Compressions of k th-order slant Toeplitz operators to model spaces

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Received November 23, 2020; revised May 21, 2021

Abstract. In this paper, we consider compressions of k th-order slant Toeplitz operators to the backward shift-invariant subspaces of the classical Hardy space H^2 . In particular, we characterize these operators using compressed shifts and finite-rank operators of special kind.

MSC: 47B32, 47B35, 30H10

Keywords: model space, compressed shift, Toeplitz operator, slant Toeplitz operator, generalized slant Toeplitz operator, truncated Toeplitz operator

1 Introduction

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and denote by $L^2 = L^2(\mathbb{T})$ the space of Lebesgue-measurable square-integrable functions on $\mathbb{T}$ and by $L^\infty = L^\infty(\mathbb{T})$ the space of Lebesgue-measurable essentially bounded functions. Moreover, let H^2 be the classical Hardy space in the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. As usual, we view H^2 as a space of functions analytic in $\mathbb{D}$ or (via radial limits) as a closed subspace of L^2 (see [11, 13] for details). We denote by P the orthogonal projection from L^2 onto H^2 .

For $\varphi \in L^\infty$, the classical Toeplitz operator T_φ is defined on H^2 by

$$T_\varphi f = PM_\varphi f, \quad f \in H^2,$$

where $M_\varphi : L^2 \rightarrow L^2$ is the multiplication operator $f \mapsto \varphi f$. For $\varphi \in L^2$, the operators T_φ and M_φ can be defined on a dense subset of H^2 and L^2 , respectively (namely, on bounded functions).

Classical Toeplitz operators, as compressions of multiplication operators to H^2 , are among the most studied linear operators on the Hardy space. Their study goes back to the beginning of the 20th century (for details and references, see, e.g., [3, 28]). In recent years, compressions of multiplication operators to model spaces were widely studied. Model spaces are the nontrivial subspaces of H^2 that are invariant for the backward shift operator $S^* = T_{\bar{z}}$. Each of these subspaces is of the form $K_\alpha = H^2 \ominus \alpha H^2$, where α is a nonconstant inner function, a function analytic and bounded in $\mathbb{D}$ ($\alpha \in H^\infty$) such that its boundary values are of modulus one a.e. on $\mathbb{T}$ (if α is constant, then $K_\alpha = \{0\}$).

Each model space K_α is a reproducing kernel Hilbert space with the reproducing kernel function k_w^α at a point $w \in \mathbb{D}$ given by

$$k_w^\alpha(z) = \frac{1 - \overline{\alpha(w)}\alpha(z)}{1 - \overline{w}z}, \quad z \in \mathbb{D}. \quad (1.1)$$

Note that for each $w \in \mathbb{D}$, we have $k_w^\alpha \in K_\alpha^\infty = K_\alpha \cap H^\infty$. As the linear span of $\{k_w^\alpha: w \in \mathbb{D}\}$ is dense in K_α , we see that K_α^∞ is a dense subset of K_α .

The orthogonal projection P_α from L^2 onto K_α is given by $P_\alpha = P - M_\alpha P M_{\overline{\alpha}}$ (see [12, Cor. 14.13]). In particular, $P_{\alpha|H^2} = I_{H^2} - T_\alpha T_{\overline{\alpha}}$.

It is easy to see that if $\alpha(z) = z^n$, then K_α is the set of all polynomials of degree at most $n-1$. In that case, $\dim K_\alpha = n$, and $\{1, z, \dots, z^{n-1}\}$ is an orthonormal basis for K_α . It is known (see [13]) that $\dim K_\alpha < \infty$ if and only if α is a finite Blaschke product. For example, if α is a finite Blaschke product with n distinct zeros $a_1, \dots, a_n$, then the set of corresponding kernel functions $k_{a_1}^\alpha, \dots, k_{a_n}^\alpha$ is a (nonorthogonal) basis for K_α (see [12, 13]).

Model spaces are important in view of their connection with various topics, one being the Sz.-Nagy–Foiás model theory [30, Chap. 13]. For more details on properties and structure of model spaces, see [13] and [12].

For an inner function α and for $\varphi \in L^2$, let

$$A_\varphi^\alpha f = P_\alpha(\varphi f), \quad f \in K_\alpha^\infty,$$

where P_α is the orthogonal projection from L^2 onto K_α . Note that A_φ^α is densely defined since K_α^∞ is a dense subset of K_α . These operators, called truncated Toeplitz operators, gained attention in 2007 with Sarason's paper [29]. Ever since, truncated Toeplitz operators have been under constant intensive study, which led to many interesting results and applications (see [8, 15] and references therein). More recently, the authors of [6, 7] and [4] introduced the so-called asymmetric truncated Toeplitz operators: for two inner functions α, β and for $\varphi \in L^2$, the asymmetric truncated Toeplitz operator $A_\varphi^{\alpha, \beta}$ is defined by

$$A_\varphi^{\alpha, \beta} f = P_\beta(\varphi f), \quad f \in K_\alpha^\infty.$$

As above, $A_\varphi^{\alpha, \beta}$ is densely defined. Moreover, $A_\varphi^{\alpha, \alpha} = A_\varphi^\alpha$. These operators were then studied in, for example, [5, 17, 22, 23, 24, 25].

Recall that if $\varphi(z) = \sum_{j=-\infty}^\infty a_j z^j \in L^\infty$, then for each $n \in \mathbb{Z}$, $M_\varphi(z^n) = \sum_{j=-\infty}^\infty a_{j-n} z^j$, and the matrix of M_φ with respect to the standard basis $\{z^n: n \in \mathbb{Z}\}$ is given by

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \dots & a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ \dots & a_1 & a_0 & a_{-1} & a_{-2} & \dots \\ \dots & a_2 & a_1 & a_0 & a_{-1} & \dots \\ \dots & a_3 & a_2 & a_1 & a_0 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Similarly, for each $n \in \mathbb{N}_0$, $T_\varphi(z^n) = \sum_{j=0}^\infty a_{j-n} z^j$, and T_φ is the operator on H^2 represented by the matrix (with respect to $\{z^n: n \in \mathbb{N}_0\}$)

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ a_1 & a_0 & a_{-1} & a_{-2} & \dots \\ a_2 & a_1 & a_0 & a_{-1} & \dots \\ a_3 & a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In [1, 2], for $k \in \mathbb{N}$, $k \geq 2$, and $\varphi(z) = \sum_{j=-\infty}^{\infty} a_j z^j \in L^\infty$, the authors define the k th-order slant Toeplitz operator U_φ on L^2 by $U_\varphi(z^n) = \sum_{j=-\infty}^{\infty} a_{kj-n} z^j$ for $n \in \mathbb{Z}$, that is, as the operator on L^2 represented by the matrix

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \dots & a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ \dots & a_k & a_{k-1} & a_{k-2} & a_{k-3} & \dots \\ \dots & a_{2k} & a_{2k-1} & a_{2k-2} & a_{2k-3} & \dots \\ \dots & a_{3k} & a_{3k-1} & a_{3k-2} & a_{3k-3} & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Equivalently, $U_\varphi : L^2 \rightarrow L^2$, $\varphi \in L^\infty$, is given by

$$U_\varphi f = W_k M_\varphi f, \quad f \in L^2, \quad \text{where} \quad W_k(z^n) = \begin{cases} z^m & \text{if } \frac{n}{k} = m \in \mathbb{Z}, \\ 0 & \text{if } \frac{n}{k} \notin \mathbb{Z}. \end{cases}$$

The authors of [1, 2] also consider V_φ , the compression of U_φ to H^2 , defined by

$$V_\varphi f = P U_\varphi f, \quad f \in H^2,$$

and represented by the matrix

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ a_k & a_{k-1} & a_{k-2} & a_{k-3} & \dots \\ a_{2k} & a_{2k-1} & a_{2k-2} & a_{2k-3} & \dots \\ a_{3k} & a_{3k-1} & a_{3k-2} & a_{3k-3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that if $\varphi \in L^2$, then U_φ and V_φ are densely defined. For $k = 2$, the operators U_φ (called slant Toeplitz operators) and their compressions to H^2 were first studied in [18, 32] (see also [19, 20, 21]). These operators have connections with wavelet theory and dynamical systems (see, e.g., [16, 20, 31]). In the more recent papers [9, 10], slant Toeplitz operators in multivariable setting are considered. In [26, 27] the authors investigate the commutativity of k th-order slant Toeplitz operators. Observe that if $k = 1$ in the above definitions, then $U_\varphi = M_\varphi$ and $V_\varphi = T_\varphi$.

Here we study compressions of k th-order slant Toeplitz operators to model spaces. For a fixed $k \in \mathbb{N}$, two inner functions α, β , and for $\varphi \in L^2$, define

$$U_\varphi^{\alpha, \beta} f = P_\beta U_\varphi f = P_\beta W_k(\varphi f), \quad f \in K_\alpha^\infty,$$

and denote by $\mathcal{S}_k(\alpha, \beta)$ the set of all the compressions $U_\varphi^{\alpha, \beta}$, $\varphi \in L^2$, that can be boundedly extended to K_α .

