition of *Extensibility*:

> EXTENSIBILITY If $[\Gamma \succ \Delta]$ is in-bounds, then so is one of $[\Gamma, \varphi \succ \Delta]$ and $[\Gamma \succ \varphi, \Delta]$.

Thus, whenever position $[\Gamma \succ \Delta]$ is in-bounds, for any sentence φ, in-boundedness is preserved by adding φ to either one's stock of assertions or one's stock of denials.

(*N.b.* that in treating the contraposition of Cut, *Extensibility* turns Cut "on its head" by characterizing conditions on in-boundedness rather than out-of-boundedness.)

This has formed a crucial element distinguishing the positions of Ripley and Restall. Ripley, for example, considers taking a position as a type of "taking a stand" with respect to its constituents.

> [I]f "taking a stand on φ" is understood to mean either asserting φ or denying φ, then Cut tells us that any in-bounds position at all can be extended to some in-bounds position that takes a stand on φ, for any φ... This is, to put it mildly, not obvious—*particularly* when we take seriously the possibility of paradoxical sentences... [for example if] there is some problem with φ itself, some reason to do with φ that it can't be either asserted or denied [6, p. 41].

I.e., to embrace Cut entails the position that for every statement φ, any in-bounds context is freely extensible either by the affirmation or denial of φ. Framed as a warrant to "take a stand" on any statement whatsoever, Cut amounts to a thesis that is quite sweeping in its scope.

As either assertion or denial of semantic paradoxes independently lead to contradiction, Ripley identifies them as obvious as counterexamples to this warrant. But Extensibility can be cast more broadly still as a thesis about the scope of acceptable topics. "Taking a stand" on a statement introduces its subject-matter to a discourse, whence *Extensibility* licenses the introduction of arbitrary subject-matter without violating the bounds. After all, to say of φ that in any context, one may take a stand without transgression *a fortiori* licenses discussion of its topic in any setting—perhaps the subject of its truth is determined ahead of time, but at the least the topic can be raised. In short, Cut-as-extensibility means that the subject-matter of every statement φ can be entered into any discourse without the risk of censure.

So framed, it seems plausible that Cut is indeed at odds with our conversational practices and the ways in which we set the bounds of discourse. Indeed, when one surveys the items of any natural language, it undoubtedly appears that there exist—for a number of reasons—statements for which their mere introduction as topics should prove to be out-of-bounds. One of the rationales for why some statements should be precluded from both assertion and rejection in a context is veridical; this is the position implicit in that of Ripley and his collaborators.

Appendix A records the formal details of *strict-tolerant* logic ST described by Cobreros, Égré, Ripley, and van Rooij in e.g. [5] and [7]. The corresponding theory of truth in ST is a paradigm case of a bounds-consequence framework that rejects Cut for veridical reasons.[2] Notably, ST's inclusion of a truth predicate makes it sufficiently expressive to include Liar sentences that are archetypal instances of sentences that can be neither affirmed nor denied on veridical grounds.

The notions of strict and tolerant truth give an operational reading in which validity of $[\Gamma \succ \Delta]$ is understood as ensuring that whenever each of Γ is strictly true, some formula from Δ must be tolerantly true, whence the provenance of the name

[2]Although [7] calls ST equipped with a truth predicate STTT (for ST *theory of truth*), the association between ST and the truth predicate is so tight that often ST is assumed to have a truth predicate.

strict-tolerant. But the interpretation is arguably far more natural and uniform against a bounds consequence reading. I.e. that validity of $[\Gamma \succ \Delta]$ corresponds to the impossibility of each of Γ being strictly true while each of Δ being strictly false immediately maps to the type of rejoinder against adopting the position at the heart of bounds consequence.

I take the strong ST semantics to correspond to the veridical conception of bounds-setting not because of the discussion of strict and tolerant truth, but rather, because of the influence that the hallmark pathologies of paradoxes have on the shape of the logic.

2.2 The Veridical Case Against Cut

Ripley [4] considers formulae like the Liar sentence—i.e., a sentence λ that asserts that it is not true. The familiar argument exposing λ as a paradox is structured to demonstrate that both the assertion of λ and its denial lead to contradiction, whence either mode of "taking a stand" would entail a commitment to a paradox. Understanding such a commitment as a veridical defect rendering a position out-of-bounds, both positions $[\lambda \succ \quad]$ and $[\quad \succ \lambda]$ will flout conversational norms.

Consider Ripley's argument against Cut in slightly more precision. ST includes a coding device assigning to each formula φ a constant $\ulcorner\varphi\urcorner$. The coding device induces fixed points, whence there exists a formula λ such that $\lambda = \neg\mathbf{T}(\ulcorner\lambda\urcorner)$. Given this λ and the transparency of the truth predicate, one can find a proof Π_0:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\lambda \succ \lambda]}\,[\text{Axiom}]}{[\lambda \succ \mathbf{T}(\ulcorner\lambda\urcorner)]}\,[\text{R}\mathbf{T}]}{[\lambda, \neg\mathbf{T}(\ulcorner\lambda\urcorner) \succ \quad]}\,[\text{L}\neg]}{[\lambda, \lambda \succ \quad]}\,[\text{Identity}]}{[\lambda \succ \quad]}\,[\text{Contract}]$$

(Note that the label [Identity] is not a rule, but a marker that the sequents $[\lambda, \neg\mathbf{T}(\ulcorner\lambda\urcorner) \succ \quad]$ and $[\lambda, \lambda \succ \quad]$ are identical.)

Analogous techniques yield a proof Π_1 of the sequent $[\quad \succ \lambda]$. Should one apply Cut to Π_0 and Π_1, one would yield:

$$\dfrac{\dfrac{\Pi_0}{[\lambda \succ \quad]} \qquad \dfrac{\Pi_1}{[\quad \succ \lambda]}}{[\quad \succ \quad]}\,[\text{Cut}]$$

Under minimal assumptions, the derivability of $[\quad \succ \quad]$ would be catastrophic. Assuming Weakening, its application to $[\quad \succ \quad]$ would immediately lead to the derivability—and hence out-of-boundedness—of any position whatsoever. Thus, one must reject Cut if one maintains that there exist any positions that are in-bounds.

Importantly, there are a host of properties equivalent to the failure of Cut; exploring these principles could lead to a more holistic understanding of bounds consequence. As an example, consider Neil Tennant's analysis in [8] of a virtually identical case in which properties of λ yield a natural-deduction proof of $\bot$. Tennant rejects this derivation as not *genuine* on the grounds that such a proof is not *normalizable*, i.e., it cannot be brought into normal form. Intriguingly, Tennant's requirement of normalizability is defended on anti-realist grounds that cohere with the concerns of bounds consequence, suggesting the potential for an investigation into normalizability through a bounds consequence lens.

A deeper investigation into the cause of the incompatibility lies outside the scope of this work, however.

3 Bounds as Content-Theoretic Demarcations

Although it is clear that veridical considerations play a governing role in circumscribing discursive bounds, it is far less clear that this dimension is exhaustive of bounds-determining semantic features of language. We now examine the role that content or subject-matter plays with respect to bounds-setting, showing that the logic of *content-responsive* bounds induces a well-behaved and elegant theory.

Modest reflection should readily uncover myriad cases in which the content expressed by a position—rather than its truth conditions alone—is a deciding feature in whether the position is out-of-bounds. Blasphemies, demeaning characterizations, profanities or obscenities, for example, are familiar examples of statements that are rendered no more suitable for having disjoined them with a true and unobjectionable statement. With respect to a host of such terms, we find that utterances in which they appear in any context constitutes grounds for censure.

Moreover, it is insufficient to explain the out-of-boundedness of, e.g., sentences including dehumanizing language by simply citing their falsity. Clearly, there are reasons to think that that in the case of such content, this censure is prior to and independent of the evaluation of the truth-value of the sentence; e.g. a linguistic commitment to avoiding dehumanizing language is arguably prior to a commitment to truth-telling. Such a case is that in which there is an implicit prohibition against explicitly denying such statements—despite their falsity—inasmuch as such a denial may be understood as an acknowledgment that their truth is up for debate or that the obviousness of their falsity is not as certain as one would like. More strongly still, many such terms or phrases are frequently deemed to be so severely out-of-bounds that their defects bleed across even the use-mention barrier, to propagate through quotation marks.

Whether the content of a particular sentence will be sufficiently problematic to preclude its introduction into a discourse is clearly context-dependent. (Of course, the matter of whether a sentence can be asserted truthfully is itself context-dependent.) We can consider a concrete example suggested to me by Greg Restall for illustration.

Common sense tells us that one is not to say the word "*bomb*"—or any synonym for "bomb," for that matter—in an airport and that to do so is to transgress a particular set of bounds. One level of this means that it is out-of-bounds to affirm a sentence like "I am carrying a bomb" while participating in such a context. But also note that it is not the truth or not of this sentence that makes this an out-of-bounds position to take; it is equally out-of-bounds to deny this sentence by uttering "I *am not* carrying a bomb." Nor is it in-bounds to affirm even the logically true sentence "Either I am carrying a bomb or I am not." So sensitive is this context that even tautologies cannot be affirmed. Clearly, in this context, it is the introduction of this content that makes the position out-of-bounds.

With this in mind, I think it is fair to suggest that if one can identify common conversational practices in which subject-matter is taken into account in the determination of bounds, then it is reasonable to investigate the corresponding types of content-responsive bounds consequence.

3.1 Monstrous Content

The defects severe enough to place positions out-of-bounds on content-theoretic grounds are diverse. One type whose out-of-boundedness is most self-evident is statements whose content is condemnable for its *harmfulness*, e.g., statements invoking dehumanizing content. In such cases, the transgression flows not from merely violating the norms of polite society but rather from the willful harm to others. In basic terms, for one to utter statements of this class is simply *monstrous*.

Although blatantly offensive or harmful defects do not exhaust the class of statements with defective content, such cases arguably make up the most prominent examples. Taking these as emblematic, I will say that all statements considered out-of-bounds as content-theoretically defective suffer from *monstrous content*. In the sequel, I let the notation `@#$%&!` serve as a placeholder for an arbitrary statement including such monstrous content.[3]

As one considers cases in which statements' subject-matter shapes the in- or out-of-boundedness of positions, one detects regularities that can be characterized by rules. Casting these regularities as rules translates content-theoretic influence over conversational bounds into inferential and metainferential principles. As an illustration, consider an instance of a classically admissible inference rule. Assume that `@#$%&!` includes monstrous content in following rule instance:

$$\frac{[\ \succ \texttt{@\#\$\%\&!}] \qquad [\texttt{@\#\$\%\&!} \succ \psi]}{[\ \succ \psi]}$$

This rule instance is clearly valid on any classical sequent calculus. Indeed, on strictly truth-functional grounds, its validity remains compelling, whether interpreted operationally or under a bounds consequence reading. Despite the monstrosity of its content, `@#$%&!` may nevertheless be truth-evaluable, whence the veridical intuitions reflected in ST should be respected.

