DOI 10.1007/s10986-023-09592-1
Lithuanian Mathematical Journal, Vol. 63, No. 2, April, 2023, pp. 190–202

Almost sure central limit theorems for the maxima of randomly chosen random variables

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Received August 1, 2022; revised March 20, 2023

Abstract. In this paper, we give an almost sure central limit theorem (ASCLT) version of a maximum limit theorem (MLT) with an arbitrary sequence $\{d_n,\, n \geqslant 1\}$ of weighted means of $\max\{X_k,\, k \in A_n\}$, where $\{X_n,\, n \geqslant 1\}$ is a sequence of independent random variables, and $\{A_n,\, n \geqslant 1\}$ is a sequence of almost surely finite random subsets of positive integers independent of $\{X_n,\, n \geqslant 1\}$. Thus we generalize the cases considered in the literature: (i) the nonrandom version of ASCLT for the MLT; (ii) the version of ASCLT for randomly indexed MLT; and (iii) the version of maximum schema of observed and unobserved random variables. We complete the paper with illustrative examples.

MSC: primary 60F05; secondary 60G70

Keywords: extreme value, almost sure central limit theorem, maximum of randomly chosen elements

1 Introduction

The central limit theorem, in its classical form, states that if $\{X_n,\, n \geqslant 1\}$ is a sequence of independent identically distributed (i.i.d.) random variables with $\mathbf{E}X_n = 0, \mathbf{E}X_n^2 = 1$, then

$$\frac{\sum_{i=1}^n X_i}{\sqrt{n}} \xrightarrow{\mathcal{D}} \Phi \quad \text{as } n \to \infty,$$

where $\Phi(x)$ denotes the standard normal distribution function, and $\xrightarrow{\mathcal{D}}$ denotes the weak convergence, whereas the maximum limit theorem states that if $\{X_n,\, n \geqslant 1\}$ is a sequence of i.i.d. random variables with distribution function F belonging to one of the classes $D_1, D_{2,\alpha}$ or $D_{3,\alpha}$ (for definition, see [13, 18] or [9, p. 92, Cases (i)–(iii)]) with some $\alpha > 0$, then there exist constants $\{a_n, b_n,\, n \geqslant 1\}$ such that

$$a_n \max_{1\leqslant j\leqslant n} X_j + b_n \xrightarrow{\mathcal{D}} G \quad \text{as } n \to \infty, \tag{1.1}$$

where G has one of the forms

$$G_1(x) = \mathrm{e}^{-\mathrm{e}^{-x}}, \qquad G_{2,\alpha}(x) = \begin{cases} 0, & x \leqslant 0, \\ \mathrm{e}^{-x^{-\alpha}}, & x > 0, \end{cases} \qquad G_{3,\alpha}(x) = \begin{cases} \mathrm{e}^{-(-x)^{\alpha}}, & x \leqslant 0, \\ 1, & x > 0, \end{cases}$$

according to $F \in D_1$, $F \in D_{2,\alpha}$, or $F \in D_{3,\alpha}$, respectively.

The simplest version of ASCLT theorem was obtained by Schatte [19] and Brosamler [2] for a sequence of i.i.d. random variables with some moment restriction (later weakened by Lacey and Phillip [12] to the existence of variance only) states that if $\{X_n,\, n \geqslant 1\}$ is a sequence of i.i.d. random variables such that $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 = 1$, then

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{i=1}^{n} \frac{1}{i} I\left[\frac{\sum_{k=1}^{i} X_k}{\sqrt{i}} < x\right] = \varPhi(x) \quad \text{a.s.}$$

ASCLT for maximum limit theorem states that if $\{X_n,\, n \geqslant 1\}$ is a sequence of i.i.d. random variables with distribution function $F \in D_1$, $F \in D_{2,\alpha}$, or $F \in D_{3,\alpha}$ for some $\alpha > 0$, then

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{i=1}^{n} \frac{1}{i} I\Big[a_i \max_{1\leqslant j\leqslant i} X_j + b_i \leqslant x\Big] = G(x) \quad \text{a.s.}$$

The maximum limit theorem (MLT) was generalized in the following directions:

(i) Some investigators tried to omit the assumption that $\{X_n,\, n \geqslant 1\}$ is an i.i.d. sequence. Loynes [14] proved MLT for the uniformly mixing strictly stationary stochastic processes. Hüsler [7] proved the MLT for nonstationary sequences but under strong conditions on common distribution with strong mixing type conditions.

(ii) Mladenović and Piterbarg [16] considered the maximum taken on observed subsets of random variables. Precisely, if $\{\epsilon_n,\, n \geqslant 1\}$ is the sequence of indicators of the events that the corresponding random variables are observed (they assumed that this sequence is independent of $\{X_n,\, n \geqslant 1\}$), then the limit theorem for the common law of $\{(\max_{1\leqslant i\leqslant n,\, \epsilon_i=1} X_i,\, \max_{1\leqslant i\leqslant n} X_i),\, n \geqslant 1\}$ with an appropriate norming and centering (assuming stationarity and some two conditions of strong mixing type) can be obtained (see also [11]).

(iii) In [1], [10], and [9], MLT (in the last two, also ASCLT) for the randomly indexed maxima of random variables was considered, that is, for $\{\max_{1\leqslant i\leqslant N_n} X_i,\, n \geqslant 1\}$ (with appropriate centering and norming), where $\{N_n,\, n \geqslant 1\}$ is a sequence of positive-integer random variables independent of $\{X_n,\, n \geqslant 1\}$

(iv) Some other common laws were also considered. For example, in [3] the common limit law was considered for appropriately centered and normed sequence $\{(\max_{1\leqslant i\leqslant n} X_i,\, \min_{1\leqslant i\leqslant n} X_i),\, n \geqslant 1\}$.

(v) MLT for multiindex fields was considered in [4] and [5].

On the other hand, in ASCLT the sequence $\{1/n,\, n \geqslant 1\}$ of norming coefficients of summands is often replaced by a general nonincreasing sequence of positive reals $\{d_n,\, n \geqslant 1\}$).