In Section 2, we investigate conditions on φ that imply that $U_\varphi^{\alpha, \beta} = 0$. We show that, in contrast with the case of U_φ and V_φ [1, 2], $U_\varphi^{\alpha, \beta}$ can be equal to the zero operator for φ not equal to zero.

In Section 3, we characterize operators from $\mathcal{S}_k(\alpha, \beta)$ using compressed shifts $S_\alpha = A_z^\alpha$ and $S_\beta = A_z^\beta$. It is well known that a bounded linear operator $T : H^2 \rightarrow H^2$ is a Toeplitz operator if and only if $T - S^* T S = 0$, where S is the shift operator $S = T_z$. A similar characterization was given in [1, 2] for V_φ . Namely, T is a compression of a k th-order slant Toeplitz operator to H^2 if and only if $T - S^* T S^k = 0$. We show, for example, that a bounded linear operator U from K_α into K_β belongs to $\mathcal{S}_k(\alpha, \beta)$ if and only if $U - S_\beta^* U (S_\alpha^*)^k$ is a special kind of operator of rank at most $k + 1$. This is done in the spirit of Sarason's characterization of truncated Toeplitz operators given in [29], where, among other results, he shows that U is a truncated Toeplitz

operator if and only if $U - S_\alpha U S_\alpha^*$ is of rank two and special kind (see also [4, 17, 24] for the asymmetric case).

In Section 4, we use the results obtained to show that for $k \leq \dim K_\alpha$, the operator $U_\varphi^{\alpha, \beta}$ is zero if and only if φ belongs to $\overline{\alpha H^2} + \bar{z}^{k-1}(W_k^* \beta) H^2$.

2 Operators from $\mathcal{S}_k(\alpha, \beta)$ equal to the zero operator, part 1

In this section, we investigate for which $\varphi \in L^2$, $U_\varphi^{\alpha, \beta} = 0$.

We start with some basic properties of the operator W_k and its adjoint W_k^* . Some of these properties can be found, for example, in [18] (for $k = 2$) and [1, 2].

Lemma 1. *Let $k \in \mathbb{N}$. Then:*

- (a) $W_k^* f(z) = f(z^k)$, $|z| = 1$, $f \in L^2$;
- (b) $W_k^*(f \cdot g) = W_k^* f \cdot W_k^* g$ for all $f, g \in L^2$ such that $f \cdot g \in L^2$;
- (c) $W_k W_k^* = I_{L^2}$, and $W_k^* W_k$ is the orthogonal projection from L^2 onto the closed linear span of $\{z^{km} : m \in \mathbb{Z}\}$;
- (d) $W_k \bar{f} = \overline{W_k^* f}$ and $W_k^* \bar{f} = \overline{W_k f}$, $f \in L^2$;
- (e) P reduces W_k , that is, $P W_k = W_k P$ and $P W_k^* = W_k^* P$;
- (f) $M_\varphi W_k = W_k M_{W_k^* \varphi}$ for every $\varphi \in L^\infty$;
- (g) for every inner function α , the function $W_k^* \alpha$ is also inner, and $P_\alpha W_k = W_k P_{W_k^* \alpha}$.

Proof. The proofs of (a)–(e) are straightforward. We only prove (f) and (g).

To show (f), fix $\varphi \in L^\infty$ and take any $f, g \in L^2$. Then by (b) and (d)

$$\langle M_\varphi W_k f, g \rangle = \langle f, W_k^*(\overline{\varphi} g) \rangle = \langle f, \overline{W_k^* \varphi} \cdot W_k^* g \rangle = \langle W_k M_{W_k^* \varphi} f, g \rangle.$$

If α is an inner function, then $W_k^* \alpha$ is also an inner function by (a). Since $P_\alpha = P - M_\alpha P M_{\overline{\alpha}}$, using (e) and (f), we obtain

$$\begin{aligned} P_\alpha W_k &= P W_k - M_\alpha P W_k M_{\overline{W_k^* \alpha}} = W_k P - M_\alpha W_k P M_{\overline{W_k^* \alpha}} \\ &= W_k (P - M_{W_k^* \alpha} P M_{\overline{W_k^* \alpha}}) = W_k P_{W_k^* \alpha}, \end{aligned}$$

that is, (g) holds. $\square$

Note that by (e), (f), and (g) we get that for $\varphi \in L^\infty$,

$$T_\varphi W_k = W_k T_{W_k^* \varphi} \quad \text{and} \quad W_k^* T_\varphi = T_{W_k^* \varphi} W_k^* \quad \text{on } H^2.$$

For the remainder of this section, fix $k \in \mathbb{N}$.

Proposition 1. *Let α and β be two inner functions, and let $\varphi \in L^2$. If $\varphi \in \overline{\alpha H^2} + (W_k^* \beta) H^2$, then $U_\varphi^{\alpha, \beta} = 0$.*

Proof. Assume that $\varphi = \overline{\alpha h_1} + W_k^* \beta \cdot h_2$ for some $h_1, h_2 \in H^2$. Then for all $f \in K_\alpha^\infty$ and $g \in K_\beta^\infty$, we have (by Lemma 1(g))

$$\begin{aligned} \langle U_\varphi^{\alpha, \beta} f, g \rangle &= \langle \varphi f, W_k^* g \rangle = \langle \overline{\alpha h_1} f, W_k^* g \rangle + \langle W_k^* \beta \cdot h_2 f, W_k^* g \rangle \\ &= \langle f, \alpha h_1 \cdot W_k^* g \rangle + \langle P_\beta W_k (W_k^* \beta \cdot h_2 f), g \rangle \\ &= \langle W_k P_{W_k^* \beta} (W_k^* \beta \cdot h_2 f), g \rangle = 0. \quad \square \end{aligned}$$

Corollary 1. *If $U \in \mathcal{S}_k(\alpha, \beta)$, then $U = U_{\varphi}^{\alpha, \beta}$ with φ from $\overline{K_{\alpha}} + K_{W_k^* \beta}$.*

Hence $U_{\varphi}^{\alpha, \beta}$ is not uniquely determined by its symbol. Moreover, the boundedness of a symbol is not necessary for the boundedness of $U_{\varphi}^{\alpha, \beta}$.

In general, the implication in Proposition 1 cannot be reversed as the following example shows.

Example 1. Let $k = 2$, $\alpha(z) = z^4$, and $\beta(z) = z^3$. In that case, for $\varphi = \sum_{n=-\infty}^{\infty} a_n z^n \in L^2$, the operator $U_{\varphi}^{\alpha, \beta}$ is represented by the matrix

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_2 & a_1 & a_0 & a_{-1} \\ a_4 & a_3 & a_2 & a_1 \end{bmatrix}.$$

Hence if, for example, $\varphi(z) = z^5$, then $U_{\varphi}^{\alpha, \beta} = 0$. But here $z^5 \notin \overline{\alpha H^2} + (W_k^* \beta) H^2 = \overline{z^4 H^2} + z^6 H^2$.

Observe that $U_{\varphi}^{\alpha, \beta}$ can be seen as a composition of W_k and an asymmetric truncated Toeplitz operator. Indeed, using Lemma 1(g), for $f \in K_{\alpha}^{\infty}$, we get

$$U_{\varphi}^{\alpha, \beta} f = P_{\beta} W_k(\varphi f) = W_k P_{W_k^* \beta}(\varphi f) = W_k A_{\varphi}^{\alpha, W_k^* \beta} f.$$

Now Proposition 1 follows from the fact that $A_{\varphi}^{\alpha, \beta} = 0$ if and only if $\varphi \in \overline{\alpha H^2} + \beta H^2$ (see [4, 22]).