This case ignores any influence of the hypothesized content-theoretic defect of `@#$%&!`, of course. The monstrosity of `@#$%&!` requires that its introduction into a discourse is out-of-bounds in any context. Consequently, adopting either of the positions $[\ \succ$ `@#$%&!`$]$ and $[$`@#$%&!` $\succ \psi]$ runs afoul of conversational norms. However, by hypothesis, `@#$%&!` alone serves as the wellspring of the defect—ψ may be an anodyne-but-false statement—whence the position $[\ \succ \psi]$ would be perfectly acceptable. In short, the above case of *modus ponens* shows that characterizing content-sensitive inferentially exerts a recognizable influence on the shape of the logic.

[3]The symbol `@#$%&!` is known as a *grawlix*, a term coined by the cartoonist Mort Walker.

3.2 The Content-Theoretic Case Against Cut

Having considered how content-sensitive bounds can influence the shape of logic, we return to the matter of Cut. Shifting the emphasis from alethic dimensions of bounds allows us to provide a stronger and more general argument against the acceptableness of Cut.

Content-theoretic considerations appear to conflict with Cut-as-extensibility with more immediacy than veridical considerations. Even extremely mundane features of conversational practice—i.e., simply observing that one cannot in practice say whatever one wants, whenever one wants—demands that taking content as a logically significant dimension places me decisively in Ripley's camp when it comes to Cut.

Ultimately, content-theoretic intuitions straightforwardly show that Cut is irreconcilable with bounds consequence. There are clearly cases in which the content of a statement is problematic enough to ensure not merely hesitation to assert it, but even to take a stand on any complex in which it appears prior to the evaluation of its veridical status. It is not the risk of asserting such statements that places them out-of-bounds—frequently, in fact, even their explicit rejection is deemed unacceptable—but rather the act of introducing their subject-matters into a discourse. It is this case on which we will focus.

Cut-as-extensibility guarantees that the in-boundedness of a position $[\Gamma \succ \Delta]$ will be preserved in one of the positions $[\Gamma, \varphi \succ \Delta]$ or $[\Gamma \succ \Delta, \varphi]$ irrespective of the choice of φ. The arbitrariness of φ is critical here. Setting aside the matter of which side of the sequent separator φ lands on, Extensibility pronounces that any statement may be incorporated in some way into an in-bounds position without running afoul of the bounds of the discourse. Although Cut does not require that everything can be affirmed without violating the bounds, it does presuppose that all subject-matter can be introduced to an in-bounds position.

The argument against Cut is simply this: There exist monstrous statements like `@#$%&!` whose introduction into discourse—in any mode—is impermissible, whence both positions [`@#$%&!` $\succ$] and [$\succ$ `@#$%&!`] are out-of-bounds. But there exist acceptable positions that one can take. If Cut-as-extensibility were admissible in a bounds-consequence interpretation of a sequent calculus—that is, were Cut to reflect a correct regularity of conversational practice—then the in-boundedness of $[\ \succ\ ]$ must be inherited by one of these positions. Thus, Cut is inconsistent with our conversational norms.

I have agreed with Ripley's argument against Cut... So how do the conceptions of the bounds make a difference? The argument from content-theoretic considerations, I believe, is more immediate, more general, and less tied to the choice of formal language involved. The veridical case against Cut requires an appeal to a device that is relatively esoteric to the layperson, that is, an appeal to the existence of fixed points of the formula scheme $\neg \mathbf{T}(\ulcorner \cdot \urcorner)$. On the other hand, appealing to a prohibition against particular subject-matters enjoys an elegance and immediacy that the veridical case lacks. Everyone, as a social creature, should be able to recognize that there are some topics that are not raised in certain contexts, irrespective of whether they are affirmed or denied. Understood as a license to freely introduce all subject matter without censure, Cut therefore stands in stark contradiction with extraordinarily entrenched societal norms.

In short, so long as subject-matter is logically relevant, anyone who can make it through a wedding or dinner party without alienating everyone around them should immediately recognize the incompatibility between bounds consequence and Cut. What remains, then, is to describe a well-behaved semantics demonstrating that there exists a plausible logical theory to which content-theoretic bounds correspond. I will devote the next section to describing such a theory.

4 The Semantics of Monstrous Content

Having argued that statements like `@#$%&!` influence the shape of admissible inferences, we now investigate the general nature of these regularities by analyzing the appropriateness of operational rules on content-theoretic considerations. This investigation essentially provides a proof-theoretic semantics for content-sensitive bounds consequence.

A characteristic feature of monstrous content is its *infectiousness*. No matter how cloaked e.g., `@#$%&!` may be within the scope of operators and unobjectionable contingent statements, any complex in which it appears inherits its monstrosities. This property demands revised truth-functions and complicates the adoption of the quantifiers.

4.1 Truth-Functions and Monstrous Content

First, consider interactions between statements possibly including monstrous content and the truth-functional connectives: $\neg$, $\vee$, and $\wedge$.

Complex statements infected by monstrous content may attribute their pathologies to the introduction of the content of their subformulae rather than the alethic mode of introduction. In the case of myriad offensive statements, merely prepending a negation leaves the offensive subject-matter intact. The monstrosity of `@#$%&!` entails that both $[\texttt{@\#\$\%\&!} \succ\ ]$ and $[\ \succ \texttt{@\#\$\%\&!}]$ are out-of-bounds and, consequently, the monstrosity lifts to the out-of-boundedness of $[\neg\texttt{@\#\$\%\&!} \succ\ ]$ and $[\ \succ \neg\texttt{@\#\$\%\&!}]$ as well.

The following rules characterize this sentiment as we've laid it out above:

$$\frac{[\Gamma, \varphi \succ \Delta] \qquad [\Gamma \succ \Delta, \varphi]}{[\Gamma, \neg\varphi \succ \Delta]}\ [\mathrm{L}\neg_w] \qquad \frac{[\Gamma, \varphi \succ \Delta] \qquad [\Gamma \succ \Delta, \varphi]}{[\Gamma \succ \Delta, \neg\varphi]}\ [\mathrm{R}\neg_w]$$

N.b. the desired rules $[\mathrm{L}\neg_w]$ and $[\mathrm{R}\neg_w]$ are consequences of the rules $[\mathrm{L}\neg]$ and $[\mathrm{R}\neg]$ of Definition 8, respectively, whence negation requires no deviation from ST.

In contrast, ensuring that binary connectives respect the aforementioned infectiousness of monstrous content demands revision to ST. The monstrosity of `@#$%&!` demands the out-of-boundedness of positions $[\texttt{@\#\$\%\&!} \vee \varphi \succ\ ]$ and $[\ \succ \texttt{@\#\$\%\&!} \wedge \varphi]$, i.e., objectionable content cannot be "insulated away" by cleverly padding the offending statement with carefully chosen and innocuous truths or falsehoods.

Fix a situation in which `@#$%&!` is monstrous and a ξ that is anodyne both truth-theoretically and content-theoretically, say, "$2 + 2 = 4$".[4] Despite the truth of the complex `@#$%&!` $\vee \xi$, truth alone is insufficient to establish the in-boundedness of [`@#$%&!` $\vee \xi \succ \quad$]. E.g., offensive literature would hardly be redeemed by the act of suffixing each sentence with "(or $2 + 2 = 4$)."

To respect this infectiousness, an adequate content-responsive notion of bounds consequence must validate the following rules for disjunction:

$$\frac{[\Gamma, \varphi_i \succ \Delta] \qquad [\Gamma \succ \varphi_i, \Delta]}{[\Gamma, \varphi_0 \vee \varphi_1 \succ \Delta]}\,[\mathrm{L}\vee_w] \qquad \frac{[\Gamma, \varphi_i \succ \Delta] \qquad [\Gamma \succ \varphi_i, \Delta]}{[\Gamma \succ \varphi_0 \vee \varphi_1, \Delta]}\,[\mathrm{R}\vee_w]$$

and rules for conjunction:

$$\frac{[\Gamma, \varphi_i \succ \Delta] \qquad [\Gamma \succ \varphi_i, \Delta]}{[\Gamma, \varphi_0 \wedge \varphi_1 \succ \Delta]}\,[\mathrm{L}\wedge_w] \qquad \frac{[\Gamma, \varphi_i \succ \Delta] \qquad [\Gamma \succ \varphi_i, \Delta]}{[\Gamma \succ \varphi_0 \wedge \varphi_1, \Delta]}\,[\mathrm{R}\wedge_w]$$

As in the case of negation, the disjunction rule $[\mathrm{R}\vee_w]$ and conjunction rule $[\mathrm{L}\wedge_w]$ are special cases of $[\mathrm{R}\vee]$ and $[\mathrm{L}\wedge]$, respectively. These regularities cohere with veridical notions of bounds, e.g., if one cannot assert φ_0 truthfully, one cannot assert $\varphi_0 \wedge \varphi_1$. Thus, their admissibility in standard ST is expected.

In contrast, the rules $[\mathrm{L}\vee_w]$ and $[\mathrm{R}\wedge_w]$ are clearly not admissible in ST. Despite the Liar's anomaly, $\lambda \vee \xi$ is not paradoxical for true ξ—it is straightforwardly and simply true. Thus, although $[\lambda \succ \quad]$ may be out-of-bounds, these veridical defects need not carry over to $[\lambda \vee \xi \succ \quad]$. In short, although "padding" may successfully dispel veridical defects, the operation is insufficient to exorcise offending content.[5]

A phenomenological point provides further empirical justification for the out-of-boundedness of the example `@#$%&!` $\vee \xi$. As described in [10], the assertion of a disjunction in many contexts has a weakly conjunctive nature, i.e., to assert a disjunction often carries a presupposition that the utterer takes both disjuncts to be conceivable. (Compare this with the reading of free choice disjunction, in which "α may φ or ψ" is taken to entail both "α may φ" and "α may ψ".) Independently of the matter of content, many cases of "monstrous" utterances receive more than a merely veridical condemnation but demand a stronger, modal condemnation. In the public forum, for example, it is not merely condemnable to assert a number of repugnant positions; it is condemnable also to signal that one is willing to consider their truth for the sake of argument.

4.2 Quantification and Monstrous Content

The interaction between content-sensitive bounds and quantifiers is complicated by the following observation: In a first-order language, there are distinct vehicles

[4]To be clear, there may be contexts in which even "$2 + 2 = 4$" would be out-of-bounds due to content-theoretic considerations. Introducing arithmetical statements in many contexts would be considered unusual; such an utterance would likely be a *faux pas* at a funeral, for example. The "monstrous" content that we have considered is just an exceptionally obvious sort of defect.

[5]A detailed discussion of this in the case of privacy is carried out in [9].

through which a sentence-part may contribute to the content of the whole. Put simply, content-theoretic contribution are carried over either predicates or terms, that is, the provenance of the monstrous content of $P(t)$ can be either P or t. An adequate theory of quantification under the specter of monstrous content demands careful consideration of each case.

We can consider the case in which a predicate acts as carrier by revisiting the earlier example of talking about bombs at an airport. Consider a predicate $\texttt{Bomb}(x)$ applying to a t when t is an explosive device. Here, the predicate is clearly the syntactic object carrying the monstrous content. The out-of-boundedness of positions $[\texttt{Bomb}(t) \succ \quad]$ and $[\quad \succ \texttt{Bomb}(t)]$ in the context of airport security follows from an incompatibility between the context and the application of the predicate $\texttt{Bomb}(x)$. The term t need not play a role.