The ASCLT for MLT is mostly independent of MLT. Usually, it suffices to consider MLT. We explain this statement by recalling [5, Theorem 2.1]:

Theorem 1. *Let $\{Y_n,\, n \geqslant 1\}$ be a sequence of a.s. bounded random variables with $\mathbf{E}Y_n = 0$, $n \in \mathbb{N}$. Assume that for some nonnegative nonincreasing sequence $\{d_n,\, n \in \mathbb{N}\}$ such that*

$$d_1 > 0, \qquad \sum_{n\in\mathbb{N}} d_n = \infty,$$

and for some $1 < \gamma < 2$ *and every* $1 \leqslant i \leqslant j$, *we have*

$$\sum_{i\leqslant k\leqslant j}\sum_{i\leqslant l\leqslant j} d_k d_l |\mathbf{E}Y_kY_l| \leqslant C\left(\sum_{k=i}^{j} d_k\right)^{\gamma}, \qquad \sum_{n=1}^{\infty}\frac{d_n}{D_n^{3-\gamma}} < \infty,$$

where $D_n = \sum_{i=1}^{n} d_i$. *Then*

$$\frac{1}{D_n}\sum_{k=1}^{n} d_k Y_k \xrightarrow{\text{a.s.}} 0 \quad \textit{as } n \to \infty.$$

If MLT (1.1) holds with a norming sequence $\{d_n,\, n \geqslant 1\}$ instead of $\{1/n,\, n \geqslant 1\}$, then we have

$$\frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbf{P}\Big[a_k \max_{1\leqslant j\leqslant k} X_j + b_k < x\Big] \to G(x) \quad \text{as } n \to \infty$$

for every $x \in \mathbb{R}$ such that G is continuous at x. Furthermore, if Theorem 1 holds with

$$Y_n = I\Big[a_n \max_{1\leqslant j\leqslant n} X_j + b_n < x\Big] - \mathbf{P}\Big[a_n \max_{1\leqslant j\leqslant n} X_j + b_n < x\Big],$$

then ASCLT for MLT also holds:

$$\frac{1}{D_n}\sum_{k=1}^{n} d_k I\Big[a_k \max_{1\leqslant j\leqslant k} X_j + b_k < x\Big] \xrightarrow{\text{a.s.}} G(x) \quad \text{as } n \to \infty,$$

In many cases, the assumptions of Theorem 1 and MLT are strictly connected.

Let $\{A_n,\, n \in \mathbb{N}\}$ be a sequence of random subsets of $\mathbb{N}$ independent of $\{X_n,\, n \in \mathbb{N}\}$. Fundamentals of random set theory can be found in Matheron's classic book [15] or in Molchanov's book [17]. For arbitrary random or nonrandom set A, we denote by $|A|$ the cardinality of A, and we also denote

$$M(A) = \begin{cases} \max_{i\in A} X_i & \text{if } A \neq \emptyset, \\ -\infty & \text{if } A = \emptyset. \end{cases} \tag{1.2}$$

All random and nonrandom sets A considered in this paper are such that $|A|$ is a random variable, and thus $M(A)$ is possibly not defined on the set of measure 0 only.

If $A \cap B = \emptyset$ a.s., then the maximum defined this way satisfies the condition

$$M(A \cup B) = \max\big\{M(A), M(B)\big\};$$

in particular, $M(A) = M(A \cup \emptyset)$.

In this paper, we prove the ASCLT for the maximum limit theorem of the form

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{i=1}^{n} d_i I\big[a_i M(A_i) + b_i \leqslant x\big] = G(x) \quad \text{a.s.}, \tag{1.3}$$

where $\{d_n,\, n \geqslant 1\}$ is a nonincreasing sequence of positive reals, $D_n = \sum_{i=1}^{n} d_i$, $n \geqslant 1$, $\{a_n,\, n \geqslant 1\}$ is a sequence of positive reals, $\{b_n,\, n \geqslant 1\}$ is a sequence of arbitrary reals, and μ is a probability measure corresponding to the distribution function G such that $\mu(x) = 0$, and the following MLT holds:

$$\lim_{n\to\infty}\mathbf{P}\big[a_n M(A_n) + b_n \leqslant x\big] = G(x).$$

Note that for a sequence of random indices $\{N_n,\, n \geqslant 1\}$ and $A_n = \{1, 2, 3, \dots, N_n\}$, the result (1.3) generalizes the ASCLT for maximum theorem in [10] and [9]. Furthermore, if $\{\epsilon_n,\, n \geqslant 1\}$ is the sequence of indicators of the event that the corresponding random variable is observed and

$$A_n = \bigcup_{\substack{\varepsilon_i = 1\\ 1 \leqslant i \leqslant n}} \{i\}, \quad n \geqslant 1, \tag{1.4}$$

then (1.3) generalizes the results presented in [16] and [11].

These results are analogous to that obtained recently by Krajka and Gdula [6] for classical ASCLT.

In the whole paper, C is a constant, possibly different in different places. For random variables X and Y, by $\pi(X, Y)$ we denote the Ky Fan metrics

$$\pi(X, Y) = \inf\bigl\{\varepsilon > 0\colon\ \mathbf{P}\bigl[|X - Y| > \varepsilon\bigr] < \varepsilon\bigr\}.$$

We use the notations $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$ for $x, y \in \mathbb{R}$. For two σ-fields σ_1 and σ_2, we also denote

$$\alpha(\sigma_1, \sigma_2) = \sup\bigl\{\bigl|\mathbf{P}[A \cap B] - \mathbf{P}[A]\mathbf{P}[B]\bigr|,\ A \in \sigma_1,\ B \in \sigma_2\bigr\},$$