Let α be an inner function. Then the operator $C_{\alpha} : L^2 \rightarrow L^2$ defined by

$$C_{\alpha} f(z) = \alpha(z) \overline{z f(z)}, \quad |z| = 1,$$

is a conjugation on L^2 (an antilinear isometric involution). Moreover, $C_{\alpha}(K_{\alpha}) = K_{\alpha}$ (see [14] and [13, Chap. 8]). It is known that all truncated Toeplitz operators are C_{α} -symmetric, that is, $C_{\alpha} A_{\varphi}^{\alpha} C_{\alpha} = (A_{\varphi}^{\alpha})^* = A_{\overline{\varphi}}^{\alpha}$ [29]. It was observed in [24] that $C_{\beta} A_{\varphi}^{\alpha, \beta} C_{\alpha} = A_{\overline{\alpha \varphi} \beta}^{\alpha, \beta}$ (see also [17]). Here we have the following:

Proposition 2. *Let α, β be two inner functions, and let $\varphi \in L^2$. Then (on K_{α}^{∞}) $C_{\beta} U_{\varphi}^{\alpha, \beta} C_{\alpha} = U_{\psi}^{\alpha, \beta}$ with $\psi = \overline{z^{k-1} \alpha \varphi} W_k^* \beta$.*

Proof. Let $\varphi \in L^2$. Then for $f \in K_{\alpha}^{\infty}$ and $g \in K_{\beta}^{\infty}$, by Lemma 1 we have

$$\begin{aligned} \langle C_{\beta} U_{\varphi}^{\alpha, \beta} C_{\alpha} f, g \rangle &= \langle C_{\beta} P_{\beta} W_k M_{\varphi} C_{\alpha} f, g \rangle = \langle C_{\beta} g, P_{\beta} W_k(\varphi C_{\alpha} f) \rangle \\ &= \langle C_{\beta} g, W_k(\varphi C_{\alpha} f) \rangle = \langle \overline{z \beta g}, W_k(\varphi \overline{z \alpha f}) \rangle \\ &= \langle W_k^*(\overline{z \beta g}), \varphi \overline{z \alpha f} \rangle = \langle \overline{z^{k-1} \alpha \varphi} W_k^* \beta \cdot f, W_k^* g \rangle \\ &= \langle P_{\beta} W_k M_{\overline{z^{k-1} \alpha \varphi} W_k^* \beta} f, g \rangle = \langle U_{\overline{z^{k-1} \alpha \varphi} W_k^* \beta}^{\alpha, \beta} f, g \rangle. \quad \square \end{aligned}$$

As a corollary of Proposition 2, we get the following:

Corollary 2. *Let α, β be two inner functions, and let U be a bounded linear operator from K_{α} into K_{β} . Then $U \in \mathcal{S}_k(\alpha, \beta)$ if and only if $C_{\beta} U C_{\alpha} \in \mathcal{S}_k(\alpha, \beta)$.*

Proposition 3. *Let α, β be two inner functions, and let $\varphi \in L^2$. If $\varphi \in \overline{\alpha H^2} + \overline{z^{k-1} (W_k^* \beta) H^2}$, then $U_{\varphi}^{\alpha, \beta} = 0$.*

Proof. Let $\varphi = \overline{\alpha h_1} + \bar{z}^{k-1} W_k^* \beta \cdot h_2$ with $h_1, h_2 \in H^2$. By Proposition 1 we have $U_{\alpha h_1}^{\alpha, \beta} = 0$ and, consequently, $U_{\varphi}^{\alpha, \beta} = U_{\bar{z}^{k-1} W_k^* \beta \cdot h_2}^{\alpha, \beta}$. Now, using Proposition 2, we obtain

$$C_{\beta} U_{\varphi}^{\alpha, \beta} C_{\alpha} = C_{\beta} U_{\bar{z}^{k-1} W_k^* \beta \cdot h_2}^{\alpha, \beta} C_{\alpha} = U_{\frac{\alpha \bar{z}^{k-1} W_k^* \beta \cdot h_2}{\bar{z}^{k-1} \alpha \bar{z}^{k-1} W_k^* \beta \cdot h_2}}^{\alpha, \beta} = U_{\frac{\alpha}{\alpha h_2}}^{\alpha, \beta} = 0,$$

from which it follows that $U_{\varphi}^{\alpha, \beta} = 0$. $\square$

Corollary 3. *If $U \in \mathcal{S}_k(\alpha, \beta)$, then $U = U_{\varphi}^{\alpha, \beta}$ with φ from $\overline{K_{\alpha}} + \bar{z}^{k-1} K_{W_k^* \beta}$.*

Note that

$$\overline{\alpha H^2} + (W_k^* \beta) H^2 = \overline{\alpha H^2} + \bar{z}^{k-1} (W_k^* \beta) z^{k-1} H^2 \subset \overline{\alpha H^2} + \bar{z}^{k-1} (W_k^* \beta) H^2.$$

Moreover, if an inner function β is such that $\beta(0) = 0$, then z^k divides $W_k^* \beta(z) = \beta(z^k)$, and $\bar{z}^{k-1} W_k^* \beta$ is also an inner function, $\bar{z}^{k-1} (W_k^* \beta) H^2 \subset z H^2$.

3 Characterizations of operators from $\mathcal{S}_k(\alpha, \beta)$

Let α and β be two nonconstant inner functions. In this section, we characterize operators from $\mathcal{S}_k(\alpha, \beta)$, $k \in \mathbb{N}$, using compressed shifts and finite-rank operators of special kind. Recall that the compressed shift S_{α} is defined as $S_{\alpha} = A_z^{\alpha} = P_{\alpha} S|_{K_{\alpha}}$. As K_{α} is S^* -invariant, we have $S_{\alpha}^* = A_z^{\alpha} = S|_{K_{\alpha}}^*$.

For all $n \in \mathbb{N}_0$ and $w \in \mathbb{D}$, the functional $f \mapsto f^{(n)}(w)$ is bounded on H^2 , and so there exists $k_{w,n} \in H^2$ such that $f^{(n)}(w) = \langle f, k_{w,n} \rangle$. It is not difficult to verify that $k_{w,n}(z) = n! z^n / (1 - \bar{w}z)^{n+1}$ and, in particular, $k_{0,n}(z) = n! z^n$. Denote $k_w(z) = k_{w,0}(z) = 1/(1 - \bar{w}z)$.

Lemma 2. *Let $k \in \mathbb{N}$. For $f \in H^2$, we have*

- (a) $(S^*)^k f(z) = \bar{z}^k f(z) - \sum_{j=0}^{k-1} (\langle f, k_{0,j} \rangle / j!) \bar{z}^{k-j}, |z| = 1;$
- (b) $W_k^* f - z^k W_k^* S^* f = \langle f, k_0 \rangle = f(0).$

Proof. Let $k \in \mathbb{N}$ and $f \in H^2$. Then $(S^*)^k f = (T_{\bar{z}})^k f = P(\bar{z}^k f)$. Since $f(z) = \sum_{j=0}^{\infty} (\langle f, k_{0,j} \rangle / j!) z^j$, we have, for $|z| = 1$,

$$\begin{aligned} (S^*)^k f(z) &= P\left(\sum_{j=0}^{\infty} \frac{\langle f, k_{0,j} \rangle}{j!} z^{j-k}\right) = \sum_{j=k}^{\infty} \frac{\langle f, k_{0,j} \rangle}{j!} z^{j-k} \\ &= \bar{z}^k f(z) - \sum_{j=0}^{k-1} \frac{\langle f, k_{0,j} \rangle}{j!} \bar{z}^{k-j}. \end{aligned}$$

Now observe that

$$\begin{aligned} W_k^* f(z) - z^k W_k^* S^* f(z) &= f(z^k) - z^k W_k^* (\bar{z} f(z) - \bar{z} f(0)) \\ &= f(z^k) - (f(z^k) - f(0)) = f(0), \end{aligned}$$

which proves (b). $\square$

As K_{α} is a closed subspace of H^2 , $f \mapsto f^{(n)}(w)$ is also bounded on K_{α} for all $n \in \mathbb{N}_0$ and $w \in \mathbb{D}$. Then $f^{(n)}(w) = \langle f, k_{w,n}^{\alpha} \rangle$ for all $f \in K_{\alpha}$, where $k_{w,n}^{\alpha} = P_{\alpha} k_{w,n}$ (see, e.g., [12, p. 577]). In particular, $k_w^{\alpha} = P_{\alpha} k_{w,0}$

is the reproducing kernel given by (1.1). Denote $\tilde{k}_{w,n}^\alpha = C_\alpha k_{w,n}^\alpha$ and $\tilde{k}_w^\alpha = \tilde{k}_{w,0}^\alpha$. Note that, a.e. on $\mathbb{T}$,

$$\tilde{k}_w^\alpha(z) = C_\alpha k_w^\alpha(z) = \frac{\alpha(z) - \alpha(w)}{z - w},$$

and this formula is true also for all $z \in \mathbb{D} \setminus \{w\}$ (and we have $\tilde{k}_w^\alpha(w) = \alpha'(w)$).