The practice in Judaism of refraining from writing or pronouncing the name of God, leading one to encounter the term 'G-d,' illustrates the case in which the term is the carrier. In this context, the out-of-boundedness of any positions $[R(\texttt{God}) \succ \quad]$ and $[\quad \succ R(\texttt{God})]$ as the positions risk flouting the religious norm. That this feature is syntax independent can be observed by examples. For one, the feature holds of any proper name of God, requiring the use of a definite description like *Adonai*. Second, other examples exhibit similar features. In a context in which a friend has suffered a recent, stressful break-up with a romantic partner $\texttt{Fred}$, to pivot to the topic of the former partner—under any name—would be a transgression.

Despite symmetry between the two cases, recognizing the distinction is critical to articulating the distinct requirements each imposes on quantification. To illustrate, consider the influence on the introduction of an existential quantifier:

$$\frac{[\texttt{Bomb}(t) \succ \quad] \qquad [\quad \succ \texttt{Bomb}(t)]}{[\exists x(\texttt{Bomb}(x)) \succ \quad]}$$

It is natural to read the above as a principle that whenever both the assertion and denial of $\texttt{Bomb}(t)$ are transgressions, even the modest assertion that there exists something satisfying $\texttt{Bomb}(x)$ is out-of-bounds.

This requires qualification on the choice of t, however. In this case, validity appears indifferent to selection of t because the monstrosity lies in the contribution of the predicate rather than a term. Thus, a requirement that t be arbitrary would be necessary to mark the predicate or matrix as the source of the monstrosity.

Conversely, when both assertion or denial of $\varphi(t)$ are out-of-bounds because of the contribution of t alone, it appears insufficient to thereby conclude that $\exists x \varphi(x)$ is itself-out-of-bounds. For example, in the example of the romantic partner $\texttt{Fred}$—who, courtesy dictates, may not be brought up in conversation—the following inference:

$$\frac{[\varphi(\texttt{Fred}) \succ \quad] \qquad [\quad \succ \varphi(\texttt{Fred})]}{[\exists x(\varphi(x)) \succ \quad]}$$

seems illegitimate. In a sensitive context in which any mention of the romantic partner $\texttt{Fred}$ will cause distress, surely $[\varphi(\texttt{Fred}) \succ \quad]$ and $[\quad \succ \varphi(\texttt{Fred})]$ are out-of-bounds. But such a case hardly calls for silence with respect to, e.g., all predicates on grounds that they could apply to $\texttt{Fred}$.

This suggests that concluding that $\exists x \varphi(x)$ is monstrous is licensed only in case the defect is borne by the contribution of $\varphi(x)$ alone. This is analogous to cases in which applying $\varphi(x)$ to any term t should result in a monstrosity, i.e., it would match the general shape of the `Bomb` case. Hence, one might codify this with a qualification on the choice of terms. Classical sequent calculi disallow the introduction of an existential quantifier in the antecedent for any term, requiring instead that the term that is being abstracted away is sufficiently generic, frequently by requiring that it is a free variable already (or that it is an arbitrary term that appears in no side formula).

Although I find the case in which the predicate is the contributor of monstrosity more compelling than that in which the term is the contributor, I will consider both cases in the sequel.

4.3 The Truth Predicate and Monstrous Content

I have suggested that propositions containing "monstrous content" are notable for bleeding through the use/mention barrier, a phenomenon that can be illustrated easily. Consider, e.g., a case in which some public figure faces a scandal in the press for having made a comment invoking some highly objectionable claim and let `@#$%&!` play the role of the offensive comment in this example. The various style manuals observed by press outlets—which are essentially compendiums regulating the setting of bounds—recommend against printing such a claim, irrespective of whether the words are being used or mentioned. Consequently, in many cases, the monstrosity of a statement `@#$%&!` is severe enough to debar its mention in even quotational contexts. In practice, such assertions may be referred to only in an oblique and indirect fashion.

The formal tools laid out in [4] are heavily motivated by resolving semantic paradoxes such as the Liar, explaining the language's inclusion of a unary truth operator **T** that applies to terms acting as "distinguished names" for formulae in the language. Where $\ulcorner\varphi\urcorner$ is the distinguished term that has as its intended denotation the formula φ, then, the intended interpretation of the formula $\mathbf{T}(\ulcorner\varphi\urcorner)$ is the statement that φ is true. The case of the truth predicate introduces formal apparatus analogous to the quotation of a sentence, providing an opportunity to get more precise with some of my remarks about the monstrosity of a statement "bleeding through the use/mention divide."

We may recast the informal observation above in these terms; a content-theoretic defect in a statement `@#$%&!` lifts to any statement including its quotation or, more precisely, is inherited by any formula in which the name $\ulcorner$`@#$%&!`$\urcorner$ appears.

I consider it to be obviously true that the "monstrosity" of uttering (or rejecting) a formula ψ should stand or fall with the "monstrosity" of the formula $\mathbf{T}(\ulcorner\psi\urcorner)$. Whenever a position $[\varphi \succ \quad]$ is out-of-bounds–on either veridical or content-theoretic grounds—then moving to the position $[\mathbf{T}(\ulcorner\varphi\urcorner) \succ \quad]$ clearly does nothing to efface or rinse away the defectiveness of φ; such a move would be rightly seen as merely "doubling down" on an inappropriate position.

As before, some facets of this case are reflected in the veridical notion of bounds. Where `@#$%&!` is an *ad hominem* smear, for example, a reputable journalist will not (or ought not, at least) include the statement $\mathbf{T}(\ulcorner$`@#$%&!`$\urcorner)$ in print due to its falsehood, in which case the grounds for this proscription could be satisfactorily explained by veridical concerns alone. However, content-responsive bounds remain more severe. The objectionable nature of @#\$%&! seems to extend even to the case in which its truth is emphatically rejected, e.g., by the sentence $\neg\mathbf{T}(\ulcorner$@#\$%&!$\urcorner)$.[6] To consider only the truth or falsity as salient bounds-setting considerations might lead one to suggest that $[\quad\succ\mathbf{T}(\ulcorner$@#\$%&!$\urcorner)]$—the uncleansed rejection of the truth of @#\$%&!—to be an acceptable position.

Thus, one of the requirements for adequacy of a sequent calculus is that if φ includes monstrous content—i.e., both $[\varphi\succ\quad]$ and $[\quad\succ\varphi]$ are out-of-bounds—whenever the distinguished term $\ulcorner\varphi\urcorner$ appears in a formula ψ, both $[\psi\succ\quad]$ and $[\quad\succ\psi]$ are likewise out-of-bounds. This can be guaranteed by the following rules:

$$\frac{[\varphi\succ\quad]\qquad[\quad\succ\varphi]}{[\mathbf{T}\ulcorner\varphi\urcorner\succ\quad]}\,[\mathrm{L}\mathbf{T}_w]\qquad\qquad\frac{[\varphi\succ\quad]\qquad[\quad\succ\varphi]}{[\quad\succ\mathbf{T}\ulcorner\varphi]}\,[\mathrm{R}\mathbf{T}_w]$$

As special cases of $[L\,\mathbf{T}]$ and $[R\,\mathbf{T}]$, these requirements are in accord with the rules of ST.

We now consider the formalization of the semantical features informed by the foregoing observations concerning the interaction of monstrous content with natural bounds of discourse.

5 Two Formal Calculi

Let us now turn to the task of formalizing the properties of content-sensitive bounds considered in Section 4. In the case of sentential connectives, argued that the alignment between the weak Kleene truth tables and the shape of content-theoretic bounds decisively favor the weak Kleene matrices depicted in Fig. 1 over the strong matrices of ST.

The intended interpretations of the truth-values 0, $\frac{1}{2}$, and 1 are *falsity*, *content-theoretically defective*, and *truth*, respectively.[7]

[6] Consequently, one might expect this bubbling up of the defective content from the formula to the distinguished term would hold even if we supplemented the formal language with a “repugnance” predicate $\mathbf{R}$ in which $\mathbf{R}(\ulcorner\varphi\urcorner)$ expresses the assertion that φ is condemnable or otherwise monstrous. Practice bears this out, *n.b.* the dilemma faced by parents and educators who must teach children to avoid certain terms without being able to identify those very terms. It is worth pointing out that, per Gupta and Martin’s results in [11], although Kripke’s fixed point models are not guaranteed in case strong Kleene is enriched with such a repugnance predicate, it can be added to weak Kleene theories of truth. A repugnance predicate is thus incompatible with stock ST but can be added to the weak version we discuss in this paper.

[7] I have described several distinct kinds of content-theoretic defects. It is reasonable to question whether the single truth-value $\frac{1}{2}$ can serve as a catch-all, i.e., why distinct types of defect do not receive distinct truth-values. The collection of such values would continue to be *infectious* in the sense that a formula must receive a value from this collection whenever any of its subformulae receive a value from this collection. Thus, the results in [12]—showing that *linearly ordered* collections of infectious truth-values can be eliminated in favor of a single value—can be easily adapted to show that a single value $\frac{1}{2}$ will suffice. I appreciate a referee’s raising this issue.

$\neg$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	0	$\frac{1}{2}$	0

$\vee$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

Fig. 1 Weak Kleene matrices

However, conclusions concerning quantification were far more preliminary. Despite my sense that the strong Kleene quantifiers are more compelling as an appropriate formalization of how content-sensitive bounds treat quantified formulae, one can find cases whose regularities match the weak quantifiers as well.

I will not make the determination here. Rather, I acknowledge the reasonableness of both options by introducing two calculi—*intermediate* ST (iST) and *weak* ST (wST)—distinguished from one another by the respective interpretation of the quantifiers. Should one believe that quantifiers in the setting of content-sensitive bounds consequence requires no revision to ST, then one has iST as an option. Should one believe otherwise—that the alignment between content-theoretic bounds and the weak Kleene scheme must boldly extend to the quantifiers—then wST captures this intuition as well.[8]

A surprising feature concerning how these systems relate to ST is worth mentioning before formally introducing the two logics. If one interprets validity in the standard fashion—that is, as preservation of designated values—the weak and strong Kleene matrices straightforwardly induce *weak Kleene logic* (Σ_0) and *strong Kleene logic* (K_3), respectively. It is easily confirmed that weak Kleene logic is strictly weaker than strong Kleene logic, i.e., the validities of Σ_0 are properly included in the validities of K_3. However, as we will see below, construed on the *strict-tolerant* notion of validity, the systems governed by the weak Kleene matrices turn out to be stronger than ST, i.e., the rules characterizing both iST and wST properly include the rules of ST described in Definition 8. Thus, it is somewhat surprising that the weak systems iST and wST should be defined by strengthening ST by the introduction of additional rules.

Finally, it is worth noting related work incorporating a strict-tolerant reading of the weak Kleene matrices. The propositional fragment of the present systems iST or wST acted as the centerpiece of [16], in which the authors argued that propositional wST provided a compelling solution to long-standing philosophical debates concerning validity in the logics of nonsense championed by Halldén and Bochvar in, e.g., [17] and [18]. First-order wST was indispensable to Paoli and Pra Baldi's recent work in

[8] The discussion in [9] considers the "immune" quantifiers introduced in [13]—so called due to their analogy to the immune matrices of [14]—to be potentially fruitful. However, evaluations on Kripke's models of [15] with the immune quantifiers will not be *monotonic*, which is a critical ingredient of proving completeness.