2 Main results

Theorem 2. *Let $\{X_n,\, n \geqslant 1\}$ be a sequence of independent random variables, let $\{A_n,\, n \geqslant 1\}$ be a sequence of random sets independent of $\{X_n,\, n \geqslant 1\}$, let $\{a_n,\, n \geqslant 1\}$ be a sequence of positive reals, and let $\{b_n,\, n \geqslant 1\}$ be a sequence of arbitrary reals such that*

$$a_n M(A_n) + b_n \xrightarrow{\mathcal{D}} G \quad \text{as } n \to \infty \tag{2.1}$$

for some nondegenerate distribution function G. Assume that for some nonnegative nonincreasing sequence of reals $\{d_n,\, n \geqslant 1\}$, $D_n = \sum_{i=1}^n d_i$ is divergent to infinity, and for some $1 < \gamma < 2$ and every positive integer $i \leqslant j$, we have

$$\sum_{i \leqslant k \leqslant j} \sum_{i \leqslant l \leqslant j} d_k d_l (\theta_{k,l} \wedge \theta_{l,k}) \leqslant C \biggl(\sum_{i \leqslant k \leqslant j} d_k\biggr)^{\gamma}, \tag{2.2}$$

where θ is defined by one of the following formulas:

$$\text{(i)} \quad \theta_{k,l} = \alpha\bigl(\sigma(A_k), \sigma(A_l)\bigr) + \pi\bigl(a_l M(A_l), a_l M(A_l \setminus A_k)\bigr), \quad k, l \in \mathbb{N},$$

or

$$\text{(ii)} \quad \theta_{k,l} = \alpha\bigl(\sigma(A_k), \sigma(A_l \setminus A_k)\bigr) + \pi\bigl(a_l M(A_l), a_l M(A_l \setminus A_k)\bigr), \quad k, l \in \mathbb{N},$$

or for some sequence of positive integers $\{\delta_k,\, k \geqslant 1\}$,

$$\begin{aligned}\text{(iii)} \quad \theta_{k,l} = {}& \alpha\bigl(\sigma(A_k), \sigma(A_l \setminus A_{k+\delta_k})\bigr) + \pi\bigl(a_l M(A_l), a_l M(A_l \setminus A_k)\bigr) \\ &+ \pi\bigl(a_l M(A_l), a_l M(A_l \setminus A_{k+\delta_k})\bigr), \quad k, l \in \mathbb{N},\end{aligned}$$

and $\sum_{n=1}^\infty d_n / D_n^{3-\gamma} < \infty$. Then

$$\lim_{n \to \infty} \frac{1}{D_n} \sum_{i=1}^n d_i I\bigl[a_i M(A_i) + b_i < x\bigr] = G(x) \quad \text{a.s.} \tag{2.3}$$

For an arbitrary sequence of random variables $\{X_n,\, n\geqslant 1\}$, a random or nonrandom but not empty ($\mathbf{P}[A_n=\emptyset]=0,\, n\geqslant 1$) sequence $\{A_n, n\geqslant 1\}$ of subsets of $\mathbb{N}$, and a sequence of reals $\{a_n,\, n\geqslant 1\}$, we denote

$$p_{k,l}=\inf_{\varepsilon>0}\bigl\{\bigl(p(l,k,\varepsilon)\wedge p(k,l,\varepsilon)\bigr)\vee\varepsilon\bigr\},\qquad p(l,k,\varepsilon)=\mathbf{P}\bigl[a_l M(A_k\cap A_l)>\varepsilon+a_l M(A_l\setminus A_k)\bigr].$$

We list some properties of $p_{k,l}$:

(i) $p_{k,l}\leqslant\pi(a_l M(A_k\cap A_l), a_l M(A_l\setminus A_k))\wedge\pi(a_k M(A_k\cap A_l), a_k M(A_k\setminus A_l))$,
(ii) $p_{k,k}=1$,
(iii) $p_{k,l}=0$ if $A_k\cap A_l=\emptyset$,
(iv) $p_{k,l}=p_{l,k}$,
(v) $p_{k,l}$ does not satisfy the triangle inequality.

We focus on property (i). The evaluation by the Ky Fan metrics (right-hand side of (i)) is simpler, but the difference of the left- and right-hand sides is essential. In Example 1, we construct a sequence of random variables $\{X_n,\, n\geqslant 1\}$, random sets $\{A_n,\, n\geqslant 1\}$, and subsequences $\{k_n, l_n,\, n\geqslant 1\}$ for which $p_{k_n,l_n}\to 0$ but $\pi(a_{l_n}M(A_{k_n}\cap A_{l_n}), a_{l_n}M(A_{l_n}\setminus A_{k_n}))\to 1$ as $n\to\infty$. It follows from the fact that for $l\gg k$, the term $M(A_l\setminus A_k)$ is relatively greater than the term $M(A_k\cap A_l)$, and thus $\mathbf{P}[a_l M(A_k\cap A_l)>\varepsilon+a_l M(A_l\setminus A_k)]$ can be small, but $\mathbf{P}[a_l M(A_l\setminus A_k)>\varepsilon+a_l M(A_l\cap A_k)]$ can be large.

Remark 1. If, under the assumptions of Theorem 2 (i) or (ii), we have

$$\sum_{i\leqslant k\leqslant j}\sum_{i\leqslant l\leqslant j} d_k d_l(\alpha_{k,l}\vee\alpha_{l,k})\leqslant C\biggl(\sum_{i\leqslant k\leqslant j} d_k\biggr)^{\gamma},$$

with $\alpha_{k,l}=\alpha(\sigma(A_k),\sigma(A_l))$ or $\alpha_{k,l}=\alpha(\sigma(A_k),\sigma(A_l\setminus A_k))$, respectively, then condition (2.2) can be replaced by

$$\sum_{i\leqslant k\leqslant j}\sum_{i\leqslant l\leqslant j} d_k d_l p_{k,l}\leqslant C\biggl(\sum_{i\leqslant k\leqslant j} d_k\biggr)^{\gamma}.$$

In case (i), this holds when $\{A_n,\, n\geqslant 1\}$ is a sequence of independent random sets, whereas in case (ii), for example, in schema considered in [16] with the i.i.d. sequence $\{\varepsilon_n,\, n\geqslant 1\}$. Then $\alpha_{k,l}=\alpha_{l,k}=0$.