Lemma 3. *Let $k \in \mathbb{N}$ and $\varphi \in L^2$. Then*

$$U_\varphi^{\alpha,\beta} - S_\beta U_\varphi^{\alpha,\beta} (S_\alpha^*)^k = k_0^\beta \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes k_{0,j}^\alpha,$$

where the equality holds on K_α^∞ with

$$\chi = P_\alpha(\overline{\varphi}) \quad \text{and} \quad \psi_j = \frac{1}{j!} S_\beta P_\beta W_k(\varphi \overline{z}^{k-j}), \quad 0 \leq j \leq k-1.$$

Proof. Let $f \in K_\alpha^\infty$ and $g \in K_\beta^\infty$. Since $S_\alpha^* = P_\alpha S_{|K_\alpha}^* = S_{|K_\alpha}^*$ (as K_α is S^* -invariant), we have $S_\alpha^* f = S^* f$ and $S_\beta^* g = S^* g$ (note that here the functions $S^* f$ and $S^* g$ are bounded). By Lemma 2(a) we get

$$\begin{aligned} & \langle (U_\varphi^{\alpha,\beta} - S_\beta U_\varphi^{\alpha,\beta} (S_\alpha^*)^k) f, g \rangle \\ &= \langle \varphi f, W_k^* g \rangle - \langle \varphi (S_\alpha^*)^k f, W_k^* S_\beta^* g \rangle = \langle \varphi f, W_k^* g \rangle - \langle \varphi (S^*)^k f, W_k^* S_\beta^* g \rangle \\ &= \langle \varphi f, W_k^* g \rangle - \langle \varphi \overline{z}^k f, W_k^* S_\beta^* g \rangle + \sum_{j=0}^{k-1} \frac{\langle f, k_{0,j}^\alpha \rangle}{j!} \langle \varphi \overline{z}^{k-j}, W_k^* S_\beta^* g \rangle \\ &= \langle \varphi f, W_k^* g - z^k W_k^* S_\beta^* g \rangle + \sum_{j=0}^{k-1} \frac{1}{j!} \langle f, P_\alpha k_{0,j}^\alpha \rangle \langle P_\beta W_k(\varphi \overline{z}^{k-j}), S_\beta^* g \rangle \\ &= \langle \varphi f, W_k^* g - z^k W_k^* S^* g \rangle + \sum_{j=0}^{k-1} \frac{1}{j!} \langle f, k_{0,j}^\alpha \rangle \langle S_\beta P_\beta W_k(\varphi \overline{z}^{k-j}), g \rangle. \end{aligned}$$

By Lemma 2(b) we have

$$\langle \varphi f, W_k^* g - z^k W_k^* S^* g \rangle = \langle \varphi f, \langle g, k_0^\beta \rangle \rangle = \langle \langle f, P_\alpha(\overline{\varphi}) \rangle k_0^\beta, g \rangle. \quad \square$$

For $k = 2$, the following fact was partly noted in [18].

Lemma 4. *Let $k \in \mathbb{N}$, $m \in \mathbb{Z}$, and let $j \in \mathbb{Z}$ be such that $|j| < k$. Then*

$$W_k M_{z^{mk+j}} W_k^* = \begin{cases} M_{z^m} & \text{if } j = 0, \\ 0 & \text{if } 0 < |j| < k. \end{cases}$$

Proof. Let $f \in L^2$, $f = \sum_{n=-\infty}^{\infty} a_n z^n$. Then

$$W_k^* f(z) = \sum_{n=-\infty}^{\infty} a_n z^{nk} \quad \text{and} \quad M_{z^{mk+j}} W_k^* f(z) = \sum_{n=-\infty}^{\infty} a_n z^{nk+mk+j}.$$

If $0 < |j| < k$, then $nk + mk + j$ is not divisible by k , and so $W_k M_{z^{mk+j}} W_k^* f = 0$. On the other hand, if $j = 0$, then

$$W_k M_{z^{mk+j}} W_k^* f = W_k \left(\sum_{n=-\infty}^{\infty} a_n z^{nk+mk} \right) = \sum_{n=-\infty}^{\infty} a_n z^{n+m} = M_{z^m} f. \quad \square$$

Lemma 5. *Let $k \in \mathbb{N}$, $\chi \in K_\alpha$, and let $\psi_0, \dots, \psi_{k-1} \in K_\beta$ be such that $\psi_0(0) = \dots = \psi_{k-1}(0) = 0$. Then for*

$$\varphi = \overline{\chi} + \sum_{j=1}^k (k-j)! (W_k^* S_\beta^* \psi_{k-j}) \cdot z^j \in L^2,$$

we have

$$U_\varphi^{\alpha,\beta} - S_\beta U_\varphi^{\alpha,\beta} (S_\alpha^*)^k = k_0^\beta \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes k_{0,j}^\alpha \quad \text{on } K_\alpha^\infty. \quad (3.1)$$

Proof. Note that for each $j \in \{1, \dots, k\}$, we have

$$(W_k^* S_\beta^* \psi_{k-j})(z) \cdot z^j = \frac{\psi_{k-j}(z^k)}{z^k} z^j \in zH^2$$

since $\psi_{k-j}(0) = 0$. This and the fact that $P_\alpha = P_\alpha P$ give

$$P_\alpha(\overline{\varphi}) = P_\alpha P(\overline{\varphi}) = P_\alpha \chi = \chi.$$

Moreover, since $P_\beta = P_\beta P$ and $PW_k = W_k P$, for $1 \leq l \leq k$, we have

$$\begin{aligned} P_\beta W_k(\varphi \overline{z}^l) &= P_\beta W_k \left(\overline{\chi} z^l + \sum_{j=1}^k (k-j)! (W_k^* S_\beta^* \psi_{k-j}) \cdot z^{j-l} \right) \\ &= P_\beta P W_k(\overline{\chi} z^l) + P_\beta W_k \left(\sum_{j=1}^k (k-j)! (W_k^* S_\beta^* \psi_{k-j}) \cdot z^{j-l} \right) \\ &= P_\beta W_k P(\overline{\chi} z^l) + P_\beta W_k \left(\sum_{j=1}^k (k-j)! M_{z^{j-l}} W_k^* S_\beta^* \psi_{k-j} \right) \\ &= P_\beta \left(\sum_{j=1}^k (k-j)! W_k M_{z^{j-l}} W_k^* S_\beta^* \psi_{k-j} \right) = (k-l)! S_\beta^* \psi_{k-l}, \end{aligned}$$

where the last equality follows from Lemma 4 and the fact that $0 < |j-l| < k$ for each $j \in \{1, \dots, k\} \setminus \{l\}$. Hence, for $0 \leq j \leq k-1$,

$$\frac{1}{j!} S_\beta P_\beta W_k(\varphi \overline{z}^{k-j}) = \frac{1}{j!} S_\beta(j! S_\beta^* \psi_j) = S_\beta S_\beta^* \psi_j = \psi_j \quad (\psi_j(0) = 0),$$

and (3.1) follows from Lemma 3. $\square$

Theorem 1. Let U be a bounded linear operator from K_α into K_β . Then $U \in \mathcal{S}_k(\alpha, \beta)$, $k \in \mathbb{N}$, if and only if there exist functions $\chi \in K_\alpha$ and $\psi_0, \dots, \psi_{k-1} \in K_\beta$ such that

$$U - S_\beta U (S_\alpha^*)^k = k_0^\beta \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes k_{0,j}^\alpha. \quad (3.2)$$

Proof. If $U \in \mathcal{S}_k(\alpha, \beta)$, then $U = U_\varphi^{\alpha, \beta}$ for some $\varphi \in L^2$, and it satisfies (3.2) by Lemma 3.

Assume that U satisfies (3.2) for some $\chi \in K_\alpha$ and $\psi_0, \dots, \psi_{k-1} \in K_\beta$. Without any loss of generality, we can additionally assume that $\psi_0(0) = \dots = \psi_{k-1}(0) = 0$ (otherwise, we would replace ψ_j and χ with

$$\psi_j - \frac{\psi_j(0)}{\|k_0^\beta\|^2} k_0^\beta \quad \text{and} \quad \chi + \sum_{j=0}^{k-1} \frac{\overline{\psi_j(0)}}{\|k_0^\beta\|^2} k_{0,j}^\alpha,$$

respectively). By (3.2), for every $l \in \mathbb{N}_0$,

$$S_\beta^l U (S_\alpha^*)^{kl} - S_\beta^{l+1} U (S_\alpha^*)^{(l+1)k} = S_\beta^l k_0^\beta \otimes S_\alpha^{kl} \chi + \sum_{j=0}^{k-1} S_\beta^l \psi_j \otimes S_\alpha^{kl} k_{0,j}^\alpha.$$

It follows that for any $n \in \mathbb{N}$,

$$U = \sum_{l=0}^n \left(S_\beta^l k_0^\beta \otimes S_\alpha^{kl} \chi + \sum_{j=0}^{k-1} S_\beta^l \psi_j \otimes S_\alpha^{kl} k_{0,j}^\alpha \right) + S_\beta^{n+1} U (S_\alpha^*)^{(n+1)k}. \quad (3.3)$$

Recall that for every $f = \sum_{n=0}^\infty a_n z^n \in H^2$, we have $\|(S^*)^m f\|^2 = \sum_{n=m}^\infty |a_n|^2 \rightarrow 0$ as $m \rightarrow \infty$. Hence for each $f \in K_\alpha$,

$$0 \leq \|S_\beta^{n+1} U (S_\alpha^*)^{(n+1)k} f\| \leq \|U\| \cdot \|(S_\alpha^*)^{(n+1)k} f\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and so

$$Uf = \sum_{l=0}^\infty \left(S_\beta^l k_0^\beta \otimes S_\alpha^{kl} \chi + \sum_{j=0}^{k-1} S_\beta^l \psi_j \otimes S_\alpha^{kl} k_{0,j}^\alpha \right) f. \quad (3.4)$$