[19], in which a new proof theory for paraconsistent weak Kleene (PWK) rests on the foundation of a weak ST sequent calculus. Finally, Fitting's [20] applies the strict-tolerant approach to the closely related logic of bilattices with cut-down operators that had been investigated in [21].

5.1 Intermediate Strict-Tolerant Logic iST

Both iST and wST capture a theory of content-sensitive bounds consequence in which the pathologies of a sentence like @#$%&! necessarily lift to (or infect) complexes in which they appear. An appeal to disjunction introduction may iron over veridical defects in a statement like @#$%&!, i.e., the disjunction @#$%&! $\vee \varphi$ has the potential to be true given a suitable φ. But the appearance of @#$%&! in the complex entails the preservation of its content in the complex, along with the out-of-boundedness of its assertion.

However, the account of bounds characterized by the logic iST limits its revisionism to the sentential connectives. One endorsing iST might note an important distinction between *disjunction introduction* and *existential generalization*. Disjunction introduction involves no bowdlerization of the individual disjuncts to which the rule is applied; the subformulae remain intact and unblemished in the complex. In contrast, introducing an existential quantifier to a formula $\varphi(t)$ potentially eliminates the term t while proceeding to a formula $\exists x\varphi(x)$. Thus, unlike the case of disjunction, the subformula $\varphi(t)$ need not appear in the complex *in toto*.

Consequently, in case t alone is the component leading to the monstrosity of a formula $\varphi(t)$, the monstrosity vanishes when t is eliminated. As a quantified formula $\exists x\varphi(x)$ need not inherit the content-theoretic defects of its substitution instances, the weak Kleene quantifiers might be considered to be inadequate devices to represent content-theoretic bounds.

It is thus natural that a sequent calculus for iST must involve revisions only to ST's account of the connectives. Essentially, we enrich the sequent calculus for ST described in Appendix A with the novel rules discussed in Section 4.1:

Definition 1 The sequent calculus iST_1 (for *intermediate* ST) is defined by adding the following rules to Definition 8:

$$\frac{[\Gamma, \varphi_i \succ \Delta] \qquad [\Gamma \succ \varphi_i, \Delta]}{[\Gamma, \varphi_0 \vee \varphi_1 \succ \Delta]}\,[\mathrm{L}\vee_w] \quad \frac{[\Gamma, \varphi_i \succ \Delta] \qquad [\Gamma \succ \varphi_i, \Delta]}{[\Gamma \succ \varphi_0 \wedge \varphi_1, \Delta]}\,[\mathrm{R}\wedge_w]$$

with the proviso that $i \in \{0, 1\}$

I have championed the use of the weak Kleene matrices depicted in Fig. 1 and believe that its ST-type interpretation (a weak ST framework) captures a content-theoretic conception of bounds-setting. Instances of logics employing the weak Kleene framework include the classical or "internal" fragments of the nonsense logics described by Bochvar in [18] or Halldén in [22].

These matrices are notable for the property of *infectiousness* of the third value, i.e., that the appearance of a subformula evaluated as $\frac{1}{2}$ in a complex entails that the complex itself is evaluated as $\frac{1}{2}$. The paradigm interpretations that support infectiousness—e.g., meaninglessness or off-topic-edness—enjoy a clear parallel with the "infectiousness" of monstrous content.

For model theory, we can define *intermediate* Kleene-Kripke models by suitable modifications to the semantics of standard ST (described in Appendix A). In particular, we make minor adjustments to Definition 9:

Definition 2 An *intermediate Kleene-Kripke* model is defined analogously to Definition 9 with the exception that sentential connectives are interpreted by the weak Kleene matrices.

Importantly, quantifiers remain interpreted by the strong Kleene quantifiers. An analogous revision to Definition 10 yields a notion of validity in iST:

Definition 3 An *intermediate* ST-*counterexample* to a position $[\Gamma \succ \Delta]$ is an intermediate Kleene-Kripke model for which $I[\Gamma] \cap \{0, \frac{1}{2}\} = \varnothing$ and $I[\Delta] \cap \{\frac{1}{2}, 1\} = \varnothing$. The sequent $[\Gamma \succ \Delta]$ is *intermediately* ST-*valid* if there are no intermediate ST-counterexamples.

In the strict-tolerant terminology, an intermediate ST counterexample to $[\Gamma \succ \Delta]$ is one in which every formula in Γ is *strictly true* (i.e., assigned 1) and every formula in Δ is *strictly false* (i.e., assigned 0).

As will be shown in Appendix B, we have soundness and completeness of iST_1 with respect to intermediate ST validity:

Observation 1 $[\Gamma \succ \Delta]$ *is intermediately* ST *valid iff* $[\Gamma \succ \Delta]$ *is derivable in* iST_1.

Moreover, like ST, we have agreement with classical logic in the **T**-free fragment of the language.

Observation 2 *If no occurrences of* **T** *appear in formulae of* $\Gamma \cup \Delta$*, then:*

$[\Gamma \succ \Delta]$ *is derivable in* iST_1 *iff* $[\Gamma \succ \Delta]$ *is derivable classically.*

This establishes that many of the facts described of ST in [4]—like the varieties of *limited Cut*—transfer directly to the case of iST.

Now, let us examine the companion to iST in which we use the weak Kleene quantifiers.

5.2 Weak Strict-Tolerant Logic wST

I have favored the strong Kleene quantifiers due to concerns about cases in which the content of a term t is monstrous; in such cases, existential generalization will remove the defect.

But perhaps it is a stretch to countenance cases in which a term—a mere proper name—is sufficiently rich in content to ensure that its expression is monstrous. E.g., suppose that one believes that the content of a proper name t is exhausted by its referent. If this in fact is the limit of the content of a term, to say that a term t is the lone source of the monstrous content expressed by $\varphi(t)$ is essentially to say that the very existence of the referent of t is monstrous.

The thesis that the monstrous content of a complex is not contributed by terms leaves only predicates as potential contributors. Despite the possible elimination of terms when introducing quantifiers, the predicates in a subformula necessarily remain untouched in the complex. Consequently, whenever φ includes monstrous content, that the predicates appearing in φ are preserved in $\exists x\varphi$ and $\forall x\varphi$ guarantee that the subformula's content-theoretic defects will also be inherited by the complex. And this type of infectiousness through quantifiers is precisely captured by the weak Kleene quantifiers.

To characterize wST requires only a modest enrichment of the theory of quantification encoded in iST:

Definition 4 The sequent calculus wST_1 (for *weak* ST) is defined by adding the following rules to Definition 1:

$$\frac{[\Gamma, \varphi(t) \succ \Delta] \qquad [\Gamma \succ \varphi(t), \Delta]}{[\Gamma, \exists x\varphi(x) \succ \Delta]}\,[\mathrm{L}\exists_w] \qquad \frac{[\Gamma, \varphi(t) \succ \Delta] \qquad [\Gamma \succ \varphi(t), \Delta]}{[\Gamma \succ \forall x\varphi(x), \Delta]}\,[\mathrm{R}\forall_w]$$

The commonly-accepted quantificational correlates to the three-valued weak Kleene matrices are described in [23] as follows, again, as operations on sets of truth-values:

$$\forall(X) = \begin{cases} 1 & \text{if } X = \{1\} \\ \frac{1}{2} & \text{if } \frac{1}{2} \in X \\ 0 & \text{otherwise} \end{cases} \quad \exists(X) = \begin{cases} 1 & \text{if } 1 \in X \text{ and } \frac{1}{2} \notin X \\ \frac{1}{2} & \text{if } \frac{1}{2} \in X \\ 0 & \text{otherwise} \end{cases}$$

Now, recall the Kleene-Kripke models of Definition 9 from Appendix A and consider the following definition:

Definition 5 A *weak Kleene-Kripke* model is defined analogously to Definition 9 with the exception that sentential connectives *and* quantifiers are interpreted by the weak Kleene interpretations.

As expected, we make reasonable modifications to Definition 10 to define wST validity:

Definition 6 A *weak* ST*-counterexample* to a position $[\Gamma \succ \Delta]$ is a weak Kleene-Kripke model for which $I[\Gamma] \cap \{0, \frac{1}{2}\} = \varnothing$ and $I[\Delta] \cap \{\frac{1}{2}, 1\} = \varnothing$. We say that $[\Gamma \succ \Delta]$ is *weakly* ST*-valid* if there are no weak ST-counterexamples.

as in the case of iST, we can infer the following:

Observation 3 $[\Gamma \succ \Delta]$ *is weakly* ST *valid iff* $[\Gamma \succ \Delta]$ *is derivable in* wST_1.

Observation 4 *If no occurrences of* $\mathbf{T}$ *appear in formulae of* $\Gamma \cup \Delta$*, then:*

$$[\Gamma \succ \Delta] \textit{ is derivable in } \mathsf{wST}_1 \textit{ iff } [\Gamma \succ \Delta] \textit{ is derivable classically.}$$

6 Discussion

I will now consider a few points for discussion, including some important directions to guide future extensions of this project and some reasonable objections to what we have described in the foregoing sections.

6.1 Unifying Veridical and Content-Theoretic Bounds

In the foregoing remarks, I have argued that conversational bounds are not determined solely by veridical features of context and a successful formalization of bounds must therefore be *content-responsive*. To precisely give an analysis of the semantic features of content-responsive bounds consequence, I have identified cases in which semantically defective statements are defective for including "monstrous" content. This has meant restricting our attention to cases in which the only explanation for contemporaneous out-of-boundedness of positions $[\varphi \succ \ ]$ and $[\ \succ \varphi]$ is due to the content of φ.

However, the reader may note that from the other sides of my mouth, I have expressed general agreement with one of the core motivating cases for ST—the case of the Liar sentence λ. At first blush, one may suspect a certain inconstancy on my part: After charging the veridical notion of bounds consequence approach with myopia, I have myself ignored the veridical case.

Consideration of the modest scope of my goals in this paper, however, should put this suspicion to rest. I have set out to demonstrate that content-theoretic conditions on bounds are at least as natural as veridical conditions with respect to their motivations and semantic theory. Examining content-theoretic bounds in isolation provided a well-behaved—if artificial—laboratory for investigating the resulting semantical and logical properties.

But it is equally obvious to me that in practice both veridical and content-theoretic concerns play a role in determining the bounds of acceptable discourse and a natural next step is to investigate the properties of a bounds consequence sensitive to both dimensions.

Describing an account of bounds consequence acknowledging the pathologies of both λ and `@#$%&!` will rest on an account of the interaction between them.[9] Consider the complex $\lambda \vee (\lambda \wedge$ `@#$%&!`$)$. From the perspective of either ST and wST, the positions $[\lambda \vee (\lambda \wedge$ `@#$%&!`$) \succ \ ]$ and $[\ \succ \lambda \vee (\lambda \wedge$ `@#$%&!`$)]$ should be jointly out-of-bounds and thus agree that $\lambda \vee (\lambda \wedge$ `@#$%&!`$)$ is pathological. Yet the nature of the pathology—veridical or content-theoretical—is left open.