The applications of three different versions of Theorem 2 depend on the structure of dependency of the sets $\{A_n,\, n\geqslant 1\}$. Case (i) is “better” to apply for weakly dependent sets $\{A_n,\, n\geqslant 1\}$, whereas cases (ii) and (iii) are more convenient in the cases where $\{A_n,\, n\geqslant 1\}$ is a sequence of increasing random sets with respect to the inclusion (i.e., $A_k\subset A_l$ a.s. for $k<l$) with independent (case (ii)) or weakly dependent (case (iii)) increments ($\{A_l\setminus A_k,\, l\geqslant k,\, l,k>1\}$). It is suitable for schema considered in [16] (definition (1.4)) with independent (case (ii)) or weakly dependent (case (iii)) $\{\varepsilon_n,\, n\geqslant 1\}$. Note that in case (ii) for weakly dependent $\{\varepsilon_n,\, n\geqslant 1\}$, (2.2) fails. Indeed, when $\{\varepsilon_n,\, n\geqslant 1\}$ are α-mixing with

$$\alpha=\inf_{j\in\mathbb{N}}\alpha\bigl(\sigma(\varepsilon_j),\sigma(\varepsilon_{j+1})\bigr)>0,$$

as $\sigma(\varepsilon_k)\subset\sigma(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k)$, $\sigma(A_k)=\sigma(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k)$, $\sigma(A_l\setminus A_k)=\sigma(\varepsilon_{k+1},\dots,\varepsilon_l)$, we have

$$\sum_{i\leqslant k\leqslant j}\sum_{i\leqslant l\leqslant j} d_k d_l(\alpha_{k,l}\wedge\alpha_{l,k})\geqslant\alpha\biggl(\sum_{i\leqslant k\leqslant j} d_k\biggr)^2>C\alpha\biggl(\sum_{i\leqslant k\leqslant j} d_k\biggr)^{\gamma}$$

for every $\gamma<2$ (recall that $D_j-D_{i-1}\to\infty$ as $j\to\infty$).

Illustrative examples of application of Theorem 2(i) and (ii) are given in Examples 2 and 3, respectively, in the last section.

3 Proofs

Denote by $\mathrm{Lip}(\mathbb{R})$ the set of bounded Lipschitz functions on $\mathbb{R}$ with the norm $\|g\|_{\mathrm{BL}} = \|g\|_\infty + \|g\|_L < \infty$, where $\|g\|_\infty$ is the supremum norm, and

$$\|g\|_L = \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|}.$$

Proof of Theorem 2. By (2.1) and $D_n = \sum_{i=1}^n d_i$, to obtain (2.3), it suffices to prove that

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{i=1}^n d_i \bigl(I\bigl[a_i M(A_i) + b_i > x\bigr] - \mathbf{P}\bigl[a_i M(A_i) + b_i > x\bigr]\bigr) = 0 \quad \text{a.s.}$$

On the other hand, from Theorem 1 and comments below this theorem it suffices to prove that

$$\sum_{i\leqslant k\leqslant j} \sum_{i\leqslant l\leqslant j} d_k d_l \bigl| \mathrm{Cov}\bigl(I\bigl[a_k M(A_k) + b_k > x\bigr], I\bigl[a_l M(A_l) + b_l > x\bigr]\bigr)\bigr| \leqslant C \biggl(\sum_{i\leqslant k\leqslant j} d_k \biggr)^\gamma. \tag{3.1}$$

Obviously, the indicator function is bounded but not continuous. However, instead, we may consider the functions

$$g_\delta(x) = \begin{cases} 0 & \text{if } x < -\frac{\delta}{2}, \\ \frac{1}{\delta}x + \frac{1}{2} & \text{if } -\frac{\delta}{2} \leqslant x \leqslant \frac{\delta}{2}, \\ 1 & \text{if } x > \frac{\delta}{2}, \end{cases}$$

since $g_\delta(x) \to I[x > 0]$ as $\delta \downarrow 0$. Thus for (3.1), we will prove that

$$\sum_{i\leqslant k\leqslant j} \sum_{i\leqslant l\leqslant j} d_k d_l \bigl| \mathrm{Cov}\bigl(Y_k(A_k), Y_l(A_l)\bigr)\bigr| \leqslant C \biggl(\sum_{i\leqslant k\leqslant j} d_k \biggr)^\gamma,$$

where $Y_i(A) = g(a_i M(A) + b_i)$ for any positive integers i and j, $1 < \gamma < 2$, and any Lipschitz function $g \in \{g_\delta,\ 0 < \delta < 1\}$.

So the rest of the proof is proceeded for the functions g of the above type.

Due to the definition of the maximum of random variables (1.2), for such function g_δ, we will consider all the evaluations and probabilities in the following part of a proof on the event $Z = \{A_l \neq \emptyset, A_k \neq \emptyset\}$ because $\mathrm{Cov}(Y_k(A_k) I[\Omega \setminus Z], Y_l(A_l) I[\Omega \setminus Z]) = 0$. To prove this fact, let us take the subset $\{\omega\colon A_l = \emptyset\}$ of $\Omega \setminus Z = [A_l = \emptyset] \cup [A_k = \emptyset]$ (by symmetry similar computations can be proceeded for the subset $\{\omega\colon A_k = \emptyset\}$). On such a subset, from the definition $M(A_l) = -\infty$, as $a_l > 0$, we have

$$Y_l(A_l) = g_\delta\bigl(a_l M(A_l) + b_l\bigr) = 0.$$

Now we will prove the following facts:

(A) For every random set B, we have

$$\mathbf{E}\bigl| \bigl(\mathbf{E}\bigl(Y_k(A_k) \bigm| A_k\bigr) - \mathbf{E} Y_k(A_k)\bigr)\bigl(Y_l(B) - \mathbf{E} Y_l(B)\bigr)\bigr| \leqslant 4\alpha\bigl(\sigma(A_k), \sigma(B)\bigr) \|g\|_\infty^2.$$

(B) For all random sets B and C,

$$\begin{aligned} &\mathbf{E}\bigl| \bigl(\mathbf{E}\bigl(Y_k(A_k) \bigm| A_k\bigr) - \mathbf{E} Y_k(A_k)\bigr)\bigl(Y_l(B) - Y_l(C)\bigr)\bigr| \\ &\quad \leqslant 2\|g\|_\infty \bigl(2\|g\|_\infty + \|g\|_L\bigr) \pi\bigl(a_l M(B), a_l M(C)\bigr), \end{aligned}$$

$$\begin{aligned}
&\mathbf{E}\big|\big(Y_k(A_k)-\mathbf{E}Y_k(A_k)\big)\big(Y_l(B)-Y_l(C)\big)\big| \\
&\qquad \leqslant 2\|g\|_\infty\big(2\|g\|_\infty+\|g\|_L\big)\pi\big(a_lM(B),a_lM(C)\big).
\end{aligned}$$

(C) We have

$$\mathbf{E}\big(\mathbf{E}\big(Y_k(A_k)\mid A_k\big)-Y_k(A_k)\big)Y_l(A_l\setminus A_k)=0.$$

Proof of fact A. From Theorem 17.2.1 of Ibragimov and Linnik [8, p. 306, Chap. 17] we have

$$\big|\mathbf{E}(UV)-\mathbf{E}U\mathbf{E}V\big|\leqslant 4\alpha_{U,V}\|U\|_\infty\|V\|_\infty$$

for any random variables U and V. Putting $U=\mathbf{E}(Y_k(A_k)|A_k)$ and $V=Y_l(B)$ and recalling that $\mathbf{E}U=\mathbf{E}Y_k(A_k)$, we get the statement.