Let now

$$\varphi = \bar{\chi} + \psi \quad \text{with} \quad \psi = \sum_{j=1}^k (k-j)! (W_k^* S_\beta^* \psi_{k-j}) \cdot z^j \in H^2.$$

By Lemma 5 we have

$$U_\varphi^{\alpha, \beta} - S_\beta U_\varphi^{\alpha, \beta} (S_\alpha^*)^k = k_0^\beta \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes k_{0,j}^\alpha \quad \text{on } K_\alpha^\infty,$$

and as above, (3.3) holds with $U_\varphi^{\alpha, \beta}$ in place of U (on K_α^∞). For $f \in K_\alpha^\infty$ and $g \in K_\beta^\infty$, we have

$$\langle S_\beta^{n+1} U_\varphi^{\alpha, \beta} (S_\alpha^*)^{(n+1)k} f, g \rangle = \langle \varphi (S_\alpha^*)^{k(n+1)} f, W_k^* (S_\beta^*)^{n+1} g \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.5)$$

Indeed, since

$$\begin{aligned}
& \langle \varphi(S_\alpha^*)^{k(n+1)} f, W_k^*(S_\beta^*)^{n+1} g \rangle \\
&= \langle \varphi(S^*)^{k(n+1)} f, W_k^*(S^*)^{n+1} g \rangle \\
&= \langle \bar{\chi} \cdot T_{\bar{z}^{k(n+1)}} f, W_k^*(S^*)^{n+1} g \rangle + \langle \psi \cdot (S^*)^{k(n+1)} f, W_k^* T_{\bar{z}^{n+1}} g \rangle \\
&= \langle P(\bar{z}^{k(n+1)} f), \chi \cdot W_k^*(S^*)^{n+1} g \rangle + \langle \psi \cdot (S^*)^{k(n+1)} f, W_k^* P(\bar{z}^{n+1} g) \rangle \\
&= \langle \bar{z}^{k(n+1)} f, \chi \cdot W_k^*(S^*)^{n+1} g \rangle + \langle \psi \cdot (S^*)^{k(n+1)} f, P W_k^*(\bar{z}^{n+1} g) \rangle \\
&= \langle \overline{\chi z^{k(n+1)}} f, W_k^*(S^*)^{n+1} g \rangle + \langle \psi \cdot (S^*)^{k(n+1)} f, \bar{z}^{k(n+1)} \cdot W_k^* g \rangle \\
&= \langle P(\overline{\chi z^{k(n+1)}} f), W_k^*(S^*)^{n+1} g \rangle + \langle (S^*)^{k(n+1)} f, P(\overline{\psi z^{k(n+1)}} \cdot W_k^* g) \rangle,
\end{aligned}$$

condition (3.5) follows from the fact that

$$|\langle P(\overline{\chi z^{k(n+1)}} f), W_k^*(S^*)^{n+1} g \rangle| \leq \|\chi f\| \cdot \|(S^*)^{n+1} g\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and

$$|\langle (S^*)^{k(n+1)} f, P(\overline{\psi z^{k(n+1)}} \cdot W_k^* g) \rangle| \leq \|(S^*)^{k(n+1)} f\| \cdot \|\psi \cdot W_k^* g\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This means that for $f \in K_\alpha^\infty$ and $g \in K_\beta^\infty$,

$$\langle U_\varphi^{\alpha, \beta} f, g \rangle = \left\langle \sum_{l=0}^{\infty} \left(S_\beta^l k_0^\beta \otimes S_\alpha^{kl} \chi + \sum_{j=0}^{k-1} S_\beta^l \psi_j \otimes S_\alpha^{kl} k_{0,j}^\alpha \right) f, g \right\rangle,$$

and by (3.4) we have

$$\langle U_\varphi^{\alpha, \beta} f, g \rangle = \langle U f, g \rangle.$$

It follows that $U = U_\varphi^{\alpha, \beta} \in \mathcal{S}_k(\alpha, \beta)$.

Corollary 4. *If a bounded linear operator $U : K_\alpha \rightarrow K_\beta$ satisfies (3.2), then $U = U_\varphi^{\alpha, \beta}$ with*

$$\varphi = \bar{\chi} + \sum_{j=0}^{k-1} \psi_j(z^k) j! \bar{z}^j.$$

Proof. Assume that U satisfies (3.2). If $\psi_0(0) = \dots = \psi_{k-1}(0) = 0$, then following the proof of Theorem 1, we get that $U = U_\varphi^{\alpha, \beta}$ with

$$\varphi = \bar{\chi} + \sum_{j=0}^{k-1} j! (W_k^* S_\beta^* \psi_j) \cdot z^{k-j} = \bar{\chi} + \sum_{j=0}^{k-1} \bar{z}^k \psi_j(z^k) j! z^{k-j} = \bar{\chi} + \sum_{j=0}^{k-1} \psi_j(z^k) j! \bar{z}^j.$$

Otherwise, as explained at the beginning of the proof of Theorem 1, replace ψ_j and χ with

$$\psi_j - \frac{\psi_j(0)}{\|k_0^\beta\|^2} k_0^\beta \quad \text{and} \quad \chi + \sum_{j=0}^{k-1} \frac{\overline{\psi_j(0)}}{\|k_0^\beta\|^2} k_{0,j}^\alpha,$$

respectively. Then, by the first part of the proof, $U = U_{\varphi_1}^{\alpha, \beta}$ with

$$\varphi_1 = \overline{\left(\chi + \sum_{j=0}^{k-1} \frac{\overline{\psi_j(0)}}{\|k_0^\beta\|^2} k_{0,j}^\alpha \right)} + \sum_{j=0}^{k-1} \left(\psi_j(z^k) - \frac{\psi_j(0)}{\|k_0^\beta\|^2} k_0^\beta(z^k) \right) j! \bar{z}^j.$$

However, in that case,

$$\begin{aligned} \varphi_1 - \varphi &= \sum_{j=0}^{k-1} \frac{\psi_j(0)}{\|k_0^\beta\|^2} \overline{k_{0,j}^\alpha} - \sum_{j=0}^{k-1} \frac{\psi_j(0)}{\|k_0^\beta\|^2} (1 - \overline{\beta(0)}\beta(z^k)) j! \bar{z}^j \\ &= \sum_{j=0}^{k-1} \frac{\psi_j(0)}{\|k_0^\beta\|^2} \overline{P_\alpha(j! z^j)} - \sum_{j=0}^{k-1} \frac{\psi_j(0)}{\|k_0^\beta\|^2} j! \bar{z}^j + \overline{\beta(0)}\beta(z^k) \sum_{j=0}^{k-1} \frac{\psi_j(0)}{\|k_0^\beta\|^2} j! \bar{z}^j \\ &= \overline{\left(P_\alpha \left(\sum_{j=0}^{k-1} \frac{\overline{\psi_j(0)}}{\|k_0^\beta\|^2} j! z^j \right) - \sum_{j=0}^{k-1} \frac{\overline{\psi_j(0)}}{\|k_0^\beta\|^2} j! z^j \right)} + \bar{z}^{k-1} W_k^* \beta \cdot \overline{\beta(0)} \sum_{j=0}^{k-1} \frac{\psi_j(0)}{\|k_0^\beta\|^2} j! z^{k-1-j} \\ &\in \overline{\alpha H^2} + \bar{z}^{k-1} (W_k^* \beta) H^2, \end{aligned}$$

and so, by Proposition 3, $U = U_{\varphi_1}^{\alpha, \beta} = U_\varphi^{\alpha, \beta}$. $\square$

Note that if $\psi_0(0) = \dots = \psi_{k-1}(0) = 0$, then the pairwise orthogonal functions $\psi_j(z^k) j! \bar{z}^j$, $0 \leq j \leq k-1$, all belong to zH^2 .

Corollary 5. *If $U \in \mathcal{S}_k(\alpha, \beta)$, $k \in \mathbb{N}$, then there exist functions $\chi \in K_\alpha$ and $\psi_0, \dots, \psi_{k-1} \in K_\beta$ such that $U = U_\varphi^{\alpha, \beta}$ with*

$$\varphi = \overline{\chi} + \psi_0(z^k) + \frac{1}{z} \psi_1(z^k) + \frac{1}{z^2} \psi_2(z^k) + \dots + \frac{1}{z^{k-1}} \psi_{k-1}(z^k)$$

and $\psi_0(0) = \dots = \psi_{k-1}(0) = 0$. Moreover, this decomposition is orthogonal.