The character of the interaction between the two types of defect will determine the resulting logic. To illustrate, let us use $\frac{1}{2}$ to represent the strong Kleene truth-value

[9]There exist traditions according to which λ should be regarded as meaningless. In these cases, as Kripke remarks, "[i]f we had regarded a Liar sentence as meaningless, presumably we would have had to regard any compound containing it as meaningless" [15, p. 700], i.e., weak Kleene would be the correct tool for both veridical and content-theoretic bounds. On such an account, there would be no need to examine the "interaction" between λ and `@#$%&!`; the results in [12] indicate that the value $\frac{1}{2}$ alone would accommodate both types of defects.

$\neg$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
$\mathfrak{u}$	$\mathfrak{u}$
0	1

$\wedge$	1	$\frac{1}{2}$	$\mathfrak{u}$	0
1	1	$\frac{1}{2}$	$\mathfrak{u}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathfrak{u}$	0
$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$
0	0	0	$\mathfrak{u}$	0

$\vee$	1	$\frac{1}{2}$	$\mathfrak{u}$	0
1	1	1	$\mathfrak{u}$	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\mathfrak{u}$	$\frac{1}{2}$
$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$
0	1	$\frac{1}{2}$	$\mathfrak{u}$	0

Fig. 2 $\mathbf{L}_{\mathfrak{ne}}$ matrices

(or veridical out-of-boundedness) and $\mathfrak{u}$ for the weak Kleene truth-value (or content-theoretic out-of-boundedness), while retaining our interpretation of the truth-values 0 and 1.[10] This licenses a concise description of the issue at hand: How ought truth functions (and quantifiers) evaluate the case in which both $\frac{1}{2}$ and $\mathfrak{u}$ are arguments?

There are two options that I find compelling, one in which we codify a natural interpretation of the interaction between the two types of pathology and another in which we remain more agnostic.

The first—most straightforward—option is to consider a semantics that extends the "infectiousness" the weak Kleene value $\mathfrak{u}$ enjoys over 0 and 1 to the strong Kleene value $\frac{1}{2}$. Such an approach would correspond to the following four-valued truth tables (which are characteristic of the system $\mathbf{L}_{\mathfrak{ne}}$ first described in [24]) described in Fig. 2.

The formal condition captures a strengthened version of an intuition that content-theoretic defectiveness overrides veridical considerations. We have seen that for many monstrous statements the out-of-boundedness of their assertion can be inferred by their content alone. In such cases, the immediacy of a statement's monstrosity obviates the need to calculate or otherwise evaluate its truth-value. Insofar as the defect of a statement like λ becomes apparent only upon the conclusion of such calculations, content-theoretic defects are arguably prior to veridical defects.

Surely, were all content-theoretic defects sufficiently obvious to eliminate the need for veridical considerations, the dominance of $\mathfrak{u}$ over $\frac{1}{2}$—and the accuracy of the above four-valued matrices—would be compelling. Yet we cannot rule out "subtle monstrosities"—statements for which the defects of their content are computationally nontrivial.

For example, consider a statement $\xi^\star$ including objectionable content that is extremely archaic in nature, e.g., an offensive idiom common hundreds of years ago. Moreover, suppose that this idiom is so obscure that most modern agents will find it nontrivial to discern its offensiveness, i.e., to recognize the monstrosity of $\xi^\star$ demands some time and effort—and possibly research—on the part of an agent. If $\xi^\star$ should also be obviously false—to the extent that recognition of its falsehood should be nearly immediate—the veridical defect of $\xi^\star$ may in general become apparent prior to its content-theoretic defect. Consequently, in practice, that the position $[\quad \succ \xi^\star]$ violates conversational norms might be attributed to veridical considerations.

[10] According to the above readings, it is natural to carry over the strict-tolerant interpretation by identifying a formula's *strict truth* as its receiving the value 1 and its *tolerant truth* as receiving any value other than 0.

As a consequence, there may exist two corner cases: Statements that are *subtly monstrous* yet *grossly paradoxical* and statements that are *grossly monstrous* yet *subtly paradoxical*. The second option, then, is to accept some level of agnosticism concerning how the two pathologies interact with one another. In such a case, although the defectiveness of statements λ and `@#$%&!` are sufficient to establish the defectiveness of $\lambda \vee (\lambda \wedge$ `@#$%&!`$)$, the nature of that defect is left underdetermined. While some audiences may shake their heads at its paradoxical nature and others may be repulsed by its profanity, all agree that its affirmation or denial are equally transgressions against the bounds of discourse.

The framework of *non-deterministic matrices* (or *nmatrices*) introduced by Avron and Lev in [25] allows a natural representation of the semantics of bounds consequence under such agnosticism. To illustrate, we will restrict ourselves to examining the case of the propositional connectives (several options for quantification may be found in e.g. [26] and [27]). One of the fundamental features of nmatrices is that truth functions map truth-values to non-empty sets of truth-values—rather than individual truth-values—which are interpreted as a range of allowable values for valuations.

That the agnosticism at the heart of the second option is limited to interactions between $\frac{1}{2}$ and $\mathfrak{u}$ has two consequences: First, in cases *lacking* such interaction, one should default to the values prescribed by the strong and weak Kleene matrices. Second, the characteristic agnosticism concerns the nature of the defect of such interactions rather the existence of a defect. Thus, for a formula like $\lambda \vee (\lambda \wedge$ `@#$%&!`$)$, the range of allowable values should be $\{\frac{1}{2}, \mathfrak{u}\}$. These considerations yield the "agnostic" nmatrices described in Fig. 3.

As usual, a valuation v is a map from a language to a set of truth-values. v will be considered *legal* if for every complex formula φ, $v(\varphi)$ belongs to the range of allowable values determined by the truth functions and the truth-values of its subformulae.

Embarking on a detailed investigation into which—or whether either—of these options is suitable is worthwhile; the problem of describing a unified theory is set aside for another day, however.

$\neg$	
1	$\{0\}$
$\frac{1}{2}$	$\{\frac{1}{2}\}$
$\mathfrak{u}$	$\{\mathfrak{u}\}$
0	$\{1\}$

$\wedge$	1	$\frac{1}{2}$	$\mathfrak{u}$	0
1	$\{1\}$	$\{\frac{1}{2}\}$	$\{\mathfrak{u}\}$	$\{0\}$
$\frac{1}{2}$	$\{\frac{1}{2}\}$	$\{\frac{1}{2}\}$	$\{\frac{1}{2}, \mathfrak{u}\}$	$\{0\}$
$\mathfrak{u}$	$\{\mathfrak{u}\}$	$\{\frac{1}{2}, \mathfrak{u}\}$	$\{\mathfrak{u}\}$	$\{\mathfrak{u}\}$
0	$\{0\}$	$\{0\}$	$\{\mathfrak{u}\}$	$\{0\}$

$\vee$	1	$\frac{1}{2}$	$\mathfrak{u}$	0
1	$\{1\}$	$\{1\}$	$\{\mathfrak{u}\}$	$\{1\}$
$\frac{1}{2}$	$\{1\}$	$\{\frac{1}{2}\}$	$\{\frac{1}{2}, \mathfrak{u}\}$	$\{\frac{1}{2}\}$
$\mathfrak{u}$	$\{\mathfrak{u}\}$	$\{\frac{1}{2}, \mathfrak{u}\}$	$\{\mathfrak{u}\}$	$\{\mathfrak{u}\}$
0	$\{1\}$	$\{\frac{1}{2}\}$	$\{\mathfrak{u}\}$	$\{0\}$

Fig. 3 "Agnostic" nmatrices

6.2 Objections

Any person who has successfully engaged in some interaction with another human being—or, if not successfully, at least has had an interaction that was not an abject catastrophe—should take it as obvious that there exist occasions in which various norms constrain our utterances so that some statements—despite their truth—are nevertheless proscribed. The old adage, for example, about discussing religion or politics at the dinner table is a normative circumscription of conversational bounds that emphasizes content over truth.

Nevertheless, one may suggest that such a case is weak and that it is obviously perilous to draw universal, semantic conclusions about consequence from antiquated etiquette manuals. But I do not think that this is too hasty or cavalier, if one grants the primary thesis about bounds consequence, namely, that conversational practices play a hand in determining the bounds.

Consider how one knows, for example, that some sentence or other is not tautologous. The most obvious method is to point to a context or situation in which the statement is false. The fact that one can point to a situation in which, say, France is a monarchy is sufficient to establish that "France is a democracy" is not universally valid. The case of religion or politics is no different; the existence of a context in which conversational bounds are not veridically determined is sufficient to demonstrate that the range of models is wider than anticipated.

Another objection may be that I have confused *terms* for *meaning* and that the fact that some statements have particular modes of presentation or names under which they are unassertable says little about the assertability of the meanings of those statements. Clearly, some obvious contenders for the label of "monstrous content" are not monstrous up to synonymy; in many cases of expletives in English—often those involving some function of the human body—one finds that a Germanic "vulgar" term has a synonymous Latinate term that is judged acceptable—and one may be led to object that this indicates that the assumption in this paper groundlessly draws semantic conclusions from either syntactic or pragmatic considerations.

Clearly, though, proscriptions against monstrous content are not explainable entirely as syntactic proscriptions. The most obvious evidence against this lies in that a number of familiar cases of offensive terms are *homographs*. Were monstrosities barred only as particular strings of characters, then their standard, non-profane counterparts could not be expressed either. But this is clearly not the case.

Nevertheless, at times it appears that some expletives are entirely synonymous with other, non-profane terms, which suggest cases in which syntax is the key differentiator. Undoubtedly, if one fixes a statement η about e.g. the excretory system appropriate for a seminar on human anatomy, one can think of a dozen *prima facie* synonymous (albeit puerile) statements η' that, despite their synonymy with η, would be out-of-bounds to utter in the same lecture hall. I do not deny that this suggests the existence of cases in which the bounds take syntactic choices into consideration. Nevertheless, the very same η and any variant η' considered by the reader would likely be inappropriate in, say, a synagogue on the basis of that shared meaning alone.

It is true that, e.g., a pejorative may have a natural counterpart in an "unobjectionable" fragment of the language but it seems that the pejorative is ampliative, that is, some additional meaning is conveyed when a pejorative is deployed by a speaker. In other words, it is far from clear that synonymy indeed holds between the pejorative and its "unobjectionable" counterpart, nor is it evident that substitutions may be made in a complex *salva significatione*.

7 Concluding Remarks

Over the course of this paper, I have sought to cite a host of very familiar, very recognizable cases in which positions are judged as out-of-bounds due to considerations of content or subject-matter. Given a cursory survey of societal expectations, it is nearly trivial to point to cases in which the a position's subject-matter is sufficiently monstrous to override any evaluation of the position's truth.

What is *not* trivial—and what I have sought to demonstrate here—is that such considerations enjoy a constancy whose patterns are as axiomatizable as veridical considerations. We have seen that the shape of these regularities closely follows the contours of weak Kleene logic, whence the analysis immediately coheres with a number of well-entrenched traditions in the philosophy of logic.

There remain several avenues for future work. There remains work to generate a unified picture of the interplay between veridical *and* content-theoretic factors. It is also worthwhile to investigate how monstrous content might impact other topic-sensitive frameworks, like Francesco Berto's topic-sensitive intentional modals of [28] or [29]. Despite these gaps, I believe that I have laid a compelling foundation for a larger project of describing a logic of bounds consequence that accounts for the rich and varied cartography of considerations by which social agents in practice determine the bounds.