Proof of fact B. Taking $\pi=\pi(a_lM(B),a_lM(C))$, this fact follows from

$$\begin{aligned}
\mathbf{E}\big|Y_l(B)-Y_l(C)\big| &\leqslant \mathbf{E}\big|Y_l(B)-Y_l(C)\mid I\big[a_l\big|M(B)-M(C)\big|>\pi\big] \\
&\quad+\mathbf{E}\big|Y_l(B)-Y_l(C)\big|I\big[a_l\big|M(B)-M(C)\big|\leqslant\pi\big] \\
&\leqslant 2\|g\|_\infty\mathbf{P}\big[a_l\big|M(B)-M(C)\big|>\pi\big]+\|g\|_L\pi \\
&\leqslant 2\|g\|_\infty\pi+\|g\|_L\pi
\end{aligned}$$

and the evaluations

$$\begin{aligned}
\big|\mathbf{E}\big(Y_k(A_k)\mid A_k\big)-\mathbf{E}Y_k(A_k)\big|&\leqslant 2\|g\|_\infty, \\
\big|Y_k(A_k)-\mathbf{E}Y_k(A_k)\big|&\leqslant 2\|g\|_\infty.
\end{aligned}$$

Proof of fact C. Let us remark that because $\{X_n,\,n\geqslant 1\}$ is a sequence of independent random variables, for arbitrary nonrandom disjoint sets B and C, the random variables $M(B)$ and $M(C)$ are independent. Because $\{A_n,\,n\geqslant 1\}$ and $\{X_n,\,n\geqslant 1\}$ are independent, for any $T\in\mathbb{N}$, we have

$$\begin{aligned}
&\big|\mathbf{E}\big(\mathbf{E}\big(Y_k(A_k)\mid A_k\big)-Y_k(A_k)\big)Y_l(A_l\setminus A_k)\big| \\
&\quad\leqslant\Bigg|\sum_{\substack{B_k,B_l\subset\mathbb{N}\\|B_k|<T,\,|B_l|<T}}\mathbf{E}\big(\mathbf{E}\big(Y_k(B_k)\mid B_k\big)-Y_k(B_k)\big)Y_l(B_l\setminus B_k)\mathbf{P}[A_k=B_k,\,A_l=B_l]\Bigg| \\
&\qquad+\mathbf{P}\big[|A_k|\geqslant T\big]+\mathbf{P}\big[|A_l|\geqslant T\big] \\
&\quad\leqslant\mathbf{P}\big[|A_k|\geqslant T\big]+\mathbf{P}\big[|A_l|\geqslant T\big],
\end{aligned}$$

and now taking the limit as $T\to\infty$, the right-hand side converges to 0 due to the tightness of random variables $|A_k|$ and $|A_l|$.

In case (i), from (B), (C), and (A) we have

$$\begin{aligned}
\operatorname{Cov}\big(Y_k(A_k),Y_l(A_l)\big)&=\mathbf{E}\big(\big(Y_k(A_k)\big)-\mathbf{E}\big(Y_k(A_k)\mid A_k\big)\big)\big(Y_l(A_l)-Y_l(A_l\setminus A_k)\big) \\
&\quad+\mathbf{E}\big(\big(Y_k(A_k)\big)-\mathbf{E}\big(Y_k(A_k)\mid A_k\big)\big)\big(Y_l(A_l\setminus A_k)-\mathbf{E}Y_l(A_l)\big) \\
&\quad+\mathbf{E}\big(\mathbf{E}\big(Y_k(A_k)\mid A_k\big)-\mathbf{E}Y_k(A_k)\big)\big(Y_l(A_l)-\mathbf{E}Y_l(A_l)\big) \\
&\leqslant C\pi\big(a_lM(A_l),a_lM(A_l\setminus A_k)\big)+C\alpha\big(\sigma(A_k),\sigma(A_l)\big) \\
&=C\theta_{k,l}.
\end{aligned}$$

In case (ii), using (B), (C), and (A), we have

$$
\begin{aligned}
\operatorname{Cov}\bigl(Y_k(A_k), Y_l(A_l)\bigr) &= \mathbf{E}\bigl(Y_k(A_k) - \mathbf{E}Y_k(A_k)\bigr)\bigl(Y_l(A_l) - Y_l(A_l \setminus A_k)\bigr) \\
&\quad + \mathbf{E}\bigl(Y_k(A_k) - \mathbf{E}(Y_k(A_k) \mid A_k)\bigr)\bigl(Y_l(A_l \setminus A_k) - \mathbf{E}Y_l(A_l)\bigr) \\
&\quad + \mathbf{E}\bigl(\mathbf{E}\bigl(Y_k(A_k) \mid A_k\bigr) - \mathbf{E}Y_k(A_k)\bigr)\bigl(Y_l(A_l \setminus A_k) - \mathbf{E}Y_l(A_l \setminus A_k)\bigr) \\
&\quad + \mathbf{E}\bigl(\mathbf{E}\bigl(Y_k(A_k) \mid A_k\bigr) - \mathbf{E}Y_k(A_k)\bigr)\bigl(\mathbf{E}Y_l(A_l \setminus A_k) - \mathbf{E}Y_l(A_l)\bigr) \\
&\leqslant C\pi\bigl(a_l M(A_l), a_l M(A_l \setminus A_k)\bigr) + C\alpha\bigl(\sigma(A_k), \sigma(A_l \setminus A_k)\bigr) \\
&= C\theta_{k,l}.
\end{aligned}
$$