Corollary 6. *Let U be a bounded linear operator from K_α into K_β . Then $U \in \mathcal{S}_k(\alpha, \beta)$, $k \in \mathbb{N}$, if and only if there exist functions $\chi \in K_\alpha$ and $\psi_0, \dots, \psi_{k-1} \in K_\beta$ such that*

$$U - S_\beta^* U S_\alpha^k = \widetilde{k}_0^\beta \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes \widetilde{k}_{0,j}^\alpha. \quad (3.6)$$

Proof. By Corollary 2, $U \in \mathcal{S}_k(\alpha, \beta)$ if and only if $C_\beta U C_\alpha \in \mathcal{S}_k(\alpha, \beta)$. By Theorem 1 the latter happens if and only if

$$C_\beta U C_\alpha - S_\beta (C_\beta U C_\alpha) (S_\alpha^*)^k = k_0^\beta \otimes \mu + \sum_{j=0}^{k-1} \nu_j \otimes k_{0,j}^\alpha \quad (3.7)$$

for some functions $\mu \in K_\alpha$ and $\nu_0, \dots, \nu_{k-1} \in K_\beta$. Since C_α is an involution, (3.7) is equivalent to

$$C_\beta^2 U C_\alpha^2 - C_\beta S_\beta C_\beta U C_\alpha (S_\alpha^*)^k C_\alpha = C_\beta (k_0^\beta \otimes \mu) C_\alpha + \sum_{j=0}^{k-1} C_\beta (\nu_j \otimes k_{0,j}^\alpha) C_\alpha.$$

By C_α -symmetry of the compressed shift, this can be written as

$$U - S_\beta^* U S_\alpha^k = \tilde{k}_0^\beta \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes \tilde{k}_{0,j}^\alpha$$

with $\chi = C_\alpha \mu \in K_\alpha$ and $\psi_j = C_\beta \nu_j \in K_\beta$ for $j = 0, 1, \dots, k-1$. $\square$

Corollary 7. *If a bounded linear operator $U : K_\alpha \rightarrow K_\beta$ satisfies (3.6), then*

$$U = U_\varphi^{\alpha,\beta} \quad \text{with } \varphi = \beta(z^k) \bar{z}^k \bar{\chi} + \bar{\alpha} \sum_{j=0}^{k-1} \psi_j(z^k) j! z^{j+1}.$$

Proof. Assume that U satisfies (3.6). A reasoning similar to that given in the proof of Corollary 6 shows that $C_\beta U C_\alpha$ satisfies (3.7) with $\mu = C_\alpha \chi$ and $\nu_j = C_\beta \psi_j$. By Corollary 4, $C_\beta U C_\alpha = U_\psi^{\alpha,\beta}$ with $\psi = \bar{\mu} + \sum_{j=0}^{k-1} \nu_j(z^k) j! \bar{z}^j$, and by Proposition 2, $U = C_\beta U_\psi^{\alpha,\beta} C_\alpha = U_\varphi^{\alpha,\beta}$ with

$$\begin{aligned} \varphi &= \overline{z^{k-1} \alpha \psi} W_k^* \beta = W_k^* \beta \left(\overline{z^{k-1} \alpha \mu} + \overline{z^{k-1} \alpha} \sum_{j=0}^{k-1} \overline{\nu_j(z^k) j! z^j} \right) \\ &= \beta(z^k) \left(\overline{z^{k-1} \alpha} \bar{\alpha} \bar{z}^k + \sum_{j=0}^{k-1} \overline{z^{k-1} \alpha} \overline{\beta(z^k) z^k \psi_j(z^k) j! z^j} \right) \\ &= \beta(z^k) \bar{z}^k \bar{\chi} + \sum_{j=0}^{k-1} \bar{\alpha} \psi_j(z^k) j! z^{j+1}. \quad \square \end{aligned}$$

Using Theorem 1 and Corollary 6, we can prove the following:

Corollary 8. *Let U be a bounded linear operator from K_α into K_β . Then $U \in \mathcal{S}_k(\alpha, \beta)$, $k \in \mathbb{N}$, if and only if there exist functions $\chi \in K_\alpha$ and $\psi_0, \dots, \psi_{k-1} \in K_\beta$ (possibly different for different conditions) such that one (and all) of the following conditions holds:*

- (a) $S_\beta^* U - U(S_\alpha^*)^k = \tilde{k}_0^\beta \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes \tilde{k}_{0,j}^\alpha$,
- (b) $S_\beta U - U S_\alpha^k = k_0^\beta \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes \tilde{k}_{0,j}^\alpha$.

Proof. Use reasoning analogous to that presented in the proof of [17, Cor. 2.4]. $\square$

Note that if $\dim K_\beta = 1$, then by Theorem 1 all bounded linear operators from K_α into K_β belong to $\mathcal{S}_k(\alpha, \beta)$ for each $k \in \mathbb{N}$.

Corollary 9. *Let α and β be two nonconstant inner functions and assume that $\dim K_\alpha = m < +\infty$. If $k \geq m$, then every bounded linear operator from K_α into K_β belongs to $\mathcal{S}_k(\alpha, \beta)$.*

Proof. Let U be a bounded linear operator from K_α into K_β . If $\dim K_\alpha = m < +\infty$, then $U - S_\beta U(S_\alpha^*)^k$ has rank at most m , which by assumption is less than or equal to k . Hence there exist functions $g_0, \dots, g_{k-1} \in K_\alpha$ and $f_0, \dots, f_{k-1} \in K_\beta$ (some of these functions possibly equal to zero) such that $U - S_\beta U(S_\alpha^*)^k = \sum_{j=0}^{k-1} f_j \otimes g_j$. Since here the kernels $k_{0,0}^\alpha, k_{0,1}^\alpha, \dots, k_{0,m-1}^\alpha$ are linearly independent and span K_α , each g_j can be written as a linear combination of these kernels, and so U satisfies (3.2). $\square$

Example 2. As in Example 1, consider $\alpha(z) = z^4$ and $\beta(z) = z^3$. Then, for $k = 5$ and $\varphi = \sum_{n=-\infty}^{\infty} a_n z^n \in L^2$, the operator $U_{\varphi}^{\alpha, \beta}$ is represented by the matrix

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_5 & a_4 & a_3 & a_2 \\ a_{10} & a_9 & a_8 & a_7 \end{bmatrix}.$$

It follows easily that every bounded linear operator from K_{α} into K_{β} belongs to $\mathcal{S}_5(\alpha, \beta)$. Note that here $U_z^{\alpha, \beta} = 0$, but

$$z \notin \overline{\alpha H^2} + \overline{z}^{k-1} (W_k^* \beta) H^2 = \overline{z^4 H^2} + z^{11} H^2.$$

Finally, we use Theorem 1 to describe some rank-one operators from $\mathcal{S}_k(\alpha, \beta)$.

Proposition 4. *Let α, β be two nonconstant inner functions, and let $k \in \mathbb{N}$. Then for each $l \in \{0, 1, \dots, k-1\}$, the rank-one operators $\tilde{k}_0^{\beta} \otimes k_{0,l}^{\alpha}$ and $k_0^{\beta} \otimes \tilde{k}_{0,l}^{\alpha}$ belong to $\mathcal{S}_k(\alpha, \beta)$.*

Proof. Let $l \in \{0, 1, \dots, k-1\}$. Since $S_{\beta} \tilde{k}_0^{\beta} = P_{\beta}(\beta - \beta(0)) = -\beta(0)P_{\beta}1 = -\beta(0)k_0^{\beta}$ (see [29, Lemma 2.2(a)]), we have

$$\begin{aligned} \tilde{k}_0^{\beta} \otimes k_{0,l}^{\alpha} - S_{\beta}(\tilde{k}_0^{\beta} \otimes k_{0,l}^{\alpha})(S_{\alpha}^*)^k &= \tilde{k}_0^{\beta} \otimes k_{0,l}^{\alpha} - (S_{\beta} \tilde{k}_0^{\beta}) \otimes (S_{\alpha}^k k_{0,l}^{\alpha}) \\ &= \tilde{k}_0^{\beta} \otimes k_{0,l}^{\alpha} + (\beta(0)k_0^{\beta}) \otimes (S_{\alpha}^k k_{0,l}^{\alpha}) \\ &= k_0^{\beta} \otimes (\overline{\beta(0)} S_{\alpha}^k k_{0,l}^{\alpha}) + \tilde{k}_0^{\beta} \otimes k_{0,l}^{\alpha}, \end{aligned}$$

and by Theorem 1, $\tilde{k}_0^{\beta} \otimes k_{0,l}^{\alpha} \in \mathcal{S}_k(\alpha, \beta)$ since it satisfies (3.2) with $\chi = \overline{\beta(0)}(S_{\alpha}^k k_{0,l}^{\alpha})$, $\psi_l = \tilde{k}_0^{\beta}$ and $\psi_j = 0$ for $j \neq l$. By Corollary 2 it follows that

$$k_0^{\beta} \otimes \tilde{k}_{0,l}^{\alpha} = C_{\beta}(\tilde{k}_0^{\beta} \otimes k_{0,l}^{\alpha})C_{\alpha} \in \mathcal{S}_k(\alpha, \beta). \quad \square$$

Corollary 10. *For each $l \in \{0, 1, \dots, k-1\}$,*

- (a) $\tilde{k}_0^{\beta} \otimes k_{0,l}^{\alpha} = U_{\varphi}^{\alpha, \beta}$ with $\varphi = W_k^* \beta \cdot l! \overline{z}^{l+k}$,
- (b) $k_0^{\beta} \otimes \tilde{k}_{0,l}^{\alpha} = U_{\psi}^{\alpha, \beta}$ with $\psi = \overline{\alpha} \cdot l! z^{l+1}$.