Appendix A: Strict-Tolerant Logic ST

Afford the truth-values 1, $\frac{1}{2}$, and 0 the standard Kleene interpretations of *true*, *neither true nor false*, and *false*, respectively. The strict-tolerant logic ST defined in [5] and [7] distinguishes between *strict* truth—cases in which a sentence affirmatively takes the value 1—and *tolerant* truth—cases of non-falsity taking either value 1 or $\frac{1}{2}$.

To precisely review ST, we first describe a first-order language with a truth predicate. We take the *signature* of the language to be arbitrary, assuming only that there exists a set **Var** of variables and a set **At** that includes atomic formulae whose terms are elements of **Var** or members of a set of standard constants, each of which is constructed in the usual fashion. One of the most noteworthy features of ST is its ability to support a *transparent truth predicate*. The inclusion of the truth predicate T demands that we include a term-forming operator $\ulcorner \cdot \urcorner$ that produces a distinguished term $\ulcorner \varphi \urcorner$ for every formula φ in the language.

Definition 7 $\mathcal{L}$ is a first-order language in a signature with $\psi \in \mathbf{At}$ and $x \in \mathbf{Var}$.[11]

$$\varphi ::= \psi | \mathsf{T}(\ulcorner\varphi\urcorner) | \neg\varphi | \varphi \vee \varphi | \varphi \wedge \varphi | \exists x\varphi | \forall x\varphi$$

Note that languages defined in this way are the same as the standard first-order language but for the inclusion of T.

We reproduce the sequent calculus for ST including transparent truth as described in [4].[12] In what follows, call a term t *novel* with respect to a sequent $[\Gamma, \varphi(t) \succ \Delta]$ or $[\Gamma \succ \Delta, \varphi(t)]$ if t appears in no side formulae.[13]

Definition 8 The logic ST corresponds to the following sequent calculus. This includes *structural rules*:

$$\frac{}{[\varphi \succ \varphi]}\ [\text{Axiom}] \qquad \frac{[\Gamma \succ \Delta]}{[\Gamma, \Gamma' \succ \Delta, \Delta']}\ [\text{Weak}]$$

$$\frac{[\Gamma, \varphi, \varphi \succ \Delta]}{[\Gamma, \varphi \succ \Delta]}\ [\text{Contract}] \qquad \frac{[\Gamma \succ \Delta, \varphi, \varphi]}{[\Gamma \succ \Delta, \varphi]}\ [\text{Contract}]$$

Rules for *propositional connectives*:

$$\frac{[\Gamma \succ \Delta, \varphi]}{[\Gamma, \neg\varphi \succ \Delta]}\ [\text{L}\neg] \qquad \frac{[\Gamma, \varphi \succ \Delta]}{[\Gamma \succ \Delta, \neg\varphi]}\ [\text{R}\neg]$$

$$\frac{[\Gamma, \varphi \succ \Delta] \qquad [\Gamma, \psi \succ \Delta]}{[\Gamma, \varphi \vee \psi \succ \Delta]}\ [\text{L}\vee] \qquad \frac{[\Gamma \succ \Delta, \varphi_i]}{[\Gamma \succ \Delta, \varphi_0 \vee \varphi_1]}\ [\text{R}\vee], i \in \{0, 1\}$$

$$\frac{[\Gamma, \varphi_i \succ \Delta]}{[\Gamma, \varphi_0 \wedge \varphi_1 \succ \Delta]}\ [\text{L}\wedge], i \in \{0, 1\} \qquad \frac{[\Gamma \succ \Delta, \varphi] \qquad [\Gamma \succ \Delta, \psi]}{[\Gamma \succ \Delta, \varphi \wedge \psi]}\ [\text{R}\wedge]$$

Rules characterizing a *transparent truth predicate*:

$$\frac{[\Gamma, \varphi \succ \Delta]}{[\Gamma, \mathbf{T}(\ulcorner\varphi\urcorner) \succ \Delta]}\ [\text{L}\mathbf{T}] \qquad \frac{[\Gamma \succ \Delta, \varphi]}{[\Gamma \succ \Delta, \mathbf{T}(\ulcorner\varphi\urcorner)]}\ [\text{R}\mathbf{T}]$$

And rules for the *quantifiers*:

$$\frac{[\Gamma, \varphi(t) \succ \Delta]}{[\Gamma, \forall x\varphi(x) \succ \Delta]}\ [\text{L}\forall], t \text{ any term} \qquad \frac{[\Gamma \succ \Delta, \varphi(s)]}{[\Gamma \succ \Delta, \forall x\varphi(x)]}\ [\text{R}\forall], s \text{ novel}$$

$$\frac{[\Gamma, \varphi(s) \succ \Delta]}{[\Gamma, \exists x\varphi(x) \succ \Delta]}\ [\text{L}\exists], s \text{ novel} \qquad \frac{[\Gamma \succ \Delta, \varphi(t)]}{[\Gamma \succ \Delta, \exists x\varphi(x)]}\ [\text{R}\exists], t \text{ any term}$$

In the *Kleene-Kripke* models of [4]—and the most frequently encountered approaches to Liar sentences, like those of Field or Priest—the semantic behavior of statements follows the *strong Kleene matrices* as described in Fig. 4.

[11] *N.b.* that the language is essentially *two-sorted*. The set of standard terms is provided by the signature and the distinguished terms $\ulcorner\varphi\urcorner$ defined in parallel with the language. As a distinguished term will not appear in any formula in **At**, there is no need to define the language with a dual recursion.

[12] We include a rule of contraction not present in Ripley's formulation to make some proofs easier; as shown in [4], the addition of contraction makes no difference.

[13] *N.b.* that it is a common alternative to require that t be a variable rather than an arbitrary "novel" term. The choice of presentation is intended to preserve maximal continuity with e.g. [4].

$\neg$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

Fig. 4 Strong Kleene matrices

The strong Kleene interpretation of the quantifiers is as follows, where the functions are operations on sets of truth-values:

$$\forall(X) = \min(X) \quad \exists(X) = \max(X)$$

Bounds consequence, as understood by Ripley, mirrors these matrices. Hence, if the position $[\varphi \succ \psi]$ is not out-of-bounds, $[\lambda \vee \varphi \succ \psi]$, too, remains in-bounds. More formally, a Kleene-Kripke model is defined in [4] as follows:

Definition 9 A Kleene-Kripke model for a language $\mathscr{L}$ is a pair $\langle D, I \rangle$ where D is a domain of elements such that $\mathscr{L} \subseteq D$ and I is an interpretation such that:

- for a term t, $I(t) \in D$
- where $\ulcorner\varphi\urcorner$ is a distinguished name for φ, $I(\ulcorner\varphi\urcorner) = \varphi$
- for an n-ary predicate R, $I(R)$ maps n-tuples from D^n to $\{0, \frac{1}{2}, 1\}$
- for atoms $P(\vec{t})$, $I(P(\vec{t})) = I(P)(I(\vec{t}))$
- sentential connectives and quantifiers are given the strong Kleene interpretation
- $I(\mathsf{T}\ulcorner\varphi\urcorner) = I(\varphi)$ for all formulae φ

Then we define strict-tolerant validity as follows:

Definition 10 An ST-*counterexample* to a position $[\Gamma \succ \Delta]$ is a Kleene-Kripke model such that $I[\Gamma] \cap \{0, \frac{1}{2}\} = \varnothing$ and $I[\Delta] \cap \{\frac{1}{2}, 1\} = \varnothing$. The position $[\Gamma \succ \Delta]$ is ST-*valid* if there are no ST-counterexamples.

The sequent calculus described in [4] is sound and complete with respect to the consequence relation induced by the strong ST framework.

Appendix B: Formal Proofs

I provide detailed proofs of the formal results in this paper, including the soundness and completeness of the calculi iST_1 and wST_1 with respect to the provided model theory.

The sequent calculus iST_1 has been defined by *supplementing* ST with principles concerning out-of-bounds positions that ST does not recognize. From an interpretative standpoint, this makes sense; there are two kinds of cases—veridical and content-theoretical—that can make the assertion of a disjunction out-of-bounds.

From a *formal* perspective, the resulting sequent calculi's possessing pairs of rules complicate the demonstration of meta-theoretic properties like completeness.

In order to make for a simpler proof, we initially will establish an equivalence between the calculi iST_1 and wST_1, on the one hand, and alternative sequent calculi iST_2 and wST_2, respectively.

Definition 11 The sequent calculus iST_2 is defined by replacing the left disjunction rules ([L∨] and [L$\vee_w$]) in iST_1 with the following rule:

$$\dfrac{[\Gamma, \varphi, \psi \succ \Delta] \qquad [\Gamma, \varphi \succ \Delta, \psi] \qquad [\Gamma, \psi \succ \Delta, \varphi]}{[\Gamma, \varphi \vee \psi \succ \Delta]}\ [\mathrm{L}\vee^\star]$$

and replacing the right conjunction rules ([R∧] and [R$\wedge_w$]) with the single rule:

$$\dfrac{[\Gamma \succ \Delta, \varphi, \psi] \qquad [\Gamma, \varphi \succ \Delta, \psi] \qquad [\Gamma, \psi \succ \Delta, \varphi]}{[\Gamma \succ \Delta, \varphi \wedge \psi]}\ [\mathrm{R}\wedge^\star]$$

Definition 12 The sequent calculus wST_2 is defined by replacing the disjunction and conjunction rules of wST_1 in the above fashion while additionally replacing the left particular quantifier rules ([L∃] and [L$\exists_w$]) with the single rule:

$$\dfrac{[\Gamma, \varphi(s), \varphi(t) \succ \Delta] \qquad [\Gamma, \varphi(s) \succ \Delta, \varphi(t)]}{[\Gamma, \exists x \varphi(x) \succ \Delta]}\ [\mathrm{L}\exists^\star]$$

(with the proviso that s is novel) and replacing the right universal quantifier rules ([R∀] and [R$\forall_w$]) with the single rule:

$$\dfrac{[\Gamma, \varphi(t) \succ \Delta, \varphi(s)] \qquad [\Gamma \succ \Delta, \varphi(t), \varphi(s)]}{[\Gamma \succ \Delta, \forall x \varphi(x)]}\ [\mathrm{R}\forall^\star]$$

(with the proviso that s is novel).

Lemma 1 *The calculi iST_1 and iST_2 are equivalent, as well as the calculi* wST_1 *and* wST_2.

Proof We prove this on induction of complexity of wST_1 and wST_2 proofs, i.e., we take as induction hypothesis that every subproof of a wST_1 proof has a corresponding wST_2 proof. As the two share axioms, the basis step is resolved. The intermediate case follows along analogous lines.