In the last case (iii), we use (B), (C), (B), and (A):

$$
\begin{aligned}
\operatorname{Cov}\bigl(Y_k(A_k), Y_l(A_l)\bigr) &= \mathbf{E}\bigl(Y_k(A_k) - \mathbf{E}Y_k(A_k)\bigr)\bigl(Y_l(A_l) - Y_l(A_l \setminus A_k)\bigr) \\
&\quad + \mathbf{E}\bigl(Y_k(A_k) - \mathbf{E}\bigl(Y_k(A_k) \mid A_k\bigr)\bigr)\bigl(Y_l(A_l \setminus A_k) - \mathbf{E}Y_l(A_l)\bigr) \\
&\quad + \mathbf{E}\bigl(\mathbf{E}(Y_k(A_k) \mid A_k) - \mathbf{E}Y_k(A_k)\bigr)\bigl(Y_l(A_l \setminus A_k) - Y_l(A_l \setminus A_{k+\delta_k})\bigr) \\
&\quad + \mathbf{E}\bigl(\mathbf{E}(Y_k(A_k) \mid A_k) - \mathbf{E}Y_k(A_k)\bigr)\bigl(Y_l(A_l \setminus A_{k+\delta_k})\bigr) - \mathbf{E}Y_l(A_l \setminus A_{k+\delta_k}) \\
&\quad + \mathbf{E}\bigl(\mathbf{E}\bigl(Y_k(A_k) \mid A_k\bigr) - \mathbf{E}Y_k(A_k)\bigr)\bigl(\mathbf{E}Y_l(A_l \setminus A_{k+\delta_k}) - \mathbf{E}Y_l(A_l)\bigr) \\
&\leqslant C\pi\bigl(a_l M(A_l), a_l M(A_l \setminus A_k)\bigr) + C\alpha\bigl(\sigma(A_k), \sigma(A_l \setminus A_{k+\delta_k})\bigr) \\
&\quad + C\pi\bigl(a_l M(A_l), a_l M(A_l \setminus A_{k+\delta_k})\bigr) \\
&= C\theta_{k,l}.
\end{aligned}
$$

Note that by symmetry

$$
\operatorname{Cov}\bigl(Y_k(A_k), Y_l(A_l)\bigr) \leqslant C(\theta_{k,l} \wedge \theta_{l,k}),
$$

which by Theorem 1 ends the proof of Theorem 2. □

Proof of Remark 1. Obviously, we can evaluate

$$
\theta_{l,k} \wedge \theta_{k,l} \leqslant (\alpha_{k,l} \vee \alpha_{l,k}) + (\pi_{k,l} \wedge \pi_{l,k}),
$$

where $\pi_{k,l} = \pi(a_l M(A_l), a_l M(A_l \setminus A_k))$. Because $M(A_l) = \max\{M(A_k \cap A_l),\ M(A_l \setminus A_k)\}$ and because

$$
\begin{aligned}
\mathbf{P}\bigl[\bigl|a_l\bigl(M(A_l) - M(A_l \setminus A_k)\bigr)\bigr| > \varepsilon\bigr] &= \begin{cases} 0 & \text{if } M(A_l \cap A_k) < M(A_l \setminus A_k), \\ \mathbf{P}[a_l(M(A_l \cap A_k) - M(A_l \setminus A_k)) > \varepsilon] & \text{otherwise} \end{cases} \\
&\leqslant \mathbf{P}\bigl[a_l M(A_k \cap A_l) > \varepsilon + a_l M(A_l \setminus A_k)\bigr],
\end{aligned}
$$

it follows that $\pi_{l,k} \leqslant \inf_{\varepsilon>0} p(l,k,\varepsilon) \vee \varepsilon$ and $\pi_{k,l} \leqslant \inf_{\varepsilon>0} p(k,l,\varepsilon) \vee \varepsilon$, and thus $\pi_{l,k} \wedge \pi_{k,l} \leqslant p_{k,l}$, which ends the proof. □

4 Examples and applications

Example 1. Let $\{a_n,\ n \geqslant 1\}$ be a sequence of positive reals, and put $\overline{a}_n = \sum_{i=1}^n a_i$. Let $\{Y_n,\ n \geqslant 1\}$ be a sequence of independent random variables with laws

$$
\mathbf{P}[Y_n \leqslant x] = \begin{cases} 1 & \text{for } x > 0, \\ \mathrm{e}^{a_n x} & \text{for } x \leqslant 0. \end{cases}
$$

Then for the sets $\{A_n = \{1,2,3,\dots,n\},\ n \geqslant 1\}$, we have $\mathbf{P}[\overline{a}_n M(A_n) < x] = G_{3,1}(x)$, and therefore for $l \geqslant k$,

$$\begin{aligned}&\mathbf{P}\big[\overline{a}_l M(A_k \cap A_l) > \varepsilon + \overline{a}_l M(A_l \setminus A_k)\big]\\ &\quad\leqslant \mathbf{P}\big[M(A_k \cap A_l) > M(A_l \setminus A_k)\big] = \int_{-\infty}^{0} \mathrm{e}^{(\overline{a}_l - \overline{a}_k)x}\, \mathrm{d}\mathrm{e}^{\overline{a}_k x} = \frac{\overline{a}_k}{\overline{a}_l}.\end{aligned}$$

In consequence,

$$p_{k,l} \leqslant \frac{\overline{a}_k \wedge \overline{a}_l}{\overline{a}_k \vee \overline{a}_l}, \quad k,l \in \mathbb{N}.$$

On the other hand,

$$\pi\big(\overline{a}_l M(A_k \cap A_l),\, \overline{a}_l M(A_l \setminus A_k)\big) = \inf_{\varepsilon>0}\left\{\left(\frac{\overline{a}_l - \overline{a}_k}{\overline{a}_l}\mathrm{e}^{-\varepsilon\overline{a}_k/\overline{a}_l} + \frac{\overline{a}_k}{\overline{a}_l}\mathrm{e}^{-\varepsilon(\overline{a}_l - \overline{a}_k)/\overline{a}_l}\right) \vee \varepsilon\right\}.$$