Proof. By Corollary 4 and the proof of Proposition 4, $\tilde{k}_0^{\beta} \otimes k_{0,l}^{\alpha} = U_{\varphi_1}^{\alpha, \beta}$ with

$$\varphi_1 = \beta(0) \overline{(S_{\alpha}^k k_{0,l}^{\alpha})} + \tilde{k}_0^{\beta}(z^k) l! \overline{z}^l.$$

Since for each $g \in K_{\alpha}$,

$$\langle S_{\alpha}^k k_{0,l}^{\alpha}, g \rangle = \langle P_{\alpha}(l! z^l), (S_{\alpha}^*)^k g \rangle = \langle l! z^l, T_{\overline{z}^k} g \rangle = \langle l! z^{l+k}, g \rangle,$$

we have $S_{\alpha}^k k_{0,l}^{\alpha} = P_{\alpha}(l! z^{l+k})$ and

$$\varphi_1 = \beta(0) \overline{P_{\alpha}(l! z^{l+k})} + (\beta(z^k) - \beta(0)) l! \overline{z}^{l+k} = W_k^* \beta \cdot l! \overline{z}^{l+k} - \beta(0) \overline{(l! z^{l+k} - P_{\alpha}(l! z^{l+k}))}.$$

Since $\varphi_1 - \varphi \in \overline{\alpha H^2}$, we get $\tilde{k}_0^{\beta} \otimes k_{0,l}^{\alpha} = U_{\varphi_1}^{\alpha, \beta} = U_{\varphi}^{\alpha, \beta}$, that is, (a) holds.

Now (b) follows from Proposition 2. $\square$

4 Operators from $\mathcal{S}_k(\alpha, \beta)$ equal to the zero operator, part 2

Let α, β be two nonconstant inner functions. In this section, we completely determine which $\varphi \in L^2$ give $U_\varphi^{\alpha, \beta} = 0$.

Note first that from Lemma 3 we can obtain the following corollary.

Corollary 11. *Let $k \in \mathbb{N}$ and $\varphi \in \overline{H^2}$. Then $U_\varphi^{\alpha, \beta} = 0$ if and only if $\varphi \in \overline{\alpha H^2}$.*

Proof. If $\varphi \in \overline{\alpha H^2}$, then $U_\varphi^{\alpha, \beta} = 0$ by Proposition 1.

On the other hand, if $U_\varphi^{\alpha, \beta} = 0$ for $\varphi \in \overline{H^2}$, then by Lemma 3

$$0 = U_\varphi^{\alpha, \beta} - S_\beta U_\varphi^{\alpha, \beta} (S_\alpha^*)^k = k_0^\beta \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes k_{0,j}^\alpha,$$

where $\chi = P_\alpha(\overline{\varphi})$ and $\psi_j = (1/j!) S_\beta P_\beta W_k(\varphi \overline{z}^{k-j})$, $0 \leq j \leq k-1$. Since here $\varphi \cdot \overline{z}^{k-j} \in \overline{\alpha H^2}$ for each $0 \leq j \leq k-1$,

$$P_\beta W_k(\varphi \overline{z}^{k-j}) = P_\beta P W_k(\varphi \overline{z}^{k-j}) = P_\beta W_k P(\varphi \overline{z}^{k-j}) = 0,$$

and so we have $\psi_0 = \psi_1 = \dots = \psi_{k-1} = 0$. Hence $\chi = P_\alpha(\overline{\varphi}) = 0$, which means that $\overline{\varphi} \in \alpha H^2$, that is, $\varphi \in \overline{\alpha H^2}$. $\square$

Let us now consider analytic symbols. Observe that for a fixed $k \in \mathbb{N}$, each $\varphi \in H^2$ can be written as

$$\varphi = W_k^* W_k \varphi + \overline{z} W_k^* W_k (z \varphi) + \dots + \overline{z}^{k-1} W_k^* W_k (z^{k-1} \varphi) = \sum_{j=0}^{k-1} \overline{z}^j W_k^* W_k (z^j \varphi).$$

Indeed, if $\varphi = \sum_{n=0}^{\infty} a_n z^n$, then for each $j \in \{0, 1, \dots, k-1\}$,

$$W_k(z^j \varphi) = W_k \left(\sum_{n=0}^{\infty} a_n z^{n+j} \right) = \begin{cases} a_0 + \sum_{m=1}^{\infty} a_{mk} z^m & \text{for } j = 0, \\ \sum_{m=1}^{\infty} a_{mk-j} z^m & \text{for } j > 0. \end{cases}$$

Hence, for each $0 < j \leq k-1$,

$$\overline{z}^j W_k^* W_k (z^j \varphi) = \overline{z}^j W_k^* \left(\sum_{m=1}^{\infty} a_{mk-j} z^m \right) = \sum_{m=1}^{\infty} a_{mk-j} \overline{z}^j z^m$$

and

$$W_k^* W_k(\varphi) = W_k^* \left(a_0 + \sum_{m=1}^{\infty} a_{mk} z^m \right) = a_0 + \sum_{m=1}^{\infty} a_{mk} \overline{z}^k z^m.$$

Thus

$$\begin{aligned} \varphi &= \sum_{n=0}^{\infty} a_n z^n = a_0 + \sum_{m=1}^{\infty} \sum_{j=0}^{k-1} a_{mk-j} \overline{z}^j z^m = a_0 + \sum_{m=1}^{\infty} a_{mk} \overline{z}^k z^m + \sum_{j=1}^{k-1} \sum_{m=1}^{\infty} a_{mk-j} \overline{z}^j z^m \\ &= W_k^* W_k(\varphi) + \sum_{j=1}^{k-1} \overline{z}^j W_k^* W_k (z^j \varphi). \end{aligned}$$

Hence, if $\varphi \in H^2$, then

$$\varphi = W_k^* \varphi_0 + \bar{z} W_k^* \varphi_1 + \cdots + \bar{z}^{k-1} W_k^* \varphi_{k-1} \quad (4.1)$$

with $\varphi_j = W_k(z^j \varphi) \in H^2$. Note that $\varphi_j \in zH^2$ for $j > 0$ and the above decomposition is orthogonal. Moreover, for arbitrary $\varphi_0 \in H^2$ and $\varphi_1, \dots, \varphi_{k-1} \in zH^2$, the function φ given by (4.1) belongs to H^2 with the decomposition again orthogonal. It follows that

$$H^2 = W_k^* H^2 \oplus \bar{z} W_k^* (zH^2) \oplus \cdots \oplus \bar{z}^{k-1} W_k^* (zH^2)$$

and

$$zH^2 = W_k^* (zH^2) \oplus \bar{z} W_k^* (zH^2) \oplus \cdots \oplus \bar{z}^{k-1} W_k^* (zH^2).$$

As $W_k^*(zH^2) = z^k W_k^* H^2$, we also have

$$H^2 = W_k^* H^2 \oplus z W_k^* H^2 \oplus \cdots \oplus z^{k-1} W_k^* H^2, \quad (4.2)$$

and so

$$zH^2 = z W_k^* H^2 \oplus z^2 W_k^* H^2 \oplus \cdots \oplus z^k W_k^* H^2. \quad (4.3)$$

Now for any inner function β , by (4.2) we have

$$(W_k^* \beta) \cdot H^2 = W_k^* (\beta H^2) \oplus z W_k^* (\beta H^2) \oplus \cdots \oplus z^{k-1} W_k^* (\beta H^2).$$

Since W_k^* is an isometry, we obtain

$$K_{W_k^* \beta} = W_k^* (K_\beta) \oplus z W_k^* (K_\beta) \oplus \cdots \oplus z^{k-1} W_k^* (K_\beta). \quad (4.4)$$

We are now ready to prove the following:

Theorem 2. Let α, β be two nonconstant inner functions, let $k \in \mathbb{N}$, and assume that $\dim K_\alpha \geq k$. Moreover, let $\varphi \in L^2$. Then $U_\varphi^{\alpha, \beta} = 0$ if and only if

$$\varphi \in \overline{\alpha H^2} + \bar{z}^{k-1} (W_k^* \beta) H^2.$$

Proof. If $\varphi \in \overline{\alpha H^2} + \bar{z}^{k-1} (W_k^* \beta) H^2$, then $U_\varphi^{\alpha, \beta} = 0$ by Proposition 3.