- To prove the interderivability between [L∨] and [L$\vee_w$] on the one hand and [L$\vee^\star$], we first show that the two wST_1 rules are emulable in wST_2 and then show that the wST_2 rule is emulable in wST_1. First, suppose that one has wST_1 proofs Ξ_0 and Ξ_1 of the sequents $[\Gamma, \varphi \succ \Delta]$ and $[\Gamma, \psi \succ \Delta]$. Then by induction hypothesis, we have proofs $\Xi_0^\star$ and $\Xi_1^\star$ in wST_2 of these sequents. We construct a proof

$$\dfrac{\dfrac{\dfrac{\Xi_0^\star}{[\Gamma, \varphi \succ \Delta]}}{[\Gamma, \varphi, \psi \succ \Delta]}\ [\mathrm{Weak}] \qquad \dfrac{\dfrac{\Xi_0^\star}{[\Gamma, \varphi \succ \Delta]}}{[\Gamma, \varphi \succ \Delta, \psi]}\ [\mathrm{Weak}] \qquad \dfrac{\dfrac{\Xi_1^\star}{[\Gamma, \psi \succ \Delta]}}{[\Gamma, \psi \succ \Delta, \varphi]}\ [\mathrm{Weak}]}{[\Gamma, \varphi \vee \psi \succ \Delta]}\ [\mathrm{L}\vee^\star]$$

Alternately, one may have wST_1 proofs Ξ_0 and Ξ_1 of the sequents $[\Gamma, \xi \succ \Delta]$ and $[\Gamma \succ \Delta, \xi]$ ' where $\xi = \varphi$ or $\xi = \psi$. Without loss of generality, we consider the former case

$$\dfrac{\dfrac{\dfrac{\Xi_0^\star}{[\Gamma, \xi \succ \Delta]}}{[\Gamma, \xi, \zeta \succ \Delta]}\,[\text{Weak}] \quad \dfrac{\dfrac{\Xi_0^\star}{[\Gamma, \xi \succ \Delta]}}{[\Gamma, \xi \succ \Delta, \zeta]}\,[\text{Weak}] \quad \dfrac{\dfrac{\Xi_1^\star}{[\Gamma \succ \Delta, \xi]}}{[\Gamma, \zeta \succ \Delta, \xi]}\,[\text{Weak}]}{[\Gamma, \xi \vee \zeta \succ \Delta]}\,[\text{L}\vee^\star]$$

Conversely, suppose that one has wST_2 proofs of $[\Gamma, \varphi, \psi \succ \Delta]$, $[\Gamma, \varphi \succ \Delta, \psi]$, and $[\Gamma, \psi \succ \Delta, \varphi]$; call these Ξ_0, Ξ_1, and Ξ_2, respectively. Then by induction hypothesis, we have proofs in wST_1 $\Xi_0^\star$, $\Xi_1^\star$, and $\Xi_2^\star$ of these sequents. From these, we are able to construct the following wST_1 proof of $[\Gamma, \varphi \vee \psi \succ \Delta]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Xi_0^\star}{[\Gamma, \varphi, \psi \succ \Delta]} \quad \dfrac{\Xi_1^\star}{[\Gamma, \varphi \succ \Delta, \psi]}}{[\Gamma, \varphi \vee \psi, \varphi \succ \Delta]}\,[\text{L}\vee_w] \quad \dfrac{\dfrac{\Xi_0^\star}{[\Gamma, \varphi, \psi \succ \Delta]} \quad \dfrac{\Xi_2^\star}{[\Gamma, \psi \succ \Delta, \varphi]}}{[\Gamma, \varphi \vee \psi, \psi \succ \Delta]}\,[\text{L}\vee_w]}{[\Gamma, \varphi \vee \psi, \varphi \vee \psi \succ \Delta]}\,[\text{L}\vee]}{[\Gamma, \varphi \vee \psi \succ \Delta]}\,[\text{Contract}]$$

- The interderivability of the right conjunction rules follows a dual argument to that for left disjunction.
- Now consider the case of $[L\exists]$, $[L\exists_w]$, and $[L\exists^\star]$. If a wST_1 proof of $[\Gamma, \exists x\varphi(x) \succ \Delta]$ terminates with an application of $[L\exists_w]$, then one has wST_1 proofs Ξ_0 and Ξ_1 of the sequents $[\Gamma, \varphi(t) \succ \Delta]$ and $[\Gamma \succ \Delta, \varphi(t)]$. By induction hypothesis, one has wST_2 proofs of these sequents, whence we can construct a wST_2 proof of $[\Gamma, \exists x\varphi(x) \succ \Delta]$.

$$\dfrac{\dfrac{\dfrac{\Xi_0^\star}{[\Gamma, \varphi(t) \succ \Delta]}}{[\Gamma, \varphi(s), \varphi(t) \succ \Delta]}\,[\text{Weak}] \quad \dfrac{\dfrac{\Xi_1^\star}{[\Gamma \succ \Delta, \varphi(t)]}}{[\Gamma, \varphi(s) \succ \Delta, \varphi(t)]}\,[\text{Weak}]}{[\Gamma, \exists x\varphi(x) \succ \Delta]}\,[\text{L}\exists^\star]$$

In case one had applied $[L\exists]$, there exists a wST_1 proof of the sequent $[\Gamma, \varphi(s) \succ \Delta]$ where s does not appear in $\Gamma \cup \Delta$. Then by induction hypothesis, we have a wST_2 proof $\Xi_0^\star$ of this sequent. Then we are able to prove $[\Gamma, \exists x\varphi(x) \succ \Delta]$ in wST_2 as follows:

$$\dfrac{\dfrac{\dfrac{\Xi_0^\star}{[\Gamma, \varphi(s) \succ \Delta]}}{[\Gamma, \varphi(s), \varphi(t) \succ \Delta]}\,[\text{Weak}] \quad \dfrac{\dfrac{\Xi_0^\star}{[\Gamma, \varphi(s) \succ \Delta]}}{[\Gamma, \varphi(s) \succ \Delta, \varphi(t)]}\,[\text{Weak}]}{[\Gamma, \exists x\varphi(x) \succ \Delta]}\,[\text{L}\exists^\star]$$

Now, suppose that one has a wST_2 proof of $[\Gamma, \exists x\varphi(x) \succ \Delta]$ terminating with an application of $[L\exists^\star]$. One must therefore have wST_2 proofs Ξ_0 and Ξ_1 of the sequents $[\Gamma, \varphi(s), \varphi(t) \succ \Delta]$ and $[\Gamma, \varphi(s) \succ \Delta, \varphi(t)]$. By induction hypothesis, there exist corresponding wST_1 proofs $\Xi_0^\star$ and $\Xi_1^\star$, from which one can construct a wST_1 derivation of $[\Gamma, \exists x\varphi(x) \succ \Delta]$:

$$\dfrac{\dfrac{\dfrac{\Xi_0^\star}{[\Gamma, \varphi(s), \varphi(t) \succ \Delta]}}{[\Gamma, \exists x\varphi(x), \varphi(t) \succ \Delta]}[\mathrm{L}\exists] \quad \dfrac{\dfrac{\Xi_1^\star}{[\Gamma, \varphi(s) \succ \Delta, \varphi(t)]}}{[\Gamma, \exists x\varphi(x) \succ \Delta, \varphi(t)]}[\mathrm{L}\exists]}{\dfrac{[\Gamma, \exists x\varphi(x), \exists x\varphi(x) \succ \Delta]}{[\Gamma, \exists x\varphi(x) \succ \Delta]}[\text{Contract}]}[\mathrm{L}\exists_w]$$

- The case of the right universal quantifier rules follows a dual argument to that used for the left existential quantifier rules. □

Lemma 2 *The rule* $[L\vee^\star]$ *preserves intermediate and weak* ST *validity.*

Proof We prove the contrapositive. Suppose that the sequent $[\Gamma, \varphi \vee \psi \succ \Delta]$ is *not* weak (intermediate, respectively) ST valid. Then there exists a three-valued valuation v mapping all elements of $\Gamma \cup \{\varphi \vee \psi\}$ to 1 and all elements of Δ to 0. Because $v(\varphi \vee \psi) = 1$, there are three possibilities concerning the values v assigns to φ and ψ, and we examine these cases individually.

- If $v(\varphi) = v(\psi) = 1$, then v maps all elements of $\Gamma \cup \{\varphi, \psi\}$ to 1 (while still mapping each formula in Δ to 0). Hence, v is a weak (intermediate) Kleene-Kripke counterexample to the sequent $[\Gamma, \varphi, \psi \succ \Delta]$.
- If $v(\varphi) = 1$ while $v(\psi) = 0$, then v assigns each formula of $\Gamma \cup \{\varphi\}$ the value 1 while assigning each formula of $\Delta \cup \{\psi\}$ the value 0, and therefore serves as a weak (intermediate) Kleene-Kripke counterexample to $[\Gamma, \varphi \succ \Delta, \psi]$.
- By swapping φ and ψ in the previous case, that $v(\psi) = 1$ while $v(\psi) = 0$ entails that v is a weak (intermediate) Kleene-Kripke counterexample to $[\Gamma, \psi \succ \Delta, \varphi]$.

These cases exhaust the possible cases for v. Hence, the weak ST invalidity of $[\Gamma, \varphi \vee \psi \succ \Delta]$ entails that at least one of the sequents $[\Gamma, \varphi, \psi \succ \Delta]$, $[\Gamma, \varphi \succ \Delta, \psi]$, and $[\Gamma, \psi \succ \Delta, \varphi]$ is also invalid. But this is the contrapositive of the rule. □

Lemma 3 *The rule* $[R\wedge^\star]$ *preserves both weak and intermediate* ST *validity.*

Proof Analogous to the previous lemma. □

Lemma 4 *The rule* $[L\exists^\star]$ *preserves weak and intermediate ST validity.*

Proof The strong Kleene interpretation of $\exists$ follows from standard ST, so we show that this holds for the weak Kleene quantifier. Suppose that $[\Gamma, \exists x\varphi(x) \succ \Delta]$ is not weak ST valid and that v is a weak Kleene-Kripke counterexample for the sequent. Let X be the set of truth-values that v assigns to formulae $\varphi(c)$. Then there are two cases: one in which $X = \{1\}$ and one in which $X = \{0, 1\}$.

- In the first case, because for any term c whose interpretation is an element of the domain $v(\varphi(c)) = 1$, one can either find a term s not appearing in $\Gamma \cup \Delta$—or introduce a new constant—such that $v(\varphi(s)) = 1$. One can also find a *second* term t for which this holds. In this case, v maps all elements of $\Gamma \cup \{\varphi(s), \varphi(t)\}$ to 1 yet continues to map all elements of Δ to 0, making it a weak Kleene-Kripke counterexample to the sequent $[\Gamma, \varphi(s), \varphi(t) \succ \Delta]$ where s does not appear in Γ or Δ.
- In the second case, one loses the assurance that for *any* term c, $v(\varphi(c)) = 1$. However, one can nevertheless guarantee that for any term t—including terms appearing in $\Gamma \cup \Delta$—either $v(\varphi(t)) = 1$ or $v(\varphi(t)) = 0$. Hence, for any term t, one of two things occurs:

  – v maps all elements of $\Gamma \cup \{\varphi(t)\}$ to 1 and all elements of Δ to 0, or
  – v maps all elements of Γ to 1 and all elements of $\Delta \cup \{\varphi(t)\}$ to 0.

But this is just to say that for any t, v is either a weak Kleene-Kripke counterexample to $[\Gamma, \varphi(t) \succ \Delta]$ or a counterexample to $[\Gamma \succ \Delta, \varphi(t)]$.