Now putting, for example, $\{a_n = 1,\ k_n = \lfloor \ln(n) \rfloor,\ l_n = n,\ n \geqslant 1\}$, we see that

$$\begin{aligned}&\lim_{n\to\infty} \pi\big(\overline{a}_{l_n} M(A_{k_n} \cap A_{l_n}), \overline{a}_{l_n} M(A_{l_n} \setminus A_{k_n})\big)\\ &\quad= \lim_{n\to\infty} \frac{n - \lfloor \ln(n) \rfloor}{n}\mathrm{e}^{-\varepsilon\frac{\lfloor \ln(n)\rfloor}{n}} + \frac{\lfloor \ln(n) \rfloor}{n}\mathrm{e}^{-\varepsilon(n - \lfloor \ln(n) \rfloor)/n} = 1,\\ &\lim_{n\to\infty} p_{k_n,l_n} = \lim_{n\to\infty} \frac{\lfloor \ln(n) \rfloor}{n}\mathrm{e}^{-\varepsilon(n - \lfloor \ln(n) \rfloor)/n} = 0.\end{aligned}$$

Example 2. Let $\{V_n,\ n \geqslant 1\}$ be a Rademacher sequence of α-mixing random variables such that $\mathbf{P}[V_n = 1] < 1$ and $\mathbf{P}[V_n = 1] = p = 1 - \mathbf{P}[V_n = 0] > 0$ with

$$\alpha_n = \sup_k \alpha\big(\sigma(V_k), \sigma(V_{k+n})\big)$$

satisfying

$$\sum_{i \leqslant k \leqslant j}\ \sum_{i \leqslant l \leqslant j} \frac{\alpha_{|k-l|}}{kl} \leqslant \ln^\gamma \frac{j}{i-1}.$$

Let $\{X_n,\ n \geqslant 1\}$ be a sequence of independent random variables, independent of $\{V_n,\ n \geqslant 1\}$, such that

$$\mathbf{P}[X_{2n} < x] = \begin{cases} 1 & \text{if } x > 0,\\ 1 - x^2 & \text{if } -1 \leqslant x \leqslant 0,\\ 0 & \text{otherwise,}\end{cases} \qquad n \geqslant 1,$$

and

$$\mathbf{P}[X_{2n-1} < x] = \begin{cases} 1, & \text{if } x > 0,\\ \mathrm{e}^{(\sqrt{n} - \sqrt{n-1})x} & \text{if } x \leqslant 0.\end{cases}$$

If

$$A_n = \begin{cases} \{1,3,5,\dots,2n-1\} & \text{if } V_n = 1,\\ \{2,4,6,\dots,2n\} & \text{if } V_n = 0,\end{cases}$$

then

$$\sqrt{n}M(A_n) \xrightarrow{\mathcal{D}} pG_{3,1}() + (1-p)G_{3,2}() = G(), \text{ say}, \quad \text{as } n \to \infty.$$

From Example 1 we have

$$\mathbf{P}\big[M(A_k \cap A_l) \geqslant M(A_l \setminus A_k) + \varepsilon,\ V_k = V_l = 1\big] \leqslant \begin{cases} \sqrt{\frac{k}{l}} & \text{if } k < l, \\ 1 & \text{otherwise,} \end{cases}$$

whereas for $k < l$,

$$\begin{aligned} &\mathbf{P}\big[M(A_k \cap A_l) \geqslant M(A_l \setminus A_k) + \varepsilon,\ V_k = V_l = 0\big] \\ &\quad\leqslant \mathbf{P}\big[M(A_k \cap A_l) \geqslant M(A_l \setminus A_k)\big] = \int_{-1}^{0} \big(1-u^2\big)^{l-k} \,\mathrm{d}\big(1-u^2\big)^k = \frac{k}{l} \leqslant \sqrt{\frac{k}{l}}, \end{aligned}$$

and

$$\begin{aligned} &\mathbf{P}\big[M(A_k \cap A_l) \geqslant M(A_l \setminus A_k) + \varepsilon,\ V_k = 0,\ V_l = 1\big] \\ &\quad= \mathbf{P}\big[M(A_k \cap A_l) \geqslant M(A_l \setminus A_k) + \varepsilon,\ V_k = 1,\ V_l = 0\big] = 0, \end{aligned}$$

as in these cases $A_k \cap A_l = \emptyset$. Thus

$$p_{k,l} \leqslant 2\sqrt{\frac{k \wedge l}{k \vee l}},$$

and by the Cauchy–Maclaurin theorem

$$\begin{aligned} \sum_{i \leqslant k \leqslant j} \sum_{i \leqslant l \leqslant j} \frac{1}{kl}\sqrt{\frac{k \wedge l}{k \vee l}} &\leqslant 2\sum_{k=i}^{j}\sum_{l=k}^{j} \frac{1}{k^{1/2} l^{3/2}} \leqslant 2\sum_{k=i}^{j} \frac{1}{\sqrt{k}} \sum_{l=k}^{\infty} \frac{1}{l^{3/2}} \\ &\leqslant C\sum_{k=i}^{j} \frac{1}{k} \leqslant C \ln \frac{j}{i-1} \leqslant C \ln^{\gamma} \frac{j}{i-1} \end{aligned}$$

for every $\gamma > 1$. Thus

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{i=1}^{n} \frac{1}{i} I\big[\sqrt{i}M(A_i) < x\big] = G(x)$$

for every $x \in \mathbb{R}$.

Example 3. Let $\{X_n,\ n \geqslant 1\}$ be an i.i.d. sequence of random variables with the exponential distribution function $F(x) = 1 - \mathrm{e}^{-x}$. Let $\{\varepsilon_n,\ n \geqslant 1\}$ be an i.i.d. sequence of random variables independent of the previous one and such that $\mathbf{P}[\varepsilon_1 = 1] = s = 1 - \mathbf{P}[\varepsilon_1 = 0]$. We put $a_n = 1$, $b_n = -\ln n$, $d_n = 1/n$, and $D_n = \ln n$. Let $\{A_n,\ n \geqslant 1\}$ be a sequence of random sets such as in (1.4).