Assume that $\varphi \in L^2$ and $U_\varphi^{\alpha, \beta} = 0$. Since

$$L^2 = \overline{H^2} \oplus zH^2 = \overline{K_\alpha} \oplus \overline{\alpha H^2} \oplus (W_k^* \beta) zH^2 \oplus zK_{W_k^* \beta},$$

we can write

$$\varphi = \overline{\varphi_1} + \overline{\alpha h_1} + (W_k^* \beta) \cdot z h_2 + z \varphi_2, \quad (4.5)$$

where $\varphi_1 \in K_\alpha$, $\varphi_2 \in K_{W_k^* \beta}$ and $h_1, h_2 \in H^2$. Since

$$(W_k^* \beta) \cdot z h_2 = \bar{z}^{k-1} (W_k^* \beta) \cdot z^k h_2 \in \bar{z}^{k-1} (W_k^* \beta) H^2,$$

by the first part of the proof (or by Proposition 3) we get $0 = U_{\varphi}^{\alpha, \beta} = U_{\overline{\varphi}_1 + z\varphi_2}^{\alpha, \beta}$. It follows from Lemma 3 that

$$0 = U_{\overline{\varphi}_1 + z\varphi_2}^{\alpha, \beta} - S_{\beta} U_{\overline{\varphi}_1 + z\varphi_2}^{\alpha, \beta} (S_{\alpha}^*)^k = k_0^{\beta} \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes k_{0,j}^{\alpha}$$

with

$$\chi = P_{\alpha}(\overline{\overline{\varphi}_1 + z\varphi_2}) = P_{\alpha}\varphi_1 = \varphi_1$$

and

$$\psi_j = \frac{1}{j!} S_{\beta} P_{\beta} W_k(\varphi \overline{z}^{k-j}) = \frac{1}{j!} S_{\beta} P_{\beta} W_k(\overline{\varphi}_1 \overline{z}^{k-j} + \overline{z}^{k-j-1} \varphi_2) = \frac{1}{j!} S_{\beta} P_{\beta} W_k(\overline{z}^{k-j-1} \varphi_2)$$

for $0 \leq j \leq k-1$. The last equality follows from the fact that

$$P_{\beta} W_k(\overline{\varphi}_1 \overline{z}^{k-j}) = W_k P_{W_k^* \beta}(\overline{\varphi}_1 \overline{z}^{k-j}) = 0.$$

Now by (4.4) we have

$$\varphi_2 = W_k^* f_1 + z W_k^* f_2 + \cdots + z^{k-1} W_k^* f_k$$

for some $f_1, f_2, \dots, f_k \in K_{\beta}$, and using Lemma 4, we get

$$\begin{aligned} \psi_j &= \frac{1}{j!} S_{\beta} P_{\beta} W_k \left(z^{j-k+1} \sum_{l=1}^k z^{l-1} W_k^* f_l \right) = \frac{1}{j!} \sum_{l=1}^k S_{\beta} P_{\beta} W_k M_{z^{j-2k+l}} W_k^* (z f_l) \\ &= \frac{1}{j!} S_{\beta} P_{\beta} W_k M_{z^{-k}} W_k^* (z f_{k-j}) = \frac{1}{j!} S_{\beta} P_{\beta} (\overline{z} z f_{k-j}) = \frac{1}{j!} S_{\beta} (f_{k-j}) \end{aligned}$$

($|j+l-k| \leq k-1$). Hence

$$k_0^{\beta} \otimes \varphi_1 + \sum_{j=0}^{k-1} \frac{1}{j!} S_{\beta} (f_{k-j}) \otimes k_{0,j}^{\alpha} = 0. \quad (4.6)$$

Since $\dim K_{\alpha} \geq k$, the functions $k_{0,j}^{\alpha}$, $j = 0, \dots, k-1$, are linearly independent, which by (4.6) means that there exists constants $c_0, \dots, c_{k-1} \in \mathbb{C}$ such that

$$S_{\beta} (f_{k-j}) = c_j \cdot k_0^{\beta} \quad \text{for each } 0 \leq j \leq k-1 \quad (4.7)$$

and

$$\varphi_1 = - \sum_{j=0}^{k-1} \frac{1}{j!} \overline{c}_j \cdot k_{0,j}^{\alpha} = - \sum_{j=0}^{k-1} \overline{c}_j \cdot P_{\alpha}(z^j). \quad (4.8)$$

Condition (4.7) means that for each $j \in \{0, \dots, k-1\}$,

$$0 = P_{\beta}(z f_{k-j}) - c_j \cdot P_{\beta} 1 = P_{\beta}(z f_{k-j} - c_j) \quad \text{and} \quad z f_{k-j} - c_j = \beta \cdot d_j$$

with $d_j \in H^2$. Moreover, this equality can be written as

$$\overline{C_{\beta} f_{k-j}} - c_j \overline{\beta} = \overline{\beta} z f_{k-j} - c_j \overline{\beta} = d_j,$$

which implies that d_j is also constant. Thus

$$f_l = \frac{d_{k-l} \cdot \beta + c_{k-l}}{z}, \quad 1 \leq l \leq k,$$

and

$$\begin{aligned} z\varphi_2 &= \sum_{l=1}^k z^l \cdot W_k^* f_l = \sum_{l=1}^k z^l \cdot W_k^* (d_{k-l} \bar{z}\beta + c_{k-l} \bar{z}) = (W_k^* \beta) \cdot \sum_{l=1}^k d_{k-l} z^l \cdot \bar{z}^k + \sum_{l=1}^k c_{k-l} \bar{z}^{k-l} \\ &= \bar{z}^{k-1} (W_k^* \beta) \cdot \sum_{l=1}^k d_{k-l} z^{l-1} + \sum_{j=0}^{k-1} c_j \bar{z}^j. \end{aligned}$$

By (4.5), (4.8), and the fact that $P_\alpha(z^j) = z^j - \alpha P(\bar{\alpha} z^j)$ we finally get

$$\begin{aligned} \varphi &= \overline{\alpha h_1} + (W_k^* \beta) \cdot z h_2 + \bar{\varphi}_1 + z\varphi_2 \\ &= \overline{\alpha h_1} + (W_k^* \beta) \cdot z h_2 + \overline{\sum_{j=0}^{k-1} \bar{c}_j (\alpha P(\bar{\alpha} z^j) - z^j)} + \bar{z}^{k-1} (W_k^* \beta) \cdot \sum_{l=1}^k d_{k-l} z^{l-1} + \sum_{j=0}^{k-1} c_j \bar{z}^j \\ &= \overline{\alpha \left(h_1 + \sum_{j=0}^{k-1} \bar{c}_j P(\bar{\alpha} z^j) \right)} - \sum_{j=0}^{k-1} c_j \bar{z}^j + \bar{z}^{k-1} (W_k^* \beta) \left(z^k h_2 + \sum_{l=1}^k d_{k-l} z^{l-1} \right) + \sum_{j=0}^{k-1} c_j \bar{z}^j \\ &= \overline{\alpha \left(h_1 + \sum_{j=0}^{k-1} \bar{c}_j P(\bar{\alpha} z^j) \right)} + \bar{z}^{k-1} (W_k^* \beta) \left(z^k h_2 + \sum_{l=1}^k d_{k-l} z^{l-1} \right), \end{aligned}$$

and so $\varphi \in \overline{\alpha H^2} + \bar{z}^{k-1} (W_k^* \beta) H^2$. $\square$

Remark.

(a) Note that if $\beta(0) = 0$, then

$$\bar{z}^{k-1} (W_k^* \beta) H^2 = W_k^* \left(\frac{\beta}{z} \right) z H^2 \subset z H^2.$$

On the other hand, if $\beta(0) \neq 0$, then

$$\bar{z}^{k-1} (W_k^* \beta) H^2 = \bar{z}^{k-1} (W_k^* \beta) (K_{z^k} \oplus z^k H^2) = ((W_k^* \beta) \overline{K_{z^k}}) \oplus ((W_k^* \beta) z H^2).$$

(b) For $k = 1$, we obtain that $U_\varphi^{\alpha, \beta} = A_\varphi^{\alpha, \beta} = 0$ if and only if

$$\varphi \in \overline{\alpha H^2} + \bar{z}^{k-1} (W_k^* \beta) H^2 = \overline{\alpha H^2} + \beta H^2,$$

which is the known result for asymmetric truncated Toeplitz operators [22].

Corollary 12. *Let α, β be two nonconstant inner functions, let $k \in \mathbb{N}$, and assume that $\dim K_\alpha \geq k$, $\beta(0) = 0$. If $\varphi \in z H^2$, then $U_\varphi^{\alpha, \beta} = 0$ if and only if*

$$\varphi \in \bar{z}^{k-1} (W_k^* \beta) H^2 = W_k^* \left(\frac{\beta}{z} \right) z H^2.$$

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