Put together, then, that $[\Gamma, \exists x\varphi(x) \succ \Delta]$ is not weak ST valid entails that either $[\Gamma, \varphi(s) \succ \Delta]$ is not valid for an s not appearing in Γ or Δ, or that for an arbitrary term t, either $[\Gamma, \varphi(t) \succ \Delta]$ is invalid or $[\Gamma \succ \Delta, \varphi(t)]$ is invalid. This proves the contraposition of the rule. □

Lemma 5 *The rule* $[R\forall^\star]$ *preserves weak and intermediate* ST *validity.*

Proof Analogous to the previous lemma. □

Theorem 1 (Soundness) iST_1 *and* wST_1 *are sound with respect to the proposed semantics.*

Proof All axioms are valid, and it is well-established that all rules for these calculi preserve validity besides [L$\vee^\star$], [R$\wedge^\star$], [L$\exists^\star$], and [R$\forall^\star$]. Lemmas 2–5 establish, then, that iST_2 and wST_2 are sound; Lemma 1 establishes that this result carries over to iST_1 and wST_1, respectively. □

Theorem 2 (Completeness) iST_1 *and* wST_1 *are complete with respect to the proposed semantics.*

Proof Theorem 2: We follow closely the completeness proof in [4], which is relatively standard. Let $[\Gamma \succ \Delta]$ be an unprovable sequent. We can construct a tree iteratively by *extending* each node sequent $[\Gamma \succ \Delta]$ with nodes above it whose sequents $[\Gamma' \succ \Delta']$ have the property that both $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. We say that a node is *closed* if $\Gamma \cap \Delta \neq \varnothing$ and *open* otherwise.

At each stage $\alpha + 1$, we apply *reduction steps* to the formulae carried over in a sequent $[\Gamma \succ \Delta]$ from stage α.

- A negation on the right is reduced in a node $[\Gamma \succ \Delta, \neg\varphi]$by extending the branch by the sequent $[\Gamma, \varphi \succ \Delta, \neg\varphi]$

- A negation on the left is reduced in a node $[\Gamma, \neg\varphi \succ \Delta]$ by extending the branch by the sequent $[\Gamma, \neg\varphi \succ \Delta, \varphi]$
- A disjunction on the right is reduced in a node $[\Gamma \succ \Delta, \varphi \vee \psi]$ by extending the branch by the sequent $[\Gamma \succ \Delta, \varphi \vee \psi, \varphi, \psi]$
- A disjunction on the left is reduced in a node $[\Gamma, \varphi \vee \psi \succ \Delta]$ by splitting the branch into *three* branches by the nodes $[\Gamma, \varphi \vee \psi, \varphi, \psi \succ \Delta]$, $[\Gamma, \varphi \vee \psi, \varphi \succ \Delta, \psi]$, and $[\Gamma, \varphi \vee \psi, \psi \succ \Delta, \varphi]$
- An existentially quantified formula on the right is reduced in a node $[\Gamma \succ \Delta, \exists x\varphi(x)]$ by the sequent $[\Gamma \succ \Delta, \exists x\varphi(x), \varphi(t)]$ where t has not yet been used in a reduction of this formula in this position
- A formula $\mathsf{T}(\ulcorner\varphi\urcorner)$ on the right is reduced in a node $[\Gamma \succ \Delta, \mathsf{T}(\ulcorner\varphi\urcorner)]$ by extending the branch with $[\Gamma \succ \Delta, \mathsf{T}(\ulcorner\varphi\urcorner), \varphi]$
- A formula $\mathsf{T}(\ulcorner\varphi\urcorner)$ on the left is reduced in a node $[\Gamma, \mathsf{T}(\ulcorner\varphi\urcorner) \succ \Delta]$ by extending the branch with $[\Gamma, \mathsf{T}(\ulcorner\varphi\urcorner), \varphi \succ \Delta]$

In the case of iST_2, assuming that we have an enumeration of terms of our language, we add:

- An existentially quantified formula on the left is reduced in $[\Gamma, \exists x\varphi(x) \succ \Delta]$ by the sequent $[\Gamma, \exists x\varphi(x), \varphi(s) \succ \Delta]$ where s is the least term not appearing in Γ, $\varphi(x)$, or Δ

while in the case of wST_2, we add instead:

- An existentially quantified formula on the left is reduced in $[\Gamma, \exists x\varphi(x) \succ \Delta]$ by splitting the branch into *two* branches with the sequents $[\Gamma, \exists x\varphi(x), \varphi(s), \varphi(t) \succ \Delta]$ and $[\Gamma, \exists x\varphi(x), \varphi(s) \succ \Delta, \varphi(t)]$ where s is the least term not appearing in Γ, $\varphi(x)$, or Δ and t has not yet been used in the reduction of the formula $\exists x\varphi(x)$ in that position

If all reduction steps are exhausted, because $[\Gamma \succ \Delta]$ is not provable, then there must remain an open branch.

Define $[\Gamma_\omega \succ \Delta_\omega]$ to be the union of all the sequents found on this open branch. *I.e.*, let Γ_ω be the union of all antecedents appearing in some sequent on the branch and let Δ_ω be the union of all succedents appearing in some sequent on the branch.

From $[\Gamma_\omega \succ \Delta_\omega]$ a model $\langle D_\omega, I_\omega\rangle$ may be defined as follows. D_ω is the set of terms appearing in some formula in $\Gamma_\omega \cup \Delta_\omega$ and the interpretation is defined so that:

- For all non-distinguished terms t, $I_\omega(t) = t$
- For all distinguished terms $\ulcorner\varphi\urcorner$, $I_\omega(\ulcorner\varphi\urcorner) = \varphi$
- For all R, $I_\omega(R)(I_\omega(t_0), ..., I_\omega(t_{n-1})) = \begin{cases} 1 & \text{if } R(t_0, ..., t_{n-1}) \in \Gamma_\omega \\ \frac{1}{2} & \text{if } R(t_0, ..., t_{n-1}) \notin \Gamma_\omega \cup \Delta_\omega \\ 0 & \text{if } R(t_0, ..., t_{n-1}) \in \Delta_\omega \end{cases}$ □

At this point, we pause the proof of Theorem 2 to establish several facts about iST, wST and the interpretation I_ω.

Lemma 6 *Both intermediate and weak Kleene-Kripke interpretations are monotonic.*

Proof The monotonicity of weak Kleene-Kripke models (and strong Kleene-Kripke models) is well-known and acknowledged in e.g. [11, 15], and [30]. Consequently the respective systems' individual truth-functions interpreting connectives and quantifiers guarantee monotonicity. This holds *a fortiori* for the truth-functions used to extend intermediate Kleene-Kripke interpretations, namely the truth-functions of the weak Kleene connectives and strong Kleene quantifiers. Thus, intermediate Kleene-Kripke models are monotonic as well. □

Lemma 7 *In the model* $\langle D_\omega, I_\omega \rangle$*,*

- $I_\omega(\varphi) = 1$ *if* $\varphi \in \Gamma_\omega$
- $I_\omega(\varphi) = 0$ *if* $\varphi \in \Delta_\omega$

Proof As in the case of Ripley's [4], induction on complexity of formulae establishes this fact. As an illustration, consider a case unique to the weak Kleene matrices in which $\varphi = \psi \vee \xi$. In this case, $\varphi \in \Gamma_\omega$ only if a reduction step had been applied to some sequent $[\Gamma, \psi \vee \xi \succ \Delta]$ on the branch. There are three possible outputs of this reduction step, one of which must have been added to the branch.

- if $[\Gamma, \psi \vee \xi, \psi, \xi \succ \Delta]$ was added, by hypothesis, $I_\omega(\psi) = 1$ and $I_\omega(\xi) = 1$
- if $[\Gamma, \psi \vee \xi, \psi \succ \Delta, \xi]$ was added, then $I_\omega(\psi) = 1$ and $I_\omega(\xi) = 0$
- if $[\Gamma, \psi \vee \xi, \xi \succ \Delta, \psi]$ was added, $I_\omega(\psi) = 0$ and $I_\omega(\xi) = 1$

In each of these cases, $I_\omega(\psi \vee \xi) = 1$, *i.e.*, $I_\omega(\varphi) = 1$. □

Although these observations are sufficient to yield a model, $\langle D_\omega, I_\omega \rangle$ may not respect the clauses for $\mathbf{T}$. An additional step is necessary to construct a provable counterexample to $[\Gamma_\omega \succ \Delta_\omega]$. The critical ingredient relies on Kremer's notion of *fixability*.

Definition 13 A three-valued interpretation I is *fixable* if $I(\mathbf{T}(\ulcorner \varphi \urcorner)) = I(\varphi)$ whenever $I(\mathbf{T}(\ulcorner \varphi \urcorner)) \neq \frac{1}{2}$.

Lemma 8 *The model* $\langle D_\omega, I_\omega \rangle$ *is fixable.*

Proof If $I_\omega(\mathbf{T}(\ulcorner \varphi \urcorner)) \neq \frac{1}{2}$ then suppose without loss of generality that $I_\omega(\mathbf{T}(\ulcorner \varphi \urcorner)) = 1$. As an atomic formula, the definition of I_ω ensures that $\mathbf{T}(\ulcorner \varphi \urcorner) \in \Gamma_\omega$. Its appearance in Γ_ω means that a sequent $[\Gamma, \mathbf{T}(\ulcorner \varphi \urcorner) \succ \Delta]$ appears at some stage in the branch to which a reduction step was applied yielding the sequent $[\Gamma, \mathbf{T}(\ulcorner \varphi \urcorner), \varphi \succ \Delta]$. Thus $\varphi \in \Gamma_\omega$ and by Lemma 7, $I_\omega(\varphi) = 1$. □

In [30], Kremer proves the following improvement of Kripke's *Minimal Fixed Point Theorem* from [15]. While Kripke's result guarantees the existence of minimal fixed points if the initial interpretation of $\mathbf{T}$ maps all distinguished terms to $\frac{1}{2}$, the model $\langle D_\omega, I_\omega \rangle$ may have already assigned values of 0 or 1 to some formulae $\mathbf{T}(\ulcorner \varphi \urcorner)$.

Theorem 3 (Kripke/Kremer) *For a three-valued logic with monotonic valuations, if a model* $\langle D, I \rangle$ *is fixable then it can be extended to a fixed-point model* $\langle D, I' \rangle$*.*

This gives us all the necessary tools to complete the proof of Theorem 2:

Proof Theorem 2, Continued: By Lemma 6 the applicable logic has monotonic valuations and by Lemma 8 the model $\langle D_\omega, I_\omega \rangle$ is fixable. Hence, by Theorem 3, I_ω can be extended to a fixed-point model $\langle D_\omega, I'_\omega \rangle$ in which $I'_\omega(\mathbf{T}(\ulcorner \varphi \urcorner)) = I'_\omega(\varphi)$ for all φ.

Thus, $\langle D_\omega, I'_\omega \rangle$ satisfies the clause for the interpretation of $\mathbf{T}$ and is in fact a model of appropriate type for $[\Gamma_\omega \succ \Delta_\omega]$. It is *a fortiori* a counterexample to $[\Gamma \succ \Delta]$, establishing completeness for iST_2 and wST_2 with respect to the intended semantics. By Lemma 1, this extends to iST_1 and wST_1. □

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