Then, as it was shown in Example 2.1 [16, p. 1979, Eq. 2.7], taking the limit as $y \to \infty$, we have

$$M(A_n) - \ln n \xrightarrow{\mathcal{D}} \mathrm{e}^{-s\mathrm{e}^{-x}}.$$

Since $A_k \subset A_l$ and $A_k \cap A_l = A_k$ for $l > k$, we have, for $l \geqslant k$,

$$p(k, l, \varepsilon) \leqslant \mathbf{P}\big[M(A_k) > M(A_l \setminus A_k)\big] = I_{k,l}, \text{ say}.$$

We have

$$I_{k,l} = \int_0^\infty \sum_{i=1}^{k} \sum_{j=0}^{l-k} \mathbf{P}\big[|A_k| = i\big] \mathbf{P}\big[|A_l \setminus A_k| = j\big] \mathbf{P}\big[M(A_k) = x \bigm| |A_k| = i\big]$$
$$\times \; \mathbf{P}\big[M(A_l \setminus A_k) \leqslant x \bigm| |A_l \setminus A_k| = j\big] \, \mathrm{d}x.$$

Because $|A_l|$ and $|A_l \setminus A_k|$ have the binomial distribution with l and $l-k$ trials, respectively, and probability of success equal to s (we denote these distributions by $B(l, s)$ and $B(l-k,\ s)$), we have

$$\mathbf{P}\big[M(A_k) = x \bigm| |A_k| = i\big] = \frac{\mathrm{d}F^i(x)}{\mathrm{d}x} = i\mathrm{e}^{-x} F^{i-1}(x),$$
$$\mathbf{P}\big[M(A_l \setminus A_k) \leqslant x \bigm| |A_l \setminus A_k| = j\big] = F^j(x),$$
$$\mathbf{P}\big[|A_l| = i\big] = \mathbf{P}\big[B(l, s) = i\big], \qquad \mathbf{P}\big[|A_l \setminus A_k| = j\big] = \mathbf{P}\big[B(l-k,\ s) = j\big].$$

Since

$$\int_0^\infty i\mathrm{e}^{-x}\big(1 - \mathrm{e}^{-x}\big)^{i+j-1} \, \mathrm{d}x = \frac{i}{i+j},$$

we have

$$I_{k,l} = \sum_{i=1}^{k} \sum_{j=0}^{l-k} \mathbf{P}\big[B(k, s) = i\big] \mathbf{P}\big[B(l-k,\ s) = j\big] \frac{i}{i+j}.$$

Furthermore, for positive integers i and j, we have

$$\frac{i}{i+j} \leqslant 1 \wedge \frac{i}{j}. \tag{4.1}$$

Let us choose t such that $1 < t < s$. Then from Chebyshev's inequality and (4.1) we have

$$I_{k,l} \leqslant \sum_{i=1}^{k} \mathbf{P}\big[B(k, s) = i\big] \left(\mathbf{P}\big[B(l-k,\ s) \leqslant \lfloor t(l-k) \rfloor\big] + \sum_{j=\lfloor t(l-k) \rfloor + 1}^{l-k} \frac{i}{j} \mathbf{P}\big[B(l-k,\ s) = j\big] \right)$$
$$\leqslant \mathbf{P}\big[B(l-k,\ s) \leqslant t(l-k)\big] + \frac{\mathbf{E}B(k, s)}{t(l-k)}$$
$$= \mathbf{P}\big[B(l-k,\ s) - \mathbf{E}B(l-k,\ s) \leqslant -(s-t)(l-k)\big] + \frac{ks}{t(l-k)}$$
$$\leqslant \mathbf{P}\big[\big|B(l-k,\ s) - \mathbf{E}B(l-k,\ s)\big| \geqslant (s-t)(l-k)\big] + \frac{ks}{t(l-k)}$$
$$\leqslant \frac{\mathrm{Var}(B(l-k,\ s))}{(s-t)^2(l-k)^2} + \frac{ks}{t(l-k)} = \frac{s(1-s)}{(s-t)^2(l-k)} + \frac{ks}{t(l-k)}.$$

Now taking $t = s/2$, we have

$$I_{k,l} \leqslant \frac{4(1-s)}{s(l-k)} + \frac{2k}{l-k} \leqslant C \frac{k}{l-k}$$

with $C = 2\max(2,\ 4(1-s)/s)$. Using the estimate

$$p_{k,l} \leqslant C \begin{cases} \frac{k}{l} & \text{if } 2k < l, \\ \frac{l}{k} & \text{if } 2l < k, \\ 1 & \text{if } l < 2k < 4l, \end{cases}$$

we have

$$\sum_{i\leqslant k\leqslant j}\sum_{i\leqslant l\leqslant j}\frac{1}{kl}p_{k,l} \leqslant \sum_{i\leqslant k\leqslant j}\sum_{2k\leqslant l\leqslant j}\frac{C}{l^2} + \sum_{i\leqslant k\leqslant j}\sum_{i\leqslant l\leqslant \lfloor k/2\rfloor}\frac{C}{k^2} + \sum_{i\leqslant k\leqslant j}\sum_{\lfloor \frac{k}{2}\rfloor\leqslant l\leqslant 2k}\frac{C}{kl}.$$

Now by the Cauchy–Maclaurin theorem we get

$$\sum_{i\leqslant k\leqslant j}\sum_{i\leqslant l\leqslant j}\frac{1}{kl}p_{k,l} \leqslant C\left(\frac{1}{2}+\frac{1}{2}+2\ln 2\right)\ln\frac{j}{i-1},$$

and thus (2.2) holds for arbitrary $1 < \gamma < 2$.

Finally, as

$$1 + C\sum_{n=2}^{\infty}\frac{1}{n\ln^{3-\gamma} n} < \infty,$$

we get

$$\lim_{n\to\infty}\frac{1}{\ln n}\sum_{i=1}^{n}\frac{1}{i}I\Big[\max_{\substack{\varepsilon_k=1\\ 1\leqslant k\leqslant i}} X_i - \ln i < x\Big] = \mathrm{e}^{-s\mathrm{e}^{-x}}.$$

Acknowledgment. The author would like to express his gratitude to the referee for extremely valuable remarks and comments improving the previous version of this paper.